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## **STATISTICAL RELIABILITY ENGINEERING**

### **PART 1: Unit Reliability Estimation**

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**PART 1:**  
**Unit Reliability Estimation**

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**2. Main Knowledge of Statistics**

**2.1 Introduction**

In reliability one deals with objects of various complexity. It is usual to speak about *systems* and *units*. We begin with consideration of a unit. We will call a unit an indivisible object in the frame of current reliability analysis. So, systems consist of units. Of course in engineering sense, a system can be considered as a “unit” if it is taken as a whole, that is, there is no need to consider its structure and its constituent parts.

Statistical reliability analysis of a unit predominantly consists of standard statistical inferences. Since this material is covered by many excellent books on statistics, we will present



only a brief review of these statistical methods as an introductory knowledge. The main goal of this book is statistical analysis of system reliability which will be given in subsequent chapters.

### 2.1.1 Reliability Indices

Engineering practice of reliability characterization requires indices which are probabilistic in nature. We deal with random variables (e.g., time to failure or repair time) and their characterization. Detailed analysis of these indices is given in the first volume, "Probabilistic Reliability Engineering" (PRE). We only describe only main indices supplying them with brief explanations here.

Any object fails after random period of operation. We call this period *time to failure* (TTF) and denote this non-negative random variable (r.v.) by  $\square$ . A complete mathematical description of TTF is given by TTF *distribution function* (d.f.):

$$F(t) = P\{\square \leq t\}.$$

Continuous r.v. can be also characterized by the *density function*

$$f(t) = \frac{d}{dt} F(t).$$

**Ошибка! Не указан аргумент ключа.**

In particular, if an analyzed random variable is time to failure, we will also call this function a *failure density*. From the definition of the density function, it follows that

$$F(t) = \int_0^t f(t)dt.$$

In reliability, one frequently uses the so-called *reliability function*, or *survival function*, defined as

$$R(t) = 1 - F(t) = P\{\square > t\}.$$

A quantile,  $t_p$ , of level  $p$  of a continuous distribution  $F(t)$  is defined as the solution of equation  $F(t_p) = p$ . A quantile of the level  $p$  shows that  $(1-p) \square 100\%$  of objects are expected to survive during time  $t_p$ .

One of the main reliability indices for unrepairable objects is the *mean time to failure* (MTTF),  $T$ , which is defined as

$$T = \int_0^{\infty} tf(t)dt.$$

As it was shown in *PRE*, the MTTF can be expressed in another (equivalent) form

$$T = \int_0^{\infty} R(t) dt.$$

For repairable objects, one also introduces the *mean time between failures* (MTBF) which is similar to MTTF. MTBF is the mathematical expectation of *time between failures* (TBF). The difference between MTTF and MTBF is explained in *PRE* with detail. We notice that in many engineering probabilistic models repair is considered "ideal", i.e., repaired item is assumed to be identical with a new one.

The *mean repair time*,  $\tau$ , is defined via the distribution of random repair time in a similar way. In practice the mean repair time is usually not derived on the basis of tests but rather based on an expert evaluation.

In addition to the indices mentioned above, repairable objects are characterized by either the so-called *availability coefficient*, or *operational availability coefficient*. The first index equals probability that a repairable object will be in the operational state at specified moment of time. The second one equals probability that an object will be operational at some moment and will have been operating without failures during a specified interval of time. (For details, see Volume 1.) Here we will deal only with the so-called *stationary availability coefficient*, i.e., the probability that an object will be operational at "the time moment in the far future". This index can be determined as

$$K = \frac{T}{T + \tau}.$$

In reliability engineering one often refers to the so-called *failure rate*, or *hazard rate* formally defined as

$$\lambda(t) = \frac{f(t)}{R(t)}.$$

In other words, the failure rate at an instant is the conditional density of TTF if it has survived up to moment  $t$ . For more explanations, note that the "element of probability" that the object, survived up to  $t$ , will have failed before moment  $t + \Delta t$  is  $\lambda(t) \Delta t$ .

The dependence of the failure rate on time is a helpful qualitative characterization of life distributions. The *increasing failure rate (IFR)* and *increasing failure rate average (IFRA)*, introduced by Barlow and Proschan, related to a wide class of distributions important in practical applications. Such distributions characterize "aging" objects whose reliability properties worsen in time. The *decreasing failure rate (DFR)* and *decreasing failure rate average (DFRA)* relates to the so-called "younging" objects whose reliability properties improves with time going. Such a phenomenon takes place in burn-in testing and in some specific situations with hardening.

Numerical values of all these reliability indices above can be experimentally checked by special testing or from the analysis of field data. Although reliability indices for units and systems are similar by their sense, statistical methods for their estimation might be different. In particular, a

very special statistical reliability problem is a system indices estimation on the basis of testing its units.

This chapter is narrowed by a brief review of statistics methods which will be necessary and/or useful for understanding of the further material.

### 2.1.2 Main Tasks of Mathematical Statistics

If you perform reliability tests of an object or observe its utilization, you might collect some statistical data and use them for characterization of the object's reliability. These collected data are realizations of some r.v.'s  $X_1, X_2, \dots, X_n$ .

Such data might represent TTF, repair time, number of cycles before failure, number of spare units used for repair and preventive maintenance, and so on. These values can be continuous or discrete depending on their nature. They are usually used for obtaining the sample mean or sample variance. The same data can be transformed into order statistics and be useful for plotting histograms and/or empirical distributions.

Another problem arises when you intend to construct confidence intervals for parameters of the distribution. You should usually have some prior information about the r.v. and possess special mathematical methods of statistical inferences.

Observing random events, you collect the number of outcomes of different types and the total number of trials. In probability theory one usually calls events as "success" and "failure". In this terms, the following so-called indicator,  $\delta_k$ , of the  $k$ th event can be introduced:  $\delta_k=1$  if success has been observed, and  $\delta_k=0$ , otherwise. After introducing the indicator, we formally can consider it as a discrete r.v. taking two meanings: 0 or 1.

Let us illustrate the role of statistics with an example of a classical Bernoulli trials. Remember that this is a series of  $n$  independent, identical trials, each of which might be a "success" with probability  $p$  or "failure" with probability  $1-p$ . In this case we can prescribe 1 to indicate success, and 0 to failure. This model is completely defined by the value of parameter  $p$  for a trial. Examples of such a situation are often met in practice. Under some specified conditions, mass production of some items is characterized by an almost stable percentage of items with fixed quality. A group of practically homogeneous objects, tested in similar conditions, is characterized by some stable frequency of successful operation. Of course, a real life differs from mathematical models: a sequence of Bernoulli trials is only an approximation for the description of these practical schemes.

If real value  $p$  is unknown but if there are experimental data, we can use methods of mathematical statistics to find various probabilistic characteristics. For instance, such expected number of successes or the probability of  $m$  successes in  $n$  trials. In this particular case of Bernoulli trials, the experimental data is the observed number of successes,  $m$ , in a series of  $n$  independent trials. On the basis of this data we need to make a conclusion about the value of unknown parameter  $p$ , which is the unknown probability of success.

In mathematical statistics following main problems are studied:

- *Point estimation of an unknown parameter.* We wish to find a function of the experimental results ( $m$  successes in  $n$  trials), which allows us to obtain a "good"

estimate of unknown parameter  $p$ . A standard estimator of the probability is the observed frequency of success:  $\hat{p} = \frac{m}{n}$ .<sup>2</sup>

- *Interval estimation of an unknown parameter.* In this case, we need to construct an interval  $[\underline{p}, \bar{p}]$ <sup>3</sup> which will cover an unknown real value of parameter  $p$  with the specified probability

$$P\{\underline{p} \leq p \leq \bar{p}\} \geq \gamma \quad (1.1)$$

where  $\gamma$  is the so-called confidence coefficient which is usually chosen close to 1.

We should emphasize that the limits of confidence interval are random because they are functions of random variables:  $\underline{p} = \underline{p}(m)$ ,  $\bar{p} = \bar{p}(m)$ .<sup>4</sup> Moreover, the confidence interval covers unknown parameter  $p$  but gives no information about its "real" position within this interval. Expression (1.1) says that in  $\gamma \cdot 100\%$  cases this confidence interval will cover unknown parameter and in  $(1-\gamma) \cdot 100\%$  cases the parameter will lie outside these limits.

- *Test of hypothesis.* One needs to check some hypothesis, for instance, that an unknown value of parameter  $p$  satisfies inequality  $p \leq p_0$  or equality  $p = p_0$  (or other conditions) where  $p_0$  given. These statements are also made with some guaranteed probabilities. The specific of test of hypothesis will be considered later.

These types of statistical inferences comprise the main body of applied statistics.

### 2.1.3 Sample

Probability theory deals with d.f. based either on measure theory concepts (following Kolmogorov), or on conception of frequency of event occurrence with potentially arbitrary number of observation (following von Mises). In statistics, one always has a *sample* of a finite size, say,  $n$ . Usually, we say about a sample from a distribution. It means that we deal with  $n$  *independent and identically distributed* (i.i.d.) r.v.'s:

$$X_1, X_2, \dots, X_n. \quad (1.2)$$

The problems in mathematical statistics is to make some inferences about the distribution to which extracted  $X_k$ 's belongs.

We extract a sample from a finite homogeneous population which is characterized by some probabilistic properties. In statistics this group of objects is called a *general population*. Usually in practice (in sociology, econometrics, medicine, telecommunications), the size of a general population,  $N$ , is assumed to be large. One takes a random sample of size  $n$  from a general population and makes a conclusion about a general population as a whole.

A sample from a general population can be taken in two main ways: with or without replacement. If sampling is performed with replacement, then the general population remains without changes at each extraction of a new item. In this case observations of r.v.'s are assumed to

be independent. If an extracted item has not been returned to the general population, the latter can change its probabilistic properties. (The general population is assumed finite in this case.) Samples without replacement are frequently used in special problems of quality control, especially in cases where tests are destructive.

For sampling without replacement, each current trial depends on the results of all previous trials. For instance, let us have a general population of 100 items among which there are exactly 3 failed items. We make a sample of size 5. Let the first pick be a failed unit. Then at the second step the probability to choose in random a failed item equals  $2/99$ . But if at the first step we have picked up a good item, then at the second step the probability to choose in random a failed item equals  $3/99$ . Further, if at the first three steps we have picked up 3 failed items, the probability to choose a failed item at any next step equals 0.

If the size of a general population is very large (but finite), there is practically no difference between these two types of sampling.

Further, we will almost exclusively consider samples from distributions, i.e., from infinite general populations.

If sample (1.2) is placed in ascending order as

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n-1)} \leq X_{(n)}$$

then  $X_{(k)}$  is called *kth order statistic*.

**Example 1.1** Five independent measurements of TTF gives the following records (in hours):  $X_1=104, X_2=95, X_3=93, X_4=101, X_5=107$ . Then the order statistics are  $X_{(1)}=93, X_{(2)}=95, X_{(3)}=101, X_{(4)}=104, X_{(5)}=107$ . □

The function

$$\hat{F}_n(x) = \frac{r(x)}{n}$$

based on a sample (1.2) is called an *empirical distribution function* (here  $r(x)$  is a number of observations  $X_i$  which are smaller than  $x$ ). An empirical d.f. can be also written with the help of the order statistic as

$$\hat{F} = \begin{cases} 0 & \text{if } x < X_1 \\ \frac{j}{n} & \text{if } X_{(j)} \leq x < X_{(j+1)}, j = 1, \dots, n \\ 1 & \text{if } X_{(n)} \leq x. \end{cases}$$

**Example 1.2** Construct the empirical d.f. for the data given in Example 1.1. The result is depicted in Figure 1.1.

.....Figure 1.1.□

Let  $F(x)$  be a true (or theoretical) d.f. Then in accordance with the Glivenko-Cantelli theorem, well-known in probability theory, the following condition:

$$\sup_x |\hat{F}_n(x) - F(x)| \rightarrow 0$$

holds with probability 1 for  $n \rightarrow \infty$ . That means that an empirical d.f. stochastically converges to its theoretical one when the sample size infinitely increases. This fact could be explained with the following simple arguments. Each value of an empirical d.f. is a frequency of the event: "a r.v. is smaller than a corresponding fixed value". The frequency converges (in probabilistic sense) to the probability with increasing the number of trials, so does the set of such frequencies (i.e., the empirical d.f.).

The following *empirical*, or *sample characteristics* can be constructed on the basis of sample data. The value of

$$\bar{X} = \frac{1}{n} \sum_{1 \leq i \leq n} X_i$$

is called the *empirical* (or *sample*) *mean*. The value of

$$s^2 = \frac{1}{n} \sum_{1 \leq i \leq n} (X_i - \bar{X})^2$$

is called the *empirical* (or *sample*) *variance*. If the sample mean indicates a "location" of the distribution, the sample variance characterizes the sample dispersion around its mean. Note that the mean is analogous to the center of mass in mechanics, and the variance to the moment of inertia. The value of  $\hat{\sigma} = \sqrt{s^2}$  is called the *empirical* (or *sample*) *standard deviation*. In some sense, the standard deviation is more convenient for sample characterization because it has the same dimension as the mean (and, consequently, the observed r.v.).

Another characteristic describing the spread of the sample is a *range of the sample* which is defined as  $R_n = X_{(n)} - X_{(1)}$ . The deficiency of this value is in its main property: with the sample size increasing this value is increasing monotone. (In principle, for distributions with an infinite area of domain, the range increases to infinity.)

As well one uses *empirical* (or *sample*) *moments*  $\hat{\mu}_r$  and *empirical central moments*  $\hat{\nu}_r$  of the  $r$ th order which are calculated by formulas

$$\hat{\mu}_r = \frac{1}{n} \sum_{1 \leq i \leq n} X_i^r,$$

$$\hat{\nu}_r = \frac{1}{n} \sum_{1 \leq i \leq n} (X_i - \bar{X})^r.$$

Obviously, the sample mean and the sample variance are particular cases of these  $\bar{X} = \hat{\mu}_1$ ,  $s^2 = \hat{\nu}_2$ .

**Example 1.3** Find the sample mean, variance, standard deviation, and range for the data represented in Example 1.1. In this particular case

$$\bar{X} = (1/5)(93+95+101+104+107) = 100;$$

$$S^2=(1/5)(7^2+5^2+1^2+4^2+7^2)=28;$$

$$\hat{\sigma} = S = \sqrt{28} \approx 5.3, 10$$

$$R_5=107-93=14. \square$$

**Remark 1.1** If value  $X_i$  repeats in the sample  $n_i$  times,  $i=1, \dots, m$ , and  $n=n_1+n_2+ \dots +n_m$  where  $m$  is the number of different  $X_i$ 's, then the formulas for the sample mean and variance can be rewritten in the form

$$\bar{X} = \frac{1}{n} \sum_{1 \leq i \leq m} n_i X_i;$$

$$s^2 = \frac{1}{n} \sum_{1 \leq i \leq m} n_i (X_i - \bar{X})^2.$$

**Example 1.4** Find the sample mean and variance of the following 100 observations:

$X_i$	1250	1270	1280	1290
$n_i$	20	25	50	5

In this case one has  $m=4$  different types of sample's values. In this case

$$\bar{X} = \frac{1}{100} [20 \cdot 1250 + 25 \cdot 1270 + 50 \cdot 1280 + 5 \cdot 1290] = 1272;$$

$$s^2 = \frac{1}{100} [20 \cdot 22^2 + 25 \cdot 2^2 + 50 \cdot 8^2 + 5 \cdot 18^2] = 125;$$

$$\hat{\sigma} = S = \sqrt{125} \approx 11.18. \blacksquare$$

## 2.2 Main Distributions

We consider here only main probability distributions used in reliability statistical problems. More detailed description of distributions, as well as interrelations between various distributions are in *PRE*.

## 2.2.1 Continuous Distributions

### Normal Distribution

A normal distribution occupies a special place in probability theory and its applications. This distribution is often called Gaussian because the great German mathematician Karl Friedrich Gauss studied its main properties and widely applied it in practice.

A number of phenomena in nature, engineering and science can be modeled with the help of this distribution. A normally distributed random value appears where a large number of independent factors influence on a considered parameter.

In probability theory there is the Central limit theorem known in several forms. All of them state the following general fact: the sum of independent r.v.'s (usually -- although not necessarily -- assumed to be identically distributed) has the asymptotically normal distribution. One of the formulations of the Central limit theorem in the Lindeberg's form states that the distribution of the sample mean asymptotically converges to the normal distribution.

The density of the normal d.f. and its main characteristics are presented in Table 1.1.

For practical use, one applies the so-called standard normal distribution. This distribution has the mean equal to 0 and variance equal to 1, i.e., its density is

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

It is clear that a general normally distributed r.v., say,  $\xi$ , is subjected to two transformations to be presented in form of a standard normally distributed r.v.  $\eta$ . First, this r.v. must be centered, that is, represented as  $\xi - \mu$ , and, second, it must be normalized, that is, its scale must be changed in accordance with the standard deviation:  $\xi = \frac{\eta - \mu}{\sigma}$ .<sup>13</sup> Thus, any normally distributed r.v. can be easily transform to the standard form.

The cumulative standard normal distribution (sometimes called the Laplace function) has form

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt. \quad (1.3)$$

There are detailed numerical tables for the standard normal distribution (one of which is given in the end of the book).

This distribution is symmetrical around  $x=0$ , and its domain is  $(-\infty, +\infty)$ . Due to symmetry of the distribution, the quantile of the level  $q$  of the standard normal distribution satisfies equation  $U_{1-q} = -U_q$ , and the cumulative d.f. satisfies equation  $\Phi(x) = 1 - \Phi(-x)$ .

### The Sum of Normally Distributed Random Variables

Let  $\xi_1, \xi_2, \dots, \xi_n$  be normally distributed r.v.'s and parameters  $\mu_i$  and  $\sigma_i$  are known for each  $\xi_i, i=1, 2, \dots, n$ . Then the sum  $\xi_1 + \xi_2 + \dots + \xi_n$  also has a normal distribution with the mean  $\mu = \mu_1 + \mu_2 + \dots + \mu_n$  and variance  $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$ .

This property of the normal distribution is true for any number of summands,  $n$ .



### Gamma Distribution

The density of this distribution is presented in Table 1.1. This distribution can be considered as a "head of a family" of distributions which are very important in reliability theory and other applications.

The following essential property of the gamma distribution is presented below.

### The Sum of Gamma Distributed Random Variables

Let  $\square_1, \square_2, \dots, \square_n$  be gamma distributed r.v.'s and parameters  $\square$  and  $a_i$  are known for each  $\square_i, i=1, 2, \dots, n$ . Then the sum  $\square_1+\square_2+\dots+\square_n$  also has the gamma distribution with parameters  $\square$  and  $a=a_1+a_2+\dots+a_n$ .

### Exponential Distribution

Exponential distribution has extremely wide application in reliability theory. This is explained by two main reasons.

From the theoretical side, the exponential distribution allows us to obtain simple analytical results for many mathematical models. But this fact alone should be a wrong excuse for its extensive use. Much more important is that from a practical viewpoint, this distribution is an appropriate reflection of many real physical phenomena. First of all, as practice shows, electronic equipment often has TTF distributed exponentially. Besides, this distribution has a close relation to a Poisson process (see Volume 1 for details). Remember that the Poisson process can be considered as a sequence of point events which distance from each other by independent exponentially distributed random time intervals. The Poisson process is a convenient model for a flow of failures of complex systems consisting of a large number of highly reliable units. This fact is confirmed by the Khinchin, Renyi, and Grigelionis-Pogozhev theorems where the procedures of thinning and superposition of point stochastic processes are analyzed (for details see Volume 1).

Notice that the exponential distribution formally is a particular case of the gamma distribution for  $a=1$ . The density of the exponential distribution is represented in Table 1.1.

The exponential distribution possesses the so-called *lack of memory* (or *Markov*) property: the conditional probability  $P(x+y|\square_x)$  does not depend on  $x$ , that is  $P(x+y|\square_x)=P(y)$ .

### Erlang Distribution

This distribution bears its name after A. K. Erlang, the Danish telephone engineer who introduced it and widely used in varied telecommunication problems. An r.v. is said to have the Erlang distribution of the  $k$ th order if it can be represented as a sum of  $k$  i.i.d. r.v.'s each of which has an exponential distribution. The form of the density function with its mathematical expectation and variance is presented in Table 1.1. Sometimes for special tasks one considers the so-called generalized Erlang distribution where r.v.'s are not identical.

### Sum of Erlang R.V.'s

Sum of the  $M$  Erlang r.v.'s of the orders  $n_1, n_2, \dots, n_M$ , respectively, has the Erlang distribution of the order  $n=n_1+n_2+\dots+n_M$ . This fact follows immediately from the definition of the Erlang distribution.

### Weibull-Gnedenko Distribution

This distribution plays an important role in many applications because its form allows to use it in many practical cases. Weibull introduced this distribution analyzing wearing failures. A year later Gnedenko published a paper dedicated to limit distributions of extremum statistics. The distribution introduced by Weibull was a particular class of limit distributions. So, excellent engineering intuition and first class mathematical research generated a new distribution with wide area of applications.

The density of this distribution, its mean and variance are presented in Table 1.1.

This distribution is widely used in engineering practice because of the obvious convenience: two parameters -- one of scale and another of shape -- allow to approximate many various empirical distributions. In particular, such different classes of distributions as IFR and DFR can be expressed with the help of the Weibull-Gnedenko distribution.

Notice that a particular case for  $\alpha=1$  corresponds to the exponential distribution.

### Uniform Distribution

An r.v.,  $X$ , is said to have the uniform distribution if it might take any value from some closed interval with equal probability (see Table 1.1). This distribution is essentially used in Monte Carlo simulation. For generating an r.v.,  $X$ , with given arbitrary d.f.  $F(t)$ , one have to solve equation  $F(X)=u$ , i.e.,  $X=F^{-1}(u)$ . Such a transformation is easily performed on a computer.

Table 1.1

#### 2.2.2 Discrete Distributions

### Binomial Distribution

The binomial distribution characterizes samples of identical and independent events each of which has two possible outcomes, say, success and failure. Let  $n$  Bernoulli trials be performed. The probability of success in a single trial equals  $p$ . Then the number of successes,  $m$ , in  $n$  trials has a binomial distribution (see Table 1.2). This distribution is often used in sample quality control of mass production. Some redundant systems are also described by this distribution.

### Joint Sample of Several Subsamples

Consider  $M$  series of Bernoulli trials of size  $n_1, n_2, \dots, n_M$ , respectively. Let probability of success in each sample is the same and equals  $p$ . In this case, the probability to observe  $m$  successes in all  $M$  samples is  $\binom{n}{m} p^m q^{n-m}$  where  $n=n_1+n_2+\dots+n_M$ . This statement follows directly from the reformulation of the problem:  $m$  samples of different sizes,  $n_i$ , can be considered as a single sample with the size equals  $n$ .

### Geometrical Distribution

Consider again a sequence of Bernoulli trials. Let  $X$  denote the random number of successes until the first failure has occurred. This random number has the geometrical distribution (see Table 1.2). This distribution describes, for instance, a random number of successful cycles of operation or switching when each of them occurs independently and with the same probability.

A geometrical distribution can be considered as a discrete analog of an exponential one.

### Negative Binomial Distribution

This distribution is, in some sense, a generalization of the geometrical distribution. Assume that a sequence of Bernoulli trials is performing. The question of interest is: What is the probability of observing the  $m$ th failure at the  $n$ th trial? This event is equivalent to the product of the two following events: in  $n-1$  first trials one observes  $m-1$  failures in any order and then with probability  $q$  the  $m$ th failure might occur at the very last trial. The probability of this combined event is

$$P_m = q \binom{n-1}{m-1} p^{n-m} q^{m-1}$$

Some combinatorial transformations (see Volume 1) leads to the standard form (see Table 1.2). This distribution is often used in quality control.

### Poisson Distribution

Consider a sequence of independent events in time, such that a time interval between two neighbor events,  $\Delta t$ , is exponentially distributed with parameter  $\lambda$ . Such a sequence of events is called a Poisson process. In engineering practice a flow of a complex system failures forms a Poisson process. The number of failures,  $m$ , occurring during a fixed interval,  $t$ , is a discrete r.v. with the Poisson distribution with parameter  $\lambda t$  (see Table 1.2). Value  $\lambda$  is called an intensity of a corresponding Poisson process. The Poisson distribution is asymptotic form for a binomial distribution if  $n \rightarrow \infty$ ,  $p \rightarrow 0$ , and at the same time  $np = \lambda t$ .

### Sum of Poisson R.V.'s

Let us consider a Poisson process with intensity  $\lambda$ . Take two non-intersecting intervals  $t_1$  and  $t_2$  and consider the numbers of events on each one of them. The number of events on the  $k$ th interval has the Poisson distribution with parameter  $\lambda t_k$ . Due to the Markov property, there is no difference if these intervals are neighbor or not. The condition of non-intersecting delivers independence of numbers of events on these intervals. Let us consider a new interval  $t = t_1 + t_2$ . It is clear that the number of events on this joint interval has the Poisson distribution with parameter  $\lambda t$ .

Obviously, the same rule expands on an arbitrary number of non-intersected intervals  $t_k$ .

### Multinomial Distribution

This distribution is a generalization of a binomial distribution. In this case, one of  $k \geq 2$  different events can be observed in each single trial. Let  $p_j$  is the probability of the event of type  $j$ ,  $j=1, 2, \dots, n$ ,  $p_1 + p_2 + \dots + p_n = 1$ . Let  $\square_j$  be the number of observations of the  $j$ th event in a series of  $n$  trials,  $\square_1 + \square_2 + \dots + \square_k = n$ . Then vector  $\square = (\square_1, \square_2, \dots, \square_k)$  has the multinomial distribution with parameters  $n$  and  $\square$  (see Table 1.2).

Table 1.2

### 2.2.3 Special Distributions

The following distributions are frequently used in solving various statistical problems.

### $\chi^2$ Distribution

Let  $\xi_1, \xi_2, \dots, \xi_n$  be i.i.d. r.v.'s each of which has a standard normal distribution. Then the sum of squares of these r.v.'s has a  $\chi^2$  distribution with  $m$  degrees of freedom. The density of this sum is represented in Table 1.3.

Notice that this distribution is a particular case of a gamma distribution with parameters  $\lambda = \frac{1}{2}, \alpha = \frac{m}{2}$ .

Due to this relation between gamma and  $\chi^2$  distributions, their quantiles can be expressed via each other as

$$\Gamma_q(\lambda, m) = \frac{\chi_q^2(2m)}{2\lambda}, \quad m = 1, 2, \dots \quad (1.13)$$

where  $\Gamma_q(\lambda, m)$  is the quantile of level  $q$  of a gamma distribution with parameters  $\lambda$  and  $m$ , and  $\chi_q^2(2m)$  is the quantile of level  $q$  of a  $\chi^2$  distribution with  $m$  degrees of freedom.

### The Sum of $\chi^2$ Distributed Random Variables

Let  $\xi_1, \xi_2, \dots, \xi_n$  be  $\chi^2$  distributed independent r.v.'s each of which has degree of freedom  $m_i, i=1, 2, \dots, n$ . Then the sum  $\xi_1 + \xi_2 + \dots + \xi_n$  also has a  $\chi^2$  distribution with the degree of freedom  $m = m_1 + m_2 + \dots + m_n$ .

### Student Distribution

This distribution is applied for finding confidence limits of the sample mean of a normal distribution if the variance of the distribution is unknown.

Let r.v.'s  $\xi$  and  $\eta$  be independent and  $\xi$  has a standard normal distribution with parameters 0 and 1, and  $\eta$  has a  $\chi^2$ -distribution with  $m$  degrees of freedom. Then r.v.  $t = \frac{\xi}{\sqrt{\eta}} \sqrt{m}$  has the

Student distribution with  $m$  degrees of freedom. The density of a Student distribution is shown in Table 1.3. There

$$B(y, z) = \int_0^1 u^{y-1} (1-u)^{z-1} du$$

is the beta-function (for details see, for instance, Rao (1965, Chapter 3).

Remark: If a sample  $X_1, X_2, \dots, X_n$  is chosen from a normal distribution then the following r.v.'s

$$\xi = \frac{1}{\sigma} \sqrt{n}(\bar{X} - \mu), \quad \eta = \frac{nS^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{1 \leq i \leq n} (X_i - \bar{X})^2$$

are independent and have, respectively, standard normal distribution and  $\chi^2$ -distribution with  $n-1$  degrees of freedom

(*ibid.*). So, the r.v.

$$T = \frac{\xi}{\sqrt{\eta}} \sqrt{n-1} = \left( \frac{\bar{X} - \mu}{S} \right) \sqrt{n-1}$$

has the Student distribution with  $n-1$  degrees of freedom. This is the basis for the construction of the standard confidence interval for an unknown mathematical expectation of a normal distribution if the variance of the distribution,  $\sigma^2$ , is also unknown. □

### Fisher Distribution

This distribution is used, in particular, for constructing the confidence limits of the ratio of variances of two normal distributions. In reliability problems, the Fisher distribution is used for constructing the confidence limits of the availability coefficient.

If r.v.'s  $\xi$  and  $\eta$  are independent and have  $\chi^2$  distributions with  $n$  and  $m$  degrees of freedom, respectively, then the r.v.  $\varphi = \frac{m\xi}{n\eta}$  has a Fisher distribution (see the density in Table 1.3).

### Kolmogorov Distribution

This distribution characterizes asymptotic (for  $n \rightarrow \infty$ ) behavior of r.v.

$$T_n = \sqrt{n} \cdot \sup_x |\hat{F}_n(x) - F(x)|$$

where  $\hat{F}_n(x)$  is empirical d.f., based on the sample of size  $n$ , and  $F(x)$  is a true (theoretical) d.f. This d.f. is given in Table 1.3.

Table 1.3

## 2.3 Point Estimation

### 2.3.1 Introduction

Let r.v.  $X$  have d.f.  $F(x, \theta)$  and density  $f(x, \theta)$  which depend on some parameter  $\theta$  whose true value is unknown. (This parameter might be a vector.) We would like to estimate this parameter on the basis of  $n$  independent observations of r.v.  $X$ . In other words, there is sample  $X_1, X_2, \dots, X_n$  of size  $n$  from d.f.  $F(x, \theta)$ .

The problem of the construction of the point estimator of

$$\hat{\theta} = \hat{\theta}(X_1, \dots, X_n) \quad (1.14)$$

parameter  $\theta$  consists of finding a function of observations

such that r.v. (or random vector)  $\hat{\theta}$  in some sense is close to an unknown true value of parameter  $\theta$  of the corresponding distribution.

A function of observations  $\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)$  is called a *statistic*. For instance, a point estimator is a statistic. This particular statistic guarantees the closeness to the unknown true parameter.

### 2.3.2 Properties of Estimators

#### Unbiased Estimator

An estimator  $\hat{\theta}$  is called an unbiased estimator of a parameter  $\theta$  if its mathematical expectation  $E\hat{\theta} = \theta$  for all  $\theta$ . In other words, it means the following. Let us estimate an unknown parameter for a sample of arbitrary fixed size. Let us repeat this procedure until the number of such samples becomes sufficiently large. Then the mean of these estimates will be approximately equal to the unknown parameter.

**Example 1.5** Consider  $n$  independent Bernoulli trials with the success probability  $p$  and failure  $q=1-p$ . Suppose  $m$  successes have been occurred.

The standard estimator of the unknown probability of success is the frequency:  $\hat{p} = \frac{m}{n}$ . This estimator is unbiased because

$$E\hat{p} = E\left(\frac{m}{n}\right) = \frac{1}{n}Em = \frac{1}{n}np = p. \blacksquare$$

**Example 1.6** Consider an estimation of the mean,  $\theta = E\theta$ , of some r.v.  $\theta$  by  $n$  independent observations  $X_1, X_2, \dots, X_n$ . The sample mean  $\bar{X}$  is the unbiased estimator of  $\theta$  because

$$E\bar{X} = E\left\{\frac{1}{n} \sum_{1 \leq i \leq n} x_i\right\} = \frac{1}{n} \sum_{1 \leq i \leq n} E x_i = \frac{1}{n} \cdot n\mu = \mu. \blacksquare$$

**Example 1.7** For data of the previous example, determine whether the sample variance,  $S^2$ , is an unbiased estimator of the variance,  $\sigma^2 = E(\theta - \theta)^2$  of r.v.  $\theta$ . For this purpose, let us write the sample variance in the form

$$S^2 = \frac{1}{n} \sum_{1 \leq i \leq n} (X_i - \bar{X})^2 = \frac{1}{n} \sum_{1 \leq i \leq n} (X_i^2 - 2\bar{X} X_i + \bar{X}^2) = \frac{1}{n} \sum_{1 \leq i \leq n} X_i^2 - \bar{X}^2. \blacksquare$$

Let us now use the well-known formula which connects the mathematical expectation and variance of any arbitrary r.v.  $\theta$  as

$$E\theta^2 - (E\theta)^2 = \text{Var}\{\theta\}$$

In the considered case, we obtain

$$E S^2 = \frac{1}{n} \sum_{1 \leq i \leq n} E X_i^2 - (E \bar{X})^2 = \frac{1}{n} \sum_{1 \leq i \leq n} (\mu^2 + \sigma^2) - [(E \bar{X})^2 + \text{Var} \bar{X}].$$

The expected value of the sample mean is  $E\{\bar{X}\} = \mu$  and its variance is

$$\text{Var} \bar{X} = \text{Var} \left\{ \frac{1}{n} \sum_{1 \leq i \leq n} X_i \right\} = \frac{1}{n^2} \sum_{1 \leq i \leq n} \text{Var} \{X_i\} = \frac{1}{n^2} \cdot n \sigma^2 = \frac{\sigma^2}{n}.$$

From here follows that

$$E S^2 = \frac{n-1}{n} \sigma^2.$$

So, the sample variance is a biased estimator of the true variance. Instead of  $S^2$ , therefore, we frequently use an estimator  $S_1^2$

$$S_1^2 = \frac{n}{n-1} S^2 = \frac{1}{n-1} \sum_{1 \leq i \leq n} (X_i - \bar{X})^2.$$

because the latter estimator is unbiased.

### Asymptotically Unbiased Estimators

An estimator  $\hat{\theta}_n$  is called an asymptotically unbiased if

$$\lim_{n \rightarrow \infty} E \hat{\theta}_n = \theta$$

for all possible values of  $\theta$ . In other words, if we produce a large number of trials then the asymptotically unbiased estimate will approximately coincide with the unknown value of the parameter.

**Example 1.8** In Bernoulli trials considered above, any estimator for the probability of success of the form

$$\hat{p} = \frac{m + C}{n + C}$$

is asymptotically unbiased. (Here  $C$  is an arbitrary positive constant.)

**Example 1.9** In the conditions of Example 1.6 the sample variance  $S^2$  is an asymptotically unbiased estimator of the population variance  $\sigma^2$ .

### Consistent Estimators

Consider a dependence of an estimator,  $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ , on a sample size. Such estimator is called consistent if it converges in probability to the true value of parameter  $\theta$  as  $n \rightarrow \infty$ , i.e.,

$$\lim_{n \rightarrow \infty} P(|\hat{\Theta}_n - \Theta| > \varepsilon) = 0$$

for any  $\varepsilon > 0$  and for any possible  $\Theta$ .

In practical terms, this is close to asymptotic unbiased.

**Example 1.10** Let us show that the estimator  $\hat{p} = \frac{m}{n}$  in Example 1.5 is consistent for the probability  $p$ .

For an arbitrary r.v.  $\eta$  with the finite mathematical expectation  $E\eta$  and variance  $Var\{\eta\}$ , the well-known Chebyshev inequality

$$P(|\eta - E\eta| > \varepsilon) \leq \frac{Var\{\eta\}}{\varepsilon^2} \quad (1.15)$$

holds. Utilizing this inequality and taking into account that

$$E\hat{p} = p, \quad Var\{\hat{p}\} = Var\left\{\frac{m}{n}\right\} = \frac{1}{n^2} Var\{m\} = \frac{p(1-p)}{n}$$

one obtains that as  $n \rightarrow \infty$

$$Pr(|\hat{p} - p| > \varepsilon) \leq \frac{Var\{\hat{p}\}}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2} \rightarrow 0$$

and the consistency of the estimator follows.  $\square$

**Example 1.11** In analogous way, we can verify that the sample mean is a consistent estimator for the mathematical expectation  $\mu = E\xi$ .

We again use the Chebyshev inequality (1.15) and the formula for variance  $Var\{\bar{X}\}$  to obtain

$$P(|\bar{X} - \mu| > \varepsilon) \leq \frac{Var\bar{X}}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \rightarrow 0. \quad \square$$

With the help of the Chebyshev inequality, we can prove a more general result concerning the consistency of asymptotically unbiased estimators.

**Theorem 1.3.1** Let  $\hat{\theta}_n$  be an asymptotically unbiased estimator of parameter  $\theta$ , such that

$$\lim_{n \rightarrow \infty} Var\hat{\theta}_n = 0.$$

Then  $\hat{\theta}_n$  is a consistent estimator.  $\square$

It is easy to see that in the case of Bernoulli trials, the consistent estimator of parameter  $p$  is not only a value  $\hat{p} = \frac{m}{n}$  but also any estimator of the form of  $\hat{p} = \frac{m+C}{n+C}$ . For the variance  $\frac{p(1-p)}{n}$ , both  $S^2$  and  $S_1^2$  are consistent.



## Efficient Estimators

Assume that we have two unbiased estimators  $\hat{\theta}_1$  and  $\hat{\theta}'_1$  for a parameter  $\theta$ . It is natural to say that the estimator  $\hat{\theta}_1$  is more efficient than  $\hat{\theta}'_1$  if the variance of the first is smaller than the variance of the second:

$$\text{Var}\{\hat{\theta}_1\} = E(\hat{\theta}_1 - \theta)^2 \leq E(\hat{\theta}'_1 - \theta)^2 = \text{Var}\{\hat{\theta}'_1\} \quad (1.16)$$

for all possible values of  $\theta$ . If in some classes of estimators there exists an estimator  $\hat{\theta}_1$  for which (1.16) holds for all of the members of this class, one says that this estimator is efficient in the chosen class of estimators.

The major approach for finding efficient estimators is based on the *Cramer-Rao inequality*.

## Cramer-Rao Inequality

Let us introduce the function,  $I(\theta)$ , of parameter  $\theta$  defined as

$$I(\theta) = E\left[\frac{\partial \ln f(x, \theta)}{\partial \theta}\right]^2 = \int_{-\infty}^{\infty} \left[\frac{\partial \ln f(x, \theta)}{\partial \theta}\right]^2 f(x, \theta) dx,$$

where  $f(x, \theta)$  is the density function. This function is called the *Fisher information*. Under some general conditions of regularity, for any unbiased estimator  $\hat{\theta}_n$  of parameter  $\theta$ , the following Cramer-Rao inequality for the variance of estimator holds for any values of  $\theta$ :

$$\text{Var}\{\hat{\theta}_n\} \geq \frac{1}{n \cdot I(\theta)}.$$

The value

$$e(\theta) = \frac{1}{n \cdot I(\theta) \cdot \text{Var}\{\hat{\theta}_n\}} \quad (1.17)$$

is called *efficiency* of the unbiased estimator. By the Cramer-Rao inequality, any estimator satisfies inequality  $0 \leq e(\theta) \leq 1$ . An unbiased estimator  $\hat{\theta}_n$  is called *efficient* if  $e(\theta) = 1$ , or in other words, if its variance  $\text{Var}\{\hat{\theta}_n\}$  exceeds the Cramer-Rao lower limit for any  $\theta$ .

**Example 1.12** For Bernoulli trials from Example 1.5, the unknown parameter is the probability of success,  $\theta = p$ . Since in this case  $f(x, \theta) = p$  for  $x=1$ , and  $f(x, \theta) = 1-p$  for  $x=0$ , the Fisher information can be written as

$$\begin{aligned} I(p) &= E\left[\frac{\partial \ln f(x, p)}{\partial p}\right]^2 \\ &= P(x=0) \cdot \left[\frac{\partial \ln f(0, p)}{\partial p}\right]^2 + P(x=1) \cdot \left[\frac{\partial \ln f(1, p)}{\partial p}\right]^2 \end{aligned}$$

$$= (1-p) \cdot \left(\frac{1}{1-p}\right)^2 + p \cdot \left(\frac{1}{p}\right)^2 = \frac{1}{p(1-p)}.$$

The variance of the estimator  $\hat{p} = \frac{m}{n}$  equals  $\frac{1}{n} p(1-p)$  (see Example 1.10), and thus it follows that the estimator  $\hat{p}$  is efficient.  $\square$

**Example 1.13** Consider a sample of size  $n$  from normal distribution with the known variance  $\sigma^2$  and density

$$f(x, \mu) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

In this case  $\mu = \mu$  and the Fisher information is expressed as

$$I(\mu) = E\left[\frac{\partial \ln f(x, \mu)}{\partial \mu}\right]^2 = E\left[\frac{(x-\mu)^2}{\sigma^2}\right]^2 = \frac{1}{\sigma^2}.$$

But the variance of the estimator  $\hat{\mu} = \frac{1}{n}(X_1 + \dots + X_n)$  equals  $\frac{\sigma^2}{n}$ . Hence from the Cramer-Rao inequality, we see that this estimator of parameter  $\mu$  is efficient.  $\square$

**Example 1.14** Consider a sample of size  $n$  from exponential distribution with density

$$f(x, \Theta) = \frac{1}{\Theta} e^{-\frac{x}{\Theta}}, \quad x > 0.$$

In this case the Fisher information is

$$I(\Theta) = E\left[\frac{\partial \ln f(x, \Theta)}{\partial \Theta}\right]^2 = E\left[\frac{(x-\Theta)^2}{\Theta^4}\right]^2 = \frac{1}{\Theta^2}.$$

The variance of the estimator  $\hat{\mu} = \frac{1}{n}(X_1 + \dots + X_n)$  equals

$$\text{Var}\{\hat{\Theta}\} = \frac{1}{n^2} \sum_{1 \leq i \leq n} \text{Var}\{X_i\} = \frac{\Theta^2}{n},$$

so this estimator of the exponent parameter is efficient.  $\square$

**Example 1.15** Consider a Poisson distribution

$$f(x, \Theta) = \frac{\Theta^x}{x!} e^{-\Theta}, \quad x = 1, 2, \dots$$

The Fisher information in this case is

$$I(\Theta) = E\left[\frac{\partial}{\partial \Theta}(x \ln \Theta - \Theta - \ln x!)\right]^2 = E\left(\frac{x}{\Theta} - 1\right)^2 = \frac{1}{\Theta}.$$

and the variance of the estimator of parameter  $\hat{\theta} = \frac{1}{n^2} \cdot (x_1 + \dots + x_n)$  equals

$$\text{Var} \hat{\Theta} = \frac{1}{n^2} \sum_{1 \leq i \leq n} \text{Var} X_i = \frac{1}{n^2} \cdot n \cdot \Theta = \frac{\Theta}{n},$$

Thus the estimator  $\hat{\theta}$  is efficient.  $\square$

### 2.3.3 Methods of Estimation

The two most frequently used methods of parameter estimation are the *method of moments* and the *method of maximum likelihood*.

#### Method of Moments

Let  $X_1, \dots, X_n$  be a sample from a distribution with density  $f(x, \square)$  which depends on a single unknown parameter  $\square$ . The moment estimate of  $\square$ , found by the method of moments, is the solution  $\hat{\theta}$  of the equation derived by setting the theoretical first moment equal to the sample first moment  $\bar{X}$ . That is, we solve

$$\mu_1 = \mu_1(\Theta) = EX(\Theta) = \bar{X}.$$

Let us take the solution of equation

$$\mu_1(\Theta) = \bar{X} \quad (1.18)$$

In other words, we take as an estimate of the parameter such a value for which the true (or "theoretical") value of the first moment (expressed as a function of  $\square$ ) coincides with its value found from experimental data.

In analogous way the method of moments is applied for the case of multiple unknown parameters. If  $\square = (\square_1, \dots, \square_k)$  is a  $k$ -dimensional parameter, then estimators  $\hat{\theta}_1, \dots, \hat{\theta}_k$  can be found from the solution of the system of  $k$  equations

$$\begin{cases} \mu_1(\Theta_1, \dots, \Theta_k) = \hat{\mu}_1 \\ \dots \\ \mu_k(\Theta_1, \dots, \Theta_k) = \hat{\mu}_k \end{cases}$$

where

$$\mu_r(\theta_1, \dots, \theta_r) = \int_{-\infty}^{\infty} x^r f(x, \theta_1, \dots, \theta_r) dx$$

is the "theoretical" moment of order  $r$ , and

$$\hat{\mu}_r = \frac{1}{n} \sum_{1 \leq i \leq n} x_i^r$$

is its empirical value found from the sample.

For  $k=2$ , the above system of equations can be written in the following form

$$\begin{cases} \mu_1(\Theta_1, \Theta_2) = \bar{X} \\ \text{Var}\{\Theta_1, \Theta_2\} = S^2 \end{cases} \quad (1.19)$$

where  $\text{Var}\{\Theta_1, \Theta_2\}$  is the true variance and  $S^2$  the empirical variance.

**Example 1.16** Consider the estimation of an unknown parameter (probability of success,  $p$ ) in Bernoulli trials. Let  $m$  be the number of observed successes in  $n$  trials. The first moment, or the expectation of  $m$ , equals

$$\mu_1 = \mu_1(p) = \text{Em} = \sum_{0 \leq m \leq n} m \binom{n}{m} p^m (1-p)^{n-m} = np.$$

Corresponding to the method of moments, the estimator  $\hat{p}$  of parameter  $p$  is thus found from equation  $np=m$ . It follows that  $\hat{p} = \frac{m}{n}$ .  $\square$

**Example 1.17** Consider a sample from the exponential distribution with density  $f(x, \lambda) = \lambda e^{-\lambda x}$ ,  $x > 0$ , where parameter  $\lambda$  is unknown. In this case the first moment equals

$$\mu_1 = \mu_1(\lambda) = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}.$$

Thus equation (1.18) has the form  $\frac{1}{\lambda} = \bar{x}$  from which it follows that  $\hat{\lambda} = \frac{1}{\hat{x}}$ .  $\square$

**Example 1.18** For the Erlang distribution of order  $r$  with density

$$f(x, \lambda) = \frac{\lambda^r x^{r-1}}{(r-1)!} e^{-\lambda x}, \quad x > 0$$

the first moment is derived as

$$\mu_1(\lambda) = \int_0^{\infty} x f(x, \lambda) dx = \frac{r}{\lambda}.$$

So, the estimator of parameter  $\lambda$  by the method of moments is found from equation  $\frac{r}{\lambda} = \bar{x}$  and,

consequently,  $\hat{\lambda} = \frac{r}{\bar{x}}$ .  $\square$

**Example 1.19** Consider a sample from the normal distribution with two unknown parameters,  $\mu$  and  $\sigma^2$ . The system of equations (1.19) in this case has the following simple form

$$\begin{cases} \mu = \bar{X} \\ \sigma^2 = S^2 \end{cases}$$

which gives  $\hat{\mu} = \bar{x}$ ,  $\hat{\sigma} = S$ .74  $\square$

**Example 1.20** Consider a sample from gamma distribution with the density

$$f(x, \lambda, \alpha) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}, \quad x > 0$$

which includes two unknown parameters  $\lambda$ ,  $\alpha$ . Using the well-known expression for gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

and recurrence relation  $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ , one obtains the following expressions for the first and second moments and variance:

$$\begin{aligned} \mu_1(\lambda, \alpha) &= \int_0^\infty \frac{\lambda^\alpha x^\alpha}{\Gamma(\alpha)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda}, \\ \mu_2(\lambda, \alpha) &= \int_0^\infty \frac{\lambda^\alpha x^{\alpha+1}}{\Gamma(\alpha)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\lambda^2}, \end{aligned}$$

$$\text{Var}(\lambda, \alpha) = \mu_2(\lambda, \alpha) - \mu_1^2(\lambda, \alpha) = \frac{\alpha}{\lambda^2}.$$

The system of equations for finding estimators of the parameters in this case has the form:

$$\begin{cases} \frac{\alpha}{\lambda} = \bar{X} \\ \frac{\alpha}{\lambda^2} = S^2 \end{cases}$$

Thus and it follows that  $\hat{\lambda} = \frac{\bar{x}}{S^2}$ ,  $\hat{\alpha} = \frac{\bar{x}}{S^2}$ .75  $\square$

The method of moments gives consistent estimators of the parameters but not always good from the efficiency viewpoint.

### Method of Maximum Likelihood

Consider a continuous distribution with density  $f(x, \theta)$ . Let  $X_1, \dots, X_n$  be a sample of size  $n$  from this distribution. The joint density of all sample data, written as a function of  $\theta$ ,

$$L(X_1, \dots, X_n, \Theta) = f(X_1, \Theta) \cdot f(X_2, \Theta) \cdot \dots \cdot f(X_n, \Theta) \quad (1.20)$$

is called the *likelihood function*.

The *maximum likelihood estimator*  $\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)$  is found as a value of parameter  $\theta$  for which the likelihood function reaches the maximum in  $\Theta$  under the condition that results of observation  $X_1, \dots, X_n$  are fixed. In other words, the maximum likelihood estimator (MLE) is found from equation

$$L(X_1, \dots, X_n, \hat{\theta}) = \max_{\theta \in \Theta} L(X_1, \dots, X_n, \theta).$$

It is typically more convenient to search for the maximum of the logarithm of the likelihood function rather than the maximum of the function itself. (The maximum of the logarithm and the function itself coincide.) Thus, under the assumption that function  $f(x, \theta)$  is differentiable with respect to  $\theta$ , one can find MLE  $\hat{\theta}$  from equation

$$\frac{\partial}{\partial \theta} \ln L(X_1, \dots, X_n, \theta) = 0. \quad (1.21)$$

Equation (1.21) is called the *likelihood equation*.

Remark: Equation (1.21) is the necessary but not sufficient condition for obtaining the maximum. However, for many distribution families used in practice, it happens that the solution is unique and delivers the desired maximum likelihood estimator.

In an analogous way, in the case of a vector parameter  $\theta = (\theta_1, \dots, \theta_k)$ , for finding the MLE of parameters  $\hat{\theta}_1, \dots, \hat{\theta}_k$ , one needs to solve a system of equations (with respect to  $\theta_1, \dots, \theta_k$ , for fixed observed values  $X_1, \dots, X_n$ ):

$$\begin{cases} \frac{\partial}{\partial \theta_1} \ln L(X_1, \dots, X_n, \theta_1, \dots, \theta_k) = 0 \\ \dots \\ \frac{\partial}{\partial \theta_k} \ln L(X_1, \dots, X_n, \theta_1, \dots, \theta_k) = 0. \end{cases}$$

The important property of the MLE is the following: If an efficient estimator exists, the maximum likelihood method delivers this estimator. In general, under some conditions, the maximum likelihood method delivers asymptotically unbiased and asymptotically efficient estimates.

**Example 1.21** Let us apply the maximum likelihood method for finding the estimator of an unknown parameter  $p$  in a set of Bernoulli trials. In this case, the likelihood function for the probability of observing  $m$  successes in a series of  $n$  trials equals

$$L(m, n) = \binom{n}{m} p^m (1-p)^{n-m}.$$

The equation of likelihood has the form

$$\frac{\partial}{\partial p} [m \ln p + (n-m) \cdot \ln(1-p)] = \frac{m}{p} - \frac{n-m}{1-p} = 0$$

so we obtain  $\hat{p} = \frac{m}{n}$ . 79 □

**Example 1.22** Let us find an estimator of parameter □ of the exponential distribution using the method of maximum likelihood. The likelihood function in this case is

$$L(X_1, \dots, X_n, \lambda) = \lambda e^{-\lambda x_1} \cdot \dots \cdot \lambda e^{-\lambda x_n} = \lambda^n \exp\left(-\lambda \sum_{1 \leq i \leq n} x_i\right).$$

The likelihood equation has form

$$\frac{\partial}{\partial \lambda} \left( n \ln \lambda - \lambda \sum_{1 \leq i \leq n} x_i \right) = \frac{n}{\lambda} - \sum_{1 \leq i \leq n} x_i = 0$$

from where

$$\hat{\lambda} = \frac{n}{\sum_{1 \leq i \leq n} x_i}. \quad \square \square$$

**Example 1.23** Using the method of maximum likelihood find the point estimators of parameters □, □ of the normal density

$$f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

In this case, the likelihood function and its logarithm are equal to

$$L = \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left[-\frac{1}{2\sigma^2} \sum_{1 \leq i \leq n} (x_i - \mu)^2\right],$$

and

$$\ln L = -n \ln \sqrt{2\pi} - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{1 \leq i \leq n} (x_i - \mu)^2.$$

There are two unknown parameters in this problem, and the system of equations for finding this MLE has the form

$$\frac{\partial}{\partial \mu} \ln L = \frac{1}{\sigma^2} \sum_{1 \leq i \leq n} (X_i - \mu) = 0$$

$$\frac{\partial}{\partial \sigma} \ln L = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{1 \leq i \leq n} (X_i - \mu)^2 = 0$$

It thus follows that

$$\hat{\mu} = \frac{1}{n} \sum_{1 \leq i \leq n} X_i, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{1 \leq i \leq n} (X_i - \bar{X})^2$$

So the MLE for the expectation  $\mu$  and variance  $\sigma^2$  of normal distribution coincide with the sample mean  $\bar{X}$  and sample variance  $S^2$ .  $\square$

### 2.3.4 Sufficient Statistics

Let  $X_1, \dots, X_n$  be a random sample of size  $n$  from a distribution with density  $f(x, \theta)$ . For the sake of simplicity, let us restrain ourselves to the case where r.v.'s are discrete. In this case,  $f(x, \theta)$  represents the probability that r.v.  $X$  has value  $x$ .

Let  $T = T(X_1, \dots, X_n)$  be some statistic, i.e., some function of the observations. Assume that after an experiment we do not know the total sample  $(X_1, \dots, X_n)$  but only the value of statistic

$$T(X_1, \dots, X_n) = t \tag{1.22}$$

Statistic  $T$  is called a *sufficient statistic* for parameter  $\theta$  (or more precisely, for the parametric family of distributions  $f(x, \theta)$ ) if for any event  $A$ , the conditional probability of occurrence of this event under condition (1.22)

$$P\{A \mid T(x_1, \dots, x_n) = t\}$$

does not depend on the value of parameter  $\theta$  for any possible value of statistic  $t$ . In the discrete case the conditional probability above can be calculated by formula

$$Pr\{A \mid T(x_1, \dots, x_n) = t\} = \frac{\sum_{T=t, \mathbf{x} \in A} L(X_1, \dots, X_n, \theta)}{\sum_{T=t} L(X_1, \dots, X_n, \theta)}$$

where  $L(X_1, \dots, X_n, \theta)$  is the likelihood function, the sum in the denominator is taken over all possible  $\mathbf{X} = (X_1, \dots, X_n)$  for which  $T(\mathbf{X}) = t$  and the sum in the numerator is taken over all  $\mathbf{x}$  such that  $T(\mathbf{X}) = t, \mathbf{x} \in A$ .

The sense of the definition given above is the following: for some known value of a sufficient statistic, changing parameter  $\theta$  does not influence the probabilities of one or another events (or, more precisely, the conditional distribution of the sample under condition (1.22)). This means that statistic  $T$  delivers the complete information about parameter  $\theta$ . The following known result gives a simple criterion for verifying whether statistic is sufficient or not.

#### Theorem 1.3.2 (Criterion of factorization)



Statistic  $T(X_1, \dots, X_n)$  is sufficient for parameter  $\theta$  if and only if the likelihood function has the form

$$L(X_1, \dots, X_n, \theta) = h(X_1, \dots, X_n) \cdot g[T(X_1, \dots, X_n), \theta]. \quad \square$$

In other words, the likelihood function can be written as a product of two factors, one of them depending only on the results of the observations (not on parameter  $\theta$ ), and another depends on parameter  $\theta$  and on observation  $X_1, \dots, X_n$  only via statistic  $T$ .

For continuous distributions with multidimensional parameter  $\theta$  and multidimensional sufficient statistic  $T$ , the principles are analogous. From the criterion of factorization, it follows that the method of maximum likelihood always leads to the expression of the parameter's estimator via the sufficient statistic. Another important property of sufficient statistics lies in the fact that if an efficient estimator of a parameter exists, then it is expressed via a sufficient statistic.

In other words, a sufficient statistic contains all necessary information about observations.

**Example 1.24** Consider the exponential distribution with density

$$f(x, \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0.$$

In this case, the likelihood function is

$$L(X_1, \dots, X_n, \theta) = f(X_1, \theta) \cdot \dots \cdot f(X_n, \theta) = \frac{1}{\theta^n} \exp\left(-\frac{1}{\theta} \sum_{1 \leq i \leq n} x_i\right)$$

From the factorization criterion, it follows that statistic  $T = X_1 + \dots + X_n$  is sufficient. In this case, an efficient estimator of parameter  $\theta$  expressed via a sufficient statistic, exists (see Example 1.6 above).  $\square$

**Example 1.25** For the normal distribution with two unknown parameters  $\mu, \sigma$ , the likelihood function has the form

$$L(X_1, \dots, X_n, \mu, \sigma) = \prod_{1 \leq i \leq n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x_i - \mu)^2}{2\sigma^2}\right] =$$

$$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \cdot \exp\left(-\frac{n\mu^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{1}{2\sigma^2} \sum_{1 \leq i \leq n} x_i^2 + \frac{\mu}{\sigma^2} \sum_{1 \leq i \leq n} x_i\right).$$

In accordance with the factorization criterion, we see that two-dimensional sufficient statistic for two-dimensional parameter  $(\mu, \sigma)$  is  $T = (T_1, T_2)$ , where

$$T_1 = \sum_{1 \leq i \leq n} x_i, \quad T_2 = \sum_{1 \leq i \leq n} x_i^2. \quad \square \square$$

## 2.4 Confidence Intervals

## 2.4.1 Introduction

*Confidence intervals* are used to estimate unknown population characteristics with a certain level of guarantee. The confidence interval is such an interval that an unknown parameter occurs within this interval with some guaranteed probability (confidence probability). Let us give a verbal explanation of this probability. If we increase the numbers of homogeneous samples (i.e. samples of identical and independent objects) and construct the confidence interval for each sample, the relative frequency of cases where the unknown parameter will be covered by these confidence intervals converges to the confidence probability.

In practice we often construct a symmetrical confidence interval. It makes an impression that the confidence interval "surrounds" the real value of the investigated parameter. It is a wrong impression. We don't know a real position of the unknown parameter within the confidence interval. Moreover, with some non-zero probability the real parameter might occur outside of the confidence interval.

Let us consider the problem in strict terms. Let  $X_1, \dots, X_n$  be a sample of size  $n$  from d.f.  $F(x, \theta)$  which depends on parameter  $\theta$  in a prior unknown way. Let us now assume that an interval  $[\underline{\theta}, \bar{\theta}]$  **Ошибка! Не указан аргумент ключа.** such that the lower and upper limits are functions of test results

$$\underline{\theta} = \underline{\theta}(X_1, \dots, X_n), \quad \bar{\theta} = \bar{\theta}(X_1, \dots, X_n)$$

and the inequality

$$P(\underline{\theta} \leq \Theta \leq \bar{\theta}) = \gamma \quad (1.23)$$

holds for all possible values of  $\theta$ . Both these limits are random because they depend on a set of random variables. These limits will change from sample to sample. (Samples are assumed homogeneous and of the same size.) Interval  $[\underline{\theta}, \bar{\theta}]$  86 is called the *confidence interval with confidence coefficient*  $\gamma$  (or, briefly,  *$\gamma$ -confidence interval*) for parameter  $\theta$ . In practice, one chooses the level of  $\gamma$  close to 1, for instance, 0.9, 0.95 or 0.99.

Based on the same statistical data, we can build a set of different confidence intervals with different confidence probabilities. Common sense hints that, for the same statistical data, better the level of guarantee (i.e., the higher confidence probability), wider is the confidence interval. On the basis of given statistical data, let us build a symmetrical confidence interval  $[\underline{\theta}, \bar{\theta}]$  87 for confidence probability  $\gamma = 0.9$ . Using the same data, we can construct confidence interval  $[\underline{\theta}, \bar{\theta}]$  88 for confidence probability  $\gamma = 0.99$ . If we choose the higher confidence probability, we must sacrifice accuracy: the confidence interval will be wider. 89

Thus, confidence interval  $[\underline{\theta}, \bar{\theta}]$  90 is an interval with random limits which are constructed on the basis of test results. This interval covers an unknown true value of parameter  $\theta$  with probability  $\gamma$ . So, in contrast to the point estimator, the confidence interval gives guarantee information about the unknown true value of a parameter though this information is "fuzzy".

In some cases (for instance, for discrete r.v.'s) it is possible to satisfy only the inequality

$$P(\underline{\Theta} \leq \Theta \leq \bar{\Theta}) \geq \gamma \quad (1.24)$$

for all possible  $\gamma$  instead of equality (1.23) because there are no lower and upper limits which deliver an exact value of  $\gamma$ . In this case, we say that interval  $[\underline{\theta}, \bar{\theta}]$  is a confidence interval with confidence coefficient not less than  $\gamma$ .

Sometimes we need to find only a one-sided interval for parameter  $\Theta$ , from below or from above. If the inequality

$$P(\underline{\Theta} \leq \Theta) \geq \gamma$$

holds, then  $\underline{\theta}$  is called the lower  $\gamma$ -confidence interval of parameter  $\Theta$ . Analogously, if

$$P(\bar{\Theta} \geq \Theta) \geq \gamma$$

holds, then  $\bar{\theta}$  is called the upper  $\gamma$ -confidence interval of parameter  $\Theta$ . For example, if we are interested in the MTTF of some equipment we should be sure that this value is not smaller than some specified level but, factually, we are less interested in the upper limit. Considering probability of failure, on the contrary, we are interested in the fact that this value is not larger than some given level.

#### 2.4.2 Construction of Confidence Intervals

One of the frequently used methods of construction of confidence intervals is based on use of some central statistic, which is a function depending on parameter  $\Theta$  and observations

$$T = T(X_1, \dots, X_n, \Theta) \quad (1.25)$$

such that its d.f.  $F(t) = P(T \leq t)$  does not depend on  $\Theta$ .

The value  $K_q$  defined from the relation

$$P(T \leq K_q) = F(K_q) = q$$

is called the quantile of level  $q$  of d.f.  $F(t)$  of r.v.  $T$ . Let us choose two small enough values  $\alpha$  and  $\beta$ , and find values  $t_1, t_2$  from the following conditions:

$$P(T \leq t_1) = F(t_1) = \alpha,$$

$$P(T > t_2) = 1 - F(t_2) = \beta.$$

For these purposes one needs to set  $t_1 = K_\alpha, t_2 = K_{1-\beta}$ . Then for the central statistic, equality

$$\Pr\{t_1 \leq T(X_1, \dots, X_n, \Theta) \leq t_2\} = \gamma \quad (1.26)$$

holds for  $\gamma = 1 - \alpha - \beta$ . Further, the lower and upper limits of the confidence interval for parameter  $\Theta$  are defined, respectively, as minimum and maximum values among all  $\Theta$  satisfying inequality

$$\tau_1 \leq T(X_1, \dots, X_n, \Theta) \leq \tau_2$$

Therefore, from (1.26), it follows that

$$\Pr(\underline{\Theta} \leq \Theta \leq \bar{\Theta}) = \gamma$$

i.e., the interval defined in such a manner is a confidence interval for parameter  $\Theta$  with confidence coefficient equal to  $\gamma = 1 - \alpha - \beta$ .

$$\begin{aligned} T(X_1, \dots, X_n, \underline{\Theta}) &= t_2 \\ T(X_1, \dots, X_n, \bar{\Theta}) &= t_1 \end{aligned} \quad (1.27)$$

If the central statistic monotonically increases in  $\Theta$ , then the confidence limits are found from the following equations:

$$\begin{aligned} T(X_1, \dots, X_n, \underline{\Theta}) &= t_1 \\ T(X_1, \dots, X_n, \bar{\Theta}) &= t_2 \end{aligned} \quad (1.28)$$

Usually in practice, a confidence interval is chosen to be symmetrical, i.e.,  $\alpha = \beta = \frac{1 - \gamma}{2}$ .

### General Method of Confidence Interval Construction

An appropriate central statistic may not always be found. This obstacle leads to the use of another -- more general -- method which is sometimes called the method of confidence sets. We consider this case for a one-dimensional parameter  $\Theta$ .

Let  $S = S(X_1, \dots, X_n)$  be some initial statistic. Most often for these purposes we use an unbiased point estimator, i.e.,  $S = \hat{\Theta}$ . For a given  $\Theta$ , a distribution function of the statistic is denoted as  $F(t, \Theta) = P(S \leq t)$ . For the sake of arguing simplicity, this function will be assumed continuous, strictly increasing in  $t$  and strictly decreasing in  $\Theta$ .

Let us set values  $t_1 = t_1(\Theta)$ ,  $t_2 = t_2(\Theta)$  corresponding to any possible value of parameter  $\Theta$ . These values are chosen from conditions

$$F(t_1, \Theta) = \alpha, \quad F(t_2, \Theta) = 1 - \beta \quad (1.29)$$

so that  $t_1(\Theta)$ ,  $t_2(\Theta)$  are, at the same time, the quantiles of levels  $\alpha$  and  $1 - \beta$  of d.f.  $F(t, \Theta)$  of statistic  $S$ . So if  $\alpha = 1 - \beta$ , equality

$$P\{t_1(\Theta) \leq S \leq t_2(\Theta)\} = \gamma$$

holds. The set of values of statistic  $S$  belonging to the interval  $[t_1(\Theta), t_2(\Theta)]$  is denoted as  $H_\Theta$  and is called a  $\gamma$ -zone of parameter  $\Theta$  (see Figure 1.2). For any possible value of parameter  $\Theta$ , the probability that statistic  $S$  will belong to the  $\gamma$ -zone equals  $\gamma$  by construction.

Figure 1.2

Further, in correspondence to an observed value of statistic  $S$ , let us define an interval of all values of  $\theta$  for which this value of  $S$  belongs to the  $\theta$ -zone (see Figure 1.2). The lower and upper limits of this interval,  $t_2(\underline{\theta})$  and  $t_1(\overline{\theta})$  are found from the following condition

$$t_2(\underline{\theta}) = S, \quad t_1(\overline{\theta}) = S$$

or by (1.29) from the equivalent conditions

$$F(S, \overline{\theta}) = \alpha, \quad F(S, \underline{\theta}) = 1 - \beta. \quad (1.30)$$

The interval constructed above is a  $\gamma$ -confidence interval for parameter  $\theta$ . Indeed, for any possible value of parameter  $\theta$  (including an unknown true), the interval  $[\underline{\theta}, \overline{\theta}]$  covers  $\theta$  if and only if an observed value of statistic  $S$  belongs to the  $\theta$ -zone  $H_\theta$  for the specified value of  $\theta$ . So, by construction of the  $\theta$ -zone, the equality

$$\Pr \{ \underline{\theta} \leq \theta \leq \overline{\theta} \} = \gamma$$

holds.

If d.f.  $F(t, \theta)$  is monotone increasing in parameter  $\theta$ , then the limits of the  $\theta$ -zone  $t_1(\theta)$ ,  $t_2(\theta)$  are monotone decreasing in  $\theta$ . Repeating the arguments above, we find that the lower and upper limits of the confidence interval in this case are defined as

$$F(S, \underline{\theta}) = \alpha, \quad F(S, \overline{\theta}) = 1 - \beta. \quad (1.31)$$

### Confidence Interval for Discrete Random Variables

The above method is used in an analogous way for the case of discrete r.v.'s. Consider, for example, a case, which is often met in practice, where statistic  $S$  takes on integer values  $0, 1, 2, \dots$ .

Let  $F(n, \theta) = P(S \leq n)$ ,  $n=0, 1, 2, \dots$ , be the d.f. of statistic  $S$ . Differing from a continuous case, the limits of the  $\theta$ -zone here have a step-like form (see Figure 1.3).

Figure 1.3

For a specified fixed value of parameter  $\theta$ , the lower limit,  $t_1(\theta)$ , of the  $\theta$ -zone is defined as the maximal number among  $n$  for which inequality

$$P(S \geq n) = 1 - F(n-1, \theta) \geq 1 - \alpha$$

holds. The lower limit,  $t_2(\theta)$ , of the  $\theta$ -zone is defined as the minimal number among  $n$  for which inequality

$$P(S \leq n) = F(n, \theta) \geq 1 - \beta.$$

holds. After this, the lower and upper limits of the confidence interval for parameter  $\theta$  with confidence coefficient not less than  $\gamma=1-\alpha-\beta$  is defined by the minimum and maximum values among all of  $\theta$  on the basis of statistic  $S$ . This which satisfy inequalities

$$t_1(\theta) \leq S \leq t_2(\theta) \quad (1.32)$$

i.e., among all  $\theta$  which belong to the  $1-\alpha$ -zone for a specified value of statistic  $S$ . Inequalities (1.32) are equivalent to

$$\begin{cases} F(S, \theta) \geq \alpha, \\ F(S-1, \theta) \leq 1-\beta. \end{cases}$$

From the above inequalities, it follows that if d.f.  $F(n, \theta)$  of statistic  $S$  is monotone decreasing in  $\theta$ , then the lower and upper limits for the  $1-\alpha$ -confidence interval for parameter  $\theta$  can be found from the solution of the system of equations

$$\begin{cases} F(S-1, \underline{\theta}) = 1-\beta, \\ F(S, \bar{\theta}) = \alpha \end{cases} \quad (1.33)$$

Analogously, if d.f.  $F(n, \theta)$  is monotone increasing in  $\theta$ , then the  $1-\alpha$ -confidence interval for parameter  $\theta$  can be found from

$$\begin{cases} F(S-1, \underline{\theta}) = \alpha, \\ F(S, \bar{\theta}) = 1-\beta. \end{cases} \quad (1.34)$$

where again  $\bar{\theta} = 1-\alpha-\underline{\theta}$ .

### Confidence Sets for Vector of Parameters

In a similar way, we can construct the confidence sets for the multi-dimensional parameter  $\theta = (\theta_1, \dots, \theta_m)$ . Let  $\mathbf{X}$  be a vector of observations, and  $P_\theta(\mathbf{X})$  be the d.f. of  $\mathbf{X}$  for a given  $\theta$ . Let set  $C_\theta$  of values  $\mathbf{X}$  be defined for each possible  $\theta$  in such a way that inequality

$$P_\theta(\mathbf{X} \in C_\theta) \geq \gamma \quad (1.35)$$

holds for all of possible  $\theta$ s. Then for each observations  $\mathbf{x}$  we find set  $D_{\mathbf{x}}$  of parameter  $\theta$  such that  $\mathbf{x} \in C_\theta$ . This procedure is a straightforward extension of the procedure above for a one-dimensional parameter. By constructing set  $D_{\mathbf{x}}$  for each fixed  $\mathbf{x}$ , events  $A = \{\mathbf{X} \in C_\theta\}$ ,  $B = \{\mathbf{X} \in D_{\mathbf{x}}\}$  are equivalent. From here, taking into account (1.35), it follows that

$$P_\theta\{\theta \in D_{\mathbf{x}}\} \geq \gamma \quad (1.36)$$

for all of possible values of parameter  $\theta$ . A collection of all sets  $D_{\mathbf{x}}$  for all possible values  $\mathbf{x}$  satisfying (1.36) is called a system of  $\gamma$ -confidence sets for parameter  $\theta$ .

Further, if we need to construct a  $\gamma$ -confidence interval for some function  $f=f(\theta)$  of the vector of parameters  $\theta$ , then such an interval can be constructed on the basis of confidence sets  $D_X$  in the following way. Let us choose the lower and upper limits of  $f$  as values

$$\underline{f} = \min f(\Theta), \quad \bar{f} = \max f(\Theta)$$

where the minimum and maximum are taken over all  $\theta$  belonging to sets  $D_X$ . Then it follows directly from (1.36) that

$$\Pr_{\theta}(\underline{f} \leq f(\Theta) \leq \bar{f}) \geq \gamma$$

for all  $\theta$ , i.e., interval  $[\underline{f}, \bar{f}]$  is the  $\gamma$ -confidence interval for  $f(\theta)$ .

This type of problem will be considered below in more detail for the case of confidence estimation of reliability of a complex system on the basis of test results of its units.

We now consider constructing confidence intervals for parameters of the most frequently used distribution functions.

### 2.4.3 Confidence Estimation of Exponential Distribution

Consider an exponential distribution with density  $f(x, \lambda) = \lambda e^{-\lambda x}$ ,  $x > 0$ . As a central statistic, we choose

$$T = 2\lambda \sum_{1 \leq i \leq n} X_i.$$

This statistic has standard  $\chi^2$  distribution with  $2n$  degrees of freedom (see Section 1.2). Equations (1.28) in this case take the form

$$2\underline{\lambda} \sum_{1 \leq i \leq n} X_i = t_1 = \chi_{\alpha}^2(2n),$$

$$2\bar{\lambda} \sum_{1 \leq i \leq n} X_i = t_2 = \chi_{1-\beta}^2(2n),$$

where  $\chi_q^2(2n)$  is the quantile of level  $q$  for a  $\chi^2$  distribution with  $2n$  degrees of freedom. From here, it follows that the lower and upper limits of the confidence interval with confidence coefficient  $\gamma=1-\alpha-\beta$  for parameter  $\lambda$  has the form

$$\underline{\lambda} = \frac{\chi_{\alpha}^2(2n)}{2 \sum_{1 \leq i \leq n} X_i} \quad \text{and} \quad \bar{\lambda} = \frac{\chi_{1-\beta}^2(2n)}{2 \sum_{1 \leq i \leq n} X_i}.$$

#### 2.4.4 Normal Distribution, Known Variance $\sigma^2$

Consider confidence interval for mean  $\mu$  of the normal distribution with known variance  $\sigma^2$ . Choose the central statistic of the form

$$T = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

which has the standard normal distribution with the mean 0 and variance 1. In this case, system (1.27) takes the form

$$\begin{aligned} \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} &= u_{1-\beta} \\ \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} &= u_{\alpha} \end{aligned}$$

where  $u_{\alpha}$  is the quantile of level  $1-\alpha$  of the standard normal distribution. Taking into account that  $u_{1-\beta} = -u_{\beta}$  for a normal distribution, we have the following lower and upper limits for the  $1-\alpha$ -confidence interval for parameter  $\mu$ :

$$\begin{aligned} \underline{\mu} &= \bar{X} - u_{\alpha} \left( \frac{\sigma}{\sqrt{n}} \right), \\ \bar{\mu} &= \bar{X} + u_{\beta} \left( \frac{\sigma}{\sqrt{n}} \right). \end{aligned}$$

#### Normal Distribution, Unknown Variance $\sigma^2$

Now consider confidence interval for the mean of a normal distribution with unknown variance  $\sigma^2$ . For this case, the central statistic is

$$T = \frac{\bar{X} - \mu}{\left( \frac{S}{\sqrt{n-1}} \right)}$$

where  $\bar{X}$  and  $S^2$  are the sample mean and variance, respectively.

This statistic has Student d.f. with  $n-1$  degrees of freedom (see Section 1.2). System of equations (1.27) in this case takes the form

$$\frac{\bar{X} - \mu}{S / \sqrt{n-1}} = t(n-1, \alpha)$$



$$\frac{\bar{X} - \mu}{S / \sqrt{n-1}} = t(n-1, 1-\beta)$$

where  $t(n-1, \alpha)$  is the quantile of level  $1-\alpha$  of Student distribution with  $n-1$  degrees of freedom. Since Student distribution is symmetrical,  $t(n-1, 1-\alpha) = -t(n-1, \alpha)$ . It follows therefore that the lower and upper limits of the confidence interval with confidence coefficient  $1-\alpha-\beta$  for parameter  $\mu$  (when the variance is unknown) can be found by formulas

$$\underline{\mu} = \bar{X} - t(n-1, \alpha) \cdot \frac{S}{\sqrt{n-1}}$$

$$\bar{\mu} = \bar{X} + t(n-1, \beta) \cdot \frac{S}{\sqrt{n-1}}.$$

### Normal Distribution, Known Mean

Consider confidence interval for the standard deviation of a normal distribution with known expectation  $\mu$ . The central statistic in this case is

$$T = \frac{1}{\sigma^2} \sum_{1 \leq i \leq n} (x_i - \mu)^2,$$

which has  $\chi^2$  distribution with  $n$  degrees of freedom (see Section 1.2). Analogous to the previous case, we obtain the following lower and upper confidence limits with confidence coefficient  $1-\alpha-\beta$  for parameter  $\sigma$ :

$$\underline{\sigma} = \sqrt{\frac{\sum_{1 \leq i \leq n} (x_i - \mu)^2}{\chi_{1-\alpha}^2(n)}}$$

$$\bar{\sigma} = \sqrt{\frac{\sum_{1 \leq i \leq n} (x_i - \mu)^2}{\chi_{\beta}^2(n)}}$$

where  $\chi_q^2(n)$  is the quantile of level  $q$  for  $\chi^2$  distribution with  $n$  degrees of freedom.

### Normal distribution, Unknown Mean

Now consider confidence interval for the standard deviation of a normal distribution with unknown expectation  $\mu$ . The central statistic in this case is

$$T = \frac{nS^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{1 \leq i \leq n} (X_i - \bar{X})^2,$$

which has a  $\chi^2$ -distribution with  $n-1$  degrees of freedom. This leads to the following lower and upper limits of the confidence interval with the confidence coefficient  $1-\alpha$  for parameter  $\sigma$ :

$$\underline{\sigma} = \frac{\sqrt{n} S}{\sqrt{\chi_{1-\alpha}^2(n-1)}}$$

$$\bar{\sigma} = \frac{\sqrt{n} S}{\sqrt{\chi_{\alpha}^2(n-1)}}$$

#### 2.4.5 Approximation Based on Levy-Lindberg Theorem

Consider a simple and constructive approximation for the confidence interval of the mean based on Levy-Lindeberg Theorem. Let  $X_1, \dots, X_n$  be a sample of  $n$  independent observations of some r.v.  $X$  with a finite and unknown mathematical expectation  $E(X) = \mu$  and variance  $E(X - \mu)^2 = \sigma^2$ . We also assume that a d.f. of the observed r.v. is unknown.

Consider statistic  $T = \frac{\bar{X} - \mu}{(\sigma / \sqrt{n})} \sqrt{n}$ . In accordance with the Levy-Lindeberg form of the

Central Limit Theorem this statistic has asymptotically normal distribution. This fact allows us consider normal approximation if  $n$  is sufficiently large. In this case, the inequalities

$$-u_{\alpha} \leq \frac{\bar{X} - \mu}{\sigma} \sqrt{n} \leq u_{\alpha} \quad (1.37)$$

hold with probability close to  $1-\alpha$ . Inequalities (1.37) are equivalent to the following ones

$$\bar{X} - u_{\alpha} \left( \frac{\sigma}{\sqrt{n}} \right) \leq \mu \leq \bar{X} + u_{\alpha} \left( \frac{\sigma}{\sqrt{n}} \right).$$

These inequalities still don't give a confidence interval for parameter  $\mu$  because the left and right parts contain an unknown parameter  $\sigma$ . Using another approximation, namely, substituting into these inequalities estimate  $\hat{\sigma} = S$  instead of  $\sigma$ , we obtain the approximate lower and upper limits of the confidence interval with the confidence coefficient  $1-\alpha$  for the mathematical expectation,  $\mu$ :

$$\underline{\mu} = \bar{X} - u_{\alpha} \cdot \frac{S}{\sqrt{n}}$$

$$\bar{\mu} = \bar{X} + u_{\beta} \cdot \frac{S}{\sqrt{n}}$$

#### 2.4.6 Clopper-Pearson Confidence Intervals

Consider construction of the confidence limits for parameter of a binomial distribution. For evaluation of a unit reliability, we often use the following procedure. A sample of size  $n$  is taken randomly from a homogeneous general population. (Sometimes this is not a sample from a population but a special lot of trial items manufactured before mass production.) This sample is tested under some specified condition. After the test completion, one observes that  $m$  items have survived. For this case, the sequence of Bernoulli trials is considered as an appropriate mathematical model.

The distribution function of statistic  $m$  (the number of successes in a series of  $n$  independent Bernoulli trials) has the form

$$F(m, p) = \sum_{0 \leq j \leq m} \binom{n}{j} p^j (1-p)^{n-j}.$$

Note that this function is decreasing in  $p$ . Applying the general formulas (1.33) obtained above, we find that the lower and upper limits of the confidence interval with the confidence coefficient  $1-\alpha$  for parameter  $p$  are found from equations

$$\sum_{m \leq j \leq n} \binom{n}{j} \underline{p}^j (1-\underline{p})^{n-j} = \beta$$

$$\sum_{0 \leq j \leq m} \binom{n}{j} \bar{p}^j (1-\bar{p})^{n-j} = \alpha.$$

Of course, solution of these equations is not a simple task, especially for high order of polynomial. In practice for this purpose one uses tables of incomplete beta function or standard computer programs.

For  $m=0$ , the lower limit equals 0, and for  $m=n$  the upper limit equals 1.

#### 2.4.7 Approximation for Binomial Distribution

Although an accurate calculation of confidence intervals with the help of the Clopper-Pearson method is possible with a computer, sometimes it is useful to have a simple approximate method of the confidence limits construction.

Let  $m$  be the number of observed successes in a series of  $n$  Bernoulli trials with an unknown parameter -- probability of success in a single trial,  $p$ . For constructing confidence interval for  $p$ , let us take value

$$T = \frac{m - np}{\sqrt{np(1-p)}}$$

as an initial statistic. In accordance with DeMoivre-Laplace Limit theorem (see *PRE*), this statistic is asymptotically normal. Thus, for large  $n$ , we can use inequalities

$$-u_\beta \leq \frac{m - np}{\sqrt{np(1-p)}} \leq u_\alpha$$

which holds approximately with the probability close to  $1 - \alpha - \beta$ . These inequalities can be rewritten in the form

$$\frac{m}{n} - u_\alpha \sqrt{\frac{p(1-p)}{n}} \leq p \leq \frac{m}{n} + u_\beta \sqrt{\frac{p(1-p)}{n}}$$

These inequalities still don't give the confidence interval for  $p$  because the left and right sides contain unknown parameter  $p$ . Therefore in practice one often substitutes estimate  $\hat{p} = \frac{m}{n}$  instead of  $p$ . As a result, one has the following lower and upper limits of the confidence interval with the confidence coefficient  $1 - \alpha - \beta$  for parameter  $p$ :

$$\underline{p} = \frac{m}{n} - u_\alpha \sqrt{\frac{m}{n^2} \left(1 - \frac{m}{n}\right)}$$

$$\overline{p} = \frac{m}{n} + u_\beta \sqrt{\frac{m}{n^2} \left(1 - \frac{m}{n}\right)}$$

These confidence intervals are approximate and can be used only if  $n$ , the size of a sample, is large.

#### 2.4.8 Confidence Interval for Parameter of Poisson Distribution

A flow of failures occurring in time for a complex system can be successfully described by Poisson process. As it was shown in Volume 1 of this book, the random number of failures having been occurred within an interval of fixed length has Poisson distribution. The problem is to construct the confidence limits for an unknown parameter of the failure flow, or for a parameter of the Poisson distribution. Consider this problem in formal terms.

Let  $d=0, 1, 2, \dots$  be an integer r.v. with the Poisson distribution (see Section 1.2) with unknown parameter  $\lambda$ . We need to construct the confidence interval for parameter  $\lambda$  on the basis of the observed value of  $d$ . The distribution function of r.v.  $d$  is defined as

$$F(d, \Lambda) = \sum_{0 \leq j \leq d} \frac{\Lambda^j}{j!} e^{-\Lambda}.$$

Note that this function is decreasing in  $\Lambda$ . Applying formulas (1.33), as before, we obtain that the lower and upper limits of the confidence interval for  $\Lambda$  with confidence coefficient equal to  $1 - \alpha - \beta$  are found from equations

$$\exp(-\underline{\Lambda}) \sum_{0 \leq j \leq d-1} \frac{\underline{\Lambda}^j}{j!} = 1 - \beta$$

$$\exp(-\bar{\Lambda}) \sum_{0 \leq j \leq d} \frac{\bar{\Lambda}^j}{j!} = \alpha$$

For  $d=0$  the lower limit is  $\underline{\Lambda} = 0$ . Notice that the left hand sides of these equations can be expressed via gamma distribution (see Section 1.2). Taking into account the relation between gamma and  $\chi^2$  distributions (see formula 1.13), the confidence interval obtained above can be rewritten in the form

$$\underline{\Lambda} = \frac{1}{2} \chi_{\beta}^2(2d),$$

$$\bar{\Lambda} = \frac{1}{2} \chi_{1-\alpha}^2(2d + 2), \quad (1.38)$$

where  $\chi_q^2(n)$  is the quantile of level  $q$  for  $\chi^2$  distribution with  $n$  degrees of freedom.

## 2.5 Test of Hypothesis

### 2.5.1 Introduction

In previous sections we considered the two main statistical inferences related to test results. They are point and confidence estimations. However, sometimes the objective is not to estimate an unknown parameter but to make a decision about some claims regarding that investigated parameter. For instance, we can formulate two contradictory claims about the parameter and the problem is to find which one is correct (with some probability, of course). Such claims might be (a) unknown parameter  $\theta$  relates to some specified level  $A$  as  $\theta < A$  (or  $\theta \leq A$ ,  $\theta > A$ ,  $\theta \geq A$ ,  $\theta = A$ ), or (b) unknown parameter  $\theta$  relates to another unknown parameter  $\theta_0$  as  $\theta < \theta_0$  (or  $\theta \leq \theta_0$ ,  $\theta > \theta_0$ ,  $\theta \geq \theta_0$ ,  $\theta = \theta_0$ ). Examples of the first case: the empirical probability of call completion obtained during some test of telephone network is compared with a specified baseline; an unknown and random strength of some mechanical construction is analyzed under a fixed load, etc. Analogous examples

for the second case: two competing telephone companies compare their probabilities of call completion; two constructions are compared by their strength under some certain load, etc. Each such claim is called a *statistical hypothesis*. The procedure for executing this statistical inference is called *tests of hypothesis*.

A test of hypotheses includes two contradictory hypotheses, one of them, which is believed to be true, is called the *null hypothesis* and is denoted by  $H_0$  and another is called *alternative hypothesis* and is denoted by  $H_a$ . For instance, the null hypothesis

$H_0: \theta = \theta_0$  means that we suggest that the true hypothesis is that parameter  $\theta$  is equal to some specified number  $\theta_0$  (which is called the *null value*). For this null hypothesis, there might be the following three possible alternative hypotheses:

- (a)  $H_1: \theta < \theta_0$ ,
- (b)  $H_1: \theta > \theta_0$ , and
- (c)  $H_1: \theta \neq \theta_0$ .

Hypothesis  $H: \theta = \theta_0$  (that is,  $\theta_0$  is a single value) is called a *simple hypothesis*. Hypothesis  $H: \theta \in D$  **Ошибка! Не указан аргумент ключа.** is called a *composite hypothesis* if  $D$  is some set of values of parameter  $\theta$  including more than a single point. Thus, alternative hypotheses (a), (b), and (c) considered above are composite ones. Of course, the null hypothesis might be one of composite hypothesis and the alternative hypothesis might be, on the contrary, a simple one.

Hypotheses considered above are called *parametrical* contrasting to *non-parametrical*. The latter will be considered in Section 1.5.6.

## 2.5.2 Two Simple Hypotheses

Let us have a sample of size  $n$  of independent observations of r.v.

$$X_1, \dots, X_n$$

with density function  $f(\mathbf{X}, \theta)$  where  $\mathbf{X}=(X_1, \dots, X_n)$  and  $\theta$  can be also a vector (multi-dimensional parameter),  $\theta=(\theta_1, \dots, \theta_k)$ .

Let us consider the simplest case where two simple hypotheses

$$H_0: \theta = \theta_0, \text{ and } H_1: \theta = \theta_1$$

are tested.

Test procedure allows us to accept or reject the null hypothesis. This procedure has two constituents: (1) a *test statistic* that is a function of the sample data on which the decision of acceptance (or rejection) is based, and (2) a *rejection region*,  $W$ , representing the set of all test statistic values for which the null hypothesis will be rejected. The rule used for hypotheses rejection is also called a *criterion of test of hypothesis*.

We might make two following types of errors:

A *type I error* ( $\alpha$ ) consisting in rejecting  $H_0$  when it is true,

A *type II error* ( $\beta$ ) consisting in acceptance  $H_0$  when it is false.

The errors of the first and second types are defined as follows:

$$\alpha = P\{(X_1, \dots, X_n) \in W \mid H_0\},$$

and

$$\beta = P\{(X_1, \dots, X_n) \in \bar{W} \mid H_1\},$$

where  $\bar{W}$  is the complementary set to  $W$  and  $P\{A \mid H_j\}$  is the probability of event  $A$  under the condition that hypothesis  $H_j$  is true,  $j=0, 1$ .

Let us introduce the *likelihood function*

$$L(X_1, \dots, X_n, \theta) = f(X_1, \theta) \cdot \dots \cdot f(X_n, \theta).$$

The errors of the both types can be written as

$$\alpha = \int \dots \int_W L(x_1, \dots, x_n, \theta_0) dx_1 \dots dx_n$$

and

$$\beta = \int \dots \int_{\bar{W}} L(x_1, \dots, x_n, \theta_1) dx_1 \dots dx_n$$

The error of type  $\beta$  is also called the *level of significance* of the criterion. The value of  $1-\beta$ , equal to the probability of rejection of hypothesis  $H_0$  when it is true, is called the *power of the criterion*.

### 2.5.3 Neumann-Pearson Criterion

The most frequent problem of the criterion construction is formulated as follows: for a fixed level of significance  $\alpha$ , construct the criterion (or, in other words, critical set  $W$ ) with the maximum power  $1-\beta$ . The solution of this problem is given by the well known result of Neumann and Pearson which for continuous variables is formulated as follows. Let us introduce the function of the sample

$$\varphi = \varphi(X_1, \dots, X_n) = \frac{L(X_1, \dots, X_n, \theta_1)}{L(X_1, \dots, X_n, \theta_0)}$$

which is the ratio of the two likelihood functions: null and alternative, respectively. The optimal, or maximally powerful for the given level of significance  $\alpha$  *Neumann-Pearson criterion* is constructed in the following way. The critical region  $W$  includes all those  $(X_1, \dots, X_n)$  for which the inequality

$$\varphi = \varphi(X_1, \dots, X_n) \geq C$$

holds. Here  $C$  is a constant which is found from the condition

$$P\{\varphi \geq C | H_0\} = \alpha,$$

that is, the level  $\alpha$  is guaranteed. In more detail, this condition can be written as

$$\int \dots \int_{\varphi(X_1, \dots, X_n) \geq C} L(x_1, \dots, x_n, \theta_0) dx_1 \cdot \dots \cdot dx_n = \alpha.$$

The Neumann-Pearson result remains analogous for discrete r.v.'s. The only difference is in a possible necessity of "randomization" of the optimal criterion (for details, see Lehmann, 1959).

**Example 1.26** Let us construct the Neumann-Pearson criterion in the case of normal distribution. Consider the test of two simple hypotheses

$$H_0: \mu = \mu_0 \text{ and } H_1: \mu = \mu_1$$

where  $\mu$  is the mean,  $\mu_0, \mu_1$  are some given values,  $\mu_0 < \mu_1$ , and the variance  $\sigma^2$  is supposed to be known.

In this case the likelihood function is

$$L(X_1, \dots, X_n) = \left( \frac{1}{\sigma\sqrt{2\pi}} \right)^n \exp\left[ -\frac{1}{2\sigma^2} \sum_{1 \leq i \leq n} (x_i - \mu)^2 \right].$$

Thus, the likelihood ratio is

$$\begin{aligned} \varphi &= \frac{L(X_1, \dots, X_n, \mu_1)}{L(X_1, \dots, X_n, \mu_0)} \\ &= \exp\left[ \frac{(\mu_1 - \mu_0)}{\sigma^2} \sum_{1 \leq i \leq n} x_i \right] \cdot \exp\left[ -\frac{n(\mu_1 - \mu_0)^2}{2\sigma^2} \right]. \end{aligned}$$

From here we see that the critical region  $W$  is given in this case as

$$\sum_{1 \leq i \leq n} X_i \geq C \quad (1.38)$$

where constant  $C$  is chosen from the condition of the given level of significance  $\alpha$ :

$$P \left\{ \sum_{1 \leq i \leq n} X_i \geq C \mid \mu = \mu_0 \right\} = \alpha. \quad (1.39)$$



Random variable  $\sum_{1 \leq i \leq n} X_i$  has normal distribution with the mean  $n\mu_0$  and variance  $n\sigma^2$  (see Section 1.2). Thus the latter equality can be rewritten as

$$1 - \Phi\left(\frac{C - n\mu_0}{\sigma\sqrt{n}}\right) = \alpha \quad (1.40)$$

where  $\Phi(\cdot)$  is the function of the standard normal distribution (Laplace function). Thus,

$$\frac{C - n\mu_0}{\sigma} \cdot \sqrt{n} = U_{1-\alpha}$$

where  $U_{1-\alpha}$  is the quantile of the standard normal distribution of the level  $1-\alpha$ . Thus, constant  $C$  which determines the critical region (1.38) can be found as

$$C = n\mu_0 + U_{1-\alpha}\sigma\sqrt{n}.$$

The error of type II is defined as

$$\beta = P\left\{\sum_{1 \leq i \leq n} x_i < C \mid \mu = \mu_1\right\} = \Phi\left\{\frac{C - n\mu_1}{\sigma\sqrt{n}}\right\}. \quad (1.41)$$

By the Neumann-Pearson lemma, this value of  $\beta$  is minimum possible for the specified value of  $\alpha$ .

**Example 1.27** Consider the previous example for the condition  $\mu_1 < \mu_0$ . Following the steps analogous to that at the Example 1.5.1, we find that the optimal Neumann-Pearson criterion with the level of significance  $\alpha$  is given by the critical region

$$\sum_{1 \leq i \leq n} x_i \leq C$$

where constant  $C$  is chosen from the condition

$$P\left\{\sum_{1 \leq i \leq n} x_i \leq C \mid \mu = \mu_0\right\} = \alpha$$

or

$$\Phi\left\{\frac{C - n\mu_0}{\sigma\sqrt{n}}\right\} = \alpha.$$

This gives

$$\frac{C - n\mu_0}{\sigma\sqrt{n}} = U_\alpha = -U_{1-\alpha}$$

and finally

$$C = n\mu_0 - U_{1-\alpha}\sigma\sqrt{n}.$$

**Example 1.28** Let us construct the optimal Neumann-Pearson criterion for parameter of exponential distribution.

Consider two simple hypotheses

$$H_0: \lambda = \lambda_0 \text{ and } H_1: \lambda = \lambda_1$$

where  $\lambda_0$  and  $\lambda_1$  are the given levels for the distribution with density  $f(x, \lambda) = \lambda e^{-\lambda x}$ ,  $x > 0$ . In this case the likelihood function is

$$L(X_1, \dots, X_n) = \lambda^n \exp\left(-\lambda \sum_{1 \leq i \leq n} X_i\right).$$

The likelihood ratio has the form

$$\varphi = \left(\frac{\lambda_1}{\lambda_0}\right)^n \exp\left[-(\lambda_1 - \lambda_0) \sum_{1 \leq i \leq n} X_i\right].$$

It follows that the critical region is defined by the inequality

$$\sum_{1 \leq i \leq n} X_i \leq C$$

where constant  $C$  is chosen from the condition

$$\Pr \left\{ \sum_{1 \leq i \leq n} x_i \leq C \mid \lambda = \lambda_0 \right\} = \alpha$$

The latter condition can be rewritten in the form

$$P \left\{ 2 \lambda_0 \sum_{1 \leq i \leq n} x_i \leq 2 \lambda_0 C \mid \lambda = \lambda_0 \right\} = \alpha.$$

Random variable  $2 \lambda_0 \sum_{1 \leq i \leq n} x_i$  for  $\lambda = \lambda_0$  has gamma distribution (see Section 1.2) with parameters  $(1/2, n)$ , which is equivalent,  $\chi^2$  distribution with  $2n$  degrees of freedom. Thus, the latter condition is equivalent to

$$H_{2n}(2 \lambda_0 C) = \alpha$$

where  $H_{2n}(\cdot)$  is the cumulative function of  $\chi^2$  distribution with  $2n$  degrees of freedom. Finally, for constant  $C$ , we have

$$C = \frac{\chi_{\alpha}^2(2n)}{2 \lambda_0}$$

where  $\chi_{\alpha}^2(2n)$  denotes the quantile of the level  $\alpha$  for  $\chi^2$  distribution with  $2n$  degrees of freedom. The minimum error of the II type in this case equals

$$\beta = P\left\{ \sum_{1 \leq i \leq n} X_i > C \mid \lambda = \lambda_1 \right\} = 1 - H_{2n}(2\lambda_1 C) = 1 - H_{2n}\left[ \chi_{\alpha}^2(2n) \cdot \left( \frac{\lambda_1}{\lambda_0} \right) \right].$$

It follows from the Neumann-Pearson statement, this value cannot be decreased for a given error of the first type,  $\alpha$ .

**Example 1.29** Let us construct the optimal Neumann-Pearson criterion for parameter of binomial distribution.

Consider two simple hypotheses

$$H_0: p=p_0 \text{ and } H_1: p=p_1$$

where  $p$  is the probability of success for a sequence of Bernoulli trials,  $p_0$  and  $p_1$  are the given values, and  $p_0 < p_1$ . Let  $X_j$  denote the result of the  $j$ th trial.  $X_j=1$  means success in the  $j$ th trial and  $X_j=0$  means failure,  $\Pr\{X_j=1\}=p$  and  $\Pr\{X_j=0\}=1-p$ .

The likelihood function in this case is

$$L(X_1, \dots, X_n, p) = \binom{n}{m} p^m (1-p)^{n-m}$$

where  $m = \sum_{1 \leq j \leq n} x_j$  is the total number of observed successes in the sequence of  $n$  trials. The likelihood ratio function is

$$\varphi = \frac{L(X_1, \dots, X_n, p_1)}{L(x_1, \dots, x_n, p_0)} = \left( \frac{p_1}{p_0} \right)^m \left( \frac{1-p_0}{1-p_1} \right)^{n-m}$$

From the latter expression, we easily find that the critical region for the Neumann-Pearson criterion has the form

$$m \geq C. \tag{1.42}$$

The DeMoivre-Laplace limit theorem states that for large  $n$ , the distribution of r.v.  $m$  is approximated by the normal distribution with the mean  $\mu=np$  and variance  $\sigma^2=np(1-p)$ . Using the normal approximation, constant  $C$  in (1.42) can be found from the condition

$$\Pr\{m \geq C \mid p = p_0\} \approx 1 - \Phi\left( \frac{C - np_0}{\sqrt{np_0(1-p_0)}} \right) = \alpha$$

and, consequently, we have

$$C = np_0 + U_{1-\alpha} \sqrt{np_0(1-p_0)}. \quad (1.43)$$

The error of type II is defined as

$$\beta = P\{m \leq C | p = p_1\} \approx \Phi\left(\frac{C - np_1}{\sqrt{np_1(1-p_1)}}\right). \quad (1.44)$$

#### 2.5.4 Sample Size

Let some values Type I and II errors ( $\alpha$  and  $\beta$ ) be specified, so that  $\alpha + \beta < 1$ . We need to determine before the test the needed sample size,  $n^*$ , to construct the criterion for testing two simple hypotheses  $H_0: \theta = \theta_0$  and  $H_1: \theta = \theta_1$  with given levels of  $\alpha$  and  $\beta$ . The quantity of  $n^*$  is defined as a minimum integer  $n$  for which the inequalities

$$\begin{aligned} P\{\varphi(X_1, \dots, X_n) \geq C | \theta = \theta_0\} &\leq \alpha \\ P\{\varphi(X_1, \dots, X_n) < C | \theta = \theta_1\} &\leq \beta \end{aligned} \quad (1.45)$$

hold for some constant  $C = C^*$ . In this case the Neumann-Pearson criterion delivering given  $\alpha$  and  $\beta$  has the critical region defined by the inequality

$$\varphi(X_1, \dots, X_n) \geq C^*.$$

**Example 1.30** Consider a sample from normal distribution. For the situation considered above in Example 1.26, from (1.40) and (1.41) we find that (1.45) in this case has the form

$$1 - \Phi\left(\frac{C - n\mu_0}{\sigma\sqrt{n}}\right) \leq \alpha,$$

$$\Phi\left(\frac{C - n\mu_1}{\sigma\sqrt{n}}\right) \leq \beta.$$

From the above formula, we obtain that the minimum sample size  $n^*$  and the corresponding critical constant  $C^*$  for given  $\alpha$  and  $\beta$  are determined from the following system of equations

$$1 - \Phi\left(\frac{C - n\mu_0}{\sigma\sqrt{n}}\right) = \alpha,$$

$$\Phi\left(\frac{C - n\mu_1}{\sigma\sqrt{n}}\right) = \beta.$$

Using quantiles of the standard normal distribution, we can write these equations in the form

$$\frac{C - n\mu_0}{\sigma\sqrt{n}} = U_{1-\alpha},$$

$$\frac{C - n\mu_1}{\sigma\sqrt{n}} = U_\beta = -U_{1-\beta}$$

Thus, we obtain a sample size as

$$n^* = \frac{\sigma^2(U_{1-\alpha} + U_{1-\beta})^2}{(\mu_1 - \mu_0)^2}. \quad (1.46)$$

For example, let hypotheses

$$H_0: \mu = \mu_0 = 3.5 \text{ and } H_1: \mu = \mu_1 = 3.8$$

should be tested for known  $\sigma = 0.8$  and given  $\alpha = 0.05$  and  $\beta = 0.1$ . Using (1.46) and taking into account that  $U_{1-\alpha} = U_{0.95} = 1.64$  and  $U_{1-\beta} = U_{0.9} = 1.28$ , we find that the sample size is  $n^* = 61$ .  $\square$

**Example 1.31** Find the sample size for testing by Bernoulli scheme. For the problem considered in Example 1.5.4, we obtain from (1.43) and (1.44) the following equation system for determination of  $n^*$  and  $C^*$ :

$$1 - \Phi\left(\frac{C - np_0}{\sqrt{np_0(1-p_0)}}\right) = \alpha$$

$$\Phi\left(\frac{C - np_1}{\sqrt{np_1(1-p_1)}}\right) = \beta$$

or in the equivalent form

$$\frac{C - np_0}{\sqrt{np_0(1-p_0)}} = U_{1-\alpha},$$

$$\frac{C - np_1}{\sqrt{np_1(1-p_1)}} = U_\beta = -U_{1-\beta}$$

and, finally,

$$n^* = \frac{[U_{1-\alpha}\sqrt{p_0(1-p_0)} + U_{1-\beta}\sqrt{p_1(1-p_1)}]^2}{(p_1 - p_0)^2}$$

This expression gives the needed sample size for testing of simple hypotheses of type  $H_0: p=p_0, H_1: p=p_1$  for a sequence of Bernoulli trials to satisfy the given probabilities of errors  $\alpha$  and  $\beta$ . Since the quantity of  $n^*$  usually is fractional, in practice we take the smallest integer larger than  $n^*$ .  $\square$

The sample size obtained with the use of the Neumann-Pearson criterion cannot be improved (decreased) if this sample size has been determined and fixed in advance. Nevertheless, the average sample size can be decreased for the same error probabilities  $\alpha$  and  $\beta$  in sequential trials where the decision about stopping the test is made during testing depending on the obtained data (see details in Part III).

### 2.5.5 Composite Parametrical Hypotheses

Let us test two composite hypotheses

$$H_0: \theta \in D_0 \text{ and } H_1: \theta \in D_1 \tag{1.47}$$

where  $D_0$  and  $D_1$  are some non-intersected regions of domain of  $\theta$ , for instance, these regions are  $\theta \leq \theta_0$  and  $\theta \geq \theta_1$  where  $\theta_0$  and  $\theta_1$  are given values,  $\theta_0 < \theta_1$ .

Parametrical hypotheses of type (1.47) can be one-parametrical (if parameter  $\theta$  is a scalar) or multi-parametrical (if parameter  $\theta$  is a vector). The criterion of testing of composite hypotheses is also defined via a critical set  $W$  of the sample  $(X_1, \dots, X_n)$ . As before, if the sample  $(X_1, \dots, X_n)$  belongs to  $W$  then the null hypothesis  $H_0$  is rejected and the alternative hypothesis is accepted. If the sample does not belong to  $W$ , then the alternative hypothesis is rejected and the null hypothesis is accepted.

The error probabilities of the I and II types have the same meaning as above and are defined as

$$\alpha(\theta) = P\{(X_1, \dots, X_n) \in W | \theta\}, \quad \theta \in D_0$$

$$\beta(\theta) = P\{(X_1, \dots, X_n) \in \bar{W} | \theta\}, \quad \theta \in D_1$$

where  $P\{\cdot | \theta\}$  is the conditional probability under condition that the true value of the parameter equals  $\theta$ . Contrary to simple hypotheses, values  $\alpha(\theta)$  and  $\beta(\theta)$  are some functions of parameter

$\theta$ . The maximum possible value of the error of type I is

$$\alpha = \max_{\theta \in D_0} \alpha(\theta)$$

and is called a *criterion scale*, or a *level of significance*.

Function

$$E(\theta) = P\{(X_1, \dots, X_n) \in W | \theta\} = \int \dots \int_W L(x_1, \dots, x_n, \theta) dx_1 \cdot \dots \cdot dx_n$$

expressing the rejection probability for hypothesis  $H_0$  dependence of parameter  $\theta$ , is called *function of criterion power (power function)*. If there exists such a criterion which maximizes function  $E(\theta)$  in all possible criteria simultaneously for all  $\theta \in D_1$  (for fixed  $\theta$ ), this criterion is called uniformly most powerful. Such criteria exist only in some particular cases for simple hypotheses (see examples below).

Probabilities of errors are expressed via the power functions as follows

$$E(\theta) = E(\theta) \text{ for } \theta \in D_0, \quad (1.48)$$

$$1 - E(\theta) = 1 - E(\theta) \text{ for } \theta \in D_1. \quad (1.49)$$

Thus the uniformly most powerful criterion (if it exists) minimizes the error of type II (for fixed  $\theta$ ) for all  $\theta \in D_1$ .

Remark. Equalities (1.48) and (1.49) are true only for values of parameter  $\theta$  indicated there. For values differing from the mentioned, the values of  $E(\theta)$  and  $1 - E(\theta)$  have no sense of probabilities of error.

Sometimes together with the power function, the so-called *operative characteristic* of a criterion is used

$$S(\theta) = P\{(X_1, \dots, X_n) \in \bar{W} | \theta\} = \int \dots \int_{\bar{W}} L(x_1, \dots, x_n, \theta) dx_1 \cdot \dots \cdot dx_n$$

that is, the probability to accept the null hypothesis,  $H_0$ , if the true value of the parameter equals  $\theta$ . Obviously, the operative characteristic and power function can be expressed via each other as  $S(\theta) = 1 - E(\theta)$ .

Let us construct the criteria for testing of composite hypotheses on normal distribution.

**Example 1.32** Consider test of simple hypothesis  $H_0: \theta = \theta_0$  vs. the composite one  $H_1: \theta > \theta_0$  where  $\theta$  is the mean of normal distribution with the known variance  $\sigma^2$ .

For any  $\theta_1 > \theta_0$ , the critical region of the most powerful Neumann-Pearson criterion of significance  $\alpha$  for simple hypotheses  $\theta = \theta_0$  against  $\theta = \theta_1$  has the form (1.38) where constant  $C$  is chosen from (1.39) or (1.40) and, consequently, does not depend on  $\theta_1$ . This means that the criterion constructed above for that simple hypotheses with the critical region

$$\sum_{1 \leq i \leq n} X_i \geq C = n\mu_0 + U_{1-\alpha} \sigma \sqrt{n} \quad (1.50)$$

is the uniformly most powerful criterion (with the significance level  $\alpha$ ) for the composite alternative hypothesis  $H_1: \mu > \mu_0$ .

**Example 1.33** Under the previous example's conditions, let us consider the test of simple hypothesis  $H_0: \mu = \mu_0$  vs. composite hypothesis  $H_1: \mu < \mu_0$ .

In this case, using the results obtained in Example 1.27, we find that the uniformly most powerful criterion of significance  $\alpha$  for this case is given by the critical region

$$\sum_{1 \leq i \leq n} X_i \leq C = n\mu_0 - U_{1-\alpha} \sigma \sqrt{n}.$$

**Example 1.34** For conditions of Example 1.32, let us consider the test of two composite hypotheses

$$H_0: \mu \leq \mu_0 \text{ and } H_1: \mu \geq \mu_1 \quad (1.51)$$

where  $\mu_0 < \mu_1$ .

Notice that for the criterion with the critical region (1.50), the probability of the error of type I

$$\alpha(\mu) = P\left\{ \sum_{1 \leq i \leq n} X_i \geq C \mid \mu \right\}$$

$$= 1 - \Phi\left(\frac{C - n\mu}{\sigma\sqrt{n}}\right) = 1 - \Phi\left(U_{1-\alpha} + (\mu_0 - \mu) \frac{\sqrt{n}}{\sigma}\right)$$

monotone increases in  $\mu$ . Thus, the maximum value of the probability of the error of the I type is determined as

$$\alpha = \max_{\mu \leq \mu_0} \alpha(\mu)$$

and achieved at the point  $\mu = \mu_0$ . It follows that this criterion used to composite hypotheses (1.51) has the significance level  $\alpha = \alpha(\mu_0)$ . Following the same arguments as in Example 1.32, we find



that the criterion with the critical region (1.50) is the uniformly most powerful criterion for the considered problem with composite hypotheses.  $\square$

**Example 1.35** Consider a test for two composite hypotheses

$$H_0: \mu = \mu_0 \text{ and } H_1: \mu > \mu_1 \quad (1.52)$$

for the mean,  $\mu$ , of normal distribution with the unknown variance  $\sigma^2$ .

In contrast to Example 1.32 hypothesis  $H_0$  in this case is also composite. For  $\mu = \mu_0$ , the statistic

$$\left( \frac{\bar{X} - \mu_0}{S} \right) \sqrt{n-1} \quad (1.53)$$

has Student distribution with  $(n-1)$  degrees of freedom (see Section 1.2 above). From here, we obtain that the criterion with the significance level  $\alpha$  for hypotheses (1.52) is given by the following critical region

$$\left( \frac{\bar{X} - \mu_0}{S} \right) \sqrt{n-1} \geq t_{1-\alpha}(n-1)$$

where  $t_{1-\alpha}(n-1)$  is the quantile of the level  $1-\alpha$  of Student distribution with  $(n-1)$  degrees of freedom.

Analogously, using the statistic (1.53), we construct the criterion for the test of composite hypotheses of the form

$$H_0: \mu = \mu_0 \text{ and } H_1: \mu < \mu_0 \quad (1.54)$$

or

$$H_0: \mu = \mu_0 \text{ and } H_1: \mu \neq \mu_0 \quad (1.55)$$

For the hypothesis (1.54) the criterion of significance  $\alpha$  is given by the critical region

$$\left( \frac{\bar{X} - \mu_0}{S} \right) \sqrt{n-1} \leq t_{\alpha}(n-1)$$

and for the alternative hypothesis (1.55) by

$$\left( \frac{|\bar{X} - \mu_0|}{S} \right) \sqrt{n-1} \geq t_{1-\frac{\alpha}{2}}(n-1) \quad \square$$

**Example 1.36** Consider a test hypothesis about equality of the mathematical expectations of two normal distributions. Let us have two independent samples  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_m$  of sizes  $n$  and  $m$  from normal distributions with parameters  $\mu_1, \sigma_1^2$  and  $\mu_2, \sigma_2^2$ , respectively. Consider the following cases of composite hypotheses where the variances  $\sigma_1^2$  and  $\sigma_2^2$  are known:

1.  $H_0: \mu_1 = \mu_2$  and  $H_1: \mu_1 > \mu_2$
2.  $H_0: \mu_1 = \mu_2$  and  $H_1: \mu_1 < \mu_2$
3.  $H_0: \mu_1 = \mu_2$  and  $H_1: \mu_1 \neq \mu_2$ .

(1.56)

The difference of the sample means  $(\bar{X} - \bar{Y})/\sigma$  has normal distribution with the mean equals  $\mu_1 - \mu_2$  and the variance equals  $(\sigma_1^2/n) + (\sigma_2^2/m)$ . It follows that if the null hypothesis is true, that is, if  $\mu_1 = \mu_2$ , statistic

$$\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \quad (1.57)$$

has normal distribution with parameters  $(0, 1)$ . Then the significance criterion,  $\alpha$ , for the problems formulated above are given by

$$1. \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \geq U_{1-\alpha}$$

$$2. \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \leq U_{\alpha}$$

$$3. \frac{|\bar{X} - \bar{Y}|}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \geq U_{1-\frac{\alpha}{2}}$$

where  $u_{\alpha}$  is the quantile of level  $\alpha$  of the standard normal distribution.

Let us now consider problems (1.56) for the case where the variances are unknown but equal, that is,  $\sigma_1^2 = \sigma_2^2 = \sigma^2$ . In this case statistics  $\frac{nS_1^2}{\sigma^2}$  and  $\frac{nS_2^2}{\sigma^2}$  have  $\chi^2$  distributions with  $(n-1)$  and  $(m-1)$  degrees of freedom, respectively. Thus, statistic  $\frac{nS_1^2}{\sigma^2} + \frac{nS_2^2}{\sigma^2}$  also has  $\chi^2$  distributions with  $(n+m-2)$  degrees of freedom. Taking into account that statistic (1.57) has the standard normal distribution for  $\mu_1 = \mu_2$ , we conclude that statistic

$$\frac{(\bar{X} - \bar{Y})\sqrt{n+m-2}}{\sqrt{\frac{1}{n} + \frac{1}{m}} \cdot \sqrt{nS_1^2 + mS_2^2}}$$

has Student distribution with  $(n+m-2)$  degrees of freedom (see Section 1.2). Thus, we find that the criterion of significance  $\square$  for the considered problems are

$$1. \frac{(\bar{X} - \bar{Y})b_{nm}}{\sqrt{nS_1^2 + mS_2^2}} \geq t_{1-\alpha}(n+m-2)$$

$$2. \frac{(\bar{X} - \bar{Y})b_{nm}}{\sqrt{nS_1^2 + mS_2^2}} \geq t_{\alpha}(n+m-2)$$

$$3. \frac{(\bar{X} - \bar{Y})b_{nm}}{\sqrt{nS_1^2 + mS_2^2}} \geq t_{1-\frac{\alpha}{2}}(n+m-2)$$

where  $b_{nm} = \sqrt{nm \cdot \frac{n+m-2}{n+m}}$  94,  $t_{\square}(n+m-2)$  is the quantile of level  $\square$  of Student distribution with  $(n+m-2)$  degrees of freedom.  $\square$

### 3. PLANS OF TESTS WITH A SINGLE CENSORSHIP

In this chapter, we consider simple test plans with a single censorship, that is, tests which are performed for a specified in advance duration of time  $T$ , or up to a fixed number of failures,  $r$ . Statistical inferences for these test plans were almost completely developed for the exponential distributions by Epstein and Sobel (1963), Epstein (1960a, 1960b, 1960c), and others. However, for many other important practical cases, there is no solution even for simple cases. Some new results for multiple censorship we will consider in the next chapter.

#### 3.1 Introduction

##### 3.1.1 Test Without Replacement.

Consider a unit whose random TTF  $\square$  has an unknown distribution  $F(t) = \mathbf{P}\{\square \leq t\}$ .  $N$  identical units are placed on test at  $t=0$ . First, consider a case where the test continues until all  $N$  units will have failed. In this case we observe a sequence of failure moments

$$0 < t_1 < t_2 < \dots < t_r < \dots < t_N \quad (2.1)$$

where  $t_r$  is the moment of the  $r$ th failure,  $1 \leq r \leq N$ .

This type of test forms the standard complete sample of  $N$  independently distributed r.v.'s  $\square$ . Test results (2.1) are ordered statistics.

In practice, we usually have no possibility to perform test until the failure of all units. A test terminates either when the  $r$ th failure has occurred at time  $t_r$ ,  $r < N$ , or when the fixed (in advance) duration  $T$  is exceeded.

Below we will use the following notation for test plans:

1. Plan **[N U r]**. Here the first symbol, **N**, denotes the number of identical units tested at  $t=0$ . The second symbol, **U**, means that the tested units are unreparable while testing. The third symbol specifies the stopping rule. In this particular case, a symbol **r** reflects that the test terminates at the moment  $t_r$ ,  $r \leq N$ , i.e., when the  $r$ th failure has happened. Thus the moment of the test termination is random. The results of the test are moments of failure's occurrence

$$0 < t_1 < t_2 < \dots < t_r.$$

(For remaining units the test is stopped at moment  $t_r$ .)

2. Plan **[N U T]**. Here the first two symbols have the same meaning as in the previous case, and symbol **T** reflects that the test continues up to the in advance fixed moment. The number of failures for this plan,  $d$ , is random. The results of the test are moments of failure's occurrence before time  $T$

$$0 < t_1 < t_2 < \dots < t_d < T$$

where  $d \leq N$ . Obviously,  $d$  is a random number.

3. Plan **[N U (r,T)]**. Here the first two symbols have the same sense as in the previous cases, and symbol **(r,T)** reflects that the test terminates at moment  $\tau = \min(t_r, T)$ , that is, either at the in advance fixed moment  $T$ , or at the moment of the  $r$ th failure occurrence depending on what happens earlier. The number of failures for this plan,  $d$ , is a r.v. restricted by  $r$ . The moment of the test termination is also random and restricted from the right. The results of the test are moments of failure occurred before time  $T$

$$0 < t_1 < t_2 < \dots < t_d < T$$

where  $d \leq r$ .

The first test plan described by statistic (2.1) is plan **[N U r]** with  $r=N$ , i.e., plan **[N U N]** corresponds to the complete (uncensored) sample of size  $N$ . Different test plans can be depicted on plane  $(t, d(t))$ , where  $d(t)$  is the number of failures occurred by in time  $t$  (Figure 2.1 - 2.3). The test terminates when the process  $d(t)$  reaches the bound of the corresponding region.

Note that for all plans of type **[• U •]** a normalized trajectory of the process  $\frac{d(t)}{N}$  **Ошибка!** **Не указан аргумент ключа.** coincides with the graph of the empirical d.f.  $\hat{F}_N(t)$  **Ошибка!** **Не указан аргумент ключа.** for all  $t$  for which this trajectory was observed. In other words, the

graph of the type mentioned above represents an empirical d.f. on time interval  $[0, \tau]$  where  $\tau$  is the stop moment.

The test termination before moment  $t_N$  is called *censorship*. The corresponding moment is called the *moment of censorship*. For each plan considered above, the moment of censorship is unique. Such plans are called plans with a single censoring. (More complex plans are considered in Chapter 3.)

Figures 2.1-2.3

### 3.1.2 Test Plans With Renewable Units

In some test plans units are renewed after failure. It is assumed that replacing units are identical to the initial ones in reliability sense. Thus, the number of tested units is constant during the entire testing period.

The test plans where units are instantly replaced after a failure are denoted as  $[\bullet \mathbf{R} \bullet]$ . The remaining attributes of the test plan notation preserve as above. So, there are the following test plans for renewable (replaceable) units.

1. Plan  $[\mathbf{N} \mathbf{R} \mathbf{r}]$ . There are  $N$  units at the initial moment  $t=0$ . All units are independent and after each failure a new units completely identical to the initial one replaces the failed one. The test terminates at the moment  $t_r$ ,  $r \leq N$ , when the  $r$ th failure has happened.

2. Plan  $[\mathbf{N} \mathbf{R} \mathbf{T}]$ . This plan differs from the previous one: the test continues up to the in advance fixed moment  $T$ .

3. Plan  $[\mathbf{N} \mathbf{R} (\mathbf{r}, \mathbf{T})]$ . Here the test terminates at moment  $\tau = \min(t_r, T)$  which we have explained above.

Test plans without replacement we will call the  $U$ -type plans, and test plans with replacement we will call the  $R$ -type plans. Graphical illustration of the  $R$ -plans coincides with that of the  $U$ -plans but in this case there is no analogy of the trajectory with an empirical d.f.

## 3.2 Exponential Distribution

Here we briefly present main results for the simplest and better investigated case where the d.f. of the unit's TTF is exponential,  $F(t) = 1 - e^{-\lambda t}$ , with unknown parameter  $\lambda$ ,  $\lambda > 0$  [Epstein and Sobel (1953); Epstein (1960a, 1960b, 1960c); Gnedenko et al. (1965)]. For this distribution all reliability indexes of interest can be expressed through parameter  $\lambda$ , therefore estimating of this parameter is the main issue.

### 3.2.1 Test Plan $[\mathbf{N} \mathbf{U} \mathbf{r}]$

In this case the results of test are moments of first  $r$  failures

$$t_1 < t_2 < \dots < t_r. \quad (2.2)$$

Due to the memoriless property of the exponential distribution, r.v.'s  $t_1, t_2 - t_1, \dots, t_r - t_{r-1}$  are independent. An r.v.  $t_j - t_{j-1}$  has the exponential distribution with parameter  $(N - j + 1)\lambda$ ,  $1 \leq j \leq r$ .

**Ошибка! Не указан аргумент ключа.** (Obviously,  $t_0=0$ .) Hence the likelihood function, i.e., the joint density function of test results (2.2), is

$$L = \prod_{1 \leq j \leq n} (N - j + 1) \lambda e^{-(N-j+1)\lambda(t_j - t_{j-1})}$$

or after simple transformations

$$L = C \lambda^r e^{-\lambda S} \tag{2.3}$$

where  $C$  is a normalizing constant,  $C = N(n-1) \dots (N-r+1)$ , and the value of  $S$  equals the total testing time of all units, that is,

$$S = \sum_{1 \leq j \leq r} (N - j + 1)(t_j - t_{j-1}) = t_1 + \dots + t_r + (n - r)t_r. \tag{2.4}$$

Thus, the likelihood function depends on test results via the value of  $S$ . It means (see above Section 1.3.4) that statistic  $S$  is a sufficient statistic for this test plan.

**Point Estimate of Parameter** □

Applying the maximum likelihood method, we obtain from (2.3) that the maximum likelihood equation has the form

$$\frac{\partial \ln L}{\partial \lambda} = \frac{r}{\lambda} - S = 0$$

and hence the maximum likelihood estimate (MLE) for □ is

$$\hat{\lambda} = \frac{r}{S}. \tag{2.5}$$

This estimate is biased. We can prove this by finding the density function of the distribution of statistic  $S$ . R.v.'s  $X_j = (N-j+1)(t_j - t_{j-1})$ ,  $1 \leq j \leq r$ , are independent and identical r.v.'s exponentially distributed with parameter  $\lambda$ . Therefore, as we can see from (2.4), the r.v.  $S$  is the sum of  $r$  independent exponentially distributed r.v.'s. From this, it follows (see Section 1.2 above) that statistic  $S$  has the gamma distribution with parameters  $(\lambda, r)$  and density function

$$f(x) = \frac{\lambda^r x^{r-1}}{(r-1)!} e^{-\lambda x}. \quad (2.6)$$

From (2.6), we can easily find the mathematical expectation of estimate  $\hat{\lambda}$  104:

$$E\{\hat{\lambda}\} = \int_0^{\infty} \frac{r}{x} \cdot \frac{\lambda^r x^{r-1}}{(r-1)!} e^{-\lambda x} dx = \left(\frac{r}{r-1}\right)\lambda.$$

Consequently, the MLE (2.5) is biased. Because of the bias, we usually use the estimate of the form

$$\hat{\lambda}' = \left(\frac{r-1}{r}\right)\hat{\lambda} = \frac{r-1}{S}$$

which is unbiased (for  $r > 1$ ). Further, we can easily show on the basis of (2.6) that the variance of the unbiased estimate is (for  $r > 2$ )

$$\text{Var}\{\hat{\lambda}'\} = \frac{\lambda^2}{r-2}.$$

### Confidence Limits of Parameter $\lambda$

Applying a general approach (see Section 1.4 above), let us take an initial centered statistic in the form

$$W = \alpha S.$$

Expressing the density function of distribution of  $S$  from formula (2.6), we can easily find that statistic  $W$  has the gamma distribution with parameters  $(1, r)$ . Let us denote the quantile of the level  $q$  for this distribution by  $\chi_q(1, r)$ . Then the following inequalities

$$\chi_{1-\alpha}(1, r) \leq \alpha S \leq \chi_{1-\beta}(1, r)$$

hold with the probability  $1 - \alpha - \beta$ . Hence the expressions for the lower and upper confidence limits with the confidence probability  $1 - \alpha - \beta$  ( $\alpha$ -confidence limits) for parameter  $\alpha$  are

$$\underline{\lambda} = \frac{1}{S} \Gamma_{\alpha}(1, r), \quad \bar{\lambda} = \frac{1}{S} \Gamma_{1-\beta}(1, r).$$

Taking into account the relation between the gamma and  $\chi^2$  distributions (see Section 1.2.3 above), we can write

$$\underline{\lambda} = \frac{1}{2S} \chi_{\alpha}^2(2r), \quad \bar{\lambda} = \frac{1}{2S} \chi_{1-\beta}^2(2r). \quad (2.7)$$

where  $\chi_q^2(2r)$  is the quantile of level  $q$  of the  $\chi^2$  distribution with  $2r$  degrees of freedom. Numerical tables for these quantiles are given in Appendix (Table E.16). Additionally, these confidence limits are often expressed as

$$\underline{\lambda} = \frac{1}{S} \Delta_{\alpha}(r-1), \quad \bar{\lambda} = \frac{1}{S} \Delta_{\beta}(r-1). \quad (2.8)$$

where  $\chi_{1-\alpha}(d)$  and  $\chi_{\alpha}(d)$  are the standard lower and upper  $\alpha$ -confidence limits,  $1 - \alpha - \beta$ , for parameter  $\alpha$  of the Poisson d.f. These limits are constructed on the basis of the observed value of  $d$  (see Section 1.4.8 above). Numerical tables of  $\chi_{1-\alpha}(d)$  are given in Appendix (Table 12.4).

**Example 2.1** We tested nine units ( $N=9$ ) by plan [N U r] until three failures ( $r=3$ ) have been occurred. The following failure moments were recorded (in hours):  $t_1=144$ ,  $t_2=182$ , and  $t_3=243$ . We



need to construct the 0.9-confidence interval for parameter  $\lambda$ . We are also interested in the corresponding  $\lambda$ -confidence limits for the MTTF,  $\tau = \frac{1}{\lambda}$ , and probability of failure-free operation (PFFO),  $P = e^{-\lambda t_0}$  for  $t_0=10$  hours.

In this case the total testing time equals

$$S = t_1 + \dots + t_r + (N-r)t_r = 144 + 182 + 243 + 6 \cdot 243 = 2027.$$

Taking  $\alpha = \frac{\gamma}{2} = \frac{0.1}{2} = 0.05$  and applying formula (2.7) and Table E.16 from Appendix, we obtain the following lower and upper  $\lambda$ -confidence limits

$$\underline{\lambda} = \frac{\chi_{0.05}^2(6)}{2S} = \frac{1.635}{2 \cdot 2027} = 0.402 \cdot 10^{-3} \frac{1}{\text{hour}},$$

$$\bar{\lambda} = \frac{\chi_{0.95}^2(6)}{2S} = \frac{12.59}{2 \cdot 2027} = 3.09 \cdot 10^{-3} \frac{1}{\text{hour}},$$

The corresponding confidence limits for  $\tau$  and  $P$  are

$$\underline{\tau} = \frac{1}{\bar{\lambda}} = 322 \text{ hours},$$

$$\bar{\tau} = \frac{1}{\underline{\lambda}} = 2480 \text{ hours},$$

and

$$\underline{P} = e^{-\bar{\lambda} t_0} = e^{-3.09 \cdot 10^{-3} \cdot 10} = 0.970,$$

$$\bar{P} = e^{-\lambda t} = e^{-0.402 \cdot 10^{-3} \cdot 10} = 0.996.$$

### 3.2.2 Test Plan [N U T]

The result of this test is represented by the set of the sequential failure moments

$$t_1 < t_2 < \dots < t_{d-1} < t_d < T$$

where  $d$  is the number of failures occurred during time  $T$ . For this plan, the number of failures,  $d$ , is a r.v.,  $d \leq N$ . As above, in the result of simple transforms, we obtain that the likelihood function in this case can be written as

$$L = C \lambda^d e^{-\lambda S} \quad (2.9)$$

where  $C = N(N-1) \dots (N-d+1)$  is a normalizing constant and  $S$  is again the total testing time of all units. The value of  $S$  is calculated as

$$S = t_1 + \dots + t_d + (N-d)T.$$

From (2.9) it follows that the sufficient statistic is the two-dimensional statistic  $(d, S)$ . The maximum likelihood equation in this case is

$$\frac{\partial \ln L}{\partial \lambda} = \frac{d}{\lambda} - S = 0$$

and, consequently, the MLE for  $\lambda$  is

$$\hat{\lambda} = \frac{d}{S}.$$

Construction of the confidence limits for parameter  $\lambda$ , based on the two-dimensional statistic  $(d, S)$ , leads to enormous volume of calculations. Besides we would like to emphasize that for the case of most practical interest, where units are highly reliable (that is, where  $\lambda T \ll 1$  and  $d \ll N$ ), the main information contains in statistic  $d$ . Therefore in practice, one has often to construct the confidence limits only on the basis of statistic  $d$ .

It is clear that r.v.  $d$  is binomial distributed with parameter  $p = 1 - e^{-\lambda T}$  [20]. Therefore the confidence limits for parameter  $\lambda$  can be easily obtained on the basis of the standard Clopper-Pearson  $\lambda$ -confidence limits for a binomial trials (see Section 1.4.6 above). The lower and upper

$\alpha$ -confidence limits for binomial parameter  $p$  (if  $d$  failures have occurred while testing  $N$  units) obtained by the Clopper-Pearson equations equal

$$\sum_{d \leq j \leq N} \binom{N}{j} \underline{p}^j (1 - \underline{p})^{N-j} = \frac{\alpha}{2}$$

(2.10)

$$\sum_{0 \leq j \leq d} \binom{N}{j} \bar{p}^j (1 - \bar{p})^{N-j} = \frac{\alpha}{2}$$

(Values of  $\underline{p}$  and  $\bar{p}$  for  $\alpha = 1 - \alpha = 0.99$  and  $\alpha = 1 - \alpha = 0.95$  are given in Tables E.14 and E.15.) Let us denote

$$\underline{\Delta}'_{1-\alpha}(d) = -\ln(1 - \underline{p}),$$

(2.11)

$$\overline{\Delta}'_{1-\alpha}(d) = -\ln(1 - \overline{p}),$$

In accordance with (2.10) the lower and upper  $\square$ -confidence limits,  $\gamma=1-\square$ , for parameter  $\square$  can be calculated using the formulas

$$\underline{\lambda} = \frac{1}{T} \underline{\Delta}'_{1-\alpha}(d), \tag{2.12}$$

$$\overline{\lambda} = \frac{1}{T} \overline{\Delta}'_{1-\alpha}(d).$$

Values of  $\underline{\Delta}'_{1-\alpha}(d)$  128 and  $\overline{\Delta}'_{1-\alpha}(d)$  129 are given in Table 12.5 for  $\gamma=1-\square=0.95$ .

**Example 2.2.2** Fifty units ( $N=50$ ) are tested during  $t^*=30$  hours by plan [N U t\*]. Six failures ( $d=6$ ) have occurred. We need to find the lower and upper  $\square$ -confidence limits,  $\square=0.95$ , for the parameter  $\square$ , MTTF  $\square=1/\square$ , and probability of successful operation  $P(t_0) = e^{-\lambda t_0}$  130 for  $t_0=4$  hours.

Applying formulas (2.12) and Table 12.5, we obtain the following lower and upper  $\square$ -confidence limits for the parameter  $\square$ :

$$\underline{\lambda} = \frac{1}{T} \underline{\Delta}'_{0.95}(6) = \frac{0.046}{30} = 15.3 \cdot 10^{-4} \left( \frac{1}{\text{hour}} \right),$$

$$\overline{\lambda} = \frac{1}{T} \overline{\Delta}'_{0.95}(6) = \frac{0.278}{30} = 92.7 \cdot 10^{-4} \left( \frac{1}{\text{hour}} \right).$$

The remaining lower and upper  $\square$ -confidence limits are

$$\underline{P}(t_0) = \frac{e^{-\lambda t_0}}{\tau} = \frac{e^{-\lambda t_0}}{\frac{1}{\lambda}} = e^{-\lambda t_0} = e^{-(92.7) \cdot 10^{-4} \cdot 4} = 0.963,$$

$$\begin{aligned} \overline{P}(t_0) &= e^{-\lambda t_0} = e^{-(15.3) \cdot 10^{-4} \cdot 4} = 0.994. \quad \square \\ \tau &= \frac{1}{\lambda} = 651 \text{ hours} \end{aligned}$$

### 3.2.3 Test Plan [N U (r, T)]

Results of this plan are given by the set of sequential moments of failures:

$$t_1 < t_2 < \dots < t_d < T$$

where  $d$  is the number of failures during time  $T$ ,  $d \leq r$ . The likelihood function has the form

$$L = C \lambda^d e^{-\lambda S}$$

where  $C = N(N-1) \dots (N-d+1)$  is the normalizing constant, and

$$\begin{aligned} S &= t_1 + \dots + t_d + (N-d)T \text{ if } d < r, \text{ and} \\ S &= t_1 + \dots + t_r + (N-r)t_r \text{ if } d = r. \end{aligned}$$

The value of  $S$ , as stated previously, represents the total testing time for all units. The sufficient statistic is again a pair  $(d, S)$ . The MLE for parameter  $\lambda$  is the same as for the previous test plans:  $\hat{\lambda} = \frac{d}{S}$ . 137

Applying the general approach given in Section 1.4, we can obtain the following confidence limits with the confidence probability  $1 - \alpha$  for parameter  $\lambda$ :

For  $d < r$  the lower and upper  $1 - \alpha$ -confidence limits are the same as for test plan [N U T], that is,

$$\underline{\lambda} = \frac{1}{T} \underline{\Delta}'_{1-\alpha}(d),$$

$$\bar{\lambda} = \frac{1}{T} \bar{\Delta}'_{1-\alpha}(d),$$

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For  $d=r$  (or for  $t_r < t^*$ ) the confidence limits for  $\lambda$  can be found from equations

$$\left( \varphi(\underline{\lambda}S, \underline{\lambda}T) = \frac{\alpha}{2}, \right. \quad (2.13)$$

$$\left. \sum_{0 \leq j \leq r-1} \binom{N}{j} (1 - e^{-\bar{\lambda}T})^j e^{-(N-j)\bar{\lambda}T} + \varphi(\bar{\lambda}S, \bar{\lambda}T) = \frac{\alpha}{2} \right)$$

where  $\varphi(x, y) = P\{S < x, t_r < y\}$ .

Finding the lower and upper  $\lambda$ -confidence limits from these equations needs cumbersome calculations and compiling special tables of function  $\varphi(x, y)$ . We would like to emphasize that in the case for highly reliable units (that is, for  $\lambda \ll 1, r \ll N$ ) which is the most interesting for practical applications, all test plans of types  $[N U r]$ ,  $[N U T]$ , and  $[N U (r, T)]$  can be considered approximately equivalent to the  $R$ -plans  $[N R r]$ ,  $[N R T]$ , and  $[N R (r, T)]$ . For this reason, in practice one often uses simple formulas for the test plan  $[N R (r, T)]$  for construction of the confidence limits  $\underline{\lambda}, \bar{\lambda}$  143 instead of the exact equations (2.13).

### 3.2.4 Test Plan $[N R r]$

Test results of this plan are given by the set of sequential moments of failures:

$$t_1 < t_2 < \dots < t_r. \quad (2.14)$$

Due to the memoryless property of the exponential distribution, r.v.'s  $t_1, t_2 - t_1, \dots, t_r - t_{r-1}$  are independent and identical r.v.'s exponentially distributed with parameter  $N\lambda$ . Thus the failures moments of (2.14) form the Poisson process with intensity  $N\lambda$ . Therefore the likelihood function has the form

$$L = N\lambda e^{-N\lambda t_1} N\lambda e^{-N\lambda(t_2-t_1)} \dots N\lambda e^{-N\lambda(t_r-t_{r-1})} = (N\lambda)^r e^{-\lambda S}$$

where  $S=Nt_r$  is, as before, the total testing time of all units. Thus, for this case, the sufficient statistic is  $S$  (or the moment of the  $r$ th failure,  $t_r$ ). The MLE is

$$\hat{\lambda} = \frac{r}{S} = \frac{r}{Nt_r}.$$

Notice that the value of  $S$  can be written for this test plan as

$$S = Nt_1 + N(t_2 - t_1) + \dots + N(t_r - t_{r-1})$$

that is,  $S$  is the sum of  $r$  independent and identical r.v.'s distributed exponentially with parameter  $\lambda$ . So,  $S$  has the distribution coinciding with that for test plan [N U r]. It follows that the confidence limits for the considered test plan [N R r] can be found by the analogous formulas:

$$\underline{\lambda} = \frac{1}{S} \Delta_{1-\alpha}(r-1),$$

$$\bar{\lambda} = \frac{1}{S} \Delta_{\beta}(r-1),$$

where  $S=Nt_r$ .

### 3.2.5 Test Plan [N R T]

Results of this plan are given by the set of sequential moments of failures:

$$t_1 < t_2 < \dots < t_d < T. \tag{2.15}$$

where  $d$  is the number of failures during time  $T$ . These moments of failures form the Poisson process with parameter  $N\lambda$ . Consequently, the likelihood function is determined by the following expression

$$L = N\lambda e^{-N\lambda t_1} N\lambda e^{-N\lambda(t_2-t_1)} \dots N\lambda e^{-N\lambda(t_d-t_{d-1})} e^{-N\lambda(T-t_d)} = (N\lambda)^d e^{-N\lambda T} \quad (2.16)$$

and the sufficient statistic is  $d$ , the number of failures having occurred during time  $T$ . From (2.16), we can obtain, as above, that the MLE of parameter  $\lambda$  has the form

$$\hat{\lambda} = \frac{d}{NT}. \quad (2.17)$$

#### Confidence Limits for $\lambda$

The number of failures during time  $T$ ,  $d$ , has the Poisson distribution with parameter  $\lambda = \lambda NT$ . Consequently, the task is reduced to the construction of the lower and upper  $\alpha$ -confidence limits for the parameter of the Poisson distribution (see Section 1.4.8 above). The lower and upper  $\alpha$ -confidence limits,  $\lambda_{1-\alpha}$  and  $\lambda_\alpha$ , for parameter  $\lambda$  can be found by the formulas

$$\lambda_{1-\alpha} = \frac{1}{NT} \Delta_{1-\alpha}(d-1), \quad (2.18)$$

$$\lambda_\alpha = \frac{1}{NT} \Delta_\alpha(d),$$

where  $\Delta_{1-\alpha}(d-1)$  and  $\Delta_\alpha(d)$  are the standard lower and upper  $\alpha$ -confidence limits,  $\alpha = 1 - \alpha$ , of the parameter of the Poisson distribution for  $d$  failures. The value of  $\Delta_\alpha(d)$  is the solution of the equation



$$e^{-\Delta} \sum_{0 \leq j \leq d} \frac{\Delta^j}{j!} = q.$$

Numerical tables of these values are given in Appendix in Table 12.4. The lower and upper  $\square$ -confidence limits (2.17) can also be expressed via quantiles of the  $\square^2$  distribution (see Section 1.4.8):

$$\underline{\lambda} = \frac{\chi_{\alpha}^2(2d)}{2NT}, \quad \bar{\lambda} = \frac{\chi_{1-\beta}^2(2d+2)}{2NT}.$$

Values  $\underline{\lambda}$  and  $\bar{\lambda}$  154 can be also used as the one-side limits:  $\underline{\lambda}$  is the one-side lower ( $I-\square$ )-confidence limit and  $\bar{\lambda}$  155 is the one-side upper ( $I-\square$ )-confidence limit for  $\square$ .

**Example 2.2.3** Ten units ( $N=10$ ) were tested according to plan [N R T] for 150 hours ( $T=150$ ). Six failures were registered,  $d=6$  failures.

Find the point estimate and lower and upper  $\square$ -confidence limits,  $\square=0.98$ , for  $\square$ ; the MTTF,  $\square=1/\square$ ; and probability of successful operation  $P(t_0) = e^{-\lambda t_0}$  156 for  $t_0=1$  hour.

Using formula (2.17), the point estimate of parameter  $\square$  is

$$\hat{\lambda} = \frac{d}{NT} = \frac{6}{10 \cdot 150} = 4 \cdot 10^{-3} \text{ (1 / hour)}.$$

Setting  $\square=1-\square-\square$ ,  $\square=0.001$  and  $\square=0.01$ , and using formula (2.18) and Table 12.4, the lower and upper  $\square$ -confidence limits for  $\square$  can be found as:

$$\hat{\lambda} = \frac{\Delta_{0.99}(5)}{10 \cdot 150} = \frac{1.79}{10 \cdot 150} = 1.19 \cdot 10^{-3} (1 / \text{hour}),$$

$$\bar{\lambda} = \frac{\Delta_{0.01}(6)}{10 \cdot 150} = \frac{14.75}{10 \cdot 150} = 9.69 \cdot 10^{-3} (1 / \text{hour}),$$

The corresponding point estimate and confidence limits for the MTTF and PFFO are

$$\hat{\tau} = \frac{1}{\hat{\lambda}} = 250 \text{hours},$$

$$\underline{\tau} = \frac{1}{\underline{\lambda}} = 103 \text{hours},$$

$$\bar{\tau} = \frac{1}{\bar{\lambda}} = 833 \text{hours},$$

$$\hat{P}(t_0) = e^{-\hat{\lambda}t_0} = 0.996,$$

$$\bar{P}(t_0) = e^{-\bar{\lambda}t_0} = 0.990,$$

### 3.2.6 Plan [N R (r,T)]

In this case the test results are the moments of failures

$$t_1 < t_2 < \dots < t_d < T$$

where  $d$  is the number of failures ( $d \leq r$ ). As above we obtain that the likelihood function in this case has the form

$$L = (N-d)^d e^{-\lambda S} \quad (2.19)$$

where  $S$  is the total testing time of all units. This value is determined by the following formulas

$$S = NT \text{ if } d < r,$$

$$S = Nt_r \text{ if } d = r.$$

According to (2.19), the sufficient statistic for this plan is the two-dimensional statistic  $(d, S)$ . The MLE of parameter  $\lambda$  has the same form as for the previous plans, that is,  $\hat{\lambda} = \frac{d}{S}$ . 162

The confidence limits for parameter  $\lambda$  for this plan are calculated exactly as for plans [N R T] and [N R r]. If  $d < r$  we use the formulas for plan [N R T]:

$$\underline{\lambda} = \frac{\Delta_{1-\alpha}(d-1)}{NT},$$

$$\bar{\lambda} = \frac{\Delta_{\beta}(d)}{NT},$$

If  $d = r$  (or, in other words,  $t_r < T$ ), the formulas for plan [N R r] are used:

$$\underline{\lambda} = \frac{\Delta_{1-\alpha}(r-1)}{S},$$

$$\bar{\lambda} = \frac{\Delta_{\beta}(r-1)}{S},$$

where  $S = Nt_r$ .

### 3.3 "Exponential" Methods for IFR Distributions

Assumption that the distribution of time to failure,  $F(t)$ , is exponential is widely used in reliability theory and its applications. This is an understandable (though not satisfactory

orforgivable) excuse because analytical approaches in this case are very simple and mathematically attractive. (The argument reminds of searching of the lost thing under a street lamp only because this is a brightly lighted place!) Nevertheless for practical purposes this assumption sometimes is far from realistic. Naturally, the question arises: how does the assumption of exponentiality effect the statistical inferences if the real distribution is not exponential? In some sense, this is a question about the stability, or robustness, of statistical inferences in relation to the form of a distribution function  $F(t)$ . This problem and related ones were studied by Barlow and Proschan (1966, 1967). Zelen and Dannemiller (1961), Barlow and Gupta (1966), Pavlov (1974, 1977), and others. In particular, these works show that some results for an exponential distribution can be extended to IFR distributions (distributions with increasing failure rate). For the sake of brevity, we will say about application of "exponential" methods (i.e., methods for estimation of the exponential distribution) for more general cases, in particular, for the estimation of the IFR class of distributions.

We will assume that a distribution function has a density:  $f(t)=F'(t)$ . Denote the reliability function by  $P(t)=1-F(t)$  and intensity function by  $\lambda(t) = \frac{f(t)}{P(t)}$ . 167 Let us also introduce the function  $\Lambda$  which is defined as cumulative hazard function  $\Lambda(t) = \int_0^t \lambda(x)dx$  168. This new function relates to the reliability function via formula  $P(t)=e^{-\Lambda(t)}$ .

A distribution function  $F(t)$  is IFR if  $\Lambda(t)$  is monotonically nondecreasing in  $t \geq 0$ . It easy to see that for IFR distributions, function  $\Lambda(t)$  is convex in  $t$  for all  $t$ , for which  $P(t) > 0$ .

The assumption that function  $\Lambda(t)$  is nondecreasing coordinates with the common understanding of aging. Indeed, for the overwhelming majority of real physical objects, time leads to deterioration of materials and worsening of its reliability properties. This is the reason why IFR distributions are widely used in reliability analysis (Barlow and Proschan (1975), Gnedenko, Belyaev and Solovyev (1965), Ushakov, ed. (1994), and others). It is necessary emphasize that the class of IFR distributions includes as particular cases such standard parametrical distributions as exponential, normal, Weibull-Gnedenko (with form parameter  $\alpha > 1$ ), gamma, and others.

### 3.3.1 Plan [N U r]

Let we need to construct the lower  $\alpha$ -confidence limit for the reliability function,  $P(t)=1-F(t)$ , on the basis of the test results by plan [N U r]:

$$t_1 < t_2 < \dots < t_r. \tag{2.20}$$

$$\underline{P}(t) = e^{-\bar{\lambda}t}$$

For an exponential function  $F(t)$  this confidence limit has the form (2.21)

where  $\bar{\lambda} = \frac{1}{S} \Delta_{1-\gamma}(r-1)$  is the upper  $\square$ -confidence limit for the parameter of the exponential distribution, and

$S = t_1 + \dots + t_r + (N-r)t_r$  is the total testing time of all units (see Section 2.2).

The lower  $\square$ -confidence limit for the IFR distribution is given by the following expression

$$P^*(t) = \begin{cases} \exp\left\{-\frac{1}{S} \Delta_{1-\square}(r-1)t\right\} & \text{if } t \leq \frac{S}{N}, \\ 0 & \text{if } t > \frac{S}{N}. \end{cases} \quad (2.22)$$

This lower limit coincides with the corresponding limit for the exponential distribution (2.21) at the beginning of time interval  $0 \leq t \leq \frac{S}{N}$ .

Another important reliability index is the time of failure-free operation,  $t_q$ , with the probability  $q$ . This value is determined from equation  $P(t_q) = q$ . It is clear that this value coincides with the quantile of level  $1-q$  of the distribution  $F(t)$ . For the exponential distribution, we can write

$e^{-\lambda t_q} = q$ , and, consequently,  $t_q = -\frac{\ln q}{\lambda}$ . The lower  $\square$ -confident limit for  $t_q$  for the exponential distribution has the form

$$\underline{t}_q = -\frac{\ln q}{\lambda} = \frac{(-\ln q) \cdot S}{\Delta_{1-\gamma}(r-1)} \quad (2.23)$$

The analogue of the limit (2.23) for  $t_q$  is the following

$$\underline{t}_q^* = \min \left\{ \frac{S}{N}, \frac{(-\ln q) \cdot S}{\Delta_{1-\gamma}(r-1)} \right\}. \quad (2.24)$$

This value delivers the lower  $\square$ -confidence limit for  $t_q$  if  $F(t)$  is IFR. The lower confidence limits in (2.22) and (2.24) for  $P(t)$  and  $t_q$  are connected by the following obvious relation

$$\underline{t}_{-q}^* = \max \{t: \underline{P}^*(t) \geq q\}.$$

The lower confidence limit in (2.24) coincides with the analogous limit for the exponential distribution in (2.23) if the level  $q$  satisfies the inequality

$$q \geq e^{-\frac{\Delta_{1,\gamma}(r-1)}{N}} \approx e^{-\frac{r}{N}}. \quad (2.25)$$

The limit of type (2.24) for the class of IFR distributions was obtained by Barlow and Proschan (1966). The limit of type (2.22) was found by Pavlov (1977). In the latter work, improved limits of types (2.22) and (2.24) were also obtained for IFR and IFRA distributions. Besides, more general test plans of type **[N U (r,T)]** were also considered there. We avoid exposure those results because they are too clumsy.

Example 2.3.1 On the basis of data given in form (2.20) we need to test hypotheses for  $P(t_0)$ :

$$H_0: P(t_0) > b \text{ vs. } H_1: P(t_0) \leq b$$

where  $b$  is some required level of the reliability index  $P(t_0)$ . Let us consider first an exponential distribution. In this case the following standard rule based on the lower  $\square$ -confidence limit of type (2.21) is used:

$$\begin{aligned} &\text{to accept } H_0 \text{ if } \underline{P}(t_0) > b, \\ &\text{to accept } H_1 \text{ if } \underline{P}(t_0) \leq b. \end{aligned} \quad (2.26)$$

In this case the probability of the II type error is not larger than  $\square = 1 - \square$ . Indeed, by the definition of the lower limit, for  $P(t_0) \leq b$  the probability of the second type error (that is, the probability to accept an item if it must be rejected) is

$$\begin{aligned} \mathbf{P}\{\text{accept } H_0\} &= \mathbf{P}\{\underline{P}(t_0) > b\} \leq \mathbf{P}\{\underline{P}(t_0) > P(t_0)\} \\ &= 1 - \mathbf{P}\{\underline{P}(t_0) \leq P(t_0)\} \leq 1 - \square. \end{aligned}$$

Let us now assume that distribution  $F(t)$  is IFR rather than exponential. Then from expressions (2.21) and (2.22) for the lower confidence limits, the rule (2.26) delivers the II type error not larger than  $\square = 1 - \square$  if the critical level  $b$  satisfies the following inequality

$$b \geq e^{-\frac{\Delta_{1,\gamma}(r-1)}{N}} \quad (2.27)$$

In practice, the required level  $b$  is usually close to 1 and the right side of (2.27) is of order

$\exp(-\frac{r}{N}) \approx 1 - \frac{r}{N}$ .<sup>182</sup> Hence condition (2.27) is not too restrictive. Thus, the decision rule (2.26) based on the exponentiality of  $F(t)$  usually preserves the supplier risk  $\alpha$  (but not the consumer risk) for distributions of the IFR class.

The analogous result can be obtained for the index of type  $t_q$ .  $\square$

Example 2.3.2 On the basis of data (2.20) let us test hypotheses

$$H_0: t_q > C \text{ vs. } H_1: t_q < C$$

where  $C$  is a required level of  $t_q$ .

Consider the decision rule for the exponential distribution based on the lower  $\alpha$ -confidence limit (2.23):

$$\begin{aligned} &\text{to accept } H_0 \text{ if } t_q > C, \\ &\text{to accept } H_1 \text{ if } t_q < C. \end{aligned} \tag{2.28}$$

For the exponential distribution, this decision rule delivers the consumer risk (the II type error) not larger than  $\alpha = 1 - \alpha$ . The proof of this fact completely coincides with the previous case. The comparison of confidence limits (2.23) and (2.24) shows that the decision rule (2.28) for the exponential distribution preserves the consumer risk for IFR distributions if the value of  $q$  satisfies the inequality (2.25). In practice, we usually choose  $q$  close to 1.  $\square$

Thus, for indexes of type  $P(t_0)$  and  $t_q$  the decision rules for the exponential distribution can be extended to IFR distributions with not too restrictive conditions, and the meaning of  $\alpha$  is preserved. Let us emphasize once more that in general it is not a correct estimate for the supplier risk (in detail this is considered in Part III).

The confidence limits (2.22) and (2.24) for the class of IFR distributions are not consistent. This deficiency, however, becomes significant only for large sample sizes. At the same time, the standard non-parametrical confidence limits for  $P(t_0)$  and  $t_q$  based on an empirical distribution function or order statistic, though being consistent, gives worse confidence limits than (2.22) and (2.24) if

- the sample size is small or/and
- an item is highly reliable (either  $t_0$  is small or  $q$  is close to 1).

Notice that the cases mentioned above are the most important in practice (see Exercises 2.3 and 2.4). The standard non-parametrical confidence limits for  $t_q$  based on order statistic have the following deficiency (see Walsh(1962), Zacks (1971) and others). For a specified confidence coefficient  $\alpha$ , the lower confidence limit for  $t_q$  can be constructed only for a large enough sample size, namely, for  $N \geq N(\alpha, q)$  where  $N(\alpha, q)$  is some threshold value. The standard non-parametrical confidence limits for  $P(t_0)$  based on an empirical distribution function (see Section 2.5 below) have an analogous deficiency: for a given sample size  $N$ , the lower  $\alpha$ -confidence limit for  $P(t_0)$  does not exceed some threshold level  $Q(N, \alpha) < 1$  for arbitrary small  $t$ . At the same time for IFR distributions  $P(t) \square 1$  if  $t \square 0$ .

### Confidence Strip for the Reliability Index

By construction, (2.22) is the lower  $\square$ -confidence limit for  $P(t)$  for any fixed moment  $t$ . It is possible to show a stronger statement: inequality

$$\underline{P}^*(t) \leq P(t) \tag{2.29}$$

simultaneously holds for all  $t \geq 0$  with the probability not less than  $\square$  if  $F(t)$  is IFR. Thus, (2.22) gives  $\square$ -confidence strip for  $P(t)$  in the case of IFR distributions. The proof of this statement follows from the fact that the confidence limit  $\underline{P}^*(t)$  for different  $t$  depends on the same statistic  $S$  (for details, see Pavlov (1977)).

Let us denote the unit's MTTF by  $\tau = \int_0^\infty P(t) dt$  184. By integration of the left side of (2.29), we obtain the following lower  $\square$ -confidence limit for the class of IFR distributions

$$\underline{\tau}^* = \int_0^\infty \underline{P}^*(t) dt = \int_0^{\frac{S}{N}} e^{-\frac{\Delta_{1-\gamma}(r-1)}{S} t} dt = C_r \cdot \frac{S}{\Delta_{1-\gamma}(r-1)} \tag{2.30}$$

where the constant is

$$C_r = 1 - e^{-\frac{\Delta_{1-\gamma}(r-1)}{N}}.$$

Let us compare this limit with the lower  $\square$ -confidence limit for  $\square$  for the exponential distribution

$$\underline{\tau} = \frac{S}{\Delta_{1-\gamma}(r-1)}. \tag{2.31}$$

The value of  $C_r$  is of the order  $C_r \square 1 - \exp(-r/N)$ . It follows that in the case of no censorship (or with insignificant one, that is, where  $r$  is close to  $N$ ) there is no practical difference between the two confidence limits:  $\underline{\tau}^*$  and  $\underline{\tau}$ . If  $r \ll N$  the difference becomes significant. The constant  $C_r$  can be



interpreted as the correction coefficient for the confidence limit for  $\square$  which should be used if you apply the "exponential methods" to a more general IFR distribution.

**Example 2.3.3** The test of four units by plan [N U r] with  $r=N=4$  gave the following failure moments (in hours):

$$t_1=40, t_2=80, t_3=110, t_4=240.$$

Then the lower  $\square$ -confidence limit (2.31), with the confidence coefficient  $\square=0.9$ , for the exponential distribution equals

$$\underline{\tau} = \frac{S}{\Delta_{1-\gamma}(r-1)} = \frac{470}{\Delta_{0.1}(3)} = \frac{470}{6.68} = 70.4 \text{ hours.}$$

The lower  $\square$ -confidence limit (2.30) for the IFR distribution equals

$$\underline{\tau}^* = C_r \cdot \underline{\tau} = \left(1 - e^{-\frac{\Delta_{0.1}(3)}{4}}\right) \cdot \underline{\tau} = 0.812 \cdot \underline{\tau} = 57.3 \text{ hours.}$$

In this case the coefficient  $C_r$  equals 0.812.  $\square$

Assume that under the assumption that  $F(t)$  is IFR, we need to check on the basis of (2.20) the following standard hypotheses related to the MTTF:

$$H_0: \square > b \text{ vs. } H_1: \square \leq b$$

where  $b$  is a specified level of the reliability index  $\square$ . Consider the following decision rule based on the lower  $\square$ -confidence limit (2.30):

$$\begin{aligned} &\text{to accept } H_0 \text{ if } \underline{\tau}^* > b, \\ &\text{to accept } H_1 \text{ if } \underline{\tau}^* \leq b. \end{aligned}$$

By definition of the lower confidence limit, this rule delivers the probability of the II type error (the consumer risk) not higher than  $\square = 1 - \square$  for an IFR distribution. This decision rule can also be expressed via the confidence limit (2.31) for the exponential distribution as

$$\begin{aligned} &\text{to accept } H_0 \text{ if } \underline{\tau} > \frac{b}{C_r} 190, \\ &\text{to accept } H_1 \text{ if } \underline{\tau} < \frac{b}{C_r} 191 \end{aligned} \tag{2.32}$$

where coefficient  $C_r$  introduced above is less than 1. It means that applying the "exponential method" we must increase the required level of reliability index equal to  $b$  in  $\frac{1}{C_r}$  192 times to keep the same II type error for the class of IFR distributions.

Example 2.3.4 Under the conditions of previous Example 2.3.3, where test was made using by the plan of type [N U r] with  $N=r=4$ , to obtain the needed level  $b$ , we need to increase the required level of reliability in  $\frac{1}{C_r} = \frac{1}{0.812} = 1.22$  193 times. □

Remark. If we use □ the decision rule (3.32) for index for the exponential distribution with no modification (i.e.,  $C_r=1$ ), the consumer risk increases. Concerning the producer risk, Barlow and Proschan (1967) showed that it might be preserved under some additional conditions on  $N$ ,  $r$  and □. For the reliability indexes of types  $P(t_0)$  and  $t_q$ , there is an inverse situation: the consumer risk is preserved and the producer risk increased. In this connection, the questions arise: How to guarantee simultaneously both types of risk for reliability indexes dealing with the class of IFR distributions? Is it necessary to increase the sample size for testing? If so, how much the sample size must be increased? These and related questions are considered in Part III of the book.

### 3.3.2 Test Plan [N R T]

Consider test plan of type [N R T] for systems consisting of units with IFR distribution of TTF (Pavlov (1974)). Using plan [N R T], at any moment  $t$  we test  $N$  identical units. Each failed unit is immediately replaced by a new one, completely identical. The test continues up to some moment  $T$  specified in advance. The result of the test are failure moments

$$t_1 < t_2 < \dots < t_d < T \quad (2.33)$$

where  $d$  is the number of failures occurred during the test.

The lower □-confidence limit for the IFR reliability function  $P(t)$  is found in the work mentioned above:

$$\underline{P}^*(t) = \begin{cases} e^{-\frac{\Delta_{1,\gamma}(d)}{S}t} & \text{if } t \leq \frac{NT}{N+D}, \\ 0 & \text{if } t > \frac{NT}{N+D}. \end{cases} \quad (2.34)$$

where  $S=NT$  is the total testing time of all units. This limit coincides for  $t \leq \frac{NT}{N+d}$  195 with the standard lower  $\square$ -confidence limit for the exponential distribution for the same plan (see Section 2.2):

$$\underline{P}(t) = e^{-\frac{\Delta_{1,\gamma}(d)}{S}t}. \quad (2.35)$$

Notice that for plan **[N R T]** the total number of units tested and the number of failure-free intervals equals  $N+d$ . Thus, the value of  $\tau^* = \frac{NT}{N+d}$  197 represents the MTTF of a unit tested. Therefore the lower confidence limit of the reliability function for the exponential d.f. remains correct for the class of IFR distributions if  $t \leq \square^*$  where  $\square^*$  is a statistical estimate of the MTTF,  $\square$ . This fact can serve as statistical analogue of the well known lower bound of the IFR reliability function with the given MTTF (see Barlow and Proschan (1967)):

$$P(t) \geq \begin{cases} e^{-\frac{t}{\tau}} & \text{if } t \leq \tau, \\ 0 & \text{if } t > \tau. \end{cases}$$

The value of  $\underline{P}^*(t)$  delivers the lower  $\square$ -confidence limit of  $P(t)$  for each fixed moment of time  $t > 0$ . Besides, there exists even more strong statement, namely, that the system of inequalities

$$\underline{P}^*(t) \leq P(t) \text{ for all } t > 0$$

holds with the probability not less than  $\square$ . This gives us an opportunity to get the corresponding confidence limits for such reliability indexes as  $t_q$  and  $\square$ .

Let us define  $\underline{t}_q^*$  as the moment when function  $\underline{P}^*(t)$  crosses the level of  $q$ :

$$\underline{t}_q^* = \max \{t: \underline{P}^*(t) \geq q\},$$

from where

$$\underline{t}_q^* = \min \left\{ \frac{(-\ln q) \cdot NT}{\Delta_{1-\gamma}(d)}, \frac{NT}{N+d} \right\}. \quad (2.36)$$

By integration (2.34), we obtain

$$\underline{\tau}^* = \int_0^{\infty} \underline{P}^*(t) dt = \int_0^{\frac{NT}{N+d}} e^{-\frac{\Delta_{1-\gamma}(d)}{NT} \cdot t} dt = C_d \frac{NT}{\Delta_{1-\gamma}(d)},$$

where

$$C_d = 1 - e^{-\frac{\Delta_{1-\gamma}(d)}{N+d}}.$$

Values of  $t_q^*$  and  $\underline{q}^*$  deliver the lower  $\alpha$ -confidence limits for indexes  $t_q$  and  $\underline{q}$ , respectively, for IFR distributions.

Example 2.3.5 On the basis of tests performed by plan [N R T] we need to test hypotheses related to the probability of failure-free operation,  $P(t_0)$ , for some fixed moment  $t$ :

$$H_0: P(t_0) > b \text{ vs. } H_1: P(t_0) \leq b$$

where  $b$  is the required level of index  $P(t_0)$ .

Consider the decision rule based on the lower  $\alpha$ -confidence limit (2.34):

$$\begin{aligned} &\text{to accept } H_0 \text{ if } \underline{P}^*(t_0) > b, \\ &\text{to accept } H_1 \text{ if } \underline{P}^*(t_0) \leq b. \end{aligned} \tag{2.38}$$

By definition of the lower confidence limit, this rule delivers the probability of the II type error (the consumer risk) not higher than  $\alpha = 1 - \alpha$  for IFR distributions. Compare this decision rule with the standard case of the exponential distribution  $F(t)$

$$\begin{aligned} &\text{to accept } H_0 \text{ if } \underline{P}(t_0) > b, \\ &\text{to accept } H_1 \text{ if } \underline{P}(t_0) \leq b. \end{aligned} \tag{2.39}$$

where  $\underline{P}(t_0)$  is the lower confidence limit (2.35) for the exponential distribution. This decision rule delivers the consumer risk equal to  $\alpha = 1 - \alpha$  for the exponential d.f.  $\alpha$

From (2.34) and (2.35), it follows that the decision rules (2.38) and (2.39) coincide if the required level of reliability,  $b$ , satisfies the inequality

$$b \geq q^* \tag{2.40}$$

where

$$q^* = \exp \left( - \min \left\{ \frac{\Delta_{1-\gamma}(d)}{N+d}, d=0,1,\dots \right\} \right).$$

Using simple transformation and taking into account that  $\Delta_{1-\gamma}(0) = \ln\left(\frac{1}{1-\gamma}\right)204$ , we obtain

$$q^* = \exp\left(-\min\left\{1, \frac{\Delta_{1-\gamma}(0)}{N}\right\}\right) = \max\left\{\frac{1}{e}, \sqrt[N]{1-\gamma}\right\}. \quad (2.41)$$

To be correct, notice that in (2.40) we find the exact lower limit of the value  $\frac{\Delta_{1-\gamma}(d)}{N+d}206$  in  $d$  rather than minimum, since this value exceeds 1 only as  $d \rightarrow \infty$ .

Thus, if the required level of reliability is sufficiently high, that is, the condition (2.40) holds, then the decision rule (2.39) preserves the consumer risk for IFR distributions. An analogous fact is true for the index of type  $t_q$ . Let us show this on the following example.

Example 2.3.6 On the basis of data obtained using plan [N R T], let us test hypotheses

$$H_0: t_q > b \text{ vs. } H_1: t_q \leq b$$

where  $b$  is the required level of  $t_q$ . Let us consider the decision rule based on the lower  $1-\alpha$ -confidence limit (2.36) for IFR distributions:

$$\begin{aligned} &\text{to accept } H_0 \text{ if } \underline{t}_q^* > b, \\ &\text{to accept } H_1 \text{ if } \underline{t}_q^* \leq b. \end{aligned} \quad (2.42)$$

and the analogous decision rule

$$\begin{aligned} &\text{to accept } H_0 \text{ if } \underline{t}_q > b, \\ &\text{to accept } H_1 \text{ if } \underline{t}_q \leq b. \end{aligned} \quad (2.43)$$

which is based on the standard lower  $1-\alpha$ -confidence limit for an exponential distribution:

$$\underline{t}_q = \frac{(-\ln q)NT}{\Delta_{1-\gamma}(d)}. \quad (2.44)$$

The confidence limits (2.44) and (2.36) coincide, and consequently, the decision rules (2.42) and (2.43) coincide if the inequality

$$q \geq q^* \tag{2.45}$$

holds. (This inequality has the same sense as above.) □

In practice, the required level of reliability is usually high, it is almost always higher than 0.9. Therefore the conditions (2.45) and (2.40) hold if, roughly speaking,  $\sqrt[3]{1-\gamma} \leq 0.9208$ , or

$$N \leq \frac{\ln\left(\frac{1}{1-\gamma}\right)}{\ln\left(\frac{1}{0.9}\right)} \approx 10 \cdot \ln\left(\frac{1}{1-\gamma}\right).$$

For the confidence coefficients from 0.8 to 0.99, the left side of the above expression equal from 20 to 40. Thus, for test plans with replacement, the "exponential methods" applied to the class of IFR distributions usually preserves the consumer risk for indexes of types  $P(t_0)$  and  $t_q$ .

### 3.4 Estimation of Unit Reliability Indexes for General Parametric Case

Let  $F(t, \theta)$  be the d.f of a unit's TTF and depend on some parameter  $\theta$ . This parameter may in general be a vector  $\theta = (\theta_1, \dots, \theta_r)$ . We denote the corresponding density function by  $f(t, \theta) = F'(t, \theta)$ . The value of parameter  $\theta$  (or components of  $\theta$ ) is unknown. We wish to determine the point estimate and confidence limits for some reliability index,  $R$ , on the basis of the test data. This index is assumed to depend on the mentioned above parameter

$$R = R(\theta) = R(\theta_1, \dots, \theta_r).$$

In practice we are usually interested in the following standard reliability indexes of a unit:

(1) The probability of unit failure-free operation (PFFO) during a required time  $t_0$ ,  $R(\theta) = 1 - F(t_0, \theta)$ .

(2) The mean time to failure (MTTF),  $R(\theta) = \int_0^{\infty} [1 - F(t, \theta)] dt$

(3) The guaranteed time to failure (or  $q$ 100th percent life),  $R(\theta) = q$ , which is determined from the equation  $1 - F(R, \theta) = q$ .

**Remark.** In Section 2.4 we consider only units with no renewal, that is, only units working until its first failure. Renewable units will be considered in Chapter 3 and in Part 2 (Reliability Estimate of Complex Systems).

### 3.4.1 Point Estimate of Reliability Index

The point estimate of reliability index  $R=R(\theta)$  is usually calculated as

$$\hat{R} = R(\hat{\theta}) = R(\hat{\theta}_1, \dots, \hat{\theta}_r)$$

where  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_r)$  is a vector of point estimate of parameters which were found on the basis of test results by some standard method, usually by the maximum likelihood method. The likelihood function is usually written for different test plans in the following ways.

For plan [N U r] the likelihood function has the form

$$L = C \prod_{1 \leq i \leq r} f(t_i, \theta) \{\bar{F}(t_r, \theta)\}^{N-r} \quad (2.46)$$

where  $t_1 < t_2 < \dots < t_{r-1} < t_r$  are the observed failure moments,  $\bar{F}(t, \theta) = 1 - F(t, \theta)$  is a complementary function, and  $C$  is a norm constant:  $C = N(N-1) \dots (N-r+1)$ . This formula for the likelihood function (as well as following below) is obtained in a similar fashion to that for the exponential distribution (see Section 2.2 above).

For plan [N U T] this formula is

$$L = C \prod_{1 \leq i \leq d} f(t_i, \theta) \{\bar{F}(T, \theta)\}^{N-d} \quad (2.47)$$

where  $d$  is the number of observed failures,  $t_1 < t_2 < \dots < t_d < T$  are the observed failure moments, and  $C$  is a norm constant (the meaning of it is insignificant).

For plan [N U (r, T)] the likelihood function is determined by (2.46) if  $d=r$  (that is, if  $t_r \leq T$ ) or by (2.47) if  $d < r$  (that is, if  $t_r > T$ ).

For plan [N R r] the likelihood function is

$$L = C \prod_{1 \leq i \leq r} f(s_i, \theta) \prod_{1 \leq j \leq N-1} \bar{F}(u_j, \theta) \quad (2.48)$$



where  $s_i$  are the intervals terminated by failure, and  $u_j$  are the incomplete intervals where the test was interrupted before a failure occurrence.

For plans with replacement **[N R T]** the likelihood function is

$$L = C \prod_{1 \leq i \leq d} f(s_i, \theta) \prod_{1 \leq j \leq N} \bar{F}(u_j, \theta) \quad (2.49)$$

where the notation coincides with the previous case.

For plan **[N R (r, T)]** the likelihood function is determined by (2.48) if  $d=r$  (that is, if  $t_r \leq T$ ) or by (2.49) if  $d < r$  (that is, if  $t_r > T$ ).

It is easy to see that the likelihood function can be written for all these cases in the uniform type as follows

$$L = C \prod_{1 \leq i \leq d} f(s_i, \theta) \prod_{1 \leq j \leq v} \bar{F}(u_j, \theta) \quad (2.50)$$

where, in addition to the previous notation,  $d$  is the number of complete intervals (up to the failure occurrence) and  $v$  is the number of incomplete intervals. The latter values might be random or be specified in advance.

In accordance with the maximum likelihood method, the estimates of parameters  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_r)$  are those values of the parameters  $\theta = (\theta_1, \dots, \theta_r)$  which deliver minimum of the likelihood function (2.50) for given fixed test results  $s_i$ ,  $u_j$ ,  $d$ , and  $v$ . As is well known, from the computational viewpoint is better to consider a logarithm of the likelihood function.

As an illustration, let us consider the two-parameter Weibull-Gnedenko distribution which is often applied to practical reliability problems. This distribution has the cumulative function and density as follows

$$F(t, \theta) = 1 - e^{-\left(\frac{t}{\beta}\right)^\alpha},$$

and

$$f(t, \theta) = \frac{\alpha t^{\alpha-1}}{\beta^\alpha} e^{-\left(\frac{t}{\beta}\right)^\alpha}$$

where  $\theta=(\alpha, \beta)$  is the vector of parameters.

This distribution was applied by Weibull for reliability engineering problems in mechanics (Weibull (1939, 1951)). Almost simultaneously Gnedenko showed that this distribution is a particular case of the class of limit distributions of extremum values of a large number of independent r.v.'s (Gnedenko (1941, 1943)).

The likelihood function (2.50) for the Weibull-Gnedenko distribution has the form

$$L = C \prod_{1 \leq i \leq d} \alpha \left( \frac{s_i^{\alpha-1}}{\beta^\alpha} \right) e^{-\left(\frac{s_i}{\beta}\right)^\alpha} \prod_{1 \leq j \leq v} e^{-\left(\frac{u_j}{\beta}\right)^\alpha}$$

Various variants of calculation of the point estimates of parameters  $\hat{\alpha}$  225 and  $\hat{\beta}$  226 for the Weibull-Gnedenko distribution are considered in Kao and John (1956), Lieberman (1960), Menon (1963), Cohen (1965), Mann (1968), Kudlaev (1986), and others.

If the point estimates  $\hat{\alpha}$  227 and  $\hat{\beta}$  228 are found, the PFFO during time  $t_0$

$$R = e^{-\left(\frac{t_0}{\beta}\right)^\alpha}$$

can be easily found as

$$\hat{R} = e^{-\left(\frac{t_0}{\hat{\beta}}\right)^{\hat{\alpha}}}.$$

In an analogous manner, we can find estimates for other reliability indexes.

### 3.4.2 Confidence Limits for Reliability Indexes

Usual general approach for calculation of approximate confidence limits for a reliability index,  $R=R(\theta)$ , consists in finding the point estimate of this index,  $\hat{R} = R(\hat{\theta})$  231 and after this a normal approximation is used for a large sample size. In this case the lower and upper  $\square$ -confidence limits for  $R$  are obtained as

$$\underline{R} = \hat{R} - u_\alpha \sqrt{V(\hat{\theta})}, \tag{2.51}$$

$$\bar{R} = \hat{R} + u_\alpha \sqrt{V(\hat{\theta})},$$

where  $u_\alpha$  is the quantile of the level  $\alpha$  of the standard normal distribution,  $V(\theta)$  is the variance of the estimate  $\hat{R}$  for the given values of parameters  $\theta=(\theta_1, \dots, \theta_r)$ , and  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_r)$  is the vector of point estimates. (Particular cases of this approach were considered in Sections 1.4.5 and 1.4.6.) The interval  $(\underline{R}, \bar{R})$  forms the approximate two-side confidence interval with the confidence coefficient equal to  $2\alpha-1$ . Besides, if the point estimates  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_r)$  are asymptotically unbiased and effective, then the confidence limits (2.51) are also asymptotically effective. (Notice that the MLEs possess such properties under some not too restrictive conditions.)

Nevertheless, this method is essentially approximate and can be applied only if the sample size is very large. We should emphasize that the error is generated by the following two main factors:

- (1) the normal approximation is used instead of an exact distribution of estimate  $\hat{R}$ , and
- (2) in (2.51) we use the estimate  $\hat{\theta}$  instead of its exact value.

We will show below that such an approximation can lead to serious mistakes if the sample size is not large. In particular, this approximation might lead to a significant increase of the lower confidence limit of reliability indexes (MTTF, or PFFO).

**Example 2.4.1** For illustration, let us consider a simple example with the exponential distribution,  $F(t, \lambda)=1-e^{-\lambda t}$ , for which the strict confidence limits are known. Consider plan [N R T] with  $N=10$ ,  $T=150$  hours. During the test we have observed  $d=6$  failures.

We are interested in the upper  $\alpha$ -confidence limit,  $\alpha=0.99$ , for parameter  $\lambda$ . (This case corresponds to the lower estimation of a reliability index). In this case the MLE of parameter  $\lambda$  has the form

$$\hat{\lambda} = \frac{d}{NT}.$$

This estimate is unbiased with the mathematical expectation  $E\hat{\lambda} = \lambda$  and variance

$$V(\hat{\lambda}) = E(\hat{\lambda} - \lambda)^2 = \frac{\lambda}{NT}.$$

Applying formula (2.51) and Table E.5 for quantiles of the standard normal distribution, we obtain the following approximate upper  $\alpha$ -confidence limit for  $\lambda$

$$\begin{aligned}\bar{\lambda} &= \hat{\lambda} + u_{\gamma} \sqrt{V(\hat{\lambda})} = \hat{\lambda} + u_{\gamma} \sqrt{\frac{\hat{\lambda}}{NT}} = \frac{d}{NT} + u_{0.99} \sqrt{\frac{d}{(NT)^2}} \\ &= \frac{6}{10 \cdot 150} + 2.32 \sqrt{\frac{6}{(10 \cdot 150)^2}} = 0.0078 \left( \frac{1}{\text{hour}} \right).\end{aligned}$$

We can find the same upper confidence limit by the strict formula (2.18). By this formula and Table 12.4, we have

$$\bar{\lambda}' = \frac{\Delta_{1-\gamma}(d)}{NT} = \frac{\Delta_{0.001}(6)}{10 \cdot 150} = \frac{14.57}{1500} = 0.0097 \left( \frac{1}{\text{hour}} \right).$$

This difference becomes more visible if we consider such a standard reliability index as MTTF,  $\tau = 1/\lambda$ . In this case the approximate lower  $\alpha$ -confidence limit,  $\alpha=0.99$  is

$$\underline{\tau} = \frac{1}{\bar{\lambda}} = \frac{1}{0.0078} = 128 \text{ hours}$$

and the strict limit is

$$\underline{\tau}' = \frac{1}{\bar{\lambda}'} = \frac{1}{0.0097} = 103 \text{ hours.}$$

Thus, the use of the normal approximation (2.51) gives some 24% of error and in the undesirable direction. For practical use it is always better to have a conservative evaluation of a reliability index.

However, except an exponential distribution, the strict confidence limits for reliability indexes can be constructed for very few particular cases. Some of them are considered below. □

### Confidence Limits for the Normal Distribution

Consider a unit which has a normal distribution of TTF

$$F(t, \theta) = \Phi\left(\frac{t - \mu}{\sigma}\right)$$

where  $\Phi(\cdot)$  is the function of the standard normal distribution (Laplace function),  $\mu$  and  $\sigma$  are the mean and standard deviation, respectively. In this case the two-dimension parameter is  $\theta = (\mu, \sigma)$ .

Consider a test by plan [N U N] which corresponds to the complete sample. The results of this test are

$$t_1 < t_2 < \dots < t_N. \quad (2.52)$$

We are interested in the confidence limits for the reliability function for the fixed  $t_0$ , that is,

$$R = 1 - \Phi\left(\frac{t_0 - \mu}{\sigma}\right) = \Phi\left(\frac{\mu - t_0}{\sigma}\right). \quad (2.53)$$

Since  $\Phi(\cdot)$  is a monotone increasing function, the problem is in finding the confidence limits for  $\frac{\mu - t_0}{\sigma}$ . 249 The standard empirical mean and variance found by the test results (2.52) are

$$\bar{x} = \frac{1}{N} \sum_{1 \leq i \leq N} t_i \quad 250 \quad \text{and} \quad S^2 = \frac{1}{N} \sum_{1 \leq i \leq N} (t_i - \bar{x})^2. \quad 251$$

Then statistic

$$W = \left(\frac{\bar{x} - t_0}{S}\right) \sqrt{N - 1} \quad (2.54)$$

has a non-centered Student distribution with  $N-1$  degrees of freedom and with the parameter of non-centrality

$$\delta = \left( \frac{\mu - t_o}{S} \right) \sqrt{N}.$$

(More details about the Student distribution see, for example, in Rao (1965), Section 3a.) Now we can construct the desired confidence limits. Let us denote the non-centered Student distribution with  $N-1$  degrees of freedom and with the parameter of non-centrality by  $G(y, \delta) = P(W < y)$ . This function is decreasing in  $\delta$ . Applying the general method-of-the confidence limit construction (see formula 1.30), we find that the lower and upper  $1-\gamma$ -confidence bounds for parameter  $\delta$  follow from the equations

$$G(W, \bar{\delta}) = 1 - \gamma,$$

$$G(W, \underline{\delta}) = \gamma,$$

where  $W$  is the value of statistic (2.54). After this the lower and upper  $1-\gamma$ -confidence limits for the reliability index (2.53) can be found by the formulas

$$\underline{R} = \Phi\left(\frac{\underline{\delta}}{\sqrt{N}}\right), \quad \bar{R} = \Phi\left(\frac{\bar{\delta}}{\sqrt{N}}\right).$$

The interval  $(\underline{R}, \bar{R})$  is the two-sided confidence interval for  $R$  with the confidence coefficient  $2\gamma-1$ .

### Confidence Limits for the Weibull-Gnedenko Distribution

Consider test of a unit by plan [N U r]. Let the unit's TTF be distributed by Weibull-Gnedenko:

$$F(t) = F(t, \alpha, \beta) = 1 - e^{-\left(\frac{t}{\beta}\right)^\alpha}$$

where  $\alpha > 0$  and  $\beta > 0$  are unknown parameters. Test results are presented by the moments of the first  $r$  failures

$$t_1 < t_2 < \dots < t_{r-1} < t_r. \quad (2.55)$$

The task is to construct the confidence limits for the reliability function for fixed  $t_0$

$$P(t_0) = 1 - F(t_0) = e^{-\left(\frac{t_0}{\beta}\right)^\alpha} \quad (2.56)$$

on the basis of the test results.

In practice, the most important is to construct the lower confidence limit for (2.56). The solution of this task is obtained in Johns and Lieberman (1966).

Instead of test results in the form of (2.55), let us introduce new r.v.'s:  $y_i = \ln\left(\frac{t_i}{t_0}\right)$ ,  $i=1, \dots, r$ . The distribution function of the r.v.  $y = \ln\left(\frac{\xi}{t_0}\right)$  can be written in the form:

$$\begin{aligned} G(u) &= P(y \leq u) = P\left(\ln\frac{\xi}{t_0} \leq u\right) = P(\xi \leq t_0 e^u) \\ &= F(t_0 e^u) = 1 - e^{-\left(\frac{t_0 e^u}{\beta}\right)^\alpha}. \end{aligned} \quad (2.57)$$

Introduce new parameters  $\mu = \ln\left(\frac{\beta}{t_0}\right)$  and  $\sigma = \frac{1}{\alpha}$ . Then the distribution function (2.57) can be written in the form

$$G(u) = 1 - \exp\left\{-e^{\frac{u-\mu}{\sigma}}\right\}$$

or, in other words,

$$G(u) = H\left(\frac{u - \mu}{\sigma}\right)$$

where  $H(z) = 1 - \exp(-e^z)$ . Thus,  $\mu$  and  $\sigma$  represent parameters of bias and scale for the distribution of the r.v.  $y$ . Notice that the reliability index defined in (2.56) is expressed via  $\mu$  and  $\sigma$  as

$$P(t_0) = \exp\left\{-\left(\frac{t_0}{\beta}\right)^\alpha\right\} = \exp\left\{-e^{-\frac{\mu}{\sigma}}\right\}. \quad (2.58)$$

So, the problem is reduced to the construction of the confidence limits for the value  $\frac{\mu}{\sigma}$ . 268

Let us introduce linear statistics

$$Z_a = \sum_{1 \leq i \leq r} a_i y_i$$

and

$$Z_b = \sum_{1 \leq i \leq r} b_i y_i$$

where  $a_i$  and  $b_i$  are some constants such that

$$\sum_{1 \leq i \leq r} a_i = 1.271 \text{ and } \sum_{1 \leq i \leq r} b_i = 0.272$$

Let us also introduce r.v.'s

$$V_a = \frac{Z_a - \mu}{\sigma}, \quad V_b = \frac{Z_b}{\sigma}. \quad (2.59)$$

These r.v.'s can be represented in the following form:



$$V_a = \sum_{1 \leq i \leq r} a_i \left( \frac{y_i - \mu}{\sigma} \right)$$

and

$$V_b = \sum_{1 \leq i \leq r} b_i \left( \frac{y_i - \mu}{\sigma} \right).$$

The distribution of r.v.'s  $\frac{y_i - \mu}{\sigma}$  does not depend on parameters  $\mu$  and  $\sigma$ . So, the two dimensional r.v.  $(V_a, V_b)$  does not depend on  $\mu$  and  $\sigma$ . This allows us to construct the confidence limits for  $\frac{\mu}{\sigma}$  and  $P(t_0)$ . From (2.59), we obtain

$$\frac{\mu}{\sigma} = \left( \frac{Z_a}{Z_b} \right) V_b - V_a.$$

For each fixed  $t$ , consider the r.v.  $tV_b - V_a$  and find the value of  $L(t)$  from the following condition

$$P\{tV_b - V_a \geq L(t)\} = \gamma \quad (2.60)$$

where  $\gamma$  is the specified confidence coefficient. Then

$$\begin{aligned} P\left\{L\left(\frac{Z_a}{Z_b}\right) \leq \frac{\mu}{\sigma}\right\} &= P\left\{\frac{Z_a}{Z_b} \leq h\left(\frac{\mu}{\sigma}\right)\right\} = P\left\{\frac{\mu + \sigma V_a}{\sigma V_b} \leq h\left(\frac{\mu}{\sigma}\right)\right\} \\ &= P\left\{\frac{\mu}{\sigma} \leq h\left(\frac{\mu}{\sigma}\right) V_b - V_a\right\} \end{aligned} \quad (2.61)$$

where  $h(\bullet)$  is the function inverse to the function  $L$ . Denoting  $t = h\left(\frac{\mu}{\sigma}\right)$  and taking into account that  $L(t) = L[h(\frac{\mu}{\sigma})] = \frac{\mu}{\sigma}$ , we obtain from (2.61) that

$$P\left\{L\left(\frac{Z_a}{Z_b}\right) \leq \frac{\mu}{\sigma}\right\} = P\{L(t) \leq tV_b - V_a\} = \gamma$$

So  $L\left(\frac{Z_a}{Z_b}\right)$  is the lower  $\gamma$ -confidence for r.v.  $\frac{\mu}{\sigma}$ . It follows from (2.58) that the lower  $\gamma$ -confidence for the reliability index  $P(t_0)$  has the form

$$P(t_0) = \exp\left\{-e^{-L\left(\frac{Z_a}{Z_b}\right)}\right\}. \quad (2.62)$$

$L(\bullet)$  can be calculated using Monte Carlo simulation.

Above procedure of the confidence limits construction depends on the choice of constants  $a_i$  and  $b_i$  while statistics  $Z_a$  and  $Z_b$  are determined. Johns and Lieberman obtained the tables for construction of optimal confidence limits which are asymptotically effective. They also give tables of the confidence limits (2.62) for various values of  $N$ ,  $r$ , and  $\gamma$ .

We present an example which permits one to compare the confidence limits obtained by the method of Johns and Lieberman with the confidence limits for an IFR distribution given in Section 2.3.

**Example 2.4.2** Ten units ( $N=10$ ) were tested by plan [N U r]. The test was terminated with occurrence of the fifth failure ( $r=5$ ). The moments of failure were (measured in hours)

$$t_1=50, t_2=75, t_3=125, t_4=250, t_5=300.$$

Assume that a unit has TTF with the Weibull-Gnedenko distribution. We need to construct the lower  $\gamma$ -confidence limit,  $\gamma=0.9$ , for the PFFO,  $P(t_0)$ , with the fixed required time of failure-free operation,  $t_0=40$  hours.

For this example, in Johns and Lieberman (1966) the lower confidence limit (2.62) was found:

$$P(t_0) = 0.796. \quad (2.63)$$

**Example 2.4.3** For the previous example, assume that the unit's TTF distribution is IFR. The total testing time of all units in this case equals

$$s=50+75+125+250+300+5 \cdot 300=2300 \text{ hours.}$$

The time interval  $t_0$  for which we estimate the unit reliability index satisfies the inequality  $40=t_0 < \frac{S}{N}$   $288=230$ . Therefore the lower confidence limit can be constructed by the "exponential methods" (see Section 2.3). Applying the formula (2.22) from the previous section and Table 12.4, we find for the same confidence coefficient  $\alpha=0.9$ :

$$\begin{aligned} \underline{P}^*(t_0) &= \exp\left\{-\frac{\Delta_{1-\gamma}(r-1)}{S}t_0\right\} = \exp\left\{-\frac{\Delta_{0.1}(4)}{2300} \cdot 40\right\} \\ &= \exp\left\{-\frac{7,99}{2300} \cdot 40\right\} = 0.869. \end{aligned} \quad \square \quad (2.64)$$

The lower confidence limit (2.64) for the class of IFR distributions is essentially higher than the lower confidence limit (2.63) which was found for the Weibull-Gnedenko distribution. Notice that the class of IFR distributions includes a family of Weibull-Gnedenko distributions (with the shape parameter  $\alpha \geq 1$ ). In our opinion, such a significant difference can be explained by the fact that  $\alpha < 1$  family of Weibull-Gnedenko distributions also contains a subclass of DFR distributions. So, the general approach can give a gain in comparison with the strict results obtained for specific cases.

### 3.5 Nonparametric Confidence Limits for Distribution Function

At the beginning of this section let us consider the test plan [N U N]. The test results in this case are moments  $t_1 < t_2 < \dots < t_N$  which form a complete sample of size  $N$ . We assume that the d.f. of unit's TTF,  $F(t)$ , is continuous. Let  $\hat{F}_N(t)$  be the empirical distribution function constructed on the basis of the test results, and

$$D_N = \max_{0 \leq t < \infty} |\hat{F}_N(t) - F(t)| \quad (2.65)$$

be the maximum deviation of the empirical d.f. from the theoretical one. It is important that the d.f. of r.v.  $D_N$  does not depend on the d.f.  $F(t)$ . To convince the reader, let us introduce a new r.v.

$\eta = F(\xi)$  which, as it is well known, has the uniform distribution on the interval  $[0, 1]$ . Then the statement above follows from the equality

$$\max_{0 \leq t < \infty} |\hat{F}_N(t) - F(t)| = \max_{0 \leq z \leq 1} |\hat{G}_N(z) - z|$$

where  $\hat{G}_N(z)$  is the empirical d.f. constructed on the basis of the sample  $\xi_i = F(t_i)$ ,  $i = 1, \dots, N$  taken from the uniform distribution.

Let the d.f. of the r.v. (2.65) be denoted by

$$H_N(u) = P\{D_N \leq u\}. \quad (2.66)$$

Distribution (2.66) is the distribution of r.v. (2.65) which is defined for any distribution of initial r.v.  $\xi$ . So, it is enough to find (2.66) for any distribution, for instance, for the uniform distribution of an r.v.  $\xi$ . Let us denote the quantile of the level  $1 - q$  for the distribution (2.66) by  $A(N, q)$ . We obtain that the inequalities

$$\hat{F}_N(t) - A(N, 1 - \gamma) \leq F(t) \leq \hat{F}_N(t) + A(N, 1 - \gamma) \quad (2.67)$$

simultaneously hold for all  $0 \leq t < \infty$  with the probability  $1 - \gamma$ . Thus, these inequalities give  $1 - \gamma$ -confidence strip for the reliability function  $P(t) = 1 - F(t)$ . In accordance with (2.67), this strip has the form

$$\hat{P}_N(t) - A(N, 1 - \gamma) \leq P(t) \leq \hat{P}_N(t) + A(N, 1 - \gamma) \quad (2.68)$$

for all  $0 \leq t < \infty$ . Here  $\hat{P}_N(t) = 1 - \hat{F}_N(t)$  is the empirical reliability function. The numerical values of  $A(N, q)$  are given in Appendix (Table E.11). For  $N \gg 1$ , the asymptotic distribution of  $\sqrt{N} D_N$  is given by the Kolmogorov distribution (see Section 1.5.6). Therefore for large sample size  $N$ , the  $1 - \gamma$ -confidence strip of the reliability function can be approximately determined by the inequalities

$$\hat{P}_N(t) - \frac{y_{1-\gamma}}{\sqrt{N}} \leq P(t) \leq \hat{P}_N(t) + \frac{y_{1-\gamma}}{\sqrt{N}}$$

where  $y_q$  is the quantile of the level  $1-q$  of the Kolmogorov distribution (the numerical tables see in Appendix, Table E.12.)

### Plans [N U r], [N U T], and [N U (r,T)]

For all these censored plans listed above, the considered confidence strips for distributions and reliability functions remain valid but only for the time interval  $0 \leq t \leq \tau$  where  $\tau$  is the moment of censorship. This moment for the plans mentioned above are determined as  $\tau = t_r$ ,  $\tau = T$ , and  $\tau = \min(t_r, T)$ . For  $t > \tau$  in the nonparametrical case, that is, where there is no assumption about a distribution belonging to any distribution family, it is impossible to declare anything except the trivial inequalities

$$0 \leq P(t) \leq \hat{P}_N(\tau) + A(N, 1-\gamma), \quad t > \tau.$$

### 3.6 Bootstrap Method of Approximate Confidence Limits

The idea of the bootstrap method will be demonstrated using the following simple example. Assume that we use plan [N U r]. A unit's TTF has the distribution,  $F(t, \theta)$ , with an unknown vector of parameters  $\theta = (\theta_1, \dots, \theta_r)$ . The results of the test are presented by the random vector

$$\mathbf{x} = (t_1, \dots, t_r)$$

where  $t_1 < t_2 < \dots < t_r$  are the moments of sequential failures.

We need to construct the confidence limits for some unit's reliability index  $R = R(\theta)$ . For instance, if we are interested in the unit's MTTF, then

$$R(\theta) = \int_0^{\infty} [1 - F(t, \theta)] dt,$$

if this reliability index is PFFO, then

$$R(\theta) = 1 - F(t_0, \theta)$$

and so on. Let  $\hat{\theta} = \hat{\theta}(x)$  and  $\hat{R} = \hat{R}(x)$  are some point estimates of  $\theta$  and  $R(\theta)$ , respectively, which are obtained on the basis of the test results  $x$ . Let us denote the d.f. of estimate  $\hat{R}$  for a given value of  $\theta$  by

$$H(t, \theta) = P_{\theta}(\hat{R} \leq t). \quad (2.69)$$

For the sake of simplicity, we assume that this function is continuous and strictly monotone increasing in  $t$ .

The lower and upper confidence limits,  $\underline{R}$  and  $\bar{R}$ , with the confidence coefficient  $1 - \alpha - \beta$  for the reliability index  $R$ , are determined from the conditions

$$H(\underline{R}, \theta) = \alpha, \quad (2.70)$$

$$H(\bar{R}, \theta) = 1 - \beta,$$

that is, they are the corresponding quantiles of the distribution  $H(t, \theta)$  of the estimate  $\hat{R}$ . However, the function  $H(t, \theta)$  depends on the parameter  $\theta$  which is unknown by the assumption. Let us set  $\theta = \hat{\theta}$  where  $\hat{\theta}$  is the estimate of  $\theta$  found on the basis of the test results. It gives us the estimate of  $H(t, \theta)$  in the form of  $H(t, \hat{\theta})$ . The final form of the confidence limits for  $\underline{R}$  and  $\bar{R}$  is presented by

$$H(\underline{R}, \hat{\theta}) = \alpha, \quad (2.71)$$

An analytical form of the d.f.  $H(t, \theta)$  is usually unknown and if known then it is rather complex. Therefore in most cases the function and confidence limits  $\underline{R}$  and  $\bar{R}$  are approximately found by Monte Carlo simulation with setting  $\theta = \hat{\theta}$ .

Thus, the procedure of this method is as follows:

(1) On the basis of real test results,  $x$ , we calculate the point estimate of the vector of unknown parameters  $\hat{\theta} = \hat{\theta}(x)$ .

(2) Then with the help of computer modeling, we find  $M$  independent realizations of the process of testing with  $\theta = \hat{\theta}$  318 :

$$(x_1, \hat{R}_1), (x_2, \hat{R}_2), \dots, (x_M, \hat{R}_M)$$

where  $x_i$  is the  $i$ th realization of modeling, and  $\hat{R}_i = \hat{R}_i(x_i)$  320 is the estimate of the reliability index obtained in the  $i$ th realization,  $1 \leq i \leq M$ .

(3) On the basis of the obtained values 321, we construct by a common way the corresponding empirical d.f.  $\hat{H}_M(t)$ . 322

Finally, its quantiles of the levels  $\alpha$  (from the left) and  $1-\alpha$  (from the right) are chosen. They are the confidence limits,  $\underline{R}$  and  $\bar{R}$  323, respectively. For instance, if  $\alpha = 0.05$  and the number of realizations  $N=1000$ , we take  $\underline{R} = \rho_{(50)}$  and  $\bar{R} = \rho_{(950)}$  324 where  $\rho_{(j)}$  is the  $j$ th order statistic.

Remark. This approach can be extended in an obvious way to a nonparametric case, if we consider  $\theta$  as the d.f. of a unit's TTF,  $F(t)$ , and  $\hat{\theta}$  325 as its empirical d.f.,  $\hat{F}_N(t)$  326 (for plan [N U N]).

The interval  $(\underline{R}, \bar{R})$  327 constructed in such a way is not, generally speaking, the confidence interval for  $R$  in a precise "frequency" sense. Indeed, it does not follow from the construction of the interval that this interval satisfies the condition

$$P_{\theta} \{ \underline{R} \leq R(\theta) \leq \bar{R} \} \geq \gamma$$

for all  $\theta$ . (Here  $\alpha = 1 - \alpha - \alpha$ .) Therefore the bootstrap limits  $(\underline{R}, \bar{R})$  329 can be interpreted as approximate confidence limits. Sometimes these limits are close to the confidence limits but sometimes they are weaker (see Example 2.6.3).

Example 2.6.1 A unit with the normal d.f of TTF is tested by plan [N U N]. The mean of this normal d.f. equals  $\mu$  and the variance equals  $\sigma^2$ , assume that  $\sigma \gg \mu^2$ . The result of this test has the form of the complete sample of size  $N$ :

$$t_1, t_2, \dots, t_N.$$

Let us construct the bootstrap limits for the parameter  $\mu$  under the assumption that the variance  $\sigma^2$  is known. The standard point estimate of the mean  $\mu$  is its empirical average

$\hat{\mu} = \frac{1}{N} \sum_{1 \leq i \leq N} t_i$ . 330 For given value of  $\mu$ , the estimate  $\hat{\mu}$  331 has the normal d.f. with the mean  $\mu$  and variance  $\sigma^2$ , that is,

$$H(t, \mu) = P_{\mu}(\hat{\mu} \leq t) = \Phi\left(\frac{t - \mu}{\sigma / \sqrt{N}}\right)$$

where  $\Phi(u)$  is the function of the standard normal distribution with parameters  $\mu=0$  and  $\sigma=1$ . Applying (2.71), we obtain the lower and upper bootstrap limits,  $\underline{\mu}$  and  $\bar{\mu}$  333, for  $\mu$  from the equations

$$\Phi\left(\frac{\mu - \hat{\mu}}{\sigma / \sqrt{N}}\right) = \alpha,$$

$$\Phi\left(\frac{\bar{\mu} - \hat{\mu}}{\sigma / \sqrt{N}}\right) = 1 - \beta.$$

From here it follows

$$\underline{\mu} = \hat{\mu} - u_{1-\alpha} \frac{\sigma}{\sqrt{N}},$$

$$\bar{\mu} = \hat{\mu} + u_{1-\beta} \frac{\sigma}{\sqrt{N}},$$

where  $u_q$  is the quantile of level  $q$  of the standard normal distribution. Thus in this case the bootstrap limits coincide with the standard exact confidence limits for  $\mu$ .  $\square$

**Example 2.6.2** Consider the conditions of the previous example with the following difference: the variance  $\sigma^2$  is unknown. In this case  $\theta = (\mu, \sigma)$  is the two dimensional parameter and the reliability index of interest is  $R=R(\theta, \theta)$ . The estimate of the vector of parameters  $\theta$  is  $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$  338

where  $\hat{\mu} = \frac{1}{N} \sum_{1 \leq i \leq N} t_i$  339 and  $\hat{\sigma} = \sqrt{\frac{1}{N-1} \sum_{1 \leq i \leq N} (t_i - \hat{\mu})^2}$  340 are the empirical average and STD,



respectively. For the given  $\square=(\square, \square)$ , the distribution of the estimate  $\hat{\mu}$  has the same form as in the previous case:

$$H(t, \mu, \sigma) = P_{\mu, \sigma}(\hat{\mu} \leq t) = \Phi\left(\frac{t - \mu}{\sigma / \sqrt{N}}\right).$$

Applying the equation (2.71), we find that the lower and upper bootstrap limits  $\underline{\mu}$  and  $\bar{\mu}$  can be found from the equations

$$\Phi\left(\frac{\underline{\mu} - \hat{\mu}}{\hat{\sigma} / \sqrt{N}}\right) = \alpha,$$

$$\Phi\left(\frac{\bar{\mu} - \hat{\mu}}{\hat{\sigma} / \sqrt{N}}\right) = 1 - \beta,$$

and from here it follows

$$\underline{\mu} = \hat{\mu} - u_{1-\alpha} \frac{\hat{\sigma}}{\sqrt{N}},$$

$$\bar{\mu} = \hat{\mu} + u_{1-\beta} \frac{\hat{\sigma}}{\sqrt{N}}.$$

In this case the bootstrap limits do not coincide with the exact confidence limits for parameter  $\square$  based on the Student distribution. Thus these limits can be considered only as approximate ones.  $\square$

**Example 2.6.3** Consider now the exponential distribution  $F(t, \square) = 1 - \exp(-t/\theta)$ . In this case, parameter  $\square$  represents the MTTF. Applying the approach described above, let us construct the bootstrap limits for the parameter  $\square$  for plan  $[N \ U \ r]$ . The standard point estimate for  $\square$  is  $\hat{\theta} = \frac{S}{r}$

348 (see Section 2.3) where  $S = t_1 + \dots + t_r + (N-r)t_r$  is the total testing time of all units during the entire test. The d.f. of the estimate  $\hat{\theta}$  349 for a specified value of  $\theta$  is given by the expression

$$H(t, \theta) = P_{\theta}(\hat{\theta} \leq t) = P_{\theta}\left(\frac{S}{r} \leq t\right) = P_{\theta}(S \leq rt).$$

Taking into account that r.v.  $S$  has the gamma distribution with the parameters  $\theta$  and  $r$  where  $\theta = 1/\lambda$  (see Section 2.3), we obtain

$$H(t, \theta) = \Gamma\left(\frac{rt}{\theta}\right) \quad (2.72)$$

where

$$\Gamma(u) = 1 - e^{-u} \sum_{0 \leq j \leq r-1} \frac{u^j}{j!}$$

is the gamma d.f.  $\square$

Thus, for this example, the d.f. of estimate  $\hat{\theta}$  353 can be easily found in an analytical form. It allows us to obtain analytical expressions for the bootstrap limits  $\underline{\theta}$  and  $\bar{\theta}$  354. In correspondence to (2.71), these limits can be found from the equations

$$H(\underline{\theta}, \hat{\theta}) = \Gamma\left(\frac{r\hat{\theta}}{\underline{\theta}}\right) = \alpha,$$

$$H(\bar{\theta}, \hat{\theta}) = \Gamma\left(\frac{r\hat{\theta}}{\bar{\theta}}\right) = 1 - \beta.$$

Finally, we have

$$\underline{\theta} = \frac{\Delta_{1-\alpha}(r-1)}{r} \hat{\theta} = \frac{\Delta_{1-\alpha}(r-1)}{r^2} S$$

$$\bar{\theta} = \frac{\Delta_{\beta}(r-1)}{r} \hat{\theta} = \frac{\Delta_{\beta}(r-1)}{r^2} S$$

where values of  $\Delta_{1-\alpha}(r-1)$  and  $\Delta_{\beta}(r-1)$  were introduced above in Section 2.3.1. At the same time, the exact confidence limits for parameter  $\theta$  for the exponential distribution and for test plan  $[N U r]$  has the form (see Section 2.3.1):

$$\underline{\theta}' = \frac{S}{\Delta_{\alpha}(r-1)} \tag{2.74}$$

$$\bar{\theta}' = \frac{S}{\Delta_{1-\beta}(r-1)}$$

From the expressions obtained above, we have that the ratio of the bootstrap limits to the confidence limits is written as follows

$$\frac{\underline{\theta}}{\underline{\theta}'} = \frac{\Delta_{1-\alpha}(r-1) \Delta_{\alpha}(r-1)}{r^2}$$

$$\frac{\bar{\theta}}{\bar{\theta}'} = \frac{\Delta_{1-\beta}(r-1) \Delta_{\beta}(r-1)}{r^2}.$$

It is easy to check, for instance, by direct calculations with the help of tables for  $\Delta_q(r-1)$  (see Table 12.4), that both ratios are less than 1. It means that the lower and upper bootstrap limits are biased to the left in respect to the exact confidence limits, that is,  $\underline{\theta} < \underline{\theta}'$  362 and  $\bar{\theta} < \bar{\theta}'$  363.

Example 2.6.4 The test was performed by plan [N U r] until the occurrence of two failures,  $d=2$ . The total unit testing time was  $S=100$  hours. Then for  $\alpha=0.9$ , and  $\beta=0.05$  using table 12.4, we find

$$\underline{\theta}' = \frac{S}{\Delta_{\alpha}(r-1)} = \frac{100}{\Delta_{0.05}(1)} = \frac{100}{4.74} = 21.1 \text{ hours}$$

$$\bar{\theta}' = \frac{S}{\Delta_{1-\beta}(r-1)} = \frac{100}{\Delta_{0.95}(1)} = \frac{100}{40.35} = 2.48 \text{ hours}$$

$$\underline{\theta} = \frac{\Delta_{1-\alpha}(r-1)}{r^2} S = \frac{0.35}{2^2} \cdot 100 = 8.75 \text{ hours}$$

$$\bar{\theta} = \frac{\Delta_{\beta}(r-1)}{r^2} S = \frac{4.74}{2^2} \cdot 100 = 118.5 \text{ hours}$$

Naturally, using the bootstrap method in this particular case has no sense since the exact confidence limits are known. However, this example clearly shows that the bootstrap method might produce a very significant bias. Thus this method should be used with a definite caution.

Nevertheless, this method has some obvious advantages: it is simple, understandable and universal. This is the reason why it is so often used in practice. This approach and some its improvements were considered by Efron (1981, 1985) and others. The bootstrap method is useful for the situations where exact confidence limits are not calculable. In particular, this method is very effective if you estimate reliability indexes of complex systems by results of testing their units (see Part 2).

## PROBLEMS to Chapter 2

2.1 Eight units ( $N=8$ ) have been tested by plan [N U r]. The test has been terminated with the occurrence of the second failure ( $r=2$ ). There were two recorded failures at  $t_1=60$  hours and  $t_2=110$  hours. A unit has the exponential distribution,  $F(t)=1-e^{-\lambda t}$ .

Construct the  $\alpha$ -confidence limit,  $\alpha=0.9$ , for the parameter  $\lambda$ , for the PFFO,  $P(t_0)$  if  $t_0=5$  hours, and for the MTTF,  $\theta$ .

2.2 The test have been performed by plan [N U T]. Ten units have been tested ( $N=10$ ). The test has been terminated at the predetermined moment,  $T=100$  hours. There have been recorded two failures at  $t_1=15$  hours and  $t_2=72$  hours. The unit has the exponential distribution,  $F(t)=1-e^{-\lambda t}$ .

Construct the point estimates and lower confidence limits with the confidence coefficient  $\alpha=0.975$  for the parameter  $\lambda$ , for the PFFO,  $P(t_0)$  if  $t_0=10$  hours, and for the MTTF,  $\theta$ .

2.3 Testing of seven units ( $N=7$ ) has been performed by plan [N U r]. The test has been terminated at the moment of occurrence of the third failure ( $r=3$ ). The moments of failure occurrence are:  $t_1=150$ ,  $t_2=250$  and  $t_3=400$  hours. The unit's TTF distribution,  $F(t)$ , is assumed to be IFR.

Construct the lower confidence limit with the confidence coefficient  $\alpha=0.975$  for the PFFO,  $P(t_0)=1-F(t_0)$  if  $t_0=20$  hours.

2.4 In the conditions of the previous exercise find the non-parametrical lower confidence limit for  $P(t_0)$  with the same confidence coefficient  $\alpha=0.975$ . Compare the result with the case where  $F(t)$  is IFR.

2.5 Assume that  $F(t)$  is IFR. In the conditions of Exercise 2.3 find the lower confidence limit for MTTF,  $\theta$ , with the confidence coefficient  $\alpha=0.975$ . Compare that results with the result for the exponential distribution.

2.6 The test of for units ( $N=4$ ) has been performed by plan [N R T]. Failed units were immediately replaced by new ones. The test has been terminated at the moment of time predetermined in advance,  $T=100$ . Six failures have been recorded ( $d=6$ ). We assume that the unit's TTF distribution,  $F(t)$ , is IFR.

Construct the lower confidence limit with the confidence coefficient  $\alpha=0.95$  for  $P(t_0)$  if  $t_0=1$  hour.

2.7 In the conditions of the previous exercise find the lower confidence limit with the confidence coefficient  $\alpha=0.95$  for the guarantee unit's TTF  $t_q$  (with the guarantee level  $q=0.9$ ).

2.8 In the conditions of Exercise 2.5 find the lower confidence limit with the confidence coefficient  $\alpha=0.95$  for the MTTF,  $\theta$ .

2.9 Ten units ( $N=10$ ) have been tested by plan [N U r]. The test has been terminated after the fifth failure ( $r=5$ ). The moments of failure occurrence are:  $t_1=50$ ,  $t_2=75$ ,  $t_3=125$ ,  $t_4=250$ , and  $t_5=300$

hours. It is known that the unit's TTF distribution,  $F(t)$ , is IFR. Find the non-parametrical lower confidence limit with the confidence coefficient  $\alpha=0.9$  for  $P(t_0)$  if  $t_0=40$  hours. Compare this result with that obtained for the Weibull-Gnedenko distribution (see Example 2.4.2 above) and IFR distribution (see Example 2.4.3).

## 4. Censored Samples

### 4.1 Introduction

Censored samples appear in practice if for some reasons, we stop testing before all tested units have failed. The procedure of test interruption may be different: test of the entire sample might be stopped at some moment of time (chosen in advance or spontaneously); test of some group of tested units might be stopped at some specified moments; finally, test of some individual units might be terminated for some reason. At the end, we know several moments of unit failures and also know that some units have their time to failure (TTF),  $t_j$ , larger than moment of their test stopping,  $\tau_j$ .

Sometimes we have information that the moment of failure belongs to the time interval  $a_j \leq t_j \leq b_j$ . Such a situation appears if we perform only periodical inspections. At moment  $b_j$  we found that the unit  $j$  has failed though at the previous inspection at moment  $a_j$  it was operational. These test plans cover most but not all possible types of censorship.

### 4.2 Independent Random Censorship

One of the most known and best investigated problem of censored testing is the following. We have a unit with unknown d.f.  $F(t)=P\{\tau \leq t\}$  of a random TTF,  $\tau$ . We are testing  $N$  identical independent units. Testing of each unit is terminated at a random time  $\tau_j$  if it has not failed. Random variables  $\tau$  and  $\tau_j$  are independent. The d.f. of  $\tau_j$  is denoted by  $G(t)=P\{\tau_j \leq t\}$ . Thus unit  $j$  is tested up to the moment

$$z_j = \min(\tau_j, \tau_j)$$

where  $\tau_j$  is the moment of the  $j$ th unit failure,  $\tau_j$  is the moment of the  $j$ th unit's test termination,  $j=1, \dots, N$ . We assume that all r.v.'s  $\tau_1, \dots, \tau_N, \tau_{j1}, \dots, \tau_{jN}$  are mutually independent. Thus, the test results are  $N$  pairs of r.v.'s

$$(z_j, I_j), \quad j=1, \dots, N \quad (3.1)$$

where  $I_j=1$  if  $\tau_j \leq \tau_j$ , that is, the  $j$ th unit has failed during the test, and  $I_j=0$  if  $\tau_j > \tau_j$ , that is, the test was terminated before a failure.

Using the test results (3.1), we need to estimate this d.f. These results can be represented in more convenient form by introducing an additional notation

$$\begin{aligned} s_j &= z_j \text{ if } I_j=1, \\ u_j &= z_j \text{ if } I_j=0. \end{aligned}$$

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Values  $s_j$  are called *complete intervals* and values  $u_j$  are called *censored ones*. In this notation (3.1) can be rewritten as

$$(s_1, \dots, s_d), (u_1, \dots, u_{N-d})$$

where  $d$  is the number of failed units and  $N-d$  is the number of units whose test was terminated.

This model is called the model of *independent random censorship*. Sometimes this model is also called the model of "*concurrent risks*" where moments of test termination are interpreted as failures of some other types.

**Example 3.1** Consider a series system consisting of two independent units. Their TTFs are denoted by  $\square$  and  $\square$ , and their d.f.'s by  $F(t)=P\{\square \leq t\}$  and  $G(t)=P\{\square \leq t\}$ , respectively. The system TTF is defined as

$$z = \min(\square, \square).$$

Assume that we wish to estimate the d.f. of the first unit, i.e.,  $F(t)$  under the following restriction: this unit can be tested only within the test of the entire system. Let  $N$  systems be under the test and every time the test is continued until system failure. We have test results in the form

$$(z_j, I_j), j=1, \dots, N$$

where  $z_j = \min(\square_j, \square_j)$  is the moment of system failure,  $\square_j$  and  $\square_j$  are the moments of unit failures within the  $j$ th system,

$$\begin{cases} 1 & \text{if } \square_j \leq \square_j \\ I_j = \{ & \\ 0 & \text{if } \square_j > \square_j. \end{cases}$$

It is clear that this is a particular case of the model of independent random censorship ("concurrent risks"). □

**Example 3.2** Consider a series system consisting of  $n$  units of the first type and  $m$  units of the second type with the d.f.'s  $F(t)=Pr\{\square \leq t\}$  and  $G(t)=Pr\{\square \leq t\}$ , respectively. All system units are independent. The moment of the system failure is determined as

$$z = \min(\square_1, \dots, \square_n, \square_1, \dots, \square_m)$$

where  $\square_k$  is the moment of failure of the  $k$ th unit of the first type,  $k=1, \dots, n$ , and  $\square_r$  is the moment of failure of the  $r$ th unit of the second type  $r=1, \dots, m$ . We are testing  $N$  identical systems until failure. Let us again estimate  $F(t)$ . The test results are represented by the set of  $Nn$  TTF intervals of units of the first type:  $d$  complete and  $(Nn-d)$  censored, where  $d$  is the number of system failures due to units of the first type.

In this case, the model of test is not based on independent random censorship; after a failure of a unit of the first type, testing of all of the remaining units of this type is terminated. It leads to the fact that the moment of the test termination of the units of the first type might depend on the failures moments of units of the same type. □

**Remark 3.1** These examples lead to the problem of estimation of unit reliability on the basis of system testing. The "inverse problems" concerning estimation of the system reliability on the basis of censored testing of its units are less investigated (see Part II of the book).

**Remark 3.2** (relating to terminology). Let  $t_j$  be the moment of the  $j$ th unit failure and  $\tau$  be the moment of censorship. Then if  $\tau < t_j$  we say about *censorship from the left*. (Examples above are of this type.) If  $t_j < T$  where  $T$  is some moment of time, we say about *censorship from the right*. Finally, if  $a < t_j < b$ , we say about *interval censorship*.

**Example 3.3** Units are tested by the plan [N U T], that is,  $N$  units to be tested starting at  $t=0$  with no replacement of failed units and the test continues until a prior specified moment  $T$ . There is no recording the moments of failures during the test. Thus, the only information is that  $d$  units have failed up to the moment  $T$ :  $t_j \leq T$ ,  $j=1, \dots, d$  (censorship from the left). All the remaining  $N-d$  units don't fail, i.e.,  $t_k > T$ ,  $k=1, \dots, N-d$  (censorship from the right).  $\square$

**Example 3.4** Consider the test plan [N U T]. There is a possibility to check the unit states only at some specified moments

$$0 < T_1 < \dots < T_n < T.$$

We have observed  $d$  failures for each of which it is known that the moment of failure  $t_j$  belongs to an interval  $(T_j, T_{j+1})$ . Thus, for  $d$  units we have interval censorship and for the remaining  $N-d$  units  $t_k > T$ , that is, there is censorship from the right.  $\square$

We see that there are many various situation which lead to the test censorship. Main attention will be paid to the models with censorship from the right.

### 4.3 Markov Model of Censored Testing without Renewal

The following testing model with censorship was considered in [Ushakov, 1980], and [Pavlov and Ushakov, 1984]. It is a natural generalization of the model of independent random censorship ("concurrent risks"). A close model was considered in [Belyaev, 1984]. Let us present an informal description of the problem (formal description is given below in Section 3.7.2 of Appendix of the current chapter).

Let a test of  $N$  identical and independent units with unknown d.f. begin at moment  $t=0$ . At some moment  $\tau_1$  we terminate the test of  $n_1$  units (operational up to this moment), then at some moment  $\tau_2$ ,  $\tau_2 > \tau_1$ , we terminate the test of  $n_2$  units, and so on. The sequential moments of time  $\tau_1, \tau_2, \dots, \tau_k, \dots$  and corresponding numbers  $n_1, n_2, \dots, n_k, \dots$  can be chosen in advance or in the process of testing (even depending on the result of the test). All of these values might be deterministic or random. Moments where we terminate testing of some number of units  $\tau_1 < \tau_2 < \dots < \tau_k < \dots$  form monotone increasing sequence of Markov random moments (see below Section 3.7.1 of Appendixes to the current chapter). The number of units,  $n_1$ , whose testing is terminated at the moment  $\tau_1$ , is random and might depend on the prehistory of test evolution up to the moment  $\tau_1$  but does not depend on the future testing process trajectory for  $t > \tau_1$ . In analogous way, the number



of units,  $n_k, k > 1$ , whose testing is terminated at the moment  $\tau_k$ , is random and might arbitrarily depend on the prehistory but does not depend on the future for  $t > \tau_k$ .

The moment of the termination of the entire test,  $\tau$ , might also be random. The total number of inner termination points might be random (although naturally restricted by  $N$ ). So, it is sufficient to consider a finite sequence of  $N$  moments:  $\tau_1 < \tau_2 < \dots < \tau_k < \dots < \tau_N \leq \tau$ . For formal description, it is convenient to assume that some  $n_k$  might be equal to 0. The number of a "real" number of termination,  $r$ , is defined as the number of such moments  $\tau_k$  for which  $n_k > 0$ . Thus the model is given by the set

$$M = \{(\tau_1, n_1), (\tau_2, n_2), \dots, (\tau_N, n_N), \tau\} \quad (3.2)$$

We will call this model of testing a Markov model. We repeat once more that the intervention in the testing process does not depend on the future.

This model generalizes the model of independent random censorship ("concurrent risks"), standard testing plans without replacement of types  $[N \ U \ T]$ ,  $[N \ U \ r]$  and  $[N \ U \ (r, T)]$  considered in Chapter 2 and factually all other testing plans without replacement. For instance, it includes the case where a sample is tested simultaneously by several different plans of class  $U$  (some by plan  $[N_1 \ U \ r_1]$ , another by plan  $[N_2 \ U \ T_2]$ , and so on).

The main characterization of the plan (3.2) is an absence of replacement of failed units. We will call this model Markov model of type  $U$  and use the abbreviation  $[MMU]$ . More general models including tests with replacement (renewal) are considered below in Section 3.6.

As above, an interval is called complete if it terminates by the unit failure, and censored otherwise. Let  $N(t)$  be the number under testing at moment  $t$ . A graph of function  $N(t)$  is a convenient form of the process presentation. The graph can be constructed by putting ordered intervals one over another. The staircase function  $N(t)$  is represented by the envelope of this set of intervals (see Figure 3.1).

Figure 3.1

The results might also be presented by the set of failure moments  $t_1, t_2, \dots, t_j, \dots$ , moments of termination  $\tau_1, \tau_2, \dots, \tau_k, \dots$ , and numbers of units whose test is terminated at those moments,  $n_1, n_2, \dots, n_k, \dots$ . Same information might be also presented by the set of complete and censored intervals:  $s_1, s_2, \dots, s_d; u_1, u_2, \dots, u_{N-d}$ , where  $d$  is the total number of failures. Notice that some  $u_k$  may coincide if  $n_k > 1$ .

## 4.4 Non-Parametrical Estimates of Reliability Function

### 4.4.1 Kaplan-Meier Estimate

Let us consider the problem of construction of the point estimate of unit reliability function,  $P(t) = 1 - F(t)$ . The d.f.  $F(t)$  is assumed continuous. Estimate (3.3) given below was found by Kaplan and Meier (1958). This estimate is often called *product limit estimator*. It can be easily extended on the general Markov model described above.

For convenience, let us introduce the following notation:

$D(t)$  is the number of failures on interval  $(0, t]$ , or in other words, the number of failure moments  $t_j$  such that  $t_j \leq t$ ;

$L(t)$  is the number of units whose test has been terminated on interval  $(0, t]$ , or in other words,

$$L(t) = \sum_{k: \tau_k \leq t} n_k$$

$N(t)$  is the number of units under test at the moment  $t$  (that is, not failed or terminated):

$$N(t) = N - D(t) - L(t).$$

All random functions  $D(t)$ ,  $L(t)$ , and  $N(t)$  are continuous in  $t$  from the right. Let us also denote

$$D(t^-) = \lim_{\varepsilon > 0, \varepsilon \rightarrow 0} D(t - \varepsilon), \quad N(t^-) = \lim_{\varepsilon > 0, \varepsilon \rightarrow 0} N(t - \varepsilon)$$

the limits of the mentioned functions from the left at the moment of time  $t$ .

Let  $0 < \tau_1 < \tau_2 < \dots < \tau_r$  be the moments of test termination for groups of units and  $l$  is the total number of such termination. Then the estimate  $\hat{P}(t)$  of the reliability function  $P(t)$  at the moment  $t$  is constructed as follows

$$\hat{P}(t) = \prod_{0 \leq k \leq m-1} \left( 1 - \frac{d_k}{N_k} \right) \left( 1 - \frac{d_m(t)}{N_m} \right)$$

for  $\tau_m < t \leq \tau_{m+1}$ ,  $m=0, 1, \dots, r$  where  $d_k = D(\tau_{k+1}) - D(\tau_k)$  is the number of failures on the  $k$ th interval  $(\tau_k, \tau_{k+1}]$  between two sequential termination points;  $N_k = N(\tau_k)$  is the number of units testing at the beginning of the  $k$ th interval  $(\tau_k, \tau_{k+1}]$ , or in other words, the number of units under the test after current test termination for a group of units at moment  $\tau_k$ ,  $k=0, 1, \dots, r$ ;  $d_m(t) = D(t) - D(\tau_m)$  is the number of failures on the interval  $(\tau_m, t]$ . Here we assume that  $\tau_0 = 0$ ,  $N_0 = N(0) = N$ , and  $\tau_{l+1} = \infty$ .

It is easy to see that (3.3) can be rewritten in the following form

$$\hat{P}(t) = \prod_{1 \leq j \leq D(t)} \left( 1 - \frac{\Delta D(t_j)}{N(t_j)} \right) \quad (3.4)$$

where  $t_j$  is the moment of the  $j$ th failure;  $N(t_j^-) = N(t_j - 0)$  is the number of units under the test just before the moment of the  $j$ th failure;  $\Delta D(t_j) = D(t_j) - D(t_j^-)$  is the number of failures at the moment  $t_j$ . If the d.f.  $F(t)$  is continuous then  $\Delta D(t_j) = 1$  with the probability 1. It means that (3.3) and (3.4) can be rewritten as

$$\hat{P}(t) = \prod_{1 \leq j \leq D(t)} \left( 1 - \frac{1}{N(t_j)} \right) \quad (3.5)$$

Sometimes it is more convenient to write (3.4) and (3.5) in the form of infinite product as

$$\hat{P}(t) = \prod_{u \leq t} \left( 1 - \frac{\Delta D(u)}{N(u)} \right)$$

where  $N(u) = N(u-0)$ , and  $\Delta D(u) = D(u) - D(u^-)$ .

**Remark 3.3** Expressions (3.3)-(3.5) are valid for all situations except the only special case where  $m=1$ ,  $\square_1 < t$  and  $N = N(\square_1) = 0$ , that is, the test of all units has been terminated at some moment  $\square_1$  before  $t$ . (For illustration, in Figure 3.2 we depicted the case with  $r=4$ .) In this case one needs to reject attempts of construction of estimate, or formally set, for instance,  $\hat{P}(t) = 0/365$ . (In the latter case, we should use a conditional agreement that ratio  $\frac{0}{0} = 1/366$ .)

Figure 3.2

**Example 3.5** Nineteen units have been tested ( $n=19$ ). The test was stopped at moment  $t$ . The test results are presented in Tables 3.1 and 3.2 where  $t_j$  are the moments of failures (in hours),  $\square_k$  are the moments of test termination for a group of  $n_k$  units. The graph for this case is depicted in Figure 3.3.

Table 3.1

Table 3.2

Figure 3.3

We need to construct the estimate  $\hat{P}(t) = 0/367$  for the reliability function  $P(t)$  for the specified moment  $t=300$  hours. In this case the total number of test termination  $r=4$ . The moment, for which the reliability function is estimated, is  $t=300$  and it satisfies the inequalities  $\square_1 < \square_2 < t < \square_3 < \square_4$ , that is,  $m=2$ . The numbers of failures between two sequential test termination moments are

$$d_0 = D(\square_1) - D(0) = D(\square_1) = 3,$$

$$d_1 = D(\square_2) - D(\square_1) = 4 - 3 = 1.$$

The number of failures on the interval  $(\square_2, t]$  equals

$$d_2(t) = D(t) - D(\square_2) = 5 - 4 = 1.$$

The values of  $N_k$  at the beginning of these intervals are

$$N_0 = N(0) = 19,$$

$$N_1 = N(\square_1) = N_0 - d_0 = 13,$$

$$N_2 = N(\square_2) = N_1 - d_1 = 10.$$

Thus, the estimate (3.3) in this case equals

$$\hat{P}(t = 300) = \left(1 - \frac{d_0}{N_0}\right) \left(1 - \frac{d_1}{N_1}\right) \left(1 - \frac{d_2(t)}{N_2}\right) = \left(1 - \frac{3}{19}\right) \left(1 - \frac{1}{13}\right) \left(1 - \frac{1}{10}\right) = 0.698.$$

Calculation with the help of formula (3.5) gives the same result

$$\hat{P}(t = 300) = \prod_{1 \leq j \leq 5} \left(1 - \frac{1}{(t_j)}\right) = \left(1 - \frac{1}{19}\right) \left(1 - \frac{1}{18}\right) \left(1 - \frac{1}{17}\right) \left(1 - \frac{1}{13}\right) \left(1 - \frac{1}{10}\right) = 0.698.$$

Notice that (3.5) usually leads to simpler calculations because it does not need any ordering of intervals  $(\square_k, \square_{k+1}]$  on the time axis.  $\square$

**Example 3.6** (The case where the moment of test termination of a group of units coincides with a moment of one of failures). In the condition of the previous example let us calculate the estimate of reliability function  $P(t)$  for the moment  $t=375$ . In this case, one of the termination moments,  $\square_3=344$ , coincides with the moment of failure,  $t_6=344$ . The calculation is analogous to the previous case. In this case the moment  $t=375$  satisfies the inequalities

$$\square_1 < \square_2 < \square_3 < t < \square_4,$$

that is,  $m=3$ . The values of  $d_k$  and  $N_k$  equal

$$\begin{aligned} d_0 &= D(\square_1) - D(0) = 3, \\ d_1 &= D(\square_2) - D(\square_1) = 1, \\ d_2 &= D(\square_3) - D(\square_2) = 2, \\ d_3(t) &= D(t) - D(\square_3) = 0; \\ N_0 &= N(0) = 19, \\ N_1 &= N(\square_1) = 13, \\ N_2 &= N(\square_2) = 10, \\ N_3 &= N(\square_3) = 3. \end{aligned}$$

The estimate (3.3) in this case is

$$\hat{P}(t = 375) = \left(1 - \frac{d_0}{N_0}\right) \left(1 - \frac{d_1}{N_1}\right) \left(1 - \frac{d_2}{N_2}\right) \left(1 - \frac{d_3(t)}{N_3}\right) = \left(1 - \frac{3}{19}\right) \left(1 - \frac{1}{13}\right) \left(1 - \frac{2}{10}\right) = 0.621.$$

Calculation with the help of formula (3.5) gives the same result

$$\hat{P}(t = 375) = \prod_{1 \leq j \leq 6} \left(1 - \frac{1}{N(t_j)}\right) = \left(1 - \frac{1}{19}\right) \left(1 - \frac{1}{18}\right) \left(1 - \frac{1}{17}\right) \left(1 - \frac{1}{13}\right) \left(1 - \frac{1}{10}\right) \left(1 - \frac{1}{9}\right) = 0.621. \quad \square$$

#### 4.4.2 Discrete Scheme of Testing

Let us consider a simple approximate method of obtaining the estimates (3.3)-(3.5) above.

Let us divide the interval  $(0, t]$  into  $M$  equal parts, each of the length  $h = \frac{t}{M}$ . Consider the case

where test termination moments of groups of units,  $\square_k$ , can take only discrete values  $h, 2h, \dots, jh, \dots$  where the size of a step  $h$  is fixed but might be infinitesimally small. Consider the conditional probability of unit failure on the interval  $(jh, jh+h)$  under condition that the unit has not failed before the moment  $jh$  or has not been removed from the test. Denote this probability by

$$q_j = \frac{P(jh) - P(jh+h)}{P(jh)}. \quad (3.6)$$

Then  $P(t)$  might be written as

$$P(t) = \prod_{0 \leq j \leq M-1} (1 - q_j). \quad (3.7)$$

Formula (3.7) can easily be obtained from the definition of the reliability function and the conditional probabilities (3.6). (See Exercise 3.1 below.)

Consider the testing process on the interval  $(jh, jh+h)$ . At the moment  $jh$  there are some random number of units  $N(jh)$  which have not failed or whose test has not been terminated. On the interval  $(jh, jh+h)$  these units are operating without external "intervention" because the next termination might occur only at the moment  $jh+h$ . So, on this interval we deal with a simple binomial scheme of testing. Namely, we have  $N(jh)$  independent trials with the probability  $q_j$  to fail at each trial. The standard estimate for the binomial parameter  $q_j$  is the value

$$\hat{q}_j = \frac{d_j}{N(jh)}$$

where  $d_j = D(jh+h) - D(jh)$  is the number of failed units onto the interval  $(jh, jh+h)$ . Substituting (3.8) into (3.7) gives the estimate of the reliability function

$$\hat{P}(t) = \prod_{0 \leq j \leq M-1} \left( 1 - \frac{d_j}{N(jh)} \right). \quad (3.9)$$

It is easy to see that this is just the given above in (3.3)-(3.5) the Kaplan-Meier estimate where the test termination moments,  $\square_k$ , are  $jh, j=1, 2, \dots$ . (The estimate for the general case can be obtained by the limit for  $h \rightarrow 0$ .)

**Remark 3.4** It can be seen from the arguments above, the discrete scheme of censored tests is a sequence of Bernoulli trials with varied sample size from step to step. For parameter  $q_j$ , the following approximate formula for small  $h$  is valid

$$q_j = \frac{P(jh) - P(jh+h)}{P(jh)} = 1 - \exp \left[ - \int_{jh}^{jh+h} \lambda(t) dt \right] \approx \lambda(jh)h$$

where  $\lambda(t)$  is the failure rate of the unit. Thus, in general case where  $h \neq 0$ , the process of testing can be interpreted as a sequence of binomial trials with a randomly changed size of a sample  $N(t)$  and infinitesimally small binomial parameter equal to the probability of failure  $q = \lambda(t)h$ . On the basis of this experiment, one needs to estimate some reliability index which depends on  $\lambda(t)$ , for instance, the reliability function

$$P(t) = \exp\left[-\int_0^t \lambda(u)du\right].$$

**Remark 3.5** Let us mention an interesting analogy with the problem of system reliability estimation on the basis of the results of unit tests. In the discrete model of censored test considered above, the estimate (point or interval) of  $P(t)$  is reduced to the estimate (3.7) of binomial parameters  $q_j$  on the basis of corresponding binomial tests

$$\{N(jh), d_j\}, j=0, 1, \dots, M-1 \quad (3.10)$$

where  $N(jh)$  is the number of tests of the parameter  $q_j$  and  $d_j$  is the number of observed failures. But this problem coincides with the problem of reliability estimation of the series system, consisting of  $M$  different units, based on the results of binomial trials (3.10) for each unit individually (see Part II below). The difference is in the fact that in the problem considered above, the sample size  $N(jh)$  for each  $j$ th parameter  $q_j$  is random and depends on other parameters test results. In system reliability estimation for different parameters  $q_j$ , sample sizes  $N_j$  are specified in advance and fixed.

Limit for  $h \rightarrow 0$  shows us that the estimate of unit reliability function,  $P(t)$ , on the basis of censored tests can be reduced to the following asymptotic problem: to construct an estimate (point or interval) for the reliability function of a series system consisting of  $M$  different units on the basis of binomial trials for each unit if the number of system's units  $M \rightarrow \infty$  and the probability of failure of each  $j$ th unit  $q_j \rightarrow 0, j=1, \dots, M$ .

Ushakov (1980) suggested another recurrent procedure of calculation of estimate  $\hat{P}(t)$  which produces the estimate coinciding with the Kaplan-Meier estimate (3.3)-(3.5). (See below Exercises 3.2 and 3.3.) Pavlov and Ushakov (1984) considered Markov model of censored tests above obtained conditions where this estimate as well as estimates (3.3)-(3.5) are unbiased.

Let us describe this recurrent procedure. As before,  $m$  is the index that satisfies the following inequalities

$$0 = t_0 < t_1 < \dots < t_m < t \leq t_{m+1} < \dots < t_r < t_{r+1} = \infty$$

where  $t_k, k=1, \dots, r$ , is the moment of test termination for a group of units,  $r$  is the total number of such moments (in this particular test). Let us introduce values  $r_m, r_{m-1}, \dots, r_0$  which are calculated by the following recurrent procedure

$$\begin{cases} r_m = d_m(t) \left(1 + \frac{n_m}{N_m}\right) \\ r_k = (r_{k+1} + d_k) \left(1 + \frac{n_k}{N_k}\right), \quad k = m-1, m-2, \dots, 0 \end{cases}$$

where values  $d_m(t)$ ,  $N_k$ ,  $d_k$  has the same meaning as in (3.3),  $n_k$  is the number of units whose test has been terminated at moment  $\square_k$ ,  $k=0, 1, \dots, m-1$ ;  $n_0=0$ . This procedure has the following simple meaning. Based on the number of failures,  $d_m(t)$ , observed on the interval  $(\square_m, t]$ , we can extrapolate the number of failures,  $r_m$ , in a natural way which may occur within this interval if  $n_m$  units, whose test was terminated at the moment  $\square_m$ , would continue to be tested. In analogous way, based on observed numbers of failures  $d_k, d_{k+1}, \dots, d_m$ , the value of  $r_k$  is the extrapolation of the number of failures which could occur on the interval  $(\square_k, t]$  if  $n_k$  units whose test has been terminated at the moment  $\square_k$  would continue to be tested. After finding all values described above, the estimate of the reliability function for the moment  $t$  can be found as

$$\hat{P}(t) = 1 - \frac{r_0}{N} \quad (3.12)$$

where  $N=N(0)$  is the initial number of units under the test. The recurrent procedure is defined for all situations except  $m=r$ ,  $\square_1 < t$  and  $N_1=N(\square_1)=0$ , that is, when at some moment  $\square_1$  before the moment  $t$  the test of all units has been terminated (see also Remark 3.1 and Figure 3.2). In this situation, as above, one should suggest no estimate, or formally takes  $\hat{P}(t) = 0.370$

#### 4.4.3 Unbiased Estimator

Let us introduce the condition

$$P\{\tau_r < t, N(\tau_r) = 0\} = 0. \quad (3.13)$$

This condition means that not all unit tests have been terminated before moment  $t$ . Let us introduce even stronger condition:

$$P\{N(\tau_r) = 0\} = 0. \quad (3.14)$$

The value of  $\square_r$  is the test termination of the last group of units. Notice that if there was no termination at all, i.e., it was a complete sample, then we assume  $r=0$ ,  $\square_r = \square_0 = 0$ ,  $n_0=0$ . The condition (3.14) means that (3.13) holds for all  $t$ . In other words, some units are always remaining under testing.

If condition (3.13) holds, the equality

$$\mathbf{E}\{r_0\} = NF(t)$$

is valid, that is, estimate (3.12) and, consequently, (3.3)-(3.5), is an unbiased estimate of the reliability function  $P(t)=1-F(t)$  at the moment  $t$ . (The proof of this statement is given in Section 3.7.3 of Appendix to the current Chapter.) If (3.14) holds, then the estimate mentioned above is unbiased for all  $t$ .

**Example 3.3.3** As a simple example let us consider test plan **[N U T]**. In this case, obviously, the condition (3.14) does not hold, and the condition (3.13) holds for  $t \leq T$ . So, the estimates considered above give an unbiased estimate for  $P(t)$  for all  $t \leq T$ . ■

**Example 3.3.4** Consider test plan **[NB U r]** where  $r < N$ . In this case for any  $t > 0$  test of all units might be terminated before  $t$  with the positive probability at the moment of the  $r$ th failure,  $t_r$ . The probability of this event equals

$$\sum_{r \leq d \leq N} \binom{N}{d} [F(t)]^d [1 - F(t)]^{N-d}$$

The condition (3.13) holds for no  $t > 0$ . ■

**Example 3.3.5** Consider now the model of independent random censoring ("concurrent risks"). For this model the probability of the left side of (3.13) also differs from 0 for any  $t > 0$  if  $G(t) > 0$ . It means, as in the previous example, the condition (3.13) does not hold, and consequently, it is impossible to declare that the estimates considered above are unbiased. ■

**Example 3.3.6** Assume that the test is performed by the following plan. At the moment  $t=0$  ten units are tested. After the first failure, test of  $n_1=2$  units is terminated. After the second failure test of  $n_2=4$  units is terminated. After this the test is continued until the failure of two remaining units. In this case the condition (3.14) holds, or in other words, the condition (3.13) holds for all  $t > 0$ . So, the estimates considered above are unbiased for all  $t > 0$ . ■

**Example 3.3.7** Let two tests be performed simultaneously: by plan **[N<sub>1</sub> U r]** where  $r < N_1$  and by plan **[N<sub>2</sub> U T]**. In this case the condition (3.13) holds and, consequently, the estimates considered above are unbiased for all  $t \leq T$ . ■

**Example 3.3.8** Let the units be tested simultaneously by several plans of type **[N<sub>j</sub> U r<sub>j</sub>]**,  $r_j < N_j$ ,  $j=1, \dots, k$ ; and several plans of type **[N<sub>j</sub>' U T<sub>j</sub>]**,  $j=1, \dots, m$ . In this case the condition (3.13) holds; consequently, the estimates considered above are unbiased for any  $t \leq \max_{1 \leq j \leq m} T_j$ . ■ 371

#### 4.4.4 Non-Parametric Estimator of "Resource Function"

In practice of reliability characterization, one often uses a *resource function*  $\square(t)$  which relates to the reliability function,  $P(t)$ , via the well known expression



$$\Lambda(t) = -\ln P(t) = \int_0^t \lambda(u) du \quad (3.16)$$

where  $\lambda(u)$  is the hazard function. (In this chapter we everywhere assume that the function  $\lambda(u)$  is continuous in  $t \geq 0$ .)

Consider the Markov testing model [MMU] corresponding to censorship without replacement of failed units. Consider an approximate derivation of the point estimate of  $\hat{\Lambda}(t)$  for the resource function. We turn our attention again to the discrete scheme of testing (see Section 3.3.2 above) where test termination moments,  $\tau_k$ , for groups of units might take only discrete meanings  $h, 2h, \dots, jh, \dots$  with some step  $h$ .

From (3.6) and (3.7) we obtain

$$\Lambda(t) = -\ln P(t) = - \sum_{0 \leq j \leq M-1} \ln (1 - q_j) \quad (3.16)$$

where

$$q_j = 1 - \exp \left[ - \int_{jh}^{jh+h} \lambda(u) du \right].$$

It is clear that if  $h \ll 0$  values  $q_j \ll 0$ . So, for small  $h$  from (3.16) the approximate equality follows

$$\Lambda(t) \approx \sum_{0 \leq j \leq M-1} q_j. \quad (3.17)$$

More precisely, as it can be shown (see Section 3.7.4 of Appendix to the current chapter) that

$$\Lambda(t) = \sum_{0 \leq j \leq M-1} q_j + \delta \quad (3.17)$$

where  $|\delta| < Ch$  and  $C$  is some constant. Thus, for a discrete testing scheme the estimation of the resource function  $\Lambda(t)$  is reduced (with some error  $\delta$  which can be chosen arbitrary small by an appropriate choice of  $h$ ) to the estimation of the sum of binomial parameters

$$\Lambda(t) = \sum_{0 \leq j \leq M-1} q_j.$$

The latter value is estimated in obvious way. One needs to substitute estimates  $\hat{q}_j$  of parameters from (3.8) to obtain

$$\hat{\Lambda}(t) = \sum_{0 \leq j \leq M-1} \hat{q}_j = \sum_{0 \leq j \leq M-1} \frac{d_j}{N(jh)}. \quad (3.19)$$

The limit in  $h \rightarrow 0$  gives

$$\hat{\Lambda}(t) = \sum_{0 \leq j \leq D(t)} \frac{1}{N(t_j)} \quad (3.20)$$

where  $t_j$  is the moment of the  $j$ th failure,  $D(t)$  is the total number of failures on the interval  $(0, t]$ ,  $N(t_j^-) = N(t_j - 0)$  is the number of units tested just before the moment of the  $j$ th failure,  $t_j$ . Indeed, from the graph of the function  $N(t)$  in Figure 3.1 we can see that right sides of (3.19) and (3.20) begin to coincide for  $h < \Delta$  where  $\Delta > 0$  is the minimum length of an interval between two neighboring failures,  $t_j$ , and termination moments,  $\Delta_k$ .

Note that the estimate (3.20) is defined for  $N(t) > 0$  or for  $\Delta \geq t$  where  $\Delta$  is the moment of termination of the entire test that means that at least one unit is still tested at moment  $t$ . If  $N(t) = 0$  (or  $\Delta < t$ ), that is, all units have failed or their testing has been terminated up to the moment  $t$ , the estimate is undefined. In this case one might reject the estimation completely, or to set, for instance, either  $\hat{\Lambda}(t) = \hat{\Lambda}(v) = 374$  or  $\hat{\Lambda}(t) = \infty = 375$ , and so on.

The estimate  $\hat{\Lambda}(t)$  (3.20) for plan [M M U] constructed in such a way can be called an empirical resource function. First estimates of type (3.20) were obtained by Nelson (1969, 1972). (For details, see also Belyaev (1984, 1987), Zamyatin (1986), and others.) Conditions for the consistency of the estimates of the reliability function  $\hat{P}(t)$  (3.17) and resource function  $\hat{\Lambda}(t)$  (3.20) for the initial number of tested units  $N = N(0) = n$  are given in Section 3.7.6 of Appendix to the current chapter.

Below we give some numerical examples of calculation of estimates of the reliability function and resource function on the basis of formula (3.20).

**Example 3.3.9** In the conditions of Example 3.3.1, let us find estimates of the reliability function  $P(t)$  and resource function  $\Lambda(t)$  for  $t=200$  and  $t=300$ . On the basis of formula (3.20) we obtain

$$\begin{aligned} \hat{\Lambda}(200) &= \sum_{1 \leq j \leq 4} \frac{1}{N(t_j)} \\ &= \frac{1}{N(34)} + \frac{1}{N(79)} + \frac{1}{N(107)} + \frac{1}{N(177)} = \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{13} = 0.244 \\ \hat{\Lambda}(300) &= \sum_{1 \leq j \leq 5} \frac{1}{N(t_j)} \\ &= \frac{1}{N(34)} + \frac{1}{N(79)} + \frac{1}{N(107)} + \frac{1}{N(177)} + \frac{1}{N(257)} = \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{13} + \frac{1}{10} = 0.344. \end{aligned}$$

Corresponding estimates of the reliability function for  $t=200$  and  $t=300$  are

$$\hat{P}(200) = e^{-\hat{\Lambda}(200)} = e^{-0.244} = 0.783,$$

$$\hat{P}(300) = e^{-\hat{\Lambda}(300)} = e^{-0.344} = 0.709.$$

The latter estimate has insignificant deviation from the estimate  $\hat{P}(300) = 0.698379$  which was found in Example 3.3.1 with the use of the Kaplan-Meier formula. †

#### 4.5 Asymptotic Confidence Limits

The problem of construction of confidence limits for the reliability function and resource function in the case of censored samples is more complex than for the case of complete sample. It explains why most often for large samples these confidence limits are constructed as asymptotical. The model of independent random censorship ("concurrent risks") is most developed. These problems were considered by Breslow and Crowley (1974), Hall and Wellner (1980), Gill (1983), Belyaev (1984, 1985), and others. For test plans with renewal of type [N R T] analogous problems were considered by Gill (1981), Belyaev (1987), and Zamyatin (1986). (See also Section 3.6.)

Consider the construction of estimates of the reliability function  $P(t)$  and resource function  $\bar{P}(t)$  for fixed moment of time  $t$ . For the model of independent random censorship, let the number of tested units be large,  $N \rightarrow \infty$ , and for the chosen  $t$  the condition  $[1-F(t)][1-G(t)] > 0$  hold. From the works mentioned above, it follows that the lower and upper  $1-\alpha$ -confidence bounds for  $P(t)$  has the form

$$\underline{\Lambda}(t) = \hat{\Lambda}(t) - u_{1-\alpha} \sqrt{\hat{V}(t)},$$

$$\bar{\Lambda}(t) = \hat{\Lambda}(t) + u_{1-\alpha} \sqrt{\hat{V}(t)}$$
(3.21)

where  $u_{1-\alpha} = \frac{1-\gamma}{2}$  is the quantile of the level  $(1-\alpha)$  of the standard normal distribution, and  $\hat{V}(t)$  is the estimate of the variance of the value  $\hat{\Lambda}(t)$  which is determined by the expression

$$\hat{V}(t) = \sum_{1 \leq j \leq D(t)} \frac{1}{[N(t_j)]^2}.$$

Corresponding lower and upper confidence limits for the reliability function  $P(t)$  are

$$\underline{P}(t) = e^{-\bar{\Lambda}(t)} \quad \text{and} \quad \bar{P}(t) = e^{-\underline{\Lambda}(t)}.$$

The confidence limits (3.21) and (3.23) are approximate and its confidence coefficient is close to the specified value  $1-\alpha$  only if the sample size,  $N$ , and the number of failures up to the moment  $t$ ,  $D(t)$ , are sufficiently large.

**Example 3.4.1** In the condition of Example 3.3.1, let us construct an approximate  $\square$ -confidence limit for  $\square=0.9$  for the reliability function  $P(t)$  and resource function  $\square(t)$  for  $t=300$ .

Using formula (3.22) we obtain the estimate of the variance

$$\begin{aligned}\hat{V}(300) &= \sum_{1 \leq j \leq 5} \frac{1}{[N(t_j)]^2} \\ &= \frac{1}{[N(34^-)]^2} + \frac{1}{[N(790^-)]^2} + \frac{1}{[N(107^-)]^2} + \frac{1}{[N(177^-)]^2} + \frac{1}{[N(257^-)]^2} \\ &= \frac{1}{19^2} + \frac{1}{18^2} + \frac{1}{17^2} + \frac{1}{13^2} + \frac{1}{10^2} = 0.0250.\end{aligned}$$

Applying formula (3.21) and using the value of the estimate  $\hat{\Lambda}(300) = 0.344383$  obtained in Example 3.3.9, we have the following lower and upper confidence limits

$$\begin{aligned}\underline{\Lambda}(300) &= \hat{\Lambda}(300) - u_{0.95} \sqrt{\hat{V}(300)} \\ &= 0.344 - 1.64 \sqrt{0.025} = 0.085, \\ \bar{\Lambda}(300) &= \hat{\Lambda}(300) + u_{0.95} \sqrt{\hat{V}(300)} \\ &= 0.344 + 1.64 \sqrt{0.025} = 0.603.\end{aligned}$$

Corresponding confidence limits for the reliability function  $P(t)$  for  $t=300$  are

$$\begin{aligned}\underline{P}(300) &= e^{-\bar{\Lambda}(300)} = e^{-0.603} = 0.547, \\ \bar{P}(300) &= e^{-\underline{\Lambda}(300)} = e^{-0.085} = 0.918. \quad \square\end{aligned}$$

#### 4.6 Accurate Confidence Limits

In practice we often meet situations where sample size are of a moderate quantity. The use of approximate formulas might lead to significant errors. In particular, the lower confidence limit of the reliability function is higher than its real value. We propose an approach (see Pavlov (1982, 1983a, 1983b, 1995) and others) which allows for models of censored tests to construct non-parametric confidence limits which are accurate, that is, they guarantee the specified level of confidence coefficient for any sample size.

Let us first consider the Markov model of censored tests without renewal considered in Section 3.2. Construction of the confidence limits and corresponding proofs are given in Section 3.7.5 in Appendix to the current chapter in more detail. Here we give only results and illustrate them with examples.

Assume that the hazard function  $\lambda(t)$  is continuous. Let  $A(t)$  be some function of time which is chosen by a researcher during the testing  $\{N(t), D(t)\}$  in such a way that the following conditions hold;

Condition 1: The inequality  $A(t) \leq 1$  for all  $t \geq 0$ .

Condition 2: Function  $A(t)$  is piecewise-continuous in  $t$ , continuous on the right and has finite limits on the left at any  $t > 0$ . Function  $A(t)$  is called "control function". Actual function can be arbitrarily chosen by the researcher in arbitrary way at any current moment  $t$  depending on all information of testing behavior on the interval  $(0, t]$ . (More detailed description of the control function see in Section 3.7.5 of Appendix to the current chapter.)

The confidential strip for the hazard rate  $\lambda(u)$  with the confidence coefficient  $\gamma$  is constructed at any current moment of time during the testing. This strip includes all functions  $\lambda(u)$ ,  $u > 0$ , which satisfy the following system of inequalities:

$$\int_0^s A(u)N(u)\lambda(u)du + \sum_{1 \leq j \leq D(s)} \ln [1 - A(t_j^-)] \leq |\ln (1 - \gamma)| \quad (3.24)$$

for all  $s \leq t$  where  $A(t_j^-) = A(t_j - 0)$  is the limit on the left of the function  $A(t)$  at the moment of the  $j$ th failure,  $t_j$ .

It is easy to see that the confidence strip (3.24) becomes tighter with increasing  $t$ . It follows from the fact that new inequalities add to (3.24) as  $t$  increases. For  $t = \tau$ , where  $\tau$  is the termination of the entire test, the formula (3.24) gives the final confidence strip for  $\lambda(u)$  with the confidence coefficient not less than  $\gamma$  by the system of inequalities

$$\int_0^s A(u)N(u)\lambda(u)du + \sum_{1 \leq j \leq D(s)} \ln [1 - A(t_j^-)] \leq |\ln (1 - \gamma)| \quad (3.25)$$

for all  $s \leq \tau$ .

Then by appropriate choice of the control function  $A(t)$  from (3.25) one can obtain the confidential strips for the resource function,  $R(t)$ , reliability function,  $P(t)$ , and the confidence limits for other reliability indexes, for instance, MTTF. Particularly, choosing  $\gamma = 1 - \alpha$  and  $A(t) = -\frac{a}{N(t)}$  384, where  $a > 0$  is some constant, we obtain the  $\alpha$ -confidence strip for  $\lambda(u)$  as the inequality

$$a \int_0^s \lambda(u)du \geq |\ln \alpha| + \sum_{1 \leq j \leq D(s)} \ln \left[ 1 + \frac{a}{N(t_j^-)} \right] \quad (3.26)$$

for all  $s \leq \tau$ . For  $s > \tau$  this inequality also holds, and for  $s \geq \tau$  its right side remains constant.

In analogous way, assuming in (3.25)  $\alpha=1-\alpha$  and  $A(t) = \frac{b}{N(t)}$  where  $b$  is a positive constant,  $0 < b \leq N(0)$ , we obtain the confidence strip for  $\lambda(u)$  with the confidence coefficient not less than  $1-\alpha$  as the inequality

$$b \int_0^s \lambda(u) du \geq |\ln \beta| - \sum_{1 \leq j \leq D(s)} \ln \left[ 1 - \frac{b}{N(t_j)} \right] \quad (3.27)$$

for all  $s \leq \tau$ .

Notice that (3.27) is valid for  $s \leq \tau = \min(\tau, \tau_b)$  where  $\tau_b = \min\{t: N(t) \leq b\}$  is the moment of crossing the level  $b$  by the function  $N(t)$ . For  $s > \tau$  in a non-parametric case, the right side of (3.27) should be assumed to be equal to  $-\alpha$ . For this purpose, we can set  $\ln z = -\alpha$  for  $z \leq 0$ .

Formulas (3.26) and (3.27) give the following confidence strip for the resource function with the confidence coefficient not less than  $1-\alpha-\beta$

$$P\{\underline{\Lambda}(t) \leq \Lambda(t) \leq \bar{\Lambda}(t) \text{ for all } t \geq 0\} \geq 1 - \alpha - \beta$$

where the lower and upper strips have the form

$$\underline{\Lambda}(t) = -\frac{|\ln \alpha|}{a} + \frac{1}{a} \sum_{1 \leq j \leq D(t)} \ln \left[ 1 + \frac{a}{N(t_j)} \right] \quad (3.28)$$

$$\bar{\Lambda}(t) = \frac{|\ln \beta|}{b} - \frac{1}{b} \sum_{1 \leq j \leq D(t)} \ln \left[ 1 - \frac{b}{N(t_j)} \right]. \quad (3.29)$$

It gives the corresponding confidence strips for the reliability function with the confidence coefficient not less than  $1-\alpha-\beta$

$$\underline{P}(t) \leq P(t) \leq \bar{P}(t), \quad t \geq 0 \quad (3.30)$$

where the lower and upper limits are determined by formulas

$$\underline{P}(t) = e^{-\bar{\Lambda}(t)}$$

and

$$\bar{P}(t) = e^{-\underline{\Lambda}(t)}.$$

In Section 3.7.6 of Appendix to the current chapter, we give the conditions which allow us to obtain consistent confidence limits for the reliability function and resource function if the initial number of tested units is  $N=N(0) > 0$ .

From the expressions given above, one can obtain the non-parametrical lower limit for the MTTF,  $\tau$ , with the confidence coefficient not less than  $1-\alpha$ . Indeed, using the formula for the MTTF expressed via the reliability function, we have

$$\underline{\mu} = \int_0^{\infty} \underline{P}(t) dt = \int_0^{\infty} e^{-\bar{\Lambda}(t)} dt.$$

In analogous way, on the basis of the confidence strip (3.30) we can construct non-parametric limits for guaranteed time,  $t_q$ , with the level of guarantee  $q$ . In this case the lower and upper limits with the confidence coefficient not less than  $1-\gamma$  are given by the following formulas

$$\begin{aligned} \underline{t}_q &= \inf \{t: \underline{P}(t) \leq q\}, \\ \bar{t}_q &= \sup \{t: \bar{P}(t) \geq q\}. \end{aligned}$$

In other words,  $\underline{t}_q$  and  $\bar{t}_q$  are defined as the moments of crossing the level  $q$  by the graphs of the lower and upper confidence limits  $\underline{P}(t)$  and  $\bar{P}(t)$ .

**Example 3.5.1** (case of no failure test) Let us choose the control function as

$$A(t) \equiv 1, t \geq 0. \quad (3.31)$$

Let there is no failures or test termination up to the moment  $t$ . Then from (3.24) follows

$$\int_0^s N(u)\lambda(u)du \leq |\ln(1-\gamma)| \text{ for all } s \leq t.$$

Since in this case  $N(u) \equiv N = N(0)$  for all  $u \leq t$ , it follows that

$$\int_0^s \lambda(u)du \leq \frac{|\ln(1-\gamma)|}{N} \text{ for all } s \leq t. \quad (3.32)$$

From (3.32) we obtain the following upper confidence limit for the resource function with the confidence coefficient not less than  $1-\gamma$

$$\Lambda(s) \leq \bar{\Lambda} = \frac{|\ln(1-\gamma)|}{N} \text{ for all } s \leq t$$

which gives the following lower  $1-\gamma$ -confidence limit for the reliability function

$$P(s) \geq \underline{P}(t) = e^{-\bar{\Lambda}} = \sqrt[N]{1-\gamma} \text{ for all } s \leq t.$$

Thus, choosing the control function  $A(t)$  from the condition 3.31), we obtain on the interval  $[0, t]$  with no failures the lower  $\gamma$ -confidence limit  $\underline{P}$  which coincides with the standard Clopper-Pearson limit for the case of testing  $N$  units with no failures.  $\square$

**Example 3.5.2** (Testing until the first failure) Let the test is performed until the first failure, i.e., the termination of the entire test is  $\tau = t_1$ . No censorship is available. Choosing again the control function as  $A(t) \equiv 1$  for all  $t \geq 0$ , from (3.24) and (3.25) we obtain

$$\int_0^s N(u)\lambda(u)du \leq |\ln(1-\gamma)| \quad \text{for all } s < v$$

or

$$\int_0^s \lambda(u)du \leq \frac{|\ln(1-\gamma)|}{N} \quad \text{for all } s < v.$$

from where the  $\gamma$ -confidence limit for the reliability function follows

$$\Lambda(s) \leq \bar{\Lambda} = \frac{|\ln(1-\gamma)|}{N} \quad \text{for all } s < v.$$

The corresponding lower  $\gamma$ -confidence limit for reliability function is

$$P(s) \geq \underline{P} = e^{-\bar{\Lambda}} = \sqrt[\gamma]{1-\gamma} \quad \text{for all } s < v. \quad \dagger$$

**Example 3.5.3** In the conditions of Example 3.3.1, let us construct the lower  $\gamma$ -confidence limit for the reliability function,  $P(t=160)$ , with the confidence coefficient not less than  $\gamma=0.95$ . Setting in (3.29) the constant  $b=10$ , we obtain the following upper confidence limit for the resource function

$$\begin{aligned} \bar{\Lambda}(160) &= \frac{|\ln 0.05|}{10} - \frac{1}{10} \sum_{1 \leq j \leq 3} \ln \left[ 1 - \frac{10}{N(t_j)} \right] \\ &= \frac{2.99}{10} - \frac{1}{10} \left[ \ln \left( 1 - \frac{10}{19} \right) + \ln \left( 1 - \frac{10}{18} \right) + \ln \left( 1 - \frac{10}{17} \right) \right] = 0.515. \end{aligned}$$

So, the lower 0.95-confidence limit for the reliability function,  $P(160)$  is

$$\underline{P}(160) = e^{-\bar{\Lambda}(160)} = e^{-0.515} = 0.597. \quad \dagger$$

Notice that on the basis of the confidence limits constructed above, one can solve the problem of test statistical hypothesis relating to different reliability indexes, for instance,

$$H_0: P(t) \geq P_0 \text{ vs. } H_1: P(t) \leq P_1$$



where  $P_0 > P_1$  are specified critical levels of the reliability index  $P(t)$ , and so on.

One of the interesting and still not appropriately solved questions is the problem of choice of an optimal - in some sense - control function  $A(t)$ . (The conditions on  $A(t)$  for which the confidence limits constructed above are consistent for the initial number of tested units  $N=N(0)$  are given in Section 3.7.6 of Appendix to the current chapter.)

We considered above only the model [MMU] but this approach can be easily extended onto more general models, including models with renewal (some of them are considered below).

## 4.7 More General Models of Censored Tests

### 4.7.1 Markov Model of Censored Tests with Renewal

Consider a test model which differs from [MMU] by the possibility to replace failed the units by identical new ones or to add new units to previously tested ones. Let us call such a model [MMR] where the last letter stands for "renewal". At the initial moment  $t=0$  there are  $N=N(0)$  identical and independent units. At some moment of time  $\tau_1 > 0$  we add  $m_2$  new units identical to the initial ones, at the moment  $\tau_2 > \tau_1$  we add  $m_1$  units, and so on. We assume that all units fail independently. So, new units are tested at moments

$$0 < \tau_1 < \tau_2 < \dots < \tau_i < \dots$$

which form Markov and "independent on the future" sequence of time moments. In other words, the moments and corresponding numbers of added units might be appointed arbitrarily, depending on any information known at the moment  $t$ . (Some comments see in Section 3.7.1 of Appendix.) So, at moment  $\tau_i$  we add  $m_i$  new units where  $m_i$  is a discrete random number which might arbitrarily depend on the current results of the test process on the interval  $[0, \tau_i]$  but does not depend on the future development of the process for  $\tau > \tau_i$ . (More accurate formal definition of  $\tau_i$  and  $m_i$  see below in Section 3.7.7.)

Thus the test model is given by the set of data

$$M = \{(\tau_1, n_1), \dots, (\tau_k, n_k), \dots; (\tau_1, m_1), \dots, (\tau_i, m_i), \dots; \tau\}$$

where  $\tau$  is the termination of the entire test. This moment is defined by some rule and in general might be a Markov random moment. Values  $(\tau_k, n_k)$  are defined as above:  $\tau_k$  is the moment of the  $k$ th test termination for a group of units and  $n_k$  is the number of units of the group,  $k=1, 2, \dots$ . If in the model [MMU] the number of the test termination was restricted by the value of  $N=N(0)$ , in the model [MMR] number of moments of test termination,  $\tau_k$ , and unit additions,  $\tau_i$ , are not restricted. The sequential moments of unit failures are denoted as above:

$$0 < t_1 < \dots < t_j < t_{j+1} < \dots$$

We also keep the notation  $D(t)$  for the number of moments  $t_j$  such that  $t_j \leq t$  (i.e., the number of failures),  $N(t)$  for the number of units which are tested at the moment  $t$

$$N(t) = N(0) + B(t) - D(t) - L(t)$$

where

$$B(t) = \sum_{i: \tau_i \leq t} m_i,$$

$$L(t) = \sum_{k: \tau_k \leq t} n_k,$$

that is,  $B(t)$  is the number of units which were put to the test before  $t$ , and  $L(t)$  is the number of units whose test has been terminated up to the moment  $t$ . All functions  $N(t)$ ,  $D(t)$ ,  $B(t)$  and  $L(t)$  are continuous from the right in  $t$ . A graphical illustration of a possible track of the function  $N(t)$  is depicted in Figure 3.4.

Figure 3.4

**Remark 3.6.1** In the previous model [MMU] all of  $N(t)$  tested units at the moment  $t$  have the same operational time equal to  $t$ . In this model -- which is more general -- the "age" of tested units are different. Denote the "age" of unit  $e$  at moment  $t$  by  $S_e(t)$ . Obviously, this value is

$$S_e(t) = t - \tau_i$$

where  $\tau_i$  is the moment when this unit was added to the set of the tested units. Denote the set of subscripts of all units tested at the moment  $t$  by  $E(t)$ . This set includes  $N(t)$  different subscripts. So, units at the moment of termination of their testing,  $\tau_k$ , have different operational time ("age"):

$$S_e(\tau_k),$$

$e \in E(\tau_k^-)$  where  $E(\tau_k^-) = E(\tau_k - 0)$  is the set of subscripts of the units which are under the testing just before the moment  $\tau_k$ . Thus, for the complete description of the model, it is necessary to point out not only the number of units whose test has been terminated but their "age":  $S_e(\tau_k)$ ,  $e \in E(\tau_k)$  where  $E(\tau_k)$  is the set of subscripts of the units whose test has been terminated at the moment  $\tau_k$ ,  $E(\tau_k) \cap E(\tau_k^-)$ . It is clear that the only case where this condition is redundant is the exponential distribution  $F(t)$  for which the distribution of residual time coincides with initial one.

This model includes the previously described model [MMU] as a particular case. This model also covers all standard plans of type **R** (with unit renewal or replacement after failure).

**Example 3.6.1** (Test plan of type [1 R T]) Consider a standard renewal process, or in our terminology the test plan [1 R T]. (See Chapter 2.) At the moment  $t=0$ , a single unit is in testing,  $N(0)=1$ . At the failure moment, a failed unit is replaced by a new one, then at the moment of the second failure, a failed unit is again replaced by a new one, and so on. It is clear that this plan is a particular case of plan [MMR] with  $N(0)=1$ ;  $n_k=0$  for all  $k=1, 2, \dots$ ; with moments of new units addition  $\tau_i=t_i$ ,  $i=1, 2, \dots$ ; and  $m_i=1$ ,  $i=1, 2, \dots$ . Obviously,  $N(t) \geq 1$  for all  $t > 0$ . The entire test termination moment is  $\tau=T$  where  $T$  is a specified constant. †

**Example 3.6.2** (Test plan of type [N R ...]) Let us consider a test consisting in  $N$  simultaneous observations of processes discussed in Example 3.6.1. We call this plan [N R ...]. This model can be considered as a particular case of the model [MMR] with  $N(0)=n$ ;  $n_k=0$ ,  $k=1, 2, \dots$ ;  $\tau_i=t_i$ ;  $m_i=1$ ,  $i=1, 2, \dots$ . At any moment of time,  $t \geq 0$ , there are  $N(t) \leq N$  tested units. Moments of the test termination depend on the particular test plan. For the plan [N R T], it is  $\tau=T$ ; for the plan [N R r], it is  $\tau=t_r$ ; and for the plan [N R (r, T)], it is  $\tau = \min(T, t_r)$ . †

The method of the construction for the confidence limits for the model [MMU] extends to the more general model [MMR] in the following way. Each unit with subscript  $e$  is set in correspondence with some "control function"  $A(e, t)$  which satisfies the same conditions as  $A(t)$  does (see Section 3.5). For each unit  $e \in E(t)$ , the meaning of  $A(e, t)$  might be chosen arbitrarily with

taking into account any previous information about the current test on the interval  $[0, t]$ . (Of course, we need to maintain conditions 1 and 2 which were formulated for control the function  $A(t)$ .) As above, the confidence strip for the hazard rate,  $\lambda(u)$ , with the confidence coefficient not less than the specified value  $\gamma$  is constructed for any  $t$  during the testing. At moment  $t$  this confident strip includes all continuous functions  $\lambda(u)$ ,  $u \geq 0$ , which satisfy the system of inequalities

$$\int_0^s \sum_{e \in E(u)} A(e, u) \lambda[S_e(u)] du + \sum_{1 \leq j \leq D(t)} \ln [1 - A(e_j, t_j)] \leq |\ln (1 - \gamma)| \quad (3.33)$$

for all  $s \leq t$  where  $S_e(u)$  is the "testing age" of unit  $e$  at the moment  $u$ ;  $A(e, t_j) = A(e, t_j - 0)$  is the limit from the left for the control function  $A(e, t)$  at the moment of the  $j$ th failure,  $t_j$ ;  $e_j$  is the subscript of the unit which fails at  $t_j$ ;  $E(u)$  is the set of unit indexes which are tested at the moment  $u$ .

From the definition follows that with increase of time  $t$  the confidence strip is monotone narrowing for any trajectory of the testing process. It follows again from the fact that with increasing  $t$ , that more inequalities are included into (3.33). At  $t = \tau$ , where  $\tau$  is the moment of the entire test termination, the system (3.33) gives the final confidence strip for the hazard rate,  $\lambda(u)$ :

$$\int_0^s \sum_{e \in E(u)} A(e, u) \lambda[S_e(u)] du + \sum_{1 \leq j \leq D(\tau)} \ln [1 - A(e_j, t_j)] \leq |\ln (1 - \gamma)| \quad (3.34)$$

for all  $s \leq \tau$ .

The proof of the confidence strips (3.33) and (3.34) for the model [MMR] is analogous to that for the model [MMU] (see Section 3.7.5 below).

As we did in the previous section, the non-parametric confidence strips can be constructed on the basis of (3.33) and (3.34) for the reliability function,  $P(t)$ , and resource function,  $\rho(t)$ , as well as the confidence limits for other reliability indexes.

## 4.8 Appendices

### 4.8.1 Markov Moments of Time

Let  $(\Omega, \mathcal{F}, P)$  be the main probabilistic space where the process of testing,  $x_t$ ,  $t \geq 0$ , is defined;  $\mathcal{F}_t$ ,  $t \geq 0$ , be a non-decreasing family of  $\sigma$ -sub-algebras; the stochastic process  $x_t$  correspond to the family of  $\mathcal{F}_t$ ,  $t \geq 0$ .

A random moment of time  $\tau = \tau(\Omega) \geq 0$  is called Markov moment with respect to the system of  $\mathcal{F}_t$ ,  $t \geq 0$ , if event  $\{\tau: \tau(\Omega) \leq t\} \in \mathcal{F}_t$  for any  $t \geq 0$  where  $\tau$  is an element of the set  $\Omega$ . Here  $\mathcal{F}_t$ -algebra  $\mathcal{F}_t$  represents a set of all events whose occurrence or non-occurrence up to the moment  $t$  are uniquely predictable. In other words, the sense of this statement is the following: a random moment  $\tau$  is Markovian if, for any  $t$ , the answer about the occurrence an event  $\{\tau \leq t\}$ , based on the prehistory of the process before the moment  $t$ , is unique.

Each Markov moment,  $\tau$ , corresponds to a  $\sigma$ -sub-algebra  $\mathcal{F}_\tau$  which is a set of all events occurrences which are uniquely predictable at the moment  $\tau$ . This  $\sigma$ -subalgebra  $\mathcal{F}_\tau$

includes all events  $A \in \mathcal{A}$  such that the event  $A \in \{\tau \leq t\} \in \mathcal{A}_t$  for any  $t \geq 0$ . If some r.v.  $\tau = \tau(\omega)$  is measurable with respect to the  $\mathcal{A}$ -algebra  $\mathcal{A}_t$ , it means that at the moment  $t$  the meaning of this r.v. is known. (More detail may be found in Doob(1953) or Shiryaev (1980).)

#### 4.8.2 Markov Censored Test Model without Renewal

Assume that at the moment  $t=0$ ,  $N$  identical and independent units enter testing. Each unit has continuous d.f.  $F(t)$  of a random TTF,  $\tau$ . At some moment of time,  $t_1 > 0$ , a group of  $n_1$  units has terminated the test, then at  $t_2 > t_1$  a group of  $n_2$  units has exit the test, and so on. Any exit moment can be arbitrarily chosen depending on the information about the "prehistory" of the testing at the current moment  $t$  but does not depend on the "future trajectory" of the testing process, i.e., for  $t > t_k$ . The number of units in a group,  $n_k$ , whose test are terminated its test at the moment  $t_k$ , is also random and can be chosen on the basis of the same information.

Let  $D(t)$  be the number of failures on the interval  $[0, t]$ , or, in other words, it is the number of moments  $t_j$  such that  $t_j \leq t$  where  $t_j$  is the moment of the  $j$ th failure. Let  $L(t) = \sum_{k: \tau_k \leq t} n_k$  be the

number of units whose test has been terminated on the interval  $[0, t]$  and  $N(t) = N - D(t) - L(t)$  be the number of units testing at the moment  $t$ . Let  $(\Omega, \mathcal{A}, P)$  be the main probabilistic space where the stochastic process  $x_t = \{N(t), D(t)\}$ ,  $t \geq 0$ , is defined in correspondence to the non-decreasing family of  $\mathcal{A}$ -algebras  $\mathcal{A}_t$ ,  $t \geq 0$ . The stochastic functions  $N(t)$  and  $D(t)$  are continuous from the left in  $t$ . Assume that the moment of the test termination of the units groups

$$t_1 < t_2 < \dots < t_k < \dots$$

are Markovian with respect to the family  $\mathcal{A}_t$ ,  $t \geq 0$ , and a r.v.  $n_k = 0, 1, 2, \dots, N$  is measurable in respect to  $\mathcal{A}_{t_k}$  relating to the Markov moment  $t_k$ ,  $k=1, 2, \dots$ . The moment of the entire test termination is denoted by  $\tau$ . We assume that this moment also is Markovian relating to the family  $\mathcal{A}_t$ ,  $t \geq 0$ . Since the initial number of units is  $N = N(0)$ , we might say that the total number of termination moments,  $t_k$ , does not exceed  $N$ . Thus in this model it is enough to consider the finite sequence of  $N$  Markov moments

$$t_1 < t_2 < \dots < t_N \leq \tau.$$

The number of "real" termination moments, i.e., such moments for which  $n_k > 0$ , might be fewer than  $N$ . So,  $r$  is a r.v.,  $r=1, 2, \dots, N$ . Thus the test model is defined by the following set

$$M = \{(t_1, n_1), (t_2, n_2), \dots, (t_N, n_N), \tau\} \quad (3.35)$$

We call this test model Markovian to emphasize that any intervention into the process (test termination of a group of units or entire test) bears the Markovian character.

This model does not assume additional switching on new units for testing. More general models that take this factor into account will be considered in Section 3.7.8.

#### 4.8.3 Unbiasedness of estimate (3.12)

Let us show that if condition (3.13) holds for the model [MMU], then

$$E\{r_0\} = NF(t) \quad (3.36)$$

that is, the estimate (3.12) is an unbiased estimate of the reliability function  $P(t)=1-F(t)$  for fixed  $t \geq 0$ . This model is defined by (3.35). Notice that since the initial number of units equals  $N$ , and  $n_k$  might take the value of 0, we can say that the last moment of test termination of a group of units,  $\tau_n$ , coincides with the moment of the complete termination of the entire test, i.e.,  $\tau_n = \tau$ . Therefore the model is defined by the set

$$M = \{(\tau_1, n_1), (\tau_2, n_2), \dots, (\tau_N, n_N)\}.$$

Consider first the case where the moments of test termination,  $\tau_k$ , take discrete values,  $t_k$ :

$$0 = t_0 < t_1 < \dots < t_m < t_{m+1} = t < t_{m+2} < \dots$$

For the sake of simplicity, let  $t$  be the moment for which the reliability function  $P(t)$  is estimated coincides with  $t_{m+1}$ . Introduce the following test model

$$M^* = \{(\tau_1^*, n_1^*), (\tau_2^*, n_2^*), \dots, (\tau_N^*, n_N^*), \dots\}.$$

where  $\tau_i^* = t_i$ , and

$$n_i^* = \sum_{1 \leq k \leq N} n_k I(\tau_k = t_i), \quad i = 1, 2, \dots$$

where  $I(A)$  is the indicator for the event  $A$ . In discrete case both models  $M$  and  $M^*$  are equivalent. The model  $M^*$  is another definition of the model  $M$  where all moments  $\tau_i^*$  are deterministic and exhaust all possible discrete set of  $\{t_i\}$  and the model is defined by the numbers of units for which the test has been terminated at these moments  $n_i^*$ ,  $i=1, 2, \dots$ . Since the r.v.  $n_k$  is measurable with respect to the  $\sigma$ -algebra  $\sigma_{\tau_k}$  corresponding to the Markov moment  $\tau_k$  ( $k=1, 2, \dots, N$ ),  $n_i^*$  by definition is measurable with respect to the  $\sigma$ -algebra  $\sigma_{t_i}$ ,  $i=1, 2, \dots$ . Let us introduce the following notation for the model  $M^*$ :  $N_i = N(t_i)$ ,  $N'_i = N_i + n_i^*$ ,  $d_i = N_i - N'_{i+1} = D(t_{i+1}) - D(t_i)$ ,  $i=0, 1, 2, \dots$ . The recurrent procedure (3.11) for the discrete model  $M^*$  has the form

$$\begin{cases} r_m = N'_m \left( \frac{d_m}{N'_m} \right), & \text{if } N'_m > 0 \\ r_m = 0, & \text{if } N'_m = N_m = d_m = 0 \end{cases}$$

$$\begin{cases} r_i = N'_i \left( \frac{d_i + r_{i+1}}{N'_i} \right), & \text{if } N'_i > 0 \\ r_0 = 0, & \text{if } N'_i = N_i = d_i = r_{i+1} = 0 \end{cases}$$

$$0 \leq i \leq m-1.$$

Let us introduce quantities

$$q_i = \frac{P(t_i) - P(t_{i+1})}{P(t_i)}$$

and

$$Q_i = \frac{P(t_i) - P(t)}{P(t_i)}$$

equal to conditional probabilities of unit's failure on intervals  $(t_i, t_{i+1})$  and  $(t_i, t)$ , respectively, under condition that the units has not failed before the moment  $t_i$ . These quantities are related to each other as

$$Q_i = q_i + (1 - q_i)Q_{i+1}, \quad i=0, 1, \dots, m \quad (3.37)$$

where  $Q_0 = F(t)$  and  $Q_{m+1} = 0$ .

In the discrete model  $M^*$ , the group of unit test terminations might occur only at moments  $t_i$ ,  $i=1, 2, \dots$ . Thus on each interval  $(t_i, t_{i+1})$  there is no "intervention" in the testing process (no termination). It means that on this interval of time the conditional distribution of the number of failures,  $d_i$ , is a standard binomial one with parameters  $N_i$  (the number of tested units) and  $q_i$  (the probability of failure). It follows that the conditional mathematical expectation can be written as

$$E\{d_i | \mathfrak{F}_i\} = N_i q_i, \quad i = 0, 1, \dots, m. \quad (3.38)$$

Here we omit the standard remark that the equality for the conditional mathematical expectation is correct with accuracy of to sets  $A \square \square$  with the null measure  $P$ . Notice also that the probabilistic measure  $P$  and corresponding mathematical expectation by this measure on  $(\square, \square)$  essentially depend on the estimated function  $F = F(t)$ , that is  $P = P_F$  and  $E = E_F$ . Subscript  $F$  is omitted for the sake of brevity. In the future we will use a shorten notation:  $\mathfrak{F}_i = \mathfrak{F}_{i,}$  390

Let us show that the conditional mathematical expectation is

$$E\{r_m | \mathfrak{F}_m\} = N'_m q_m = N'_m Q_m.$$

For  $N'_m = 0$  this equality directly follows from the definition of r.v.  $r_m$ . Let  $N'_m > 0$ . Then by the condition (3.13) the inequality  $N_m > 0$  holds. Since quantities  $N'_m$  and  $N_m$  are  $\square_m$ -measurable, then

$$E\{r_m | \mathfrak{F}_m\} = E\left\{N'_m \frac{d_m}{N_m} | \mathfrak{F}_m\right\} = \frac{N'_m}{N_m} E\{d_m | \mathfrak{F}_m\}$$

and with taking into account (3.38), it follows

$$E\{r_m | \mathfrak{F}_m\} = N'_m q_m = N'_m Q_m. \quad (3.39)$$

Let us show now that the equality

$$E\{r_i | \mathfrak{F}_i\} = N'_i Q_i \quad (3.40)$$

follows from the equality

$$E\{r_{i+1} | \mathfrak{F}_{i+1}\} = N'_{i+1} Q_{i+1}. \quad (3.41)$$

(Here  $i = m-1, m-2, \dots, 1, 0$ .) Since  $N'_0 = N_0 = N$ , and taking into account (3.39), one has the proof of the equality (3.36) for the discrete model  $M^*$ .

Let (3.40) hold. Then for  $N'_i=0$  the equality (3.41) directly follows from the definition of r.v.  $r_i$ . Let  $N'_i>0$ . From condition (3.13) the equality  $N'_i>0$  also holds. Since the r.v.  $N'_i$  and  $N_i$  are  $\mathfrak{F}_i$ -measurable, using (3.38) we have

$$\begin{aligned} E\{r_i|\mathfrak{F}_i\} &= E\left\{N'_i \frac{d_i + r_{i+1}}{N_i} \middle| \mathfrak{F}_i\right\} = \frac{N'_i}{N_i} E\{d_i + r_{i+1}|\mathfrak{F}_i\} \\ &= \frac{N'_i}{N_i} E\{d_i|\mathfrak{F}_i\} + \frac{N'_i}{N_i} E\{r_{i+1}|\mathfrak{F}_i\} = N'_i q_i + \frac{N'_i}{N_i} E\{r_{i+1}|\mathfrak{F}_i\}. \end{aligned} \quad (3.41)$$

Since  $\mathfrak{F}_i \subset \mathfrak{F}_{i+1}$ , then the following equality holds for the conditional mathematical expectation

$$\begin{aligned} E\{r_{i+1}|\mathfrak{F}_i\} &= E\{E\{r_{i+1}|\mathfrak{F}_{i+1}\}|\mathfrak{F}_i\} = E\{N'_{i+1} Q_{i+1}|\mathfrak{F}_i\} \\ &= Q_{i+1} E\{N_i - d_i|\mathfrak{F}_i\} = Q_{i+1} (N_i - N_i q_i). \end{aligned}$$

Taking (3.37) into account, we have

$$E\{r_i|\mathfrak{F}_i\} = N'_i [q_i + Q_{i+1} (1 - q_i)] = N'_i Q_i$$

where  $i=m-1, m-2, \dots, 1, 0$ . It proves (3.36) for the discrete model.

In general case, let us consider a sequence of dividing of the time axis by discrete moments:

$$S_j = \{0 = t_0 < t_1 < \dots < t_j < \dots\}$$

where  $t_{ij} = i \left(\frac{h}{2^j}\right)$ ,  $i=0, 1, \dots$ ;  $h = \frac{t}{L}$ ;  $L$  is an arbitrary integer positive number,  $S_j \subset S_{j+1}$ ,  $j=1, 2, \dots$ . Let for the  $j$ th dividing be

$$\tau_{kj} = \left[ \frac{\tau_k 2^j}{h} \right]^* \left( \frac{h}{2^j} \right)$$

where  $[z]^*$  denotes the closest integer more than (or equal to)  $z$ . Consider the process  $y(t)$  which denotes the total number of units being tested (including those whose testing has been terminated) and have not failed up to the moment  $t$ . By definition, the trajectory of this process is continuous in  $t$  on the right. Let us introduce the event

$$B_{kj} = \{y(\tau_k) = y(\tau_{kj})\},$$

which corresponds to the fact that there is no failure on interval  $(\tau_k, \tau_{kj}]$  including those whose tests have been terminated. Since the trajectory of the process  $y(t)$  is monotone decreasing,

$$B_{kj} \subset B_{k,j+1}, \quad j = 1, 2, \dots$$

Then let us set

$$n_{kj} = n_k I(B_{kj}), \quad k=1, \dots, N; \quad j=1, 2, \dots.$$

R.v.  $n_{kj}$  is measurable relative to a  $\sigma$ -algebra  $\mathfrak{F}_{\tau_{kj}}$  which is tied to the Markov moment  $\sigma_{kj}$  because  $\sigma_{k \leq \tau_{kj}}$  and  $\mathfrak{F}_{\tau_k} \subset \mathfrak{F}_{\tau_{kj}}$ .

Now introduce the Markov test model

$$M_j^* = \{(\sigma_{kj}, n_{kj}), \quad k=1, \dots, N\}.$$

For this model moments of intermediate terminations  $\sigma_{kj}$  might take only discrete values  $t_{ij}$ ,  $i=1, 2, \dots$  which corresponds to the  $j$ th division of  $S_j$ . Let  $1 - \left(\frac{r_{0j}}{N}\right)$  be the estimate of the reliability function  $P(t)$  for model  $M_j^*$ . Then, as it was proved before, for any  $j=1, 2, \dots$  the following equality

$$E\{r_{0j}\} = NF(t)$$

holds. The sequence  $r_{0j} \rightarrow r_0$  as  $j \rightarrow \infty$  everywhere on the set

$$B = \bigcap_{1 \leq j < \infty} \bigcap_{1 \leq j < \infty} \bigcap_{1 \leq k \leq N} B_{kl} = \{y(\tau_k) = y(\tau_k + 0), \quad k = 1, \dots, N\}.$$

Because of the continuity of trajectories of the process  $y(t)$  on the left in  $t$ , the set  $B$  has the probability 1. Taking into account the restrictiveness of the sequence  $\sigma_{0j}$ :  $0 \leq r_{0j} \leq N$ , it follows that

$$E\{r_0\} = NF(t).$$

#### 4.8.4 The Proof of Formula (3.18)

Parameter  $q_j$  can be written in the form  $q_j = 1 - e^{-z_j}$  where

$$z_j = \int_{j^h}^{j^{h+h}} \lambda(u) du, \quad j = 0, 1, \dots, M-1. \quad (3.42)$$



From the continuity of function  $\varphi(u)$  its bounds on interval  $[0, t]$  follows:  $\varphi(u) \leq C < \infty$  for all  $0 \leq u \leq t$ . It gives, taking into account (3.42), that  $z_j \leq Ch$  for all  $j=0, 1, \dots, M$ ;  $h = \frac{t}{M}$ ,  $M=1, 2, \dots$ .

It follows

$$q_j = z_j + \varepsilon_j$$

where  $|\varepsilon_j| \leq C'h^2$ ,  $C' = \frac{C^2}{2}$  for all  $j=0, 1, \dots, M-1$ ,  $h = \frac{t}{M}$ ,  $M=1, 2, \dots$ . Summation in  $j$  of the left and right sides of (3.42) gives

$$\Lambda(t) = \sum_{0 \leq j \leq M-1} z_j = \sum_{0 \leq j \leq M-1} q_j + \delta$$

where

$$|\delta| \leq \sum_{0 \leq j \leq M-1} |\varepsilon_j| \leq C' M h^2 = C'' h$$

where in turn,  $C'' = C't$  which proves (3.18).

#### 4.8.5 Non-Parametric Confidence limits for Test Markov Model [MMU]

We further assume that the  $\varphi(t)$  is continuous in  $t \geq 0$ . The class of all such functions is denoted by  $\mathcal{W}$ . Let us also denote  $\mathcal{Q} = \{\varphi(t), t \geq 0\}$ . The probabilistic measure on  $(\mathcal{Q}, \mathcal{Q})$  for given  $\varphi \in \mathcal{W}$  is denoted by  $P_\varphi$  and the mathematical expectation (by measure  $P_\varphi$ ) by  $E_\varphi$ .

Let

$$x^t = \{x_u, 0 \leq u \leq t\} = \{[N(u), D(u)], 0 \leq u \leq t\}$$

be a collection of all observed statistical data at the moment  $t$ , that is, the trajectory of the process  $x_u = [N(u), D(u)]$  on interval  $[0, t]$ . For the test model  $x^t$ , it is a set of all failure moments,  $t_j$ , intermediate termination,  $\tau_k$ , and respective numbers of  $n_k$ , which are observed up to the moment  $t$ .

Assume that at any current moment of time, we construct the confidence set on the basis of information  $x^t$

$$H_t = H_t(x^t) \subset \mathcal{W} \tag{3.43}$$

for  $\varphi \in \mathcal{W}$ . A collection of all sets (3.43) is called a system of  $\mathcal{Q}$ -confidence sets for  $\mathcal{Q}$  if the following conditions hold:

- (1) For any trajectory of the testing process  $x_t$ ,  $0 \leq t < \infty$  and any moments of time  $u \leq t$  the relation

$$H_t(x^t) \subset H_u(x^u)$$

holds, that is the confidence set  $H_t$  becomes monotonically narrower with  $t$  increasing for any trajectory of observation.

(2) For any Markovian (in respect to the family  $\{P_\lambda, \lambda \geq 0\}$ ) termination moment  $\tau$ , such that  $P_\lambda(\nu < \tau) = 1$  for all  $\lambda \in W$ , the inequality

$$P_\lambda(\lambda \in H_\nu) \geq \gamma$$

holds for all  $\lambda \in W$  where  $H_\nu = H_\nu(x^\nu)$  is the confidence set constructed up to the moment  $\nu$ . The sense of this condition is in the fact that we can terminate the entire test at any moment  $\nu$  and the set  $H_\nu$  constructed up to this moment is  $\gamma$ -confidence for  $\lambda$ .

Now we give the procedure of construction of the system of sequential confidence sets for the considering model of censored tests (see Pavlov (1982, 1983a, 1983b)). Let  $A(t)$  be a stochastic function related to the family  $\{P_\lambda, \lambda \geq 0\}$ , and satisfying the conditions:

(1)  $A(t) \leq 1$  for all  $t \geq 0$ .

(2)  $A(t)$  is linear-spline function continuous on the right and having finite limits on the left at any point  $t > 0$ .

(3) On any finite interval  $0 \leq u \leq t$  function  $A(u)$  is restricted from below:

$$P_\lambda\{A(u) \geq C\} = 1 \text{ for all } 0 \leq u \leq t$$

for all  $\lambda \in W$ .

Let  $\gamma$  be the given level of the confidence coefficient,  $0 < \gamma < 1$ . Let each  $\lambda \in W$  corresponds to a set of events  $\{\varphi_\lambda^t, t > 0\}$ , where events  $\varphi_\lambda^t$  is defined in the following way:

$$\varphi_\lambda^t = \left\{ \int_0^t A(u)N(u)\lambda(u)du + \sum_{1 \leq j \leq D(s)} \ln[1 - A(t_j)] \leq |\ln(1 - \gamma)| \text{ for all } s \leq t \right\}. \quad (3.44)$$

Introduce subsets

$$H_t \subset W, \quad t > 0 \quad (3.45)$$

which we define on the basis of events  $\varphi_\lambda^t, t > 0$ , as follows

$$H_t = \{ \lambda \in W : I(\varphi_\lambda^t) = 1 \}$$

where  $I(\varphi_\lambda^t)$  is the indicator of event  $\varphi_\lambda^t$ . So,  $H_t$  is the set of all  $\lambda \in W$  such that event  $\varphi_\lambda^t \in \mathfrak{F}_t$  for this trajectory of the testing process takes place. In accordance with this definition, the set  $H_t$  is written as

$$H_t = \left\{ \lambda : \int_0^s A(u)N(u)\lambda(u)du + \sum_{1 \leq j \leq D(s)} \ln[1 - A(t_j)] \leq |\ln(1 - \gamma)| \text{ for all } s \leq t \right\}.$$

Let us now show that the system of sets (3.45) constructed in this way is the system of sequential  $\gamma$ -confidence sets for  $\lambda \in W$ . For this it is enough to show that

$$P_\lambda(\lambda \in H_t) = P_\lambda(\varphi_\lambda^t \geq \gamma) \quad (3.46)$$

for any  $\square \square W$  and any fixed finite  $t > 0$ . Indeed, the events  $\varphi_\lambda^t$  and  $\square \square H_t$  are equivalent by construction. Then let for some  $\square \square W$  the inequality (3.46) holds for any  $t > 0$ . Let us show that then

$$P_\lambda(\lambda \in H_v) \geq \gamma$$

for any Markovian moment of the total termination  $\square$  such that

$$P_\lambda(v < \infty) = 1. \quad (3.47)$$

Introduce event  $\varphi_\lambda^t \square \square \square$  which is defined as

$$\varphi_\lambda = \lim_{t \rightarrow \infty} \downarrow \varphi_\lambda^t = \bigcap_{t > 0} \varphi_\lambda^t$$

From (3.46) it follows that

$$P(\varphi_\lambda) = \lim_{t \rightarrow \infty} P_\lambda(\varphi_\lambda^t \geq \gamma)$$

For any finite  $t > 0$ , the relation  $\varphi_\lambda \subset \varphi_\lambda^t$  holds, and it follows that

$$\varphi_\lambda \cap (v < \infty) \subset \varphi_\lambda^t \cap (v < \infty).$$

Besides, due to equivalency of events  $\varphi_\lambda^t$  and  $\square \square H_t$ , the relation

$$(\lambda \in H_v) = \bigcap (v < \infty) = \varphi_\lambda^v \cap (v < \infty)$$

holds. Taking into account (3.47), we have

$$\begin{aligned} P_\lambda(\lambda \in H_v) &= P_\lambda\{(\lambda \in H_v) \cap (v < \infty)\} = \\ &= P_\lambda\{\varphi_\lambda^v \cap (v < \infty)\} \geq P_\lambda\{\varphi_\lambda \cap (v < \infty)\} = P_\lambda\{\varphi_\lambda\} \geq \gamma. \end{aligned}$$

Notice that in the considered model, the condition  $P_{\square}(\square < \square) = 1$  automatically holds if  $F(\square) = 1$ , because any total test termination does not last beyond the failure of the last unit.

Thus, it is enough to show that inequality (3.46) holds for any fixed, finite  $t > 0$  and for any  $\square \square W$ . Let us first prove this inequality for a discrete model  $M_j^*$  introduced in Section 3.7.3. For this model, the intermediate termination moments,  $\square_k$ , might take only discrete values  $t_{ij}$ ,  $i=1, 2, \dots, j=1, 2, \dots, n$ . Introduce the following additional notation for this model:

$N_j(t)$  is the number of units on test up to the moment  $t$ ;

$d_{ij}$  is the number of failures on the interval  $(t_{ij}, t_{i+1,j})$ ,

$$\square_{ij} = \begin{cases} 1 & \text{if } d_{ij} > 0, \\ 0 & \text{if } d_{ij} = 0 \end{cases}$$

is an indicator showing that at least one failure has occurred on this interval;

$$q_{ij} = \frac{P(t_{ij}) - P(t_{i+1,j})}{P(t_{ij})} = 1 - \exp \left[ - \int_{t_{ij}}^{t_{i+1,j}} \lambda(u) du \right] \quad (3.48)$$

is the conditional probability of the unit failure on the interval  $(t_{ij}, t_{i+1,j})$  under condition that until the moment  $t_{ij}$  the unit has not failed, and

$$\pi_{ij} = P_{\lambda}(\delta_{ij} = 1 | \mathfrak{S}_{t_{ij}}) = E_{\lambda}(\delta_{ij} | \mathfrak{S}_{t_{ij}}) \quad (3.49)$$

is the conditional probability of at least one failure on the interval  $(t_{ij}, t_{i+1,j})$  under condition that there is information about entire previous trajectory of the process on interval  $[0, t_{ij}]$ .

In discrete model  $M_j^*$ , the testing process develops without external interventions (intermediate test terminations) on each interval  $(t_{ij}, t_{i+1,j})$ . It means that the conditional distribution of the number of failures  $d_{ij}$  on this interval (under condition that all previous information about process on interval  $[0, t_{ij}]$  is known) has the standard binomial distribution with two parameters: the number of trials,  $N_j(t_{ij})$ , and the probability of failure,  $q_{ij}$ . Thus the conditional probability (3.49) is defined as

$$\pi_{ij} = 1 - (1 - q_{ij})^{N_j(t_{ij})} = 1 - \exp \left( - N_j(t_{ij}) \int_{t_{ij}}^{t_{i+1,j}} \lambda(u) du \right) \quad (3.50)$$

Introduce the following random sequence for the discrete model  $M_j^*$

$$\xi_{nj} = \prod_{1 \leq i \leq n-1} [1 + A(t_{ij})(\pi_{ij} - \delta_{ij})], \quad n = 1, 2, \dots \quad (3.51)$$

with  $\square_{l_j}=1$ .

In accordance with (3.49) the following equality

$$E_\lambda(\xi_{n+1,j} | \mathfrak{F}_{t_{nj}}) = \xi_{nj}, \quad n = 0, 1, \dots$$

holds. Thus the sequence of  $\square_{nj}$ ,  $n=1, 2, \dots$  forms a martingale in respect to the family  $\mathfrak{F}_{t_{nj}}$ ,  $n=1, 2, \dots$  with the mathematical expectation  $E_\lambda \square_{nj}=1$ ,  $n=1, 2, \dots$ . Because of continuity of  $\square(u)$  this function is restricted on each finite interval for  $0 \leq u \leq t$  and it follows that  $\pi_{ij} \leq C \left(\frac{h}{2^j}\right)$  for all  $i=0, 1, \dots, r_j$  where  $r_j=L \cdot 2^j$  and  $C$  is a constant. Taking into account that  $A(t)$  is restricted and  $\square_{ij} \leq 1$ , it follows that the martingale (3.51) is non-negative beginning from some subscript  $j_1$ :  $\square_{ij} \geq 0$ ,  $j \geq j_1$  for all  $n=1, 2, \dots, r_j$ . Using known Doob-Kolmogorov inequality (see, for instance, Doob(1953) or Shiryaev (1980)), we find for  $j \geq j_1$

$$P_\lambda \left\{ \max_{1 \leq n \leq l_j} \xi_{nj} \geq b \right\} \leq \frac{1}{b}$$

where level  $b > 0$ . Taking in this inequality  $b = \frac{1}{1-\gamma}$ , we obtain

$$P_\lambda \left\{ \max_{1 \leq n \leq r_j} \xi_{nj} \leq \frac{1}{1-\gamma} \right\} \geq \gamma \quad (3.52)$$

Let us introduce an event

$$\varphi_{\lambda,j}^t = \left\{ \max_{1 \leq n \leq r_j} \xi_{nj} \leq \frac{1}{1-\gamma} \right\}$$

and let  $\overline{\varphi}_\lambda^t$  and  $\overline{\varphi}_{\lambda,j}^t$  be complementary events to  $\varphi_\lambda^t$  and  $\varphi_{\lambda,j}^t$  respectively.

Let the event  $\overline{\varphi}_\lambda^t$  take place, that is,

$$I(\overline{\varphi}_\lambda^t) = 1 \quad (3.53)$$

The latter equality is equivalent to the condition that for some  $s \leq t$  the inequality

$$\int_0^s A(u)N(u)\lambda(u)du + \sum_{1 \leq j \leq D(s)} \ln[1 - A(t_j)] > |\ln(1 - \gamma)| \quad (3.54)$$

holds.

Let us now show that beginning from some  $j_2$  the event  $\bar{\varphi}_{\lambda,j}^t$  414 takes place:

$$I(\bar{\varphi}_{\lambda,j}^t) = 1, \quad j \geq j_2 \quad (3.55)$$

In other words, we need to show that (3.53) implies (3.55). It will follow that for  $j \square \square$

$$\bar{\varphi}_{\lambda}^t \subset \liminf \bar{\varphi}_{\lambda,j}^t$$

or

$$\varphi_{\lambda}^t \supset \limsup \varphi_{\lambda,j}^t$$

which, with (3.53), proves the inequality (3.46).

Introduce a value

$$n(j) = \left[ \frac{s 2^j}{h} \right]^*$$

where  $s$  is defined in (3.54). It is enough to show that if (3.54) holds, then from some  $j_3$  the inequality

$$\square_{n(j),j} > (1 - \square)^{-1}, \quad j \geq j_3 \quad (3.56)$$

This inequality can be rewritten in the following form

$$\Sigma'_j + \Sigma''_j > |\ln(1 - \gamma)| \quad (3.57)$$

where the first sum equals

$$\Sigma'_j = \sum_{\substack{1 \leq i \leq n(j)-1 \\ \delta_{ij}=0}} \ln [1 + A(t_{ij})\pi_{ij}],$$

where the sum is taken for all indexes  $i=1, \dots, n(j)-1$  for which  $\square_{ij}=0$ , and the second sum equals

$$\Sigma_j'' = \sum_{\substack{1 \leq i \leq n(j)-1 \\ \delta_{ij}=1}} \ln [1 + A(t_{ij})(\pi_{ij}-1)],$$

where the sum is taken for all indexes  $i=1, \dots, n(j)-1$  for which  $\delta_{ij}=1$ .

Let us now show that for  $j \geq j_1$  the left side of (3.57) converges to the left side of (3.54) everywhere on the set  $B$  introduced in Section 3.7.3. Since this set has probability equal to 1, the following statement follows. From (3.50), function  $\pi(u)$  is restricted on the interval  $0 \leq u \leq t$ , and values  $N(u)$  and  $N_j(u)$  are restricted by constant  $N(0)$ , the following equality

$$\pi_{ij} = N_j(t_{ij}) \int_{t_{ij}}^{t_{i+1,j}} \lambda(u) du + \varphi \quad (3.58)$$

holds where  $\delta_{ij} \leq C \delta^2$ ,  $C$  is some constant, and  $\delta = t_{i+1,j} - t_{ij} = \frac{h}{2^j}$ . So, the first sum, with the conditions that  $A(u)$  is restricted, can be written as

$$\Sigma'_j = \int_0^s A(u) N_j(u) \lambda(u) du + \varphi_1$$

where  $\delta_{ij} \leq C_1 \delta$ . Taking into account that, on set  $B$ , the value of  $N_j(u)$  differs from  $N(u)$  beginning from some  $j$  only on some finite number of intervals of length  $\delta$ , we can write

$$\Sigma'_j = \int_0^s A(u) N_j(u) \lambda(u) du + \varphi_2$$

for  $j \geq j_4$  where  $\delta_{ij} \leq C_2 \delta$ . Thus, for  $j \geq j_4$ , the first sum converges

$$\Sigma'_j \rightarrow \int_0^s A(u) N_j(u) \lambda(u) du$$

everywhere on the set  $B$ .

Further, from (3.58) it follows that  $\delta_{ij} \leq C_3 \delta$  for all  $i=1, \dots, r_j$ . Taking into account the restrictiveness of  $A(u)$  on the interval  $0 \leq u \leq t$ , we have

$$\Sigma_j'' = \sum_{\substack{1 \leq i \leq n(j)-1 \\ \delta_{ij}=1}} \ln [1 - A(t_{ij})] + \varphi_3$$

for  $j \geq j_5$  where  $\delta_{ij} \leq C_4 \delta$ . Thus, for  $j \geq j_5$ , on the set  $B$  the second sum converges

$$\Sigma''_j \rightarrow \sum_{1 \leq j \leq D(s)} \ln[1 - A(t_j)].$$

It proves the initial statement.

**Remark 3.7.1** Main condition on which the above proof is based is the formula (3.50) for the conditional probability (3.49). In more descriptive form, it means the following: Let us know that the intermediate termination cannot be performed on the time interval  $(t, t+h)$ . Then the conditional probability of unit failure on this interval under condition that it has not failed on interval  $(0, t]$  equals

$$q(t, t+h) = \frac{P(t) - P(t+h)}{P(t)} = 1 - \exp\left[-\int_t^{t+h} \lambda(u) du\right]$$

independent of the "prehistory" of the testing process up to the moment  $t$ . In other words, there is no dependence on the intermediate termination or failures of other units on interval  $(0, t]$ . This and the assumption of the independence of tested units gives us that the conditional distribution of the number of failures  $d=D(t+h)-D(t)$  on this interval is binomial with parameters  $N=N(t)$ , the number of independent trials, and  $q=q(t, t+h)$ , the probability of failure. The same condition (formula (3.38)) was the key for the proof of unbiasedness of the estimate  $\hat{P}(t)$  in Section 3.7.3.

**Remark 3.7.2** For the more general test model, [MMR], where new tested units might appear at moments  $\tau_1, \tau_2, \dots$  (see Section 3.6), the analogous statement is as follows. Let it be known that on the time interval  $(t, t+h)$  no external interventions are possible (no intermediate termination, no addition of new units). Then the conditional probability of unit failure under condition that it has not failed or its testing has not been terminated up to this moment equals

$$q = \frac{P[S(t)] - P[S(t)+h]}{P[S(t)]} = 1 - \exp\left[-\int_{S(t)}^{S(t)+h} \lambda(u) du\right] \tag{3.59}$$

where  $S(t)$  is the time for which the unit was operating up to the moment  $t$ , that is,  $S(t)=t-\tau$  where  $\tau$  is the moment when this unit began to be tested. On the basis of this condition, the proof of the confidence limits (3.33) and (3.34) for the model [MMR] can be performed in the manner completely analogous to [MMU]. The only additional condition for this case is  $E_{\tau} B(t) < \infty$ ,  $\tau \in W$  for all  $\tau > 0$  where  $B(t)$  is the number of new units which began to be tested on the time interval  $[0, t]$ .

#### 4.8.6 Consistency of Confidence Limits (3.24) and (3.25)



Let  $t > 0$  be some fixed moment. Consider a sequence of Markov test models  $[MMU]_r$ ,  $r=1, 2, \dots$ . Characteristics for the  $r$ th model is denoted by subscript  $r$ :  $N_r(t)$ ,  $D_r(t)$ , and so on. Let the initial number of tested units be

$$N_r = N_r(0) \rightarrow \infty$$

for  $r \rightarrow \infty$ , with the following condition (for fixed  $t$ )

$$\frac{N_r \sqrt{N_r}}{N_r^2(t)} \xrightarrow{p} 0. \quad (3.60)$$

Assume that the intermediate termination moments can take only discrete values  $v_j = ih$ ,  $i=1, 2, \dots$  with some fixed infinitesimally small  $h > 0$ . For the sake of simplicity, assume that  $h = \frac{t}{K}$  where  $K$  is an integer number. Set  $\alpha = 1 - \alpha$  and choose function  $A(t)$  as follows

$$A(t) = -\frac{a\sqrt{N_r}}{N_r(t)}, \quad t \geq 0$$

where  $a$  is a positive constant. From (3.24) we have the following lower  $(1 - \alpha)$ -confidence limit for the resources function

$$\underline{\Lambda}_r(t) = -\frac{|\ln \alpha|}{a\sqrt{N_r}} + \frac{1}{a\sqrt{N_r}} \sum_{1 \leq j \leq D_r(t)} \ln \left[ 1 + \frac{a\sqrt{N_r}}{N_r(t_j)} \right]. \quad (3.62)$$

In an analogous way, setting  $\alpha = 1 - \alpha$  and

$$A(t) = \frac{b\sqrt{N_r}}{N_r(t)}, \quad t \geq 0 \quad (3.63)$$

where  $b$  is a positive constant, we obtain the following upper  $(1 - \alpha)$ -confidence limit for the resource function

$$\bar{\Lambda}_r(t) = -\frac{|\ln \beta|}{b\sqrt{N_r}} - \frac{1}{b\sqrt{N_r}} \sum_{1 \leq j \leq D_r(t)} \ln \left[ 1 - \frac{b\sqrt{N_r}}{N_r(t_j)} \right]. \quad (3.64)$$

Let us show that for  $r \rightarrow \infty$  there is the following convergence

$$\underline{\Lambda}_r(t) \xrightarrow[p]{} \Lambda(t) \quad (3.65)$$

$$\overline{\Lambda}_r(t) \xrightarrow[p]{} \Lambda(t) \quad (3.66)$$

First let us prove (3.65). The first term in (3.62) is a constant converging to 0. The second term after some transformations can be rewritten as

$$\frac{1}{a\sqrt{N_r}} \sum_{1 \leq j \leq D_r(t)} \ln \left[ 1 + \frac{a\sqrt{N_r}}{N_r(t_j)} \right] = \hat{\Lambda}_r(t) + \delta$$

where

$$\begin{aligned} \hat{\Lambda}_r &= \sum_{1 \leq j \leq D_r(t)} \frac{1}{N_r(t_j)}, \\ |\delta| &\leq \frac{a\sqrt{N_r}}{2} \sum_{1 \leq j \leq D_r(t)} \frac{1}{N_r^2(t_j)}. \end{aligned} \quad (3.67)$$

From the latter inequality, taking into account that  $N_r(t_j) \geq N_r(t)$  for all  $j=1, \dots, D_r(t)$ , we have

$$|\delta| \leq \frac{a\sqrt{N_r}}{2} \frac{D_r(t)}{N_r^2(t)}.$$

Thus now it is enough to show that

$$\frac{D_r(t)\sqrt{N_r}}{N_r^2(t)} \xrightarrow[p]{} 0 \quad (3.68)$$

$$\hat{\Lambda}_r(t) \xrightarrow[p]{} \Lambda(t). \quad (3.69)$$

Convergence of (3.68) directly follows from the condition (3.60) if one takes into account that  $D_r(t) \leq N_r$ . Let us show the convergence of (3.69). The value  $\hat{\Lambda}_r(t)$  can be written in the form

$$\hat{\Lambda}_r(t) = \sum_{1 \leq i \leq K} \eta_i, \quad (3.70)$$

where

$$\eta_i = \sum_{j \in J_i} \frac{1}{N_r(t_j)}$$

and the sum is taken over the set of subscripts

$$J_i = \{j: v_i - h < t_j \leq v_i\}, \quad i = 1, \dots, K.$$

So, it is enough to show that

$$\eta_i \xrightarrow[p]{} \Lambda(v_i) - \Lambda(v_i - h)$$

for all  $i=1, \dots, K$ . Since on the interval  $(v_i - h, v_i)$  there are no intermediate terminations,

$$\eta_i = \sum_{N_r^- \leq n \leq N_i^+} \frac{1}{n}$$

where  $N_i^- = N(v_i)$  and  $N_i^+ = N(v_i - h)$ . For this value the following limits

$$\int_{N_j^-}^{N_i^+} \frac{dx}{x} \leq \eta_i \leq \int_{N_i^-}^{N_i^+} \frac{dx}{x}$$

Denoting the number of failures on the interval  $(v_i - h, v_i)$  by  $d_i = N_i^+ - N_i^-$ , we have

$$\ln\left(1 - \frac{d_i}{N_i^+}\right)^{-1} \leq \eta_i \leq \ln\left(1 - \frac{d_i}{N_i^+}\right)^{-1} + \ln\left(1 - \frac{1}{N_i^-}\right)^{-1} \quad (3.71)$$

For assumed conditions,  $N_i^- \xrightarrow[p]{} \infty$  and  $N_i^+ \xrightarrow[p]{} \infty$ ,  $i=1, \dots, K$ , it follows

$$\frac{1}{N_i^-} \xrightarrow[p]{} 0, \quad \frac{d_i}{N_i^+} \xrightarrow[p]{} q_i \quad (3.72)$$

where  $q_i = 1 - \exp[-\lambda(v_i - h) - \lambda(v_i)]$  is the conditional probability of unit failure on the interval  $(v_i - h, v_i)$  under condition that up to that interval the unit has not failed. From (3.71) and (3.72) it follows the convergence of (3.69) and (3.65).

The proof of the convergence of (3.66) is completely analogous.

**Remark 3.7.3** The main condition (3.60) for the consistency of the confidence limits means that the number of units whose testing has been terminated up to the moment  $t$  must not be too large, or, in other words,  $N_r(t)$  must grow to  $\infty$  faster than  $N_r^{(3/4)}$ .<sup>426</sup> This condition automatically holds for the model of independent random censoring ("concurrent risks") for all  $t$  such that  $[1-F(t)][1-G(t)] > 0$ , because in this case

$$\frac{N_r(t)}{N_r} \xrightarrow{p} [1-F(t)][1-G(t)]$$

from where it follows that

$$\frac{N_r \sqrt{N_r}}{N_r^2(t)} = \frac{1}{\sqrt{N_r}} \left[ \frac{N_r}{N_r(t)} \right]^2 \xrightarrow{p} 0.$$

Notice that in the conditions mentioned above, the estimate of the resource function  $\hat{\Lambda}(t)$ <sup>427</sup> and reliability function  $\hat{P}(t)$ <sup>428</sup> are consistent for  $N_r \rightarrow \infty$ , though for their consistency more weak condition is satisfactory:  $N_r(t) \xrightarrow{p} \infty$ <sup>429</sup>, or, in other words,

$$P\{N_r(t) < C\} \rightarrow 0 \text{ for any finite } C.$$

#### 4.8.7 Markov Test Model [MMR] with Censorship and Renewal

Let us add to the model [MMU] the following two sequences:

$$\begin{aligned} \tau_1 < \tau_2 < \dots < \tau_i < \tau_{i+1} < \dots \\ m_1, m_2, \dots, m_i, m_{i+1}, \dots \end{aligned}$$

where  $\tau_i$  is the sequence of monotonically increasing Markov (in respect to the family  $\mathfrak{F}_t$ <sup>430</sup>,  $t \geq 0$ ) moments, and  $m_i = 0, 1, 2, \dots$  is  $\mathfrak{F}_{\tau_i}$ <sup>431</sup>-measurable r.v.,  $i=1, 2, \dots$ . Here  $\tau_i$  is a moment of adding  $m_i$  new identical units to the tested ones.

The total test termination moment,  $\tau$ , is also Markovian in respect to the family  $\mathfrak{F}_t$ <sup>432</sup>,  $t \geq 0$ , such that  $P_\tau(\tau < \tau) = 1$ ,  $\tau \in \mathcal{W}$ . Thus, this model is defined by the following set

$$M = \{(\tau_1, n_1), \dots, (\tau_k, n_k), \dots; (\tau_1, m_1), \dots, (\tau_i, m_i), \dots; \nu\}$$

(3.73)

where  $\tau_k$  is the moment of the testing termination for a group of  $n_k$  units,  $k=1, 2, \dots$  (see Section 3.7.2).

One more difference of this model from [MMU] is in the fact that the sequence  $(\tau_k, n_k)$ , as well as the sequence  $(\tau_i, m_i)$ , is, in general, infinite.

It is clear that the model [MMR] includes the model [MMU] as a particular case for  $m_i = 0$ ,  $i=1, 2, \dots$ .

Notice also that in the model [MMU] all operating units at any moment  $t$  have the same "age" (operational time) equal to  $t$ . In the more general model [MMR]  $N(t)$  units currently testing at the time moment  $t$  have different "age"

$$S_e(t), e \in E(t)$$

where  $e$  is the subscript of the unit,  $S_e(t)$  is the 'age' of the  $e$ th unit at time  $t$ ,  $E(t)$  is the set of subscripts of the units testing at moment  $t$ . The set  $E(t)$  consists of  $N(t)$  subscripts, so for defining the test model, the set (3.73) must be widened: for each intermediate termination moment,  $\square_k$ , we need to specify the set of the unit's subscripts rather than just their quantity,  $n_k$ :

$$\{e_r(\square_k), r=1, \dots, n_k\}$$

where  $e_r(\square_k)$  is the subscript of the  $r$ th unit whose test has been terminated at the moment  $\square_k$ ,  $e_r(\square_k) \in E(\square_k)$ . Of course, it is possible to show the "age" of all units which are still being tested after the moment  $\square_k$ :

$$S_e(\square_k), e \in E(\square_k) \cap E(\square_k^-).$$

### Problems for Chapter 3

**Exercise 3.1** Prove the formula for the reliability function

$$P(t) = \prod_{0 \leq j \leq M-1} (1 - q_j) \quad (3.74)$$

where  $t = Mh$ ,

$$q_j = P(jh < \xi \leq jh + h | \xi > jh) = \frac{P(jh) - P(jh + h)}{P(jh)}$$

$j=0, 1, \dots, M-1$ , i.e.,  $q_j$  is the conditional probability of unit failure on the interval  $(jh, jh+h]$  under condition that up to the moment  $jh$  the unit has not failed. (Here  $\xi$  denotes unit's TTF.)

**Exercise 3.2** In the conditions of Example 3.3.1 calculate the estimate of the reliability function  $P(t)$  for  $t=300$  with the help of the recurrent procedure (3.11).

**Exercise 3.3** In the conditions of Example 3.3.1 calculate the estimate of the reliability function  $P(t)$  for  $t=375$  with the help of the recurrent procedure (3.11).

**Exercise 3.4** Consider the following Markov test model [MMU] with censorship and without renewal. At moment  $t=0$  fifteen identical units are tested ( $N=15$ ). The test results are represented in Tables 3.3 and 3.4. To find the estimate of the reliability function  $P(t=200)$ .

Table 3.3

Table 3.4

**Exercise 3.5** In the conditions of the previous Exercise find the estimate for the reliability function  $P(t=120)$ .

**Exercise 3.6** In the conditions of Exercise 3.4 find the estimates for the reliability function  $P(t=150)$  and resource function  $\square(t=150)$ .

**Exercise 3.7** In the conditions of Example 3.3.1, using asymptotically normal approximation (see Section 3.4), construct the lower confidence limit with confidence coefficient  $\square=0.95$  for the reliability function  $P(t=160)$ .

**Exercise 3.8** In the conditions of Exercise 3.4, using the method described in Section 3.5, find the lower confidence limit with the confidence coefficient  $\alpha=0.9$  for the reliability function  $P(t=150)$  and resource function  $\square(t=150)$ .

## 5. Bayes Methods of Reliability Estimation

### 5.1 Introduction

To illustrate the main idea of the Bayes method we consider the following standard case. Let us have a unit with d.f of TTF  $F(t, \theta)$  and density  $f(t, \theta)$  where  $\theta=(\theta_1, \dots, \theta_m)$  is parameter (in general case, a vector) from some space  $\Theta$ . Let  $\mathbf{z}$  be a vector of test results obtained on the basis of some test plan, and  $L(\mathbf{z}|\theta)$  is a likelihood function or, in other words, the density of distribution of test results  $\mathbf{z}$  for given value of parameter  $\theta$ . For instance, if  $\mathbf{z}$  is a complete sample of size  $n$ , that is,  $\mathbf{z}=(x_1, \dots, x_n)$  where  $x_i$  are independent values of observed TTF, then likelihood function has the form

$$L(\mathbf{z}|\theta) = \prod_{1 \leq i \leq n} f(x_i, \theta)$$

Following Bayes approach we consider parameter  $\theta$  a r.v. with some density  $h(\theta)$  which is called a prior density of distribution. This density, roughly speaking, reflects our prior (before testing) knowledge about the parameter  $\theta$ . The main idea of the Bayes approach consists in finding conditional density of distribution  $h(\theta|\mathbf{z})$  under condition that the test results is  $\mathbf{z}$ :

$$h(\theta|\mathbf{z}) = \frac{h(\theta)L(\mathbf{z}|\theta)}{\varphi(\mathbf{z})} \quad (4.1)$$

where  $\varphi(\mathbf{z})$  is density of distribution of  $\mathbf{z}$ , taking into account a prior distribution  $h(\theta)$ :

$$\varphi(\mathbf{z}) = \int_{\Theta} h(\theta)L(\mathbf{z}|\theta)d\theta$$

In formula (4.1) function  $h(\theta)L(\mathbf{z}|\theta)$  is a joint density of distribution of a pair  $(\theta, \mathbf{z})$ , and function  $\varphi(\mathbf{z})$  is marginal density of distribution of  $\mathbf{z}$ . For given fixed test results  $\mathbf{z}$ , the value of function  $h(\theta|\mathbf{z})$  is proportional to the product of the prior density and likelihood function,  $h(\theta)L(\mathbf{z}|\theta)$ . Value of  $\varphi(\mathbf{z})$  is a norm multiplier which is chosen in such a way that the integral of function  $h(\theta|\mathbf{z})$  by parameter  $\theta$  is equal to 1.

Relation (4.1) is the basis one for the Bayes method. Function  $h(\theta|\mathbf{z})$  is called a posterior density of distribution of parameter  $\theta$  for given test results  $\mathbf{z}$ . This density reflects our knowledge about parameter  $\theta$  after the test. For instance, if  $D$  is some area in parameter space, then value  $\int_D h(\theta|\mathbf{z})d\theta$  can be interpreted as a measure of our assurance in the fact that parameter  $\theta$  belongs to the area  $D$  if the test results is  $\mathbf{z}$ .

A posterior density (4.1) is the basis for obtaining statistical decisions about parameter  $\theta$ . We will consider the construction of Bayes estimates and confidence limits.

## 5.2 Point Estimates and Confidence Limits

### 5.2.1 Bayes Point Estimates

Let  $R=R(\theta)$  be a unit reliability index which is a function of vector parameter  $\theta$ . Let  $\hat{R} = \hat{R}(\mathbf{z})$  be a point estimate for  $R$ . A natural measure of quality of estimate  $\hat{R}$  can be chosen to be the mathematical expectation of the square of deviation of estimate  $\hat{R}$  from  $R$ , that is, value

$$E(\hat{R} - R)^2 = \int_{\Theta} h(\theta) d\theta \int_Z [\hat{R}(\mathbf{z}) - R(\theta)]^2 L(\mathbf{z}|\theta) d\mathbf{z} \quad (4.2)$$

where  $Z$  is a set of possible test results  $\mathbf{z}$ . The problem is to find function  $\hat{R}(\mathbf{z})$  for which value (4.2) is minimum.

Taking into account the definition of the posterior density (4.1), we can present (4.2) in the form

$$E(\hat{R} - R)^2 = \int_Z \varphi(\mathbf{z}) d\mathbf{z} \int_{\Theta} [\hat{R}(\mathbf{z}) - R(\theta)]^2 h(\theta|\mathbf{z}) d\theta.$$

Hence, for solving the problem formulated above, we need to find, for each fixed test results  $\mathbf{z}$ , minimum in  $\hat{R}$  of function

$$G(\hat{R}) = \int_{\Theta} [\hat{R} - R(\theta)]^2 h(\theta|\mathbf{z}) d\theta$$

This function can be written in the form

$$G(\hat{R}) = \hat{R}^2 - 2\hat{R} \int_{\Theta} R(\theta) h(\theta|\mathbf{z}) d\theta + \int_{\Theta} R^2(\theta) h(\theta|\mathbf{z}) d\theta$$

Calculating derivative  $G'(\hat{R})$  and setting it equal to 0, we see that minimum of function  $G(\hat{R})$  is reached at

$$\hat{R} = \int_{\Theta} R(\theta) h(\theta|\mathbf{z}) d\theta \quad (4.3)$$



The right side of this equation is a posterior mathematical expectation for  $R(\theta)$ . Thus the best Bayes point estimate of reliability index  $R=R(\theta)$ , which minimizes the standard deviation (4.2) of  $R(\theta)$  for given test results  $\mathbf{z}$ .

Notice that the choice of the type of measure of quality of estimate  $\hat{R}$  essentially influences on the type of "the best" est. For instance, if we choose the mathematical expectation of module of deviation of estimate  $\hat{R}$  from  $R$ , as the measure of quality, that is, value

$$E|\hat{R} - R| = \int_{\Theta} h(\theta) d\theta \int_{\mathbf{z}} |\hat{R}(\mathbf{z}) - R(\theta)| L(\mathbf{z}|\theta) d\mathbf{z}$$

then we can show that the best Bayes point estimate of  $\hat{R} = \hat{R}(\mathbf{z})$  is the median of the posterior distribution of  $R(\theta)$  for fixed test results  $\mathbf{z}$  (see below the Problem 4.1 to the chapter).

### 5.2.2 Sufficient Statistics

Let  $T=T(\mathbf{z})$  be a sufficient statistic, then (see Section 1.3.4) likelihood function can be written in the form of product of two multipliers

$$L(\mathbf{z}|\theta) = C(\mathbf{z}) g[T(\mathbf{z}), \theta]$$

where function  $C(\mathbf{z})$  depends only on the test results  $\mathbf{z}$  but does not depend on parameter  $\theta$ , and function  $g[T(\mathbf{z}), \theta]$  depends on parameter  $\theta$  and test results  $\mathbf{z}$  via statistic  $T(\mathbf{z})$ . Substituting this expression into (4.1), we have the following formula for the posterior density of distribution:

$$h(\theta|\mathbf{z}) = \frac{h(\theta) \cdot C(\mathbf{z}) \cdot g[T(\mathbf{z}), \theta]}{C(\mathbf{z}) \int_{\Theta} h(\theta) g[T(\mathbf{z}), \theta] d\theta} = \frac{h(\theta) \cdot g[T(\mathbf{z}), \theta]}{\int_{\Theta} h(\theta) g[T(\mathbf{z}), \theta] d\theta}$$

From here it follows that the posterior distribution density depends on test results  $\mathbf{z}$  via a sufficient statistic  $T=T(\mathbf{z})$ . In other words, if different test results,  $\mathbf{z}_1$  and  $\mathbf{z}_2$  are such that  $T(\mathbf{z}_1)=T(\mathbf{z}_2)$ , then the posterior distribution densities  $h(\theta|\mathbf{z}_1)$  and  $h(\theta|\mathbf{z}_2)$  coincide. Thus all Bayes statistical inferences (point estimates, confidence limits, etc.) depend only on sufficient statistic  $T=T(\mathbf{z})$ . For instance (see below Section 4.4), in the "exponential model" below, two-dimensional statistic  $T=(d,S)$  is sufficient. (Here  $d$  is the number of failures during the test and  $S$  is the total test time.)

### 5.2.3 Bayes confidence limits

Let  $R=R(\theta)$  be a reliability index. Introduce function

$$\Phi(t|\mathbf{z}) = P\{R(\theta) \leq t|\mathbf{z}\} = \int_{R(\theta) \leq t} h(\theta|\mathbf{z}) d\theta \quad (4.4)$$

which represents a posterior d.f. of reliability index  $R(\theta)$  for a given test results  $\mathbf{z}$ . Bayes confidence limits for  $R(\theta)$  are determined as corresponding quantiles of d.f. (4.4). Namely, Bayes Lower Confidence Level  $\underline{R} = \underline{R}(\mathbf{z})$  with confidence coefficient  $1 - \alpha$  for  $R(\theta)$  can be found from the solution of equality

$$\Phi(\underline{R}|\mathbf{z}) = \alpha \quad (4.5)$$

Bayes UCL  $\bar{R} = \bar{R}(\mathbf{z})$  with confidence coefficient  $1 - \beta$  for  $R(\theta)$  is found from

$$\Phi(\bar{R}|\mathbf{z}) = 1 - \beta \quad (4.6)$$

From (4.5) and (4.6) also follows that the following equation is true:

$$P\{\underline{R} \leq R(\theta) \leq \bar{R}|\mathbf{z}\} = \int_{\underline{R} \leq R(\theta) \leq \bar{R}} h(\theta|\mathbf{z}) d\theta = 1 - \alpha - \beta$$

that is, interval  $(\underline{R}, \bar{R})$  is Bayes confidence limit for  $R(\theta)$  with confidence coefficient  $1 - \alpha - \beta$ .

Let us now consider some applications of Bayes method.

### 5.3 Binomial model

In the result of the test we know the number of failures in  $N$  independent tests,  $z=d$ . Parameter  $\theta=q$  is the probability of failure in a single test. In this case likelihood function, that is, the probability of observation of  $d$  failures in a series of  $N$  trials, has the form

$$L(\mathbf{z}|\theta) = L(d|q) = \binom{N}{d} q^d (1-q)^{N-d}.$$

Let a prior distribution density,  $h(q)$ , of parameter  $q$  be given. By the main formula (4.1) the posterior distribution density of parameter  $q$  for fixed  $d$  is determined as

$$h(q|d) = \frac{h(q)q^d(1-q)^{N-d}}{\int_0^1 h(q)q^d(1-q)^{N-d} dq} \quad (4.7)$$

where  $0 < q < 1$ . Let a prior distribution  $h(q)$  have the form of the standard beta distribution with parameters  $(a, b)$ , that is,

$$h(q) = \frac{q^{a-1}(1-q)^{b-1}}{B(a,b)} \quad (4.8)$$

where  $B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$  is beta function.

Substituting (4.8) into (4.7), we obtain

$$h(q|d) = \frac{q^{a+d-1} (1-q)^{b+N-d-1}}{B(a+d, b+N-d)}, \quad (4.9)$$

that is, posterior distribution density (4.9) in this case is also a density of beta distribution though with new parameters  $(a+d, b+N-d)$ . It shows in a very transparent form how test results  $(N, d)$  influences on the prior distribution (4.8).

Remark. If prior and posterior distributions belongs to the same family of distributions, such a family is called adjoin. For the "binomial model" a family of beta distributions is adjoin. In "exponential model" such adjoin distribution family is gamma (see Section 4.4 below).

Applying (4.3), we obtain Bayes point estimate  $\hat{q}$ , a posterior mean of parameter  $q$ :

$$\hat{q} = \int_0^1 qh(q|d)dq = \frac{a+d}{a+b+N} \quad (4.10)$$

Formula for variance of the posterior distribution of parameter  $q$  can be also easily obtained:

$$E[(q - \hat{q})^2 | d] = \int_0^1 (q - \hat{q})^2 h(q|d)dq = \frac{(a+d)(b+N-d)}{(a+b+N)^2 (a+b+N+1)}.$$

Using (4.5) and (4.6), we obtain that Bayes LCL,  $\underline{q}$ , with confidence coefficient  $1-\alpha$  and Bayes UCL,  $\bar{q}$ , with confidence coefficient  $1-\beta$  for parameter  $q$  can be found from

$$I_{\underline{q}}(a+d, b+N-d) = \alpha \quad (4.11)$$

$$I_{\bar{q}}(a+d, b+N-d) = 1 - \beta \quad (4.12)$$

where  $I_q(a,b)$  is incomplete beta distribution with parameters  $a$  and  $b$  :

$$I_q(a,b) = \frac{1}{B(a,b)} \int_0^q t^{a-1} (1-t)^{b-1} dt \quad (4.13)$$

Notice that standard Clopper-Pearson confidence limits for binomial parameter  $q$  are found from (see Section 1.4.6 above) from

$$\sum_{d \leq j \leq N} \binom{j}{N} \underline{q}^j (1 - \underline{q})^{N-j} = \alpha$$

$$\sum_{0 \leq j \leq d} \binom{j}{N} \bar{q}^j (1 - \bar{q})^{N-j} = \beta.$$

Taking into account

$$\sum_{0 \leq j \leq d} \binom{j}{N} \underline{q}^j (1 - \underline{q})^{N-j} = 1 - I_{\underline{q}}(d + 1, N - d),$$

we can express Clopper-Pearson equations via incomplete beta function in the form

$$I_{\underline{q}}(d, N - d + 1) = \alpha \quad (4.14)$$

$$I_{\bar{q}}(d + 1, N - d) = 1 - \beta \quad (4.15)$$

Solution of these equations calculated on the basis of statistic  $(d, N)$  are denoted by  $\underline{q}_{1-\alpha}(N, d)$  and

for Clopper-Pearson LCL (with confidence coefficient  $1 - \alpha$ ) and UCL (with confidence coefficient  $1 - \beta$ ), respectively. Notice that left sides of these equations are defined not only for integer values of  $N$  and  $d$ .

From comparison (4.11) and (4.12) with (4.14) and (4.15), we see that Bayes confidence limits  $\underline{q}$  and  $\bar{q}$  relate to the Clopper-Pearson confidence limits as follows:

$$\underline{q} = \underline{q}_{1-\alpha}(N + a + b - 1, d + a) \quad (4.16)$$

$$\bar{q} = \bar{q}_{1-\beta}(N + a + b - 1, d + a - 1) \quad (4.17)$$

These equations allow us to calculate Bayes confidence limits with the help of standard tables of binomial distribution (or tables of beta functions).

Formulas (4.61) and (4.17) can be interpreted in the following way. A knowledge of the prior distribution (4.8) of parameter  $q$ , roughly speaking, is equivalent to a situation where we have information about  $a+b-1$  additional trials with  $a$  observed failures. In the case of calculation of the

upper confidence level,  $\bar{q}$ , we have information about additional  $a+b-1$  during which  $a-1$  failures have occurred. In an analogous way, the point Bayes estimate (4.10) can also be interpreted.

Example 4.3.1. Let the parameters of the prior beta distribution be  $a=2$  and  $b=9$  and test results be the following:  $N=8$  and  $d=1$ . Using (4.10), we obtain in this case that Bayes point estimate  $\hat{q}$  for reliability index  $q$  (the failure probability) is

$$\hat{q} = \frac{d+a}{N+a+b} = \frac{3}{19} = 0.158.$$

Using formulas (4.16) and (4.17) and table E.14 of Appendix, we found that 0.99-confidence limits ( $\underline{q}, \bar{q}$ ) for parameter  $q$  (for  $\alpha=\beta=0.005$ ) are:

$$\begin{aligned} \underline{q} &= \underline{q}_{1-\alpha}(N+a+b-1, d+a) = \underline{q}_{0.995}(18, 3) = 0.020 \\ \bar{q} &= \bar{q}_{1-\beta}(N+a+b-1, d+a-1) = \bar{q}_{0.995}(18, 2) = 0.422 \end{aligned}$$

Remark. The examples above illustrate both advantages and disadvantages of Bayes approach.

Indeed, if we could choose an prior distribution (4.8) so that ratio  $\frac{a}{a+b}$  is close to the real value of parameter  $q$ , then we obtain, roughly speaking, an equivalent number of trials  $N+(a+b-1)$  instead of initial value of  $N$ . However, if this choice is not felicitous, then we need a large number of trials  $N$  to correct the results due to the wrong choice.

### 5.4 Exponential Model

Let unit TTF have exponential d.f.  $F(t, \theta) = 1 - e^{-\theta t}$ . In this case  $\theta = \lambda$ . Assume that test was performed by plan [N U r], that is, for the number of failures fixed in advance (see Section 2.1). In this case test results are expressed as  $z=S$  where  $S$  is the total test time determined by formula (2.4) and likelihood function is

$$L(S|\lambda) = C \lambda^r e^{-\lambda S}$$

where  $C$  is a norm constant (see Section 2.2.1 above).

Assume that a prior distribution density  $h(\lambda)$  of parameter  $\lambda$  is given. Then in accordance with (4.1), the posterior distribution density of  $\lambda$  has the form

$$h(\lambda|S) = \frac{h(\lambda) \lambda^r e^{-\lambda S}}{\int_0^{\infty} h(\lambda) \lambda^r e^{-\lambda S} d\lambda} \quad (4.18)$$

Take the standard gamma distribution (see Section 1.2 above and Table 1.1) with parameters  $(U, a)$  as a prior distribution  $h(\lambda)$ , that is,

$$h(\lambda) = \frac{u^a \lambda^{a-1}}{\Gamma(a)} e^{-u\lambda} \quad (4.19)$$

where  $\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$  is gamma function. Then from (4.18), we obtain that the posterior distribution density is

$$h(\lambda|S) = \frac{(u+S)^{a+r} \lambda^{a+r-1} e^{-(u+S)\lambda}}{\Gamma(a+r)}$$

from where we see that it is also gamma distributed with parameters  $(u+S, a+r)$ . Thus, the exponential family is adjoin for the "exponential model" considered above.

Using (4.3) we obtain that Bayes point estimate (the posterior mean) for parameter  $\lambda$  has the form

$$\hat{\lambda} = \int_0^{\infty} \lambda h(\lambda|S) d\lambda = \frac{a+r}{u+S} \quad (4.20)$$

From (4.5) and (4.6) we find that Bayes LCL  $\underline{\lambda}$  (with confidence coefficient  $1-\alpha$ ) and Bayes UCL  $\bar{\lambda}$  (with confidence coefficient  $1-\beta$ ) for parameter  $\lambda$  can be found as

$$\underline{\lambda} = \frac{\Gamma_{\alpha}(1, a+r)}{u+S}, \quad \bar{\lambda} = \frac{\Gamma_{1-\beta}(1, a+r)}{u+S}$$

where  $\Gamma_{\epsilon}(1, a+r)$  is the quantile of level  $\epsilon$  for gamma distribution with parameters  $(1, a+r)$ . These confidence limits can be also expressed via quantiles of chi square distribution (see Section 1.2.3 above):

$$\underline{\lambda} = \frac{\chi_{\alpha}^2(2a+2r)}{2(u+S)}, \quad \bar{\lambda} = \frac{\chi_{1-\beta}^2(2a+2r)}{2(u+S)}, \quad (4.21)$$

where  $\chi_{\epsilon}^2(m)$  is the quantile of level  $\epsilon$  for chi square distribution with  $m$  degrees of freedom.

Comparing these formulas with the analogous ones in Section 2.2.1, we see that using prior distribution (4.19) with parameters  $(u, a)$  corresponds to additional tests with the total test time  $u$  and the number of failure  $a$ .

**Example 4.4.1** Let the parameters of the prior distribution be  $u=150$  hours and  $a=1$ , and test results be  $S=295$  hours and  $r=1$ . Using (4.20) and (4.21) we obtain that Bayes point estimate (the posterior mean) for parameter  $\lambda$  is equal to

$$\hat{\lambda} = \frac{a+r}{u+S} = \frac{1+1}{150+295} = 45 \cdot 10^{-4} \text{ (1/hour)}$$

Bayes 0.9-LCL,  $\underline{\lambda}$ , and 0.9-UCL,  $\bar{\lambda}$ , for parameter  $\lambda$  are

$$\underline{\lambda} = \frac{\chi_{0.1}^2(4)}{2(150+295)} = \frac{1.064}{890} = 12 \cdot 10^{-4} \text{ 1/hour}$$

$$\bar{\lambda} = \frac{\chi_{0.9}^2(4)}{2(150+295)} = \frac{7.78}{890} = 87.5 \cdot 10^{-4} \text{ 1/hour}$$

#### Test Plan [N U T]

Consider now plan [N U T], that is test for time  $T$  fixed in advance time  $T$ . In this case test results are given by a pair  $z=(S,d)$  where, as before,  $S$  is the total test time and  $d$  is the number of failures.  $fl$  is given by the expression

$$L(S,d|\lambda) = C \lambda^d e^{-\lambda S} \quad (4.22)$$

where  $C=C(d)=N(n-1) \cdot \dots \cdot (N-d+1)$  depends on the test results ( $d$ ) but does not depend on parameter  $\lambda$ . Consequently, pair  $(S,D)$  is sufficient statistic (see Section 2.2.2 above).

Let  $h(\lambda)$  be a prior distribution density of parameter  $\lambda$ . Then from (4.1) and (4.22) follows that the posterior distribution density has the form

$$h(\lambda|S,d) = \frac{h(\lambda) \lambda^d e^{-\lambda S}}{\int_0^{\infty} h(\lambda) \lambda^d e^{-\lambda S} d\lambda} \quad (4.23)$$

It is easy to see that posterior densities (4.18) and (4.23) coincide if to take  $d=r$ . It means that all results for plan [N U r] considered above can be extended to plan [N U T] for  $d=r$ .

#### Other test Plans.

The same conclusions can be obtained for other test plans of type U (no renewal) and type R (with renewal). Indeed, for these plans likelihood function has the form

$$L(S,d|\lambda) = C \lambda^d e^{-S\lambda} \quad (4.24)$$

where  $C$  is a constant or a function depending on the test results but does not depend on parameter  $\theta$  (see Section 2.2). Notice that for plans  $[\mathbf{N}\mathbf{U}\mathbf{r}]$  and  $[\mathbf{N}\mathbf{R}\mathbf{r}]$ , where tests continue until the number of failures equal to  $r$ , fixed in advance. We assume  $d=r$ . From (4.1) and (4.24) follows that for all these test plans a posterior distribution density of parameter  $\theta$  has the same form (4.23). Respectively, the conclusion made on the basis of the posterior distribution density, in particular, formulas (4.20) and (4.21) for Bayes point estimate and confidence limits for test plan  $[\mathbf{N}\mathbf{U}\mathbf{r}]$ , can be expanded on test plans mentioned above for  $d=r$ .

This important property of the Bayes approach ("insensitivity" in respect to the test plans) is expanded onto a more general case where a distribution of TTF is not necessarily exponential.

### 5.5 General Parametric Model

Consider a general case where TTF has d.f.  $F(t, \theta)$  with density  $f(t, \theta)$  where  $\theta$  is some (in general case, a vector) parameter taking its values from set  $\Theta$ . Let us denote  $\bar{F}(t, \theta) = 1 - F(t, \theta)$  a complementary distribution function (in other terms, reliability function). For all test plans of type  $\mathbf{U}$  (no renewal) and type  $\mathbf{R}$  (with renewal), considered above, likelihood function has the form (see Section 2.4):

$$L(z|\theta) = C \cdot \prod_{1 \leq i \leq d} f(s_i, \theta) \prod_{1 \leq j \leq v} \bar{F}(u_j, \theta) \quad (4.25)$$

where  $\mathbf{z} = (d, s_1, \dots, s_d, v, u_1, \dots, u_v)$  is a set of all test results,

$s_i$  are complete intervals of observations (i.e. those which are terminated by a failure),  $1 \leq i \leq d$ ,  
 $u_j$  are incomplete intervals of observations (i.e. those which are terminated before failure has occurred),  $1 \leq j \leq v$ ,

$d$  is the number of failures (coincides with the number of complete intervals of observations),

$v$  is the number of units whose test was terminated before failure (coincides with the number of censored test intervals),

$C$  is a function which can depend on the test results but does not depend on parameter  $\theta$ .

Numbers  $d$  and  $v$  might be determined in advance or random depending on the test plan (see Section 2.4).

Assume that a prior distribution density  $h(\theta)$  of parameter  $\theta$  is given. From (4.1) and (4.25) we can obtain that for all mentioned test plans the posterior distribution density  $h(\theta|\mathbf{z})$  of parameter  $\theta$  for a given test results  $\mathbf{z}$  has the same form:

$$h(\theta|\mathbf{z}) = \frac{h(\theta)}{\varphi(\mathbf{z})} \cdot \prod_{1 \leq i \leq d} f(s_i, \theta) \prod_{1 \leq j \leq v} \bar{F}(u_j, \theta) \quad (4.26)$$

where



$$\varphi(\mathbf{z}) = \int_{\Theta} h(\theta) \cdot \prod_{1 \leq i \leq d} f(s_i, \theta) \prod_{1 \leq j \leq v} \bar{F}(u_j, \theta) d\theta.$$

Thus, for fixed test results  $\mathbf{z}$  the posterior distribution density (4.26) and all statistical inferences do not depend on the test plan. So, the Bayes approach is insensitive to test plans.

In conclusion, notice that the Bayes approach possesses of some essential advantages: simplicity, universality and inner logic. However, this method has its weak sides, determined by an arbitrary choice of prior distributions.

Since the choice of the prior distribution is completely in the hand of researcher, there is a possibility of "adapting" the final numbers to almost any "needed" results. This dependence on subjective viewpoint of a researcher is a serious disadvantage of the method. At the same time, this method very effective for "aggregation" of statistical data obtained from different sources. The problem of choice of a prior distributions discussed in detail by Cox and Hinkley (1978).

Bayes approach is widely discussed last years. Probably, first intensive wave of works dedicated applications of the Bayes approach to reliability problems began in late 60's and 70's: Springer and Thompson (1966, 1967, 1968), Fergusson (1973), Cole (1975), Mastran and Singpurwalla (1976, 1978) Smith and Springer (1976). Among recent works, we would like to mention Barlow (1985), Martz and Waller (1982, 1990), Martz, Waller and Fickas (1988). Some monographs on the theme also could be mentioned: Mann, Schaefer and Singpurwalla (1974), Belyaev (1982), Savchuk (1989).

#### Problems to Chapter 4

- 4.1. Find Bayes point estimate  $\hat{R} = \hat{R}(\mathbf{z})$  for  $R=R(\theta)$  which minimizes the posterior mathematical expectation of module  $|\hat{R} - R|$ .
- 4.2. Let (see Section 4.3) a prior distribution parameter  $q$  (failure probability) in a "binomial model" be uniform in interval  $(0, 1)$ . For this case find the Bayes posterior distribution, point estimate (posterior mean) and confidence limits for parameter  $q$ .
- 4.3. Assume that in a "binomial model" we know in advance that failure probability  $q$  satisfies inequality  $q \leq q^*$  where  $q^* \leq 1$ . A prior distribution of parameter  $q$  is chosen uniform in interval  $(0, q^*)$ . Find for this case Bayes point estimate and confidence limits for parameter  $q$ .
- 4.4. Let a prior distribution parameter  $q$  (failure probability) in a "binomial model" be uniform in interval  $(0, 1)$  and test results be the following:  $d=1$  (number of failures) and  $N=10$  (number of tested units). Find Bayes point estimate and confidence limits with confidence coefficient 0.95 for parameter  $q$ .
- 4.5. Solve the previous problem for the same test results if it is known in advance that failure probability  $q$  satisfies inequality  $q \leq q^* = 1/2$  and a prior distribution of parameter  $q$  is chosen uniform in interval  $(0, q^*)$ .

4.6. Let a prior distribution of parameter  $\lambda$  (failure rate) an "exponential model" be gamma (4.19) with parameters  $\mu=200$  hours and  $\alpha=2$ . The test results are the following:  $S=400$  hours and  $d=1$ . Find the posterior distribution density, Bayes point estimate and confidence limits with confidence coefficient 0.9 for parameter  $\lambda$ .

## 6. ACCELERATED TESTING

### 6.1 INTRODUCTION

The true level of modern hardware reliability can be estimated only on the basis of tests under specified conditions, or on the basis of real data. To estimate the reliability of a large population of items, one needs to test a number of items and to treat the data thus obtained using statistical methods. This seemingly straightforward way of reliability estimation is often difficult in practice due to monetary and time restrictions. Let us assume that an item has the exponential TTF distribution with parameter  $\lambda = 10^{-9}$  1/hour. For satisfactory statistical estimation, one needs to observe, say, 10 failures. To obtain 10 failures, approximately,  $10^{10}$  item-hours are required. Thus, one unit should be tested for approximately 10 million years (!); or one million items should be tested for 10 years (!). Neither approach makes any practical sense.

As another example, consider an item with an extremely stable performance parameter which determines the item's reliability. (For instance, it can be a quartz timer in a synchronized system.) To estimate a time when the examined parameter deviates from its tolerance, one also needs either many items tested or enormous testing time.

Such situations lead to the need for development of some special methods for reliability estimation. One of the most important practical methods, used in these situations, is the so called accelerated life testing (ALT).

There are two main ways to accelerate tests:

- to put the test items under conditions much more severe than the nominal (operational) ones, which will result in faster failure occurrence;
- or/and to choose much more rigid tolerances and consider crossing of these limits as a degradation (conditional) failure.

These approaches are based on the natural assumption that increasing stress (as well as rigid tolerances) results in decreasing the time of failure-free operation. Both approaches need credible techniques for extrapolation from the "accelerated domain" into "the normal domain" and they are based on the hypothesis of similarity of the item behavior under accelerated and normal (use) stress conditions. In mechanics, for instance, there exists a special approach, the so-called, similarity theory, which is dedicated to such theoretical constructions and has many practical applications. Unfortunately, a hidden danger of erroneous conclusions always exists if one makes "too brave" an extrapolation.

Indeed, increasing the stress factors (shock, vibration, temperature, humidity) we can completely change the failure mechanisms, so that all of our predictions might turn out to be

useless.

For the case of rigid tolerances we meet analogous situations. A parameter of interest can be very stable within a narrow band around the nominal level, and, at the same time, there might be a catastrophic failure mode because of various influences of positive feedback.

In this chapter we will formally consider the first mentioned above case, though some results might be naturally extended to the second case.

## 6.2 Basic Notions and Probabilistic Model

### 6.2.1 TTF Distributions and Accelerated Life Reliability Model

A *reliability model* in accelerated life testing (*AL reliability model*) is usually defined as a relationship between the time to failure (TTF) distribution of a device and *stress factors*, such as load, cycling rate, temperature, humidity, voltage, etc. The AL reliability models are based on the considerations of physics of failure. From the mathematical point of view, an AL reliability model can be considered as a deterministic transformation of TTF.

### 6.2.2 Stress Severity in Terms of TTF Distribution

Let  $F_1(t; z_1)$  and  $F_2(t; z_2)$  be time to failure cumulative distribution functions (CDF) of the item under the constant stress conditions  $z_1$  and  $z_2$  respectively<sup>1</sup>. The stress condition  $z_2$

$$F_2(t; z_2) > F_1(t; z_1) \quad (5.1)$$

is more severe than  $z_1$ , if for all values of  $t$

This inequality means that a more severe stress condition accelerates the time to failure, so that the reliability of the item under stress condition  $z_2$  is less than the reliability under stress condition  $z_1$ .

### 6.2.3 Time-transformation Function for the Case of Constant Stress

For monotonically increasing CDFs  $F_1(t; z_1)$  and  $F_2(t; z_2)$ , if constant stress condition  $z_1$  is less severe than  $z_2$  and  $t_1$  and  $t_2$  are the times at which  $F_1(t_1; z_1) = F_2(t_2; z_2)$ , there exists a function  $g$  (for all  $t_1$  and  $t_2$ ) such that  $t_1 = g(t_2)$  and

$$F_2(t_2; z_2) = F_1(g(t_2), z_1) \quad (5.2)$$

---

<sup>1</sup>Stress condition,  $z$ , in general case is a vector of the stress factors.

Because  $F_1(t;z) < F_2(t;z)$ ,  $g(t)$  must be an increasing function with  $g(0)=0$  and  $\lim_{t \rightarrow \infty} g(t) = \infty$ .

The function  $g(t)$  is called the *acceleration* or the *time transformation function*.

As it was mentioned above, AL reliability model is a deterministic transformation of time to failure. Two main time transformations are considered in Life Data Analysis. These transformations are known as the *Accelerated Life (AL) Model* and the *Proportional Hazards (PH) Model*.

#### 6.2.4 Accelerated Life Model (Linear Time-transformation Function)

Without loss of generality, one may assume that  $z = 0$  for the normal (use) stress condition. Denote a failure time CDF under normal stress condition by  $F_0(\cdot)$ . The AL time transformation is given in terms of  $F(t;z)$  and  $F_0(\cdot)$  by the following relationship [Cox and Oaks, 1984]

$$F(t;z) = F_0[t\psi(z, A)] \quad (5.3)$$

where  $y(z,A)$  is a positive function connecting time to failure with a vector of stress factors  $z$ ; and  $A$  is a vector of unknown parameters; for  $z = 0$ ,  $y(z,A)$  is assumed equal to 1.

The corresponding relationship for the probability density function is given by

$$f(t;z) = f_0[t\psi(z, A)]\psi(z, A) \quad (5.3')$$

Relationship (5.3) is a scale transformation. It means that a change in stress does not result in a change of the shape of the distribution function, but changes its scale only. Relationship (5.3) can be written in terms of the acceleration function as follows:

$$g(t) = \psi(z, A)t \quad (5.4)$$

In other words, relationship (5.3) is equivalent to the linear one with time acceleration function.

The distributions considered are geometrically *similar* each other. They are called belonging to the class of TTF distribution functions which is closed with respect to scale [Leemis, 1995]. The similarity property is extremely useful in physics and engineering. Because it is difficult to imagine that any change of failure modes or mechanism would not result in a change in the shape of the failure time distribution, the relationship (5.3) can be also considered as a principle of failure mechanism conservation or a *similarity* principal. The analysis of some sets of real life data using the statistical procedures described below (*Criteria of Linearity of Time Transformation Function*) often show that the similarity of time to failure distributions really exists, so that a violation of the similarity can identify a change in a failure mechanism.

The relationship for the 100 $p$ th percentile of time to failure  $t_p(z)$  can be obtained from (5.3) as

$$t_p(z, B) = \frac{t_p^0}{\psi(z, A)} \quad (5.5)$$

where  $t_p^0$  is the 100 $p$ th percentile for the normal stress condition  $z = 0$ .

The relationship (5.4) is the *percentile AL reliability model* and it is usually written in the form

$$t_p(z, B) = \eta(z, B) \quad (5.6)$$

The AL reliability model is related to the relationship for percentiles, (5.5), as

$$\eta(z, B) = \frac{t_p^0}{\psi(z, A)} \quad (5.7)$$

Corresponding relationship for failure rate can also be obtained from (5.3) as

$$\lambda(t; z) = \psi(z, A) \lambda^0[t\psi(z, A)], \quad (5.8)$$

It is easy to see that the relationship for percentiles is the simplest; that's why it is being used in the following sections on AL data analysis.

### 6.2.5 Cumulative Damage Models and Accelerated Life Model

Some known cumulative damage models can result in the similarity of TTF distributions under quite reasonable restrictions. As an example, consider the Barlow and Proschan model resulting in an aging TTF distribution. They consider an item subjected to shocks appearing randomly in time, is considered. Let these shocks arrive according to the Poisson process with constant intensity  $\lambda$ . Each shock cause a random amount  $x_i$  of damage, where  $x_1, x_2, \dots$  are independently distributed with a common cumulative distribution function,  $F(x)$ , (damage distribution function). The item fails when accumulated damage exceeds a threshold  $X$ . The TTF CDF is given by

$$H(t) = \sum_{k=0}^{\infty} e^{-\lambda t} F^{(k)}(X) \frac{(\lambda t)^k}{k!}$$

where  $F^{(k)}$  is  $k$  times convolution of  $F(x)$ . It was shown by Barlow and Proschan that for *any* damage distribution function  $F(x)$ , the TTF distribution function has increasing failure rate on the average (IFRA).

Now consider an item under the stress conditions characterized by different stress intensities  $\lambda_i$  and different damage distribution functions  $F_i(x)$ . It can be also shown that the similarity of the corresponding TTF distribution functions,  $H_i(t)$ , (or the failure mechanism conservation (Equation (5.3))) will hold for all these stress conditions ( $\lambda_i, F_i(x)$ ), if they have the same damage CDF,  $F_i(x) = F(x)$ ,  $i = 1, 2, \dots$ . A similar example from fracture mechanics is

considered in [Crowder et al, 1991].

### 6.2.6 Proportional Hazards Model

For the PH model the basic relationship analogous to (5.3) is given by

$$F(t; z) = 1 - [1 - F_0(t)]^{\psi(z, A)}, \quad (5.9)$$

or, in terms of reliability function, R, as

$$R(t; z) = R_0(t)^{\psi(z, A)} \quad (5.9')$$

The proper proportional hazards model is known as the relationship for hazard rate, which can be obtained from (5.9) as

$$\lambda(t; z) = \psi(z, A) \lambda^0(t), \quad (5.10)$$

where  $\psi(z, A)$  is usually chosen as a log-linear function.

The PH model does not normally retain the shape of the CDF, and the function  $\psi(z)$  no longer has a simple relationship to the acceleration function, nor a clear physical sense. That is why the PH model is not as popular in reliability applications as the AL model.

Nevertheless it can be shown [Cox, and Oaks, 1984] that for the Weibull distribution (and only for the Weibull distribution) the PH model coincides with the AL model.

It should be noted that the AL model time transformation is more popular for reliability applications, while the PH model is widely used in biomedical life data analysis.

### 6.2.7 Some Popular AL (Reliability) Models for Percentiles

Most commonly used models for the percentiles (including median) are log-linear models. Two such models are the Power Rule Model and the Arrhenius Reaction Model. The Power Rule Model for 100 $p$ th percentile is given by:

$$t_p(x) = \frac{a}{x^c}, \quad a > 0, c > 0, x > 0, \quad (5.11)$$

where  $x$  is a mechanical or electrical stress,  $a$  and  $c$  are constants. In reliability of electrical insulation and capacitors,  $x$  is, usually, applied voltage. In estimating fatigue life the model is used as the analytical representation of the  $S$ - $N$  or *Wöhler curve*, where  $S$  is a stress amplitude and  $N$  is life in cycles to failure, i.e., in the form:

$$N = k S^{-b}, \quad (5.11')$$

where  $b$  and  $k$  are material parameters estimated from test data. Because of probabilistic nature of fatigue life at any given stress level, one has to deal with not one  $S-N$  curve, but with a family of  $S-N$  curves, so that each curve is related to a probability of failure as a parameter. These curves are called  $S-N-P$  curves, or curves of constant probability of failure on a stress-versus life plot. It should be noted that relationship (5.11') is only empirical. [Sobczyk and Spencer, 1992].

The Arrhenius Reaction Rate Model is the following relationship between life and absolute temperature,  $T$ :

$$t_p(T) = a \exp\left(\frac{E_a}{T}\right), \quad (5.12)$$

where  $E_a$  is activation energy. This model is the most widely used one to examine the effect of temperature on reliability. Originally, it was introduced as a chemical reaction rate model.

The model combining the above models is given by

$$t_p(x, T) = a x^{-c} \exp\left(\frac{E_a}{T}\right), \quad (5.13)$$

In the fracture mechanics of polymers this model is known as Bruhanova's and Bartenev's model. It is also used as a model for the electromigration failures in aluminum thin films of integrated circuits; in this case  $x$  is current density.

Another popular AL reliability model is Jurkov's model:

$$t_p(x, T) = t_0 \exp\left(\frac{E_a - \gamma x}{T}\right) \quad (5.14)$$

This model is considered as an empirical relationship reflecting the thermal fluctuation character of long-term strength [Regel, et. al., 1974; Goldman, 1994]. For mechanical long-term strength, parameter  $t_0$  is a constant which is numerically close to the period of thermal atomic oscillations ( $10^{-11}$  -  $10^{-13}$  s),  $E_a$  is the effective activation energy numerically close to vaporization energy for metals and to chemical bond energies for polymers, and  $\gamma$  is a structural coefficient. The model is widely used for reliability problems of mechanical and electrical long-term strength.

The *a priori* choice of a model (or some competing models) is being made using physical considerations. Meanwhile, statistical data analysis of ALT results and failure mode and effects analysis (FMEA) afford to check the adequacy of the model chosen, or to discriminate the best model among the competing ones.

## 6.2.8 Accelerated Life Model for Time-dependent Stress

Models considered in the previous section are for stress constant in time. The case of time-dependent stress is not only more general, but also of more practical importance because of applications in engineering reliability are not limited to ALT problems. As an example, consider the time-dependent stress analog of the model (5.11'). The stress amplitude,  $S$ , experienced by a structural element often varies during its service life, so that the straightforward use of Equation (5.11') is not possible. In such situations the so-called Palmgren-Miner rule is widely used to estimate the fatigue life. The rule treats the fatigue fracture as a result of a *linear accumulation* of partial fatigue damage fractions. According to the rule, the damage fraction,  $\Delta_i$ , at any stress level  $S_i$  is proportional to the ratio  $\frac{n_i}{N_i}$ , where  $n_i$  is the number of cycles of operation under stress level  $S_i$ , and  $N_i$  is the total number of cycles to failure (life) under the *constant* stress level  $S_i$ , i.e.,

$$\Delta_i = \frac{n_i}{N_i}, \quad n_i \leq N_i.$$

Total accumulated damage,  $D$ , under different stress levels  $S_i$  ( $i=1, 2, \dots$ ) is defined as

$$D = \sum_i \Delta_i = \sum_i \frac{n_i}{N_i}$$

It is assumed that failure occurs if  $D \geq 1$ .

Accelerated life tests with time dependent stress such as step-stress and ramp tests are also of a great importance. For example, one of the most common reliability tests of thin silicon dioxide films in metal-oxide-semiconductor integrated circuits is the so-called ramp-voltage test. In this test, the oxide film is stressed to a breakdown by a voltage which increases linearly with time [Chan, 1990].

Let  $z(t)$  be a time-dependent stress vector such that  $z(t)$  is integrable. In this case, the relationship (5.3) can be written in the form:

$$F\{t; [z(\cdot)]\} = F_0[\Psi(t)] \quad (5.15)$$

where

$$\Psi(t^{(z)}) = \int_0^{t^{(z)}} \psi[z(s), A] ds$$

and  $t^{(z)}$  is the time related to an item under the stress condition  $z(t)$ .

The relationship (5.15) was given by D. R. Cox [Cox, 1984]. Based on this relationship



Cox also obtained the analogous relationships for the probability density function and hazard rate function.

The corresponding relationship for the 100 $p$ th percentile of time to failure  $t_p[z(t)]$  can be obtained from (5.15) as

$$t_p^0 = \int_0^{t_p[z(t)]} \psi[z(s), A] ds.$$

Using (5.6) and (5.7) the last relationship can be rewritten as

$$1 = \int_0^{t_p[z(t)]} \frac{1}{t_p^0 \{\psi[z(s), A]\}^{-1}} ds \equiv \int_0^{t_p[z(t)]} \frac{1}{\eta[z(s), B]} ds \quad (5.16)$$

### 6.2.9 AL Reliability Model for Time-dependent Stress and Miner's Rule

It should be noted that relationship (5.16) is an exact nonparametric probabilistic continuous form of the Palmgren-Miner rule mentioned in the previous section. So, the problem of using AL tests with time-dependent stress is identical to the problem of cumulative damage addressed by the Palmgren-Miner rule. Moreover, there exists a useful analogy between mechanical damage accumulation and electrical breakdown; for example, Jurkov's model is used as the relationship for mechanical as well as for long-term electrical strength.

In the theory of cumulative damage, a certain *damage measure*  $D(t)$  is introduced ( $0 \leq D(t) \leq 1$ ).

Assuming that  $D(t)$  depends on its value at some initial time  $t_0$  and on an *external action*,  $Q(t)$ , the following general equation for  $D(t)$  is being postulated [Sobczyk and Spencer, 1992]:

$$\frac{dD(t)}{dt} = f[D(t), Q(t)] \quad (5.17)$$

where  $f(D, Q)$  is a nonnegative function which satisfies the conditions ensuring the existence and uniqueness of the solution of Equation (5.17). The equation is regarded as a *kinetic equation* for damage evaluation [Bolotin, 1989].

If the right side of Equation (5.17) is independent of  $D(t)$ , the solution of the equation with the initial condition  $D(0) = 0$  is the *linear damage accumulation* model given by

$$D(t) = \int_0^t f[Q(\tau)]d\tau \quad (5.18)$$

The time  $T$  at which the damage reaches its critical value corresponds to the condition  $D(t) = 1$ . Using the notation  $t(q) = \frac{1}{f(Q)}$ , one gets

$$\int_0^T \frac{d\tau}{t[Q(\tau)]} = 1 \quad (5.19)$$

Equation (5.19) formally coincides with Equation (5.16), so it is clear that AL Model is the linear damage accumulation model. Nevertheless Equation (5.19) is deterministic and, from engineering point of view, it is also not clear how to measure or estimate the *external action*,  $Q(t)$ , what the function  $f[Q(t)]$  is, and how to validate the correctness of Equation (5.19).

On the contrary, Equation (5.16) depicting the general case of time-dependent stress accelerated life model, is expressed in terms of quantiles of time to failure and usual (constant stress) AL reliability models. The correctness of the equation can be tested using the statistical procedures considered in Section 5.2.

There could be two main kinds of application of Equation (5.16): fitting an AL reliability model (estimating the vector of parameters,  $B$ , of percentile reliability model,  $\eta(z, B)$ , on the basis of AL tests with time-dependent stress), and reliability (percentiles of time to failure) estimation (when the model is known) for the given time-dependent stress, in the stress domain, where the similarity of time to failure distributions exists (conservation of failure mechanisms holds).

### 6.3 Accelerated Life Test Data Analysis

#### 6.3.1 Exploratory Data Analysis (Criteria of Linearity of Time Transformation Function for Constant Stress)

The possibility to verify the correctness of relationship (5.3) experimentally is not only important for failure mechanism study but it has a great practical importance, because almost all the statistical procedures for AL test planning and data analysis (for both the constant and time dependent stress) are based on the assumption (5.3).

Several techniques can be used for verification of the linearity of the time transformation function or the verification of the PM rule. Let's start with the historically first criterion which shows similarity of TTF distribution functions and the linear damage accumulation model. This criterion requires two special tests [Gugushvili et al., 1975]. During the first test a sample is tested at  $z_1$  constant stress level for a time  $t_1$  at which  $z_1$  is changed to a constant stress  $z_2$  for a

time  $t_2$ . During the second test another sample is first tested under  $z_2$  for  $t_2$  and then it is tested under the stress level  $z_1$  for time  $t_1$ . The time transformation function will be linear in time, if the reliability functions of the items after the first and the second test are equal, i.e., a change of loading order does not change the cumulative damage. We will come back to this criterion while considering exploratory data analysis for the case of time-dependent stress, because this test is based on the time-dependent loading.

The second criterion is associated with the variation coefficient (i.e., standard deviation to average ratio,  $s/m$ ) of TTF. It is easy to show that if the time transformation function is linear for the constant stress levels  $z_1, z_2, \dots, z_k$ , the variation coefficient of TTF will be the same for all these stress levels. Let  $x$  be TTF under stress condition  $z_i$  and  $y$  be TTF under stress condition  $z_j$ . Under the basic AL model assumption (5.3) these random variables are related to each other as  $y = kx$ , where  $k$  is a constant. The corresponding mean values,  $E(x)$  and  $E(y)$  are related to each other as  $E(y) = kE(x)$ . The analogous relationship for the standard deviations,  $s(x)$  and  $s(y)$  is, obviously:  $s(y) = ks(x)$ , so that the variation coefficients,  $\frac{s(y)}{E(y)}$  and  $\frac{s(x)}{E(x)}$  are equal.

Thus, the analysis of the variation coefficient not only provides information about aging of the TTF distribution but it helps one to understand if the conservation of failure mechanisms holds true. (Recall that  $s/m < 1$  for IFR and IFRA TTF distributions,  $s/m = 1$  for the exponential distribution, and  $s/m > 1$  for DFR and DFRA distributions). Analogously, it can be shown that under the same assumption, the variance of the logarithm of times to failure will be the same for these stress levels. Consider the times to failure  $x$  (under stress condition  $z_i$ ) and  $y$  (under stress condition  $z_j$ ). Under AL model assumption (5.3), the logarithms of these random variables are related to each other as

$$\log y = \log x + \log k,$$

where  $k$  is the same constant. Taking variances of both sides of the equation above, one gets

$$\text{Var}(\log y) = \text{Var}(\log x).$$

If TTF is lognormal, the Bartlett's and Cochran's tests can be used for checking if the variances are constant.

The third criterion is based on the use of quantile-quantile plots. The quantile-quantile plot is a curve, such that the coordinates of every point are the TTF quantiles (percentiles) for investigated pairs of stress conditions. If the time transformation function is linear in time (i.e., (5.3) holds), the quantile-quantile plot will be a straight line going through the origin. The corresponding data analysis is realized in the following way. All sample quantiles of a given constant stress condition are plotted on one axis and the sample quantiles of another stress condition are plotted on the other axis. Using the points obtained (a pair of quantiles of the same level gives a point), the straight regression line can be fitted. The time transformation function will be considered as linear, provided one gets linear dependence between the sample quantiles, and if the hypothesis that the intercept of the fitted line is equal to zero, is not rejected.

### **Example 5.1** (*Fatigue Life Data*)

Consider the Birnbaum-Saunders et al. data [Bogdanoff and Kozin, 1985]. "The test specimens

were 6061-T6 aluminum strips, 0.061 in. thick, 4.5 in. long, and 0.5 in. wide. The specimens were cut parallel to the direction of rolling of the sheet stock. The specimens were mounted in simple supported bearings and deflected at the center with a Teflon clamp in reverse bending. The center was deflected 18 times per second and three stress amplitudes were used; these amplitudes were 21, 26, and 31 Kpsi. There were 101 specimens at 21 Kpsi, 102 specimens at 26 Kpsi, and 101 specimens at 31 Kpsi. Specimens were tested to failure. Life in  $10^3$  units were recorded." The sample means,  $m$ , the sample variations,  $s^2$ , and the sample variation coefficients,  $s/m$ , are summarized in Table 5.1.

Table 5.1

First of all, note that all the sample variation coefficients are less than 1, so the time to failure distributions for all the stress levels might be considered as aging (IFR or IFRA). The variation coefficients for the amplitude 26 and 31 Kpsi are approximately equal, so these stress conditions might be considered as leading to the same failure mechanisms. This conclusion can be supported by quantile-quantile plot analysis: all the quantile-quantile plots show strong linear dependence (all the sample correlation coefficients are about 0.99), but zero (insignificant) intercept gives the quantile-quantile plot for the amplitudes 26 and 31 Kpsi only. Thus the TTF distribution functions for these amplitudes are similar (AL model is applicable) and the failure mechanism conservation takes place in the amplitude range (26 - 31) Kpsi. ■

### 6.3.2 Statistical Methods of Reliability Prediction on the Basis of AL Tests with Constant Stress

Statistical methods of reliability prediction on the basis of AL tests can be divided into parametric and nonparametric ones. In the first case TTF distribution is related to a particular parametric distribution - Normal, Exponential, Weibull; in the second case the only assumption is the assumption about a particular class of TTF distribution - continuous, IFR, IFRA.

The most commonly used parametric methods are the parametric regression (Normal and Lognormal, Exponential, Weibull and Extreme Value), Least Square Method and Maximum Likelihood Methods [Lawless, 1982; Nelson, 1990; Leemis, 1995], which are briefly discussed below. We also consider the nonparametric regression procedure for the percentile AL reliability model fitting for the cases of constant stress and time-dependent stress [Kaminskiy, 1994].

#### Maximum Likelihood Approach

Consider the ML approach to statistical analysis of AL models based on the Weibull TTF distribution as a typical example.

The reliability function for the Weibull distribution is given by:

$$R(t) = \exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right] \quad (5.20)$$

where  $\alpha$  is a scale parameter and  $\beta$  is a shape parameter.

Under the AL model assumption, the scale parameter is considered as a function of stress factors,  $z$ . Consider the case when this function (reliability model) is log-linear one, i.e.:

$$\log \alpha(Z, B) = ZB \quad (5.21)$$

where  $Z = (z_0, z_1, \dots, z_p)$  is a vector of stress factors, and  $B = (b_0, b_1, \dots, b_p)^T$  is a vector of model parameters,  $z_0 \equiv 1$ .

For the following discussion it is better to deal with logarithm of TTF. Denote  $y = \log t$ . It is easy to show that  $y$  has the Type I (Gumbel) Extreme value distribution for minimum. The PDF of the distribution is given by:

$$f(y) = \frac{1}{\sigma} \exp \left[ \frac{y - \log \alpha}{1/\beta} - \exp \left( \frac{y - \log \alpha}{1/\beta} \right) \right], \quad (5.22)$$

$$-\infty < y < \infty.$$

Using the AL model in the form of Equation (5.3'), the stress-dependent PDF can be written as

$$f(y-z) = \frac{1}{\sigma} \exp \left[ \frac{y-ZB}{\sigma} - \exp \left( \frac{y-ZB}{\sigma} \right) \right], \quad (5.23)$$

$$\sigma = \frac{1}{\beta}$$

Let  $y_i$  be either a logarithm of time to failure or a logarithm of censoring time associated with a stress condition  $Z_i$ . Denote the sets of observations for which  $y_i$  is a logarithm of TTF and a logarithm of right censoring time by  $U$  and  $C$  respectively. The likelihood function for the given observations and PDF (5.23) is [Lawless, 1982 ]:

$$L(B, \sigma) = \prod_{i \in U} \frac{1}{\sigma} \exp \left[ \frac{y_i - Z_i B}{\sigma} - \exp \left( \frac{y_i - Z_i B}{\sigma} \right) \right] \prod_{i \in C} \exp \left[ - \exp \left( \frac{y_i - Z_i B}{\sigma} \right) \right] \quad (5.24)$$

$$\log L(B, \sigma) = -r \log \sigma + \sum_{i \in U} \frac{y_i - Z_i B}{\sigma} - \sum_{i=1}^n \exp \left( \frac{y_i - Z_i B}{\sigma} \right)$$

The maximum likelihood equations are

$$\frac{\partial \log L}{\partial b_l} = - \frac{1}{\sigma} \sum_{i \in U} z_{il} + \frac{1}{\sigma} \sum_{i=1}^n z_{il} e^{x_i}, \quad l = 0, 1, \dots, p \quad (5.25)$$

$$\frac{\partial \log L}{\partial \sigma} = - \frac{r}{\sigma} - \frac{1}{\sigma} \sum_{i \in U} x_i + \frac{1}{\sigma} \sum_{i=1}^n x_i e^{x_i}; \quad x_i = \frac{1}{\sigma} (y_i - z_i B)$$

The likelihood function can also be used for discriminating between two competing models [Crowder, et al., 1991]. Let us have a model  $M_1$  with  $p_1$  parameters and a model  $M_2$  with  $p_2$  parameters, and let  $p_2 > p_1$ . Let  $L_1$  and  $L_2$  be the maximized value of  $\log L$  for the models  $M_1$  and  $M_2$  respectively. The *likelihood ratio* statistic  $W = 2(\log L_1 - \log L_2)$  has an approximate  $\chi^2$  distribution with  $(p_2 - p_1)$  degrees of freedom. Large values of  $W$  provide evidence against the null hypothesis that both models supply the same goodness of fit.

To get the observed variance-covariance matrix (or the observed information matrix) after solving the system (5.25), one needs the second derivatives of the loglikelihood function considered, which are given by

$$\begin{aligned} \frac{\partial^2 \log L}{\partial b_l \partial b_l} &= -\frac{1}{\sigma^2} \sum_{i \in U} z_{il} z_{is} e^{x_i} \quad l, s = 0, 1, \dots, p \\ \frac{\partial^2 \log L}{\partial \sigma^2} &= \frac{r}{\sigma^2} - \frac{2}{\sigma^2} \sum_{i \in U} x_i - \frac{2}{\sigma^2} \sum_{i=1}^n x_i e^{x_i} - \frac{1}{\sigma^2} \sum_{i=1}^n x_i^2 e^{x_i} \quad (5.26) \\ \frac{\partial^2 \log L}{\partial B_l \partial \sigma} &= \frac{1}{\sigma^2} \sum_{i \in U} z_{il} \frac{1}{\sigma^2} \sum_{i=1}^n z_{il} e^{x_i} - \frac{1}{\sigma^2} \sum_{i=1}^n z_{il} x_i e^{x_i} \end{aligned}$$

The system (5.25) can only be solved numerically. The Newton-Raphson method is usually recommended. It is not as simple a problem as it is sometimes stated [Lawless, 1982]. Moreover, using different software realizations of the same optimization method, one usually gets different solutions. In the following section the least square method as a shortcut procedure for AL data analysis is discussed. This method is not as effective as the maximum likelihood one, but it is robust in the sense that it is not associated with numerical optimization, so using different software realizations one always gets the same results. Moreover, even if a good software tool for the maximum likelihood method is available, it is very important to have good starting values; the least squares estimates can be used as such starting values.

### Least Squares Estimation

The relationship (5.5) from Section 5.1.4 can be written in terms of random variables as

$$T = \frac{T_0}{\psi(z)}$$

where  $T_0$  has TTF CDF  $F_0(\cdot)$ . Denote the expectation of  $\log T_0$  by  $\mu_0$ . Using the equation above one can write

$$\log T = \mu_0 - \log \psi(z) + \varepsilon \quad (5.27)$$

where  $\varepsilon$  is a random error of zero mean with a distribution not depending on  $z$ .

If  $\log \psi(z)$  is, again, a linear function (the case of loglinear reliability model), i.e.,

$$\log \psi(Z, B) = ZB \quad (5.28)$$

Equation (5.27) can be written as

$$\log T = \mu_0 - ZB + \varepsilon \quad (5.29)$$

which is linear model satisfying the conditions of Gauss-Markov Theorem.

When TTF samples are uncensored, the regression equation for observations  $T_i, Z_i (i = 1, 2, \dots, n)$  is

$$\log T_i = \mu_0 - Z_i B + \varepsilon_i \quad (5.30)$$

where for any TTF distribution  $e_i (i = 1, 2, \dots, n)$  are independent and identically distributed with an unknown variance and known distribution (if the distribution of TTF is known). Thus, on one hand, the Least Squares technique for AL data analysis can be used as a nonparametric one, on the other hand, if TTF distribution is known, one can try a parametric approach. The Lognormal TTF distribution is an example of the last case, which is reduced to standard normal regression, which makes clear the popularity of the Lognormal distribution in AL practice.

**Example 5.2.** (Class-H Insulation Data [ Nelson, 1990])

The data are hours to failure of 40 units which were divided in four samples of equal size (10). The samples were tested at 190, 220, 240, and 260°C. The test purpose was to estimate the median life at the design temperature 180°C. The test results are given in the following table.

Table 5.2

In Table 5.3 some statistics useful for the exploratory data analysis are summarized.

Table 5.3.

It should be noted that the data considered are not distinct times to failure but grouped (inspection) data, nevertheless, the test results under 260 °C look suspicious. If the data were

treated as having the lognormal distribution the Bartlett's test would immediately show that this stress level (260 °C) results in significantly greater variance compared with other stress levels. Analysis of failure modes for this insulation shows that failure mode at 260 °C is different compared with failure modes under lower temperatures [Nelson, 1990].

The Arrhenius model for the logarithms of time to failure can be easily fitted using any software tool having linear least squares or regression procedure. Using the four temperature levels, the fitting results in the following estimates of the model parameters:  $a = 6.90 \cdot 10^{-4}$  hours, and the activation energy  $E_a = 7531^\circ \text{K} \gg 0.65 \text{eV}$ , which give the prediction for median life at the design temperature  $180^\circ \text{C}$ ,  $t_{50\%}(180^\circ \text{C}) \gg 11500$  hours. The proportion of the variance of logarithm of TTF explained by the model (adjusted  $R^2$ ) is 91%. Deleting the sample obtained under  $260^\circ \text{C}$  (which is quite reasonable) results in the following estimates of the model parameters:  $a = 1.50 \cdot 10^{-4}$  hours, and the activation energy  $E_a = 8260^\circ \text{K} \gg 0.71 \text{eV}$ , which give a little more optimistic prediction for the median life at the design temperature  $180^\circ \text{C}$ ,  $t_{50\%}(180^\circ \text{C}) \gg 12500$  hours and better proportion of the variance explained by the model (adjusted  $R^2$ ) is 95%. ■

### **Simple Percentile Regression**

The procedure is based on the following formal assumptions:

1. AL time transformation (5.3) is true and the AL reliability model,  $\eta(z, B)$ , for a quantile,  $t_p$ , (100 $p$ th percentile) is a given function of the stress factors  $z$  with an unknown vector of parameters,  $B$ , (Equation (5.6)).
2. The TTF distributions for all the stress conditions  $z_i$  ( $i = 1, \dots, k$ ) are IFRA distributions having continuous density functions  $f(t; z_i)$ .
3. The test results are Type II right censored samples, with the number of uncensored failure times  $r_i$  ( $i = 1, \dots, k$ ) and the sample sizes  $n_i$  are large enough to estimate the  $t_p$  as the sample percentile  $\hat{t}_p$  :

$$\hat{t}_p = \begin{cases} t_{(\lfloor n_i p \rfloor)} & \text{if } n_i p \text{ is not integer} \\ \text{any value from interval } [t_{(n_i p)}, t_{(n_i p + 1)}] & \text{if } n_i p \text{ is integer} \end{cases}$$

where  $t_{(x)}$  is the failure time (order statistic), and  $\lfloor x \rfloor$  means *the greatest integer which does not exceed x*.

4. The sample sizes are large enough that the asymptotic normal distribution of the above estimate can be used. This normal distribution has the mean equal to  $t_p$ , and the variance equal to  $\frac{p(1-p)}{nf^2(t_p)}$ , where  $n$  is the sample size.

The goal is to estimate vector  $B$  of the parameters of model (5.6) and to predict the percentile at the normal (or any given) stress condition on the basis of AL tests at different constant stress conditions  $z_1, \dots, z_k$ , where  $k$  is greater than the dimension of the vector  $B$ , i.e. ( $k > \dim B$ ).



Based on the preceding assumptions, the statistical model corresponding to Equation (5.6) can be written in a typical regression form as

$$\hat{t}_p(z_i, B) = \eta(z_i, B) + \varepsilon_i, \quad (5.31)$$

where  $\varepsilon_i$  are normally distributed with mean 0 and variance  $\frac{p(1-p)}{n_i f^2(t_p)}$ , i.e., using standard

notation  $N\left(0, \frac{p(1-p)}{n_i f^2(t_p)}\right)$ . Note that the distribution of error,  $\varepsilon$ , is dependent (through TTF probability density function,  $f(t_p)$ ) on the particular TTF distribution, i.e., the model is not distribution free.

Rewrite the model (5.31) in the following multiplicative form

$$\hat{t}_p(z_i, B) = \eta(z_i, B) \cdot \varepsilon, \quad (5.32)$$

where  $\varepsilon$  has distribution

$$N\left(1, \frac{p(1-p)}{n_i \eta^2(z_i, B) f^2[\eta(z_i, B)]}\right)$$

Now we try to transform the multiplicative model (5.32) to the model with normally distributed additive error, i.e., to the standard normal regression.

Taking the logarithm<sup>2</sup>, model (5.21) can be written as

$$\log \hat{t}_p(z_i, B) = \log \eta(z_i, B) + \log(1 + \varepsilon_i), \quad (5.33)$$

where  $\varepsilon_1$  has distribution

$$N\left(0, \frac{p(1-p)}{n_i \eta^2(z_i, B) f^2[\eta(z_i, B)]}\right) \quad (5.34)$$

Using AL model relationships for the probability density function (5.3') and for percentiles (5.5), it is easy to show that  $\eta(z, B) f[\eta(z, B)]$  is a constant, so, if the sample sizes  $n_i$  ( $i = 1, 2, \dots, k$ ) are

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<sup>2</sup>Percentile reliability models,  $\eta(z, B)$ , usually are not linear, but loglinear.

equal, the variances in (5.34) are also equal.

To avoid the distribution dependence on the variance in (5.34), let's find the nonparametric nonrandom lower bound for  $f(t_p)$ . The bound is given by the following

**Corollary.** If  $F(t)$  is IFRA cumulative distribution function with probability density  $f(t)$  and  $F(t_p) = p$ , then

$$f(t_p) \geq -\frac{(1-p)\log(1-p)}{t_p}$$

*Proof.* Consider the function

$$G(t) = 1 - \exp(\alpha t)$$

where

$$\alpha = \frac{-\log(1-p)}{t_p}$$

By Theorem 5.3, Vol.1, [Barlow and Proschan, 1975] the function  $F(t) - G(t)$  has not more than one change of sign. If this change occurs, plus will be changed for minus. Since  $F(t)$  and  $G(t)$  have the same percentile  $t_p$ , this change of sign can occur at the point  $t_p$  only. The result follows. Now demand the satisfying of the inequality  $\varepsilon_1 \ll 1$ . The inequality will be satisfied, if the standard deviation of  $\varepsilon_1$  is much less than unity, i.e., for each  $n_i$  ( $i = 1, 2, \dots, k$ )

$$\left( \frac{p}{n_i(1-p)\log^2(1-p)} \right)^{\frac{1}{2}} \ll 1 \quad (5.35)$$

In this case model (5.32) can be written as

$$\log \hat{t}_p(z, B) = \log \eta(z, B) + \varepsilon_2 \quad (5.36)$$

where  $\varepsilon_2$  has distribution  $N(0, \sigma^2)$  and  $\sigma^2$  is unknown constant, if the sample sizes  $n_i$  ( $i = 1, 2, \dots, k$ ) are equal, otherwise the weights proportional to the values of  $n_i$  should be used.

Thus the multiplicative model (5.32) can be transformed to the model with additive normally distributed error (i.e., to the standard normal regression model), if inequality (5.35) is satisfied. This inequality is the restriction superimposed on the test sample size. As far as the procedure considered is already based on the asymptotic properties of sample percentile

distribution this restriction should be easily satisfied. So, the problem of the reliability prediction is reduced to the estimation of the parameters of a normal regression followed by point and interval predicting, ranging the stress factors according to their influence on reliability using stepwise regression methods, etc. Using this approach the standard regression experiment design can be applied to AL test planning.

**Example 5.3. (Capacitor Breakdown Data)**

Certain capacitors were tested by using voltage in conjunction with temperature as accelerating stress factors. One hundred capacitors were tested at each voltage-temperature combination and each test was terminated after not less than 11 failures had been observed. The purpose of the test was to predict the 10th percentile of TTF distribution under the stress condition 63 V and 100°C. The test plan and results are given in Table 5.4.

Table 5.4

Model (5.14) was used as the relationship between the 10th percentile and the stress factors (voltage and temperature) The estimation of the model parameters resulted in the following relationship:

$$t_p(V,T) = a \exp\left(\frac{E_a - BV}{T}\right)$$

where  $a = 7.735 \cdot 10^{-15}$  1/hour;  $E_a = 16099$  °K (or 1.39eV);  $B = 4.6256$  °K/Volt;  $V$  is voltage in Volts and  $T$  is temperature in °K.

The results of prediction obtained on basis of this model are given in the right column of Table 4. The multiple correlation coefficient for this model is greater than 0.99. To estimate the influence of the voltage on the capacitor's life, the Arrhenius model was fitted for the same data set. The correlation coefficient for the fitted Arrhenius model turned out to be 0.633 only. So, it is easy to conclude that both factors, the temperature and the voltage, are significant. The same conclusion could be drawn using F-ratio for residual variances for these models. This example is a simple illustration of step-wise regression idea. ■

### 6.3.3 Exploratory Data Analysis for Time-Dependent Stress

The first criterion considered in Section 5.2.1 is the criterion for the particular time-dependent stress. In general case, the value of the integral in Equation (5.16) does not change, when a stress history  $z(s), t_p \geq s \geq 0$ , has been changed for  $z(t_p - s), t_p \geq s \geq 0$ ; it means that time is reversible under AL Model. Based on this property, it is not very difficult to realize the verification of the AL Model. For example, each sample which is going to be tested under time-dependent stress can be divided in two equal parts, so that the first subsample could be tested under forward stress history, meanwhile the second subsample is tested under the backward stress history.

### 6.3.4 Statistical Estimation of AL Reliability Models on the Basis of AL Tests with Time-dependent Stress

Using Equation (5.16) the time-dependent analog of the model (5.32) can be written as

$$t_p^0 = \int_0^{\hat{t}_p[z(t)]} \psi[z(s), A] ds \quad (5.37)$$

where  $\hat{t}_p[z(t)]$  is the sample percentile for an item under the stress condition (loading history)  $z(t)$ .

The problem of estimating the vector  $A$  and  $t_p^0$  in this case cannot be reduced to parameter estimation for a standard regression model as in the previous case of constant stress.

Consider  $k$  different time dependent stress conditions (loading histories)  $z_i(t)$ ,  $i = 1, 2, \dots, k$ , ( $k > (\dim A) + 1$ ), under which the test results are (as in the previous case) Type II censored samples and the number of uncensored failure times and the sample sizes are large enough to estimate the  $t_p$  as the sample percentile  $\hat{t}_p$ . In this situation the parameter estimates for the AL reliability model (of the vector  $A$  and  $t_p^0$ ) can be obtained using a least square method solution of the following system of integral equations:

$$t_p^0 = \int_0^{\hat{t}_p[z_i(t)]} \psi[z_i(s), A] ds, \quad (5.38)$$

$i = 1, 2, \dots, k.$

#### Example 5.4.

Assume a model (5.13) for the 10th percentile of time to failure  $t_{0.1}$  of a ceramic capacitor in the form

$$t_{0.1}(U, T) = a U^{-c} \exp\left(\frac{E_a}{T}\right), \quad (5.39)$$

where  $U$  is applied voltage and  $T$  is absolute temperature.

Consider a time-step-stress AL test plan using step-stress voltage in conjunction with constant temperature as accelerating stress factors. A test sample starts at a specified low voltage  $U_0$  and it is tested for a specified time  $\Delta t$ . Then the voltage is increased by  $\Delta U$ , and the sample is tested at  $U_0 + \Delta U$  during  $\Delta t$  etc., i.e.,

$$U(t) = U_0 + \Delta U n(t / \Delta t)$$

where  $En(x)$  means "nearest integer not greater than  $x$ ". The test will be terminated after the portion  $p \geq 0.1$  of items fails. So, the test results are the sample percentiles at each voltage-temperature combination. The test plan and simulated results with  $\Delta U = 10V$ ,  $\Delta t = 24h$  are given in Table 5.5.

Table 5.5

For the example considered the system of integral equations (5.38) takes the form:

$$a = \int_0^{t_{0,i}} \exp\left(-\frac{E_a}{T_i}\right) [U(s_i)]^c ds,$$

$$i = 1,2,3,4.$$

Solving this system for the data above yields the following estimates for the model (5.39):  $a = 2.23 \cdot 10^{-8} hV^{1.88}$ ,  $E_a = 1,32 \cdot 10^4 K$ ,  $c = 1.88$ , which are not bad compared with the following values of the parameters used for simulating the data:  $a = 2.43 \cdot 10^{-8} h/V^{1.87}$ ,  $E_a = 1,32 \cdot 10^4 K$ ,  $c = 1.87$ . ■

**Part 2:**  
**System Reliability Estimation**

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**7. TESTING WITH NO FAILURES**

**7.1 Introduction**

One of the main problems in reliability engineering is estimating the system's reliability in early phases of design. At this stage a designer only knows the results of reliability tests on separate parts and components of the system. This problem is also very crucial if a designer deals with a continuously developing system such as telecommunication or power networks. Its parts (subsystems, interfaces, terminals, etc.) are operating within an existing system and reliability data are collected. New items can be specially tested in advance. Thus different statistical data concerning units are available. The problem is to figure out how the system reliability will change if the system changes its configuration or incorporates some new units.

This chapter is dedicated to the investigation of highly reliable systems or equipment. In this case, one knows the total testing time for each of the system's parts and that there were no failures for any of the considered system's part, or unit. A significant factor in this case is the possibility of obtaining best lower confidence limits for a wide class of complex structures.

**7.2 Series System**

One of the most commonly used and theoretically well investigated test plans is the *binomial test plan*. This plan is described in the following way. Assume that a system has  $m$

different units and  $p_i$  is the probability of failure-free operation (PFFO) of unit  $i$  during the specified time period,  $t$ . A system reliability index  $R$  can be defined as a function  $R=R(\mathbf{p})$ , which depends on unit reliability values  $p_i$ , where  $\mathbf{p}=(p_1, \dots, p_m)$ . Values of  $p_i$ 's are *a priori* unknown, but we know the results of tests:  $N_i$  units of type  $i$  were tested, and  $d_i$  failures were observed. We are interested in the estimation of an unknown index  $R$  by test results  $\mathbf{d}=(d_1, \dots, d_m)$ .

A value  $\varphi=\varphi(\mathbf{d})$ , which depends on test results, is called a point unbiased estimate of  $R=R(\mathbf{p})$  if for all possible  $\mathbf{p}$

$$E_p\{\varphi\}=R(\mathbf{p})$$

where  $E_p\{\varphi\}$  is the mean of the estimate  $\varphi$  for a given  $\mathbf{p}$ . Usually, one uses the variance of an unbiased estimate of  $\varphi$

$$D_p\{\varphi\}=E_p\{[\varphi-R(\mathbf{p})]^2\}$$

as a measure of its effectiveness. The variance characterizes a deviation of the estimate around a real value of  $R$  (which is unknown).

One often uses confidence intervals also. Remember that an interval  $[\varphi(\mathbf{d}), \bar{\varphi}(\mathbf{d})]$  is called the confidence interval with the confidence level  $\square$  for  $R(\mathbf{p})$  if for all  $\mathbf{p}$

$$P_p\{\varphi(\mathbf{d}) \leq R(\mathbf{p}) \leq \bar{\varphi}(\mathbf{d})\} \geq \square.$$

Sometimes the main interest is only in one sided limit of the confidence interval which guarantees the value of the reliability index. So, for many systems, it is important to be sure that the PFFO is not lower than some specified level. Another example is the *coefficient of unavailability*, which is expected to be not larger than some given level. Value  $\varphi(\mathbf{d})$  is called a one-side lower  $\square$ -confidence limit for  $R(\mathbf{p})$ , if for all  $\mathbf{p}$

$$P_p\{\varphi(\mathbf{d}) \leq R(\mathbf{p})\} \geq \square.$$

In an analogous way, value  $\bar{\varphi}(\mathbf{d})$  is called a one-side upper  $\square$ -confidence limit for  $R(\mathbf{p})$ , if for all  $\mathbf{p}$

$$P_p\{R(\mathbf{p}) \leq \bar{\varphi}(\mathbf{d})\} \geq \square.$$

As a sample, let us consider one of the simplest system structures - a series connection of units (see Figure 6.1).

Figure 6.1

Such a system fails if any of its unit has failed, that is, if a system consists of independent units

$$R(\mathbf{p}) = \prod_{1 \leq i \leq m} p_i.$$

The most effective unbiased estimate for the PFFO of unit  $i$ ,  $p_i$ , is a value

$$\hat{p}_i = 1 - \frac{d_i}{N_i}.$$

An unbiased estimate with the minimal variance for the system PFFO is defined as

$$\hat{P} = \prod_{1 \leq i \leq m} \left(1 - \frac{d_i}{N_i}\right).$$

In practice, a system with highly reliable units are typical, and, consequently, numbers of failures,  $d_i$ , are small. This leads to large values of the variance of the point estimate, which means that the resulting point estimate is very unstable: values may significantly changed from test to test. Notice that a value of a point estimate has no information about a confidence of the obtained result. For instance, the same unbiased estimate will be obtained for two different cases: (a) 1 failure is observed per 10 tested units and (b) 10 failures is observed per 100 tested units. However, even a simple common sense tells us that the second case delivers more confidence in the results.

It leads to the necessity of characterization by confidence limits in addition to unbiased estimates. But, if constructing of unbiased estimates is a standard task, then the construction of confidence limits is more sophisticated.

In this chapter we are dealing with a no-failure-test case:  $d=0$ , i.e., where each  $d_i=0$ . At the same time, numbers of tested units of different types,  $N_i$ , are different.

Probably, the first strong solutions of this problem was obtained by Mirnyi and Solovyev (1964). They constructed the lower confidence limit for a series system (see Figure 6.1) for no-failure-tests. In this case, the lower confidence limit  $R^*$  with the confidence level  $\gamma$  for the system's PFFO is defined as

$$\underline{R} = \min_i \underline{p}_i \quad (6.1)$$

where

$$\underline{p}_i = (1 - \gamma)^{\frac{1}{N_i}}$$

is the standard lower  $\gamma$ -confident Clopper-Pearson's limit for a single unit  $i$  (see p.1 in Appendix to the chapter). It is clear that this produces

$$\underline{R} = (1 - \gamma)^{\frac{1}{N^*}}$$

where

$$N^* = \min_i N_i$$

is the minimal number of tested units.

At a first glance (6.1) seems paradoxical. For an explanation of this result on an intuitive level, let us imagine the following situation. We are testing  $N_i$  units of each of  $m$  types,  $1 \leq i \leq m$ , and no failure has occurred. Now, let us assemble series systems so that each system consists of units of different types. We are able to assemble only  $N^*$  complete series systems, that is, the number of such completed systems, consisting of required units, cannot exceed the minimum number of units. Imagine that we have tested these units assembled in system instead testing of separate units. Then none of these system would have failed during tests. (We suppose that assembling into a system

does not worsen units' reliability.) All of the remaining units, which cannot be assembled into a complete system, give us no additional information about a possible system behavior. Thus, the above described no-failure-test of  $N=(N_1, N_2, \dots, N_m)$  units is equivalent to the no-failure-test of  $N^*$  series systems. These arguments explain the Mirnyi-Solovyev's result.

Now consider the construction of a lower confidence limit of the PFFO for more complex systems when no unit failures are observed.

### 7.3 General Expression for Best Lower Confidence Limit

#### 7.3.1 Systems With a Monotone Structure

The state of unit  $i$  of a system at moment  $t, t > 0$ , can be described with the help of a Boolean variable  $X_i(t)$ , such that  $X_i(t)=0$ , if the unit is failed, and  $X_i(t)=1$  if the unit is operational. A system state can be described with a vector  $\mathbf{X}=(X_1, \dots, X_m)$ , which is defined in space  $\Omega$  consisting of  $2^m$  discrete points. For independent units, the probability that the system is in state  $\mathbf{X}$  is defined as

$$P\{\mathbf{X}\} = \prod_{1 \leq i \leq m} p_i^{x_i} (1 - p_i)^{1 - x_i}$$

where  $p_i = P\{X_i=1\} = E\{X_i\}$  is the  $i$ th unit's PFFO, and  $E\{X_i\}$  is the mean of  $X_i$ . Introduce now the structural function of a system  $\square(\mathbf{X})$ , such that  $\square(\mathbf{X})=1$ , if a system is operational, and  $\square(\mathbf{X})=0$  otherwise. Of course, such a description of a system is possible only if the system's failure criterion is strictly defined. In this case, all space  $\Omega$  can be divided into two disjoint subspaces  $G$  and  $\bar{G}$ , such that

$$\Psi(\mathbf{X}) = \begin{cases} 1 & \text{if } \mathbf{X} \in G \\ 0 & \text{if } \mathbf{X} \in \bar{G}. \end{cases}$$

Such a description of complex system reliability is used by many researchers. For example, see [2-5]. In [5] a system structure is called *monotone*, if it satisfies the following conditions:

1)  $\square(\mathbf{X}) \geq \square(\mathbf{Y})$  (6.2)  
if  $X_i \geq Y_i$  for all  $1 \leq i \leq m$ , i.e., a unit failure cannot lead to a system state improvement.

2)  $\square(\mathbf{1})=1, \square(\mathbf{0})=0$   
where  $\mathbf{1}=(1, \dots, 1)$  and  $\mathbf{0}=(0, \dots, 0)$ . In other words, if all of the system units are operational, the system itself is operational; if all of the system units have failed, the system itself has failed.

The system's PFFO is determined by the vector  $\mathbf{p}=(p_1, \dots, p_m)$  and can be expressed with the help of the structural function, as follows



$$R(\mathbf{p}) = Pr\{X \in G\} = E\Psi(X) = \sum_{X \in \Omega} \Psi(X) \prod_{1 \leq i \leq m} p_i^{X_i} (1 - p_i)^{1-X_i} \quad (6.3)$$

It is clear that condition (6.2) leads to the monotonicity of the system's PFFO  $R(\mathbf{p})$  by each  $p_i$ . This fact can be shown in the following way. Let us fix all parameters except  $p_1$ . Expression (6.3) can be presented in the following form:

$$R(\mathbf{p}) = p_1 [C_1(p_2, \dots, p_m) - C_0(p_2, \dots, p_m)] + C_0(p_2, \dots, p_m)$$

where

$$C_0(p_2, \dots, p_m) = \sum_{X_2, \dots, X_m} \Psi(0, X_2, \dots, X_m) \prod_{2 \leq i \leq m} p_i^{X_i} (1 - p_i)^{1-X_i}$$

is the reliability function under condition that the first unit is absolutely unreliable, and

$$C_1(p_2, \dots, p_m) = \sum_{X_2, \dots, X_m} \Psi(1, X_2, \dots, X_m) \prod_{2 \leq i \leq m} p_i^{X_i} (1 - p_i)^{1-X_i}$$

is the PFFO under the condition that unit  $i$  is absolutely reliable. By condition (6.2),  $C_1 \geq C_0$  and, consequently,  $R(\mathbf{p})$  monotone increases when  $p_i$  increases.

Examples of a system with a monotone structure are considered below.

### 7.3.2 Best Lower Confidence Limit for No-Failure-Test

Assume again that  $N_i$  units of type  $i$ ,  $i=1, 2, \dots, m$ , were tested and no failures have been observed for any units, that is,  $d_i=0$ ,  $1 \leq i \leq m$ . The lower  $\square$ -confidence limit ( $\underline{R}$ ) for the system's PFFO can be found as the solution of the following optimization problem (see details in Section 2.2):

$$\min_{\mathbf{p} \in H_0} R(\mathbf{p}) = \underline{R} \quad (6.4)$$

where the minimum is taken from set  $H_0$  of points  $\mathbf{p}=(p_1, \dots, p_m)$  which

$$\prod_{1 \leq i \leq m} p_i^{N_i} \geq 1 - \gamma \quad (6.5)$$

$$0 \leq p_i \leq 1, \quad 1 \leq i \leq m. \quad (6.6)$$

In corresponding with inequality (6.5), set  $H_0$ , where we search minimum of  $R(\mathbf{p})$ , is such that for all values of the parameters, included in set  $H_0$ , the probability to observe the event of type  $d_1 = \dots = d_m = 0$  is not less than the level of significance  $\varepsilon = 1 - \square$ . Inequalities in (6.6) are obvious and

follows from the definition of parameters  $p_i$ . As the lower confidence limit for the system PFFO, we take a minimum possible value  $R(\mathbf{p})$  in set  $H_0$ .

Assume that function  $R(\mathbf{p})$  is monotone in each  $p_i$ . One can show (see p.2 of Appendix) that  $\underline{R}$  found in (6.4) is the best  $\square$ -confidence lower limit for  $R(\mathbf{p})$  under the condition that  $d_1 = \dots = d_m = 0$ .

Monotone increments in the system's structural function by each parameter means that, in principle, the best lower confidence limit can be found as the solution of the above-mentioned non-linear optimization problem (6.4). It is interesting to note that the only upper confidence limit for PFFO is trivial:  $\bar{R} = 1$ .

#### 7.4 Structures with Convex Cumulative Hazard Function

To find the maximum limit in (6.4)-(6.6), it is convenient to introduce variables

$$p_i = e^{-z_i}, \quad 1 \leq i \leq m. \quad (6.7)$$

For systems with PFFO equal to  $R(\mathbf{p})$ , let us introduce the function

$$f(\mathbf{z}) = f(z_1, \dots, z_m) = -\ln R(e^{-z_1}, \dots, e^{-z_m}) \quad (6.8)$$

which is called a *cumulative hazard function*. Transformation (6.8) is used in [3, 4] for the analysis of systems with complex monotone structures. A cumulative hazard function is increased in each  $z_i$ , if PFFO  $R(\mathbf{p})$  is increased in each  $p_i$ .

Transformations from (6.7) through (6.8) have the following meaning. Let all of the system's units have an exponential distribution of time-to-failure (TTF):

$$p_i(t) = e^{-\lambda_i t}$$

where  $\lambda_i$  is the failure rate of unit  $i$ ,  $1 \leq i \leq m$ . In this case  $z_i$ 's coincide (with accuracy to the coefficient  $t$ ) with parameters  $\lambda_i$ 's. Now let us express the system's PFFO with the help of parameters  $\lambda_i$ :

$$R = e^{-f(\lambda_1 t, \dots, \lambda_m t)}.$$

From the latter expression, one can see that the cumulative hazard function has the meaning of system's failure rate, which is expressed via unit parameters. Thus, for a series system, we can write

$$f(\mathbf{z}) = z_1 + \dots + z_m \quad (6.9)$$

which corresponds to the well-known fact that the failure rate of a series system equals the sum of the unit's failure rates.

If units have a distribution of TTF differing from exponential, the meaning of the system's cumulative hazard function remains the same if, instead of parameters  $\lambda_i$ , one uses parameters of a type

$$\Lambda_i = \frac{1}{t} \int_0^t \lambda_i(u) du$$

where  $\lambda_i(t)$  is the failure rate of unit  $i$ . A value of  $\Lambda_i$  has the meaning of the average failure rate of unit  $i$  on time interval  $(0, t)$ .

The optimization problem (6.4)-(6.6) is quite easy to solve if the system's cumulative hazard function is convex in  $\mathbf{z}=(z_1, \dots, z_m)$ . With the use of new variables  $\mathbf{z}$ , the problem (6.4) can be written as follows:

$$\text{to find } \underline{R} = \min e^{-f(\mathbf{z})} \quad (6.10)$$

under linear restrictions

$$\sum_{1 \leq i \leq m} N_i z_i \leq A, \quad z_i > 0, \quad (6.11)$$

where  $A = -\ln(1 - \bar{\gamma})$ . As soon as function  $e^{-f}$  monotone decreases by  $f$ , computation of minimum in (6.10) is equivalent to the problem of finding

$$\bar{f} = \max f(\mathbf{z}) \quad (6.12)$$

with the same restrictions. The lower confidence limit  $\underline{R}$  after this can be found as  $\underline{R} = \exp(-\bar{f})$ . A region given by restrictions (6.11) is convex. If, in addition, the cumulative hazard function  $f(\mathbf{z})$  is also convex, then in correspondence with the well-known results of convex programming (see p.3 of Appendix), the maximum of (6.12) under the restrictions of (6.11) is located in one of  $m$  "corner" points of type

$$z^i = (0, \dots, 0, \frac{A}{N_i}, 0, \dots, 0), \quad 1 \leq i \leq m,$$

where all coordinates, except one, are 0's. Thus, for systems with a convex cumulative hazard function, the solution is given by a simple expression

$$\bar{f} = \max_i f(0, \dots, 0, \frac{A}{N_i}, 0, \dots, 0),$$

from which we obtain

$$\underline{R} = \min_i R(1, \dots, 1, \frac{p_i}{N_i}, 1, \dots, 1), \quad (6.13)$$

where

$$p_i = e^{-\frac{A}{N_i}} = (1 - \gamma)^{\frac{1}{N_i}}$$

is the  $\alpha$ -confidence Clopper-Pearson limit for  $p_i$ . The results of the previous section are as follows: limit (6.13) is the best  $\alpha$ -confidence Clopper-Pearson limit for the system's PFFO  $R(\mathbf{p})$ .

By its nature, (6.13) is similar to the Mirnyi and Solov'yev result for a series system and includes the latter as a particular case. The procedure can be described in the following way: first, one computes  $m$  estimates for the system under the condition that unit  $i$ ,  $1 \leq i \leq m$ , has the PFFO equal to its lower  $\alpha$ -confidence limit  $\underline{p}_i$ ; then the lowest value is considered as the lower  $\alpha$ -confidence limit for the system's PFFO.

Let us now consider the main cases when the cumulative hazard function of the system is convex. We have already mentioned that the Mirnyi-Solov'yev's result is valid for a series system where reliability function is  $R(\mathbf{p}) = \prod_{1 \leq i \leq m} p_i$ .

#### 7.4.1 Series connection of groups of identical units

Let a system consists of several groups of units in series. Different groups consist of different units but each group includes identical units in series: there are  $n_i$  units of type  $i$ , each of them with PFFO equals to  $p_i$ . In this case

$$R(\mathbf{p}) = \prod_{1 \leq i \leq m} p_i^{n_i}$$

where  $m$  is a number of different types of system's units. The cumulative hazard function of the system

$$f(\mathbf{z}) = \sum_{1 \leq i \leq m} n_i z_i$$

is linear as it was in the previous case. For this case, (6.13) gives  $\underline{R} = \min_{1 \leq i \leq m} (1 - \alpha)^{\frac{n_i}{N_i}}$ .

#### 7.4.2 Series-Parallel System with Identical Redundant Units

Consider a system consisting of  $m$  series redundant groups. Each group  $i$ ,  $1 \leq i \leq m$ , consists of  $n_i$  parallel identical units with PFFO equals to  $p_i$  (see Figure 6.2). The system fails if at least one redundant group has failed. A redundant group fails if all of its units have failed.

Figure 6.2

In this case

$$R(\mathbf{p}) = \prod_{1 \leq i \leq m} [1 - (1 - p_i)^{n_i}],$$

$$f(z) = \sum_{1 \leq i \leq m} \varphi_i(z_i)$$

where

$$\varphi_i(z_i) = -\ln [1 - (1 - e^{-z_i})^{n_i}].$$

By direct differentiation, one can show that  $\varphi_i''(z_i) \geq 0$ ,  $1 \leq i \leq m$ , and, consequently, the cumulative hazard function  $f(z)$  is convex because it is a sum of convex functions. In this case, the general expression (6.13) gives the best lower  $\square$ -confidence limit for the PFFO of a series-parallel system:

$$\underline{R} = \min_{1 \leq i \leq m} \{1 - [1 - (1 - \gamma)^{\frac{1}{N_i}}]^{n_i}\}$$

This result was obtained in [6].

#### 7.4.3 Series Connection of "K out of N" Subsystems

A more general case, in comparison with the previous one, concerns a system in which the redundant group  $i$  fails if at least  $K_i$  of its  $n_i$  units have failed. The previous structures are specific examples of this general case:  $K_i=1$  in the first case and  $K_i=n_i$  in the second case,  $1 \leq i \leq m$ . In this general case

$$R(p) = \prod_{1 \leq i \leq m} \sum_{0 \leq j \leq K_i-1} \binom{n_i}{j} (1-p_i)^j p_i^{n_i-j},$$

$$f(z) = \sum_{1 \leq i \leq m} \varphi_i(z_i),$$

$$\varphi_i(z_i) = -\ln \sum_{0 \leq j \leq K_i-1} \binom{n_i}{j} (1 - e^{-z_i})^j e^{-n_i z_i + j z_i}.$$

As previously, by a direct differentiation, one can show that  $\varphi_i''(z_i) \geq 0$  for all  $1 \leq i \leq m$ , and, consequently, the cumulative hazard function  $f(z)$  is convex, being a sum of convex functions. Expression (6.13) in this case gives

$$\underline{R} = \min_{1 \leq i \leq m} \sum_{0 \leq j \leq K_i-1} \binom{n_i}{j} (1 - \underline{p}_i)^j \underline{p}_i^{n_i-j}.$$

When the system's cumulative hazard function is not convex, the problem becomes more difficult.

### 7.5 Series-Parallel Structure with Different Redundant Units

A system consists of  $m$  series connection of redundant groups (see Figure 6.2). Group  $i$  consists of  $n_i$  units each of them with its own PFFO:

$$p_1^{n_1}, p_2^{n_2}, \dots, p_m^{n_m}$$

The failure criteria for a redundant group and for the system remain the same as before. As one can see, here we rejected an assumption about the units identity within a redundant group. The PFFO and the cumulative hazard function of the system are

$$R(p) = \prod_{1 \leq i \leq m} [1 - \prod_{1 \leq j \leq n_i} (1 - p_i^j)], \quad (6.14)$$

$$f(z) = - \sum_{1 \leq i \leq m} \ln [1 - \prod_{1 \leq j \leq n_i} (1 - e^{-z^j})],$$

The problem reduces to the computation of the maximum of the cumulative hazard function of (6.14) under the following restrictions:

$$z_{ij} \geq 0, \quad 1 \leq j \leq n_i, \quad 1 \leq i \leq m;$$

$$\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n_i} N_{ij} z_{ij} \leq -\ln(1 - \gamma) \quad (6.15)$$

where  $N_{ij}$  and  $p_i^j$  are the numbers of tested units and the PFFO for unit  $i$  within redundant group  $j$ , respectively;  $z_{ij} = -\ln p_i^j$ ,  $p = \{p_i^j\}$ ,  $z = \{z_{ij}\}$ .

We use an auxiliary problem: To find

$$\varphi_i(x_i) = \max \prod_{1 \leq j \leq n_i} (1 - p_i^j) \quad (6.16)$$

under the restrictions

$$\prod_{1 \leq j \leq n_i} (p_i^j)^{N_{ij}} \geq 1 - e^{-x_i}, \quad 0 \leq p_i^j \leq 1, \quad 1 \leq j \leq n_i, \quad (6.17)$$

An equivalent problem in variables  $z_{ij}$  is: To find

$$\varphi_i(x_i) = \max \prod_{1 \leq j \leq n_i} (1 - e^{-z_{ij}}) \quad (6.18)$$

under restrictions

$$\begin{aligned} z_{ij} &\geq 0, \quad 1 \leq j \leq n_i, \\ \sum_{1 \leq j \leq n_i} N_{ij} z_{ij} &= x_i. \end{aligned} \quad (6.19)$$

Directly, from the definition of the auxiliary problem (6.16)-(6.19), it follows that the desired maximum of function (6.14) under restrictions (6.15) can be computed as

$$\max_z f(z) = \max_x \left\{ - \sum_{1 \leq i \leq m} \ln [1 - \varphi_i(x_i)] \right\} \quad (6.20)$$

where maximum is found under restrictions

$$\begin{aligned} x_i &\geq 0, \quad 1 \leq i \leq m, \\ \sum_{1 \leq i \leq m} x_i &\leq \ln(1 - \gamma) \end{aligned} \quad (6.21)$$

The following statements are formulated in the form of theorems for purpose of brevity.

**Theorem 6.1** Solution of auxiliary problem (6.18)-(6.19) has the form:

$$\varphi_i(x_i) = \prod_{1 \leq j \leq n_i} \frac{t(x_i)}{t(x_i) + N_{ij}} \quad (6.22)$$

where  $t(x_i)$  is the solution of equation

$$\sum_{1 \leq j \leq n_i} N_{ij} \ln \left( 1 + \frac{t}{N_{ij}} \right) = x_i \quad (6.23)$$

relatively to  $t > 0$ . (For proof of the theorem, see p.4 of the Appendix to the chapter.)

**Theorem 6.2** Function  $\varphi_i(x_i) = -\ln [1 - \varphi_i(x_i)]$  is monotone increasing and convex for  $x_i \geq 0, 1 \leq i \leq m$ . (For proof of the theorem, see p.5 of the Appendix.)

It is easy to see that Theorem 6.2 provides the solution of the problem (6.14) - (6.15). Indeed, by the theorem, the function in the right side of (6.20) is convex in  $\mathbf{x} = (x_1, \dots, x_m)$  being a sum of convex

functions and, consequently, the maximum of (6.20) is reached at one of  $m$  "corner" points of the type

$$(0, \dots, 0, -\ln(1-\alpha), \dots, 0, \dots, 0).$$

Thus,

$$\max_z f(z) = -\ln \left\{ \min_{1 \leq i \leq m} [1 - \varphi_i(-\ln(1-\gamma))] \right\}$$

and the final form of the best lower  $\alpha$ -confidence limit for the system's PFFO follows from this:

$$\underline{R} = \min_{1 \leq i \leq m} \{1 - \varphi_i[-\ln(1-\gamma)]\} \quad (6.24)$$

where  $\varphi_i(\cdot)$  is defined from (6.22)-(6.23). Since the left side of (6.23) is monotone increasing for  $t \geq 0$ , numerical computation of (6.23) and the further computation of  $\varphi_i$  is not a difficult task.

### 7.5.1 Parallel connection

Let us separately consider an important particular case of a parallel system, which represents the above-considered system for  $m=1$  (see Figure 6.3).

Figure 6.3.

The system's PFFO of  $n$  units connected in parallel is defined by

$$r(\mathbf{p}) = 1 - \prod_{1 \leq j \leq n} (1 - p_j)$$

where  $p_j$  is the  $j$ th unit PFFO. From (6.22) - (6.24) for  $m=1$ ,  $n_i=n$ , and  $N_{ij}=N_j$ , we obtain that the best  $\alpha$ -confidence limit for  $r(\mathbf{p})$  is determined as

$$r^* = 1 - [-\ln(1-\gamma)] = 1 - \prod_{1 \leq j \leq n} \frac{t}{t + N_j} \quad (6.25)$$

where  $N_j$  is the number of tested units of type  $j$ , and value  $t$  is determined from equation

$$\sum_{1 \leq j \leq n} N_j \ln \left( 1 + \frac{t}{N_j} \right) = -\ln(1-\gamma) \quad (6.26)$$

with the left side monotone increasing in  $t$ .

In a specific case of equal numbers of tested units,  $N_j=N$ ,  $1 \leq j \leq n$ , the result obtained from (6.25)-(6.26) in [7] is as follows:



$$r_* = 1 - [1 - (1 - \gamma)^{\frac{1}{nN}}]^n \quad (6.27)$$

for the lower  $\square$ -confidence limit of the PFFO of a parallel system under the condition of equality of all  $N_i$ .

Note that the value of  $r_* = 1 - \varphi_i[-\ln(1 - \square)]$  in (6.24) gives the best lower  $\square$ -confidence limit for the PFFO of each separately considered redundant group  $i$ . Thus, (6.24) has the following meaning. For computation of the best lower  $\square$ -confidence limit for the PFFO of a series-parallel system, one needs to compute the best  $\square$ -confidence limit of the type (6.25) - (6.26) for each redundant group and then to take the minimum among them. In this sense, the procedure remains similar to the Mirnyi-Solovyev's procedure for a series system, including the latter as a particular case for  $n_i = 1$ ,  $1 \leq i \leq m$ .

**Example 6.1** A system consists of three series redundant groups. The number of redundant units in these groups are:  $n_1 = 2$ ,  $n_2 = 3$  and  $n_3 = 3$ . The system as a whole was tested  $N = 6$  times and no failures were observed. The lower 90%-confidence limits of the PFFO of different redundant groups by (6.27)

$$r_{i*} = 1 - [1 - (1 - \gamma)^{\frac{1}{Nn_i}}]^{n_i}, \quad 1 \leq i \leq 3$$

are 0.970, 0.998 and 0.998, respectively. The smallest number gives us the best lower 90%-confidence limit for the system's PFFO, i.e.,  $\underline{R} = 0.970$ .

Let us consider a case when all of unit within the group are identical:  $p_{11} = p_{12} = p_1$ ,  $p_{21} = p_{22} = p_{23} = p_2$ ,  $p_{31} = p_{32} = p_{33} = p_3$ . In this case, the best lower  $\square$ -confidence limit for the system's PFFO is computed by (6.14)

$$\underline{R} = \min_{1 \leq i \leq m} \{1 - [1 - (1 - \gamma)^{\frac{1}{Nn_i}}]^{n_i}\}$$

where one needs to take  $N_i = Nn_i$ , because this is a total number of tested units of type  $i$ . The limit obtained in this case obviously coincides with the value of  $\underline{R} = 0.97$  computed above. This means that in this case an assumption about identity of redundant units did not improve the confidence estimate of the system's PFFO.

## 7.5.2 Parallel-series system

Let us consider a system consisting of  $m$  series subsystems connected in parallel (see Figure 6.4).

Figure 6.4

Subsystem  $i$  consists of  $n_i$  units. The system's PFFO can be determined by:

$$R(p) = 1 - \prod_{1 \leq i \leq m} [1 - \prod_{1 \leq j \leq n_i} p_{ij}]$$

where  $p_{ij}$  is the PFFO of unit  $j$  within series subsystem  $i$ ,  $1 \leq i \leq m$ ;  $1 \leq j \leq n_i$ .

The problem is to find the minimum of function  $R(p)$  under the restrictions of type (6.5)-(6.6), or in variables of type

$$z_{ij} = -\ln p_{ij}$$

$$\max \prod_{1 \leq i \leq m} \left[ 1 - \exp \left( - \sum_{1 \leq j \leq n_i} z_{ij} \right) \right] \quad (6.28)$$

with restrictions (6.15) where  $N_{ij}$  is the number of tested units with subscript "ij". Let us denote

$$g_i(x_i) = \max \sum_{1 \leq j \leq n_i} z_{ij} \quad (6.29)$$

where the maximum is taken under the following restrictions:

$$z_{ij} \geq 0, \quad 1 \leq j \leq n_i;$$

$$\sum_{1 \leq j \leq n_i} N_{ij} z_{ij} \leq N_i x_i \quad (6.30)$$

where

$$N_i = \min_{1 \leq j \leq n_i} N_{ij}$$

is the minimal number of tested units within series subsystem  $i$ .

Computation of maximum in (6.28) under restrictions (6.15) is reduced to the problem: find

$$\max \prod_{1 \leq i \leq m} [1 - e^{-g_i(x_i)}]$$

under restrictions

$$x_i \geq 0, \quad 1 \leq i \leq m;$$

$$\sum_{1 \leq i \leq m} N_i x_i \leq -\ln(1 - \gamma) \quad (6.31)$$

In correspondence with (6.29)-(6.30), and  $g_i(x_i) = x_i$ , the problem is reduced to the computation of

$$\max \prod_{1 \leq i \leq m} [1 - e^{-x_i}] \quad (6.32)$$

under restrictions (6.31). This problem was solved in Theorem 6.1. Hence, the lower confidence limit of the system's PFFO can be written in the following form:

$$\underline{R} = 1 - [-\ln (1 - \gamma)] \quad (6.33)$$

where

$$\varphi(u) = \prod_{1 \leq i \leq m} \frac{t(u)}{t(u) + N_i}$$

and  $t(u)$  is the solution of equation

$$\sum_{1 \leq i \leq m} N_i \ln \left( 1 + \frac{t}{N_i} \right) = u.$$

Thus the best lower confidence limit for the PFFO of a parallel-series system is found as the limit of type (6.25)-(6.26) for a parallel system when a number of tested units of type  $i$  equals the minimum number of tested units within series subsystem  $i$  of the original system. Particular cases for  $m=1$  (a series system) and for  $n_1=\dots=n_m=1$  (a parallel system) completely coincide with the results of [1] and [7].

## 7.6 Systems with a Complex Structure (Identical Tests)

All of the structures considered above can be easily analyzed because of the specific nature of their reliability functions. In general, for systems with an arbitrary monotone structure there is no solution. But, nevertheless, a solution exists for a wide class of complex structures  $K'$  (see Definition 6.4 below) for a very important practical case where the number of tested units of each type is the same ( $N_1=\dots=N_m=N$ ). Note that this case coincides with  $N$  tests of the entire system. Of course, a system can be tested in an incomplete structure (not all units can be installed). It can occur if the system develops in time and enlarges from stage to stage during the practical utilization (for instance, telecommunication networks, power systems, etc. which are continuously developing). Later, new units that will be installed in the system are separately tested and the statistical data obtained is incorporated with the previously available data.

Let  $m$  equal the total number of the system's units and  $S=(i_1, \dots, i_n)$ . There are some subsets of the system's units,  $n \leq m$ . Denote a vector of a system state  $\mathbf{e}(i_1, \dots, i_n)$ , if units  $i_1, \dots, i_n$  are operational and the remaining have failed, and a vector of a system state  $\bar{\mathbf{e}}(i_1, \dots, i_n)$ , if units  $i_1, \dots, i_n$  have failed and remaining are operational. (In some sense these two types of states are "mirror".) The following two definitions are well-known [4]:

**Definition 6.1** A subset of units  $A=(i_1, \dots, i_n)$  is called a path of a two-pole graph if  $\square[\mathbf{e}(i_1, \dots, i_n)]=1$ , i.e., a system is operational if all units of the path are operational.

**Definition 6.2** A subset of units  $B=(i_1, \dots, i_n)$  is called a cut of a two-pole graph if  $\square[\bar{e}(i_1, \dots, i_n)] = 0$ , i.e., a system is failed if all units of the cut have failed.

Let us select the system cut that contains the smallest number of units. We will call this cut a main cut  $S'$  and we denote a number of its units as  $b$ . Let us consider a structure which is obtained as a parallel connection of all units of the main cut. This structure corresponds to the initial system under the assumption that all of its remaining units are absolutely reliable. It is clear that because of the monotonicity property, the new structure is more reliable than the original one:

$$R(p) \leq R'(p) = 1 - \prod_{i \in S'} p_i \quad (6.34)$$

for any values of unit reliability parameters  $p=(p_1, \dots, p_m)$ .

We now find a substructure of the original system structure which deliver the lower limit.

**Definition 6.3** Paths  $A_1, \dots, A_k$  are called independent (non-overlapping) if they do not contain common units, i.e.,

$$A_i \cap A_j = \emptyset$$

for any  $i \neq j, (i, j) \in \{1, 2, \dots, k\}$ .

Among all of the sets of independent paths we select one which contains a maximal number of independent paths. We denote this number  $a$ . There may be several such sets of independent paths. We denote a set of all of such sets of independent paths by  $\square$ . We denote set  $k$  of independent paths by  $A_1^k, \dots, A_a^k$ , where  $k$  belongs to  $\square$ . Now consider a structure that is obtained as a parallel connection of these independent paths. This structure can be obtained from the original one by the assumption that all of the remaining units of the original structure are absolutely unreliable. (The same assumption can be obtained if these units are deleted from the original structure.) Again, it follows from definition of the system monotonicity that we obtain the lower limit

$$R'_{(k)}(p) = 1 - \prod_{1 \leq i \leq a} [1 - \prod_{j \in A_i^k} p_j] \leq R(p) \quad (6.35)$$

Inequalities (6.34) and (6.35) are true for all values of reliability parameters  $p=(p_1, \dots, p_m)$ , thus it follows that

$$\max_{j \in \Delta} \min_{H_0} R''_{(k)}(p) \leq \min_{H_0} R(p) \leq \min_{H_0} R'(p)$$

where  $H_0$  is a set given by restrictions (6.5) and (6.6). Consequently,

$$\underline{R}'' \leq \underline{R} \leq \underline{R}'$$

where  $\underline{R}''$ ,  $\underline{R}'$  and  $\underline{R}$  are the best  $\square$ -confidence limits of the two majorant structures and the original one, respectively. Note that values of  $\underline{R}'$  and  $\underline{R}''$  can be easily computed with the use of previously obtained results. Using (6.27) and (6.33) we can finally obtain

$$1 - [1 - (1 - \gamma)^{\frac{1}{na}}]^a \leq \underline{R} \leq 1 - [1 - (1 - \gamma)^{\frac{1}{nb}}]^b \quad (6.36)$$

where the left and right sides are the lower  $\square$ -confidence limits for a parallel system consisting of  $a$  and  $b$  units, respectively.

Obviously, a number of units in the main cut coincides with a maximal number of independent paths.

**Definition 6.4** Let us call a system structure  $K'$ , if

$$a=b. \quad (6.37)$$

For a structure of class  $K'$  inequalities (6.36) give the best lower  $\square$ -confident limit for the system's PFFO:

$$\underline{R} = 1 - [1 - (1 - \gamma)^{\frac{1}{na}}]^a \quad (6.38)$$

which coincides with a similar limit for the main cut of a system.

It is easy to find that class  $K'$  includes all of the above-considered structures of series-parallel and parallel-series types. Condition (6.37) is not true for all monotone structures. For instance, in Section 6.2 we considered so-called "k out of n" structure. For a structure with  $n=3$  and  $k=2$ , we have  $a=1$  and  $b=2$ . But for most monotone structures this condition is true.

**Example 6.1** Consider a system represented in Figure 6.5.

Figure 6.5

One can find from this figure that  $a=b=3$ . The main cut of a system is represented in the figure by shadowed units.  $\square$

**Example 6.2** A radial-ring structure is represented in Figure 6.6.

Figure 6.6.

Let us consider the probability of a successful transmission of a signal between the central unit and the shadowed peripheral unit as a system reliability index. In this case  $a=b=4$ , if transmission is possible in both directions, and  $a=b=3$ , if transmission is possible only from a peripheral unit to the central one.  $\square$

**Example 6.3** A lattice-type structure is represented in Figure 6.7.

Figure 6.7.

Let us consider the probability of a successful transmission of a signal between two shadowed units as a system's reliability index. In this case  $a=b=2$ .  $\square$

All of these non-trivial structures allow us to use a simple expression (6.38) to obtain the best lower confidence limit. Notice that, in these cases, direct attempt to solve the optimization problem (6.4)-(6.6) lead to huge calculations and usually are unsuccessful.

### 7.6.1 Computation of Confidence Limit for a System Bases on a Known Limit for another system

Assume that the same set of units with reliability parameters  $\mathbf{p}=(p_1, \dots, p_m)$  is used to build two different structures with reliability functions  $R(\mathbf{p})$  and  $R'(\mathbf{p})$ . Let us call these structures main and auxiliary, respectively. Assume that for the PFFO of the auxiliary system we know lower  $\square$ -confidence limit

$$\underline{R}'(\mathbf{d})=\underline{R}'(d_1, \dots, d_m).$$

This limit can be found by any known method.

We will consider the following problem: to find the lower  $\square$ -confidence limit  $R^*$  for the PFFO of the main system if the limit for the auxiliary system is known.

Let us introduce a system of sets

$$H_d=\{\mathbf{p}: \underline{R}'(\mathbf{p}) \geq \underline{R}'(\mathbf{d})\}. \quad (6.39)$$

By definition of a  $\square$ -confidence limit  $\underline{R}'(\mathbf{d})$ , the following inequality is true for all  $\mathbf{p}=(p_1, \dots, p_m)$

$$P_p \{ \mathbf{p} \in H_d \} = P_p \{ \underline{R}'(\mathbf{d}) \leq R'(\mathbf{p}) \} \geq \gamma \quad (6.40)$$

from where it follows that sets (6.39) form a system of  $\square$ -confidence sets, and value

$$\underline{R}(\mathbf{d}) = \min_{H_d} R(\mathbf{p}) \quad (6.41)$$

gives the lower  $\square$ -confidence limit for the main system's PFFO. Notice that all of these arguments are correct not only for binomial test plans and results in the form  $\mathbf{d}=(d_1, \dots, d_m)$ , but they are also correct for arbitrary plans with results  $\mathbf{x} \in X$  with substitution in (6.39)-(6.41) notation  $\mathbf{d}$  for  $\mathbf{x}$ .

Let us consider a case when the auxiliary structure is series with the reliability function

$$R'(p) = \prod_{1 \leq i \leq m} p_i.$$

By (6.39)-(6.41), the problem of construction of the lower  $\square$ -confidence limit for the main system's PFFO is equivalent to the problem of finding the following minimum:

$$\underline{R} = \min_p R(p)$$

under restrictions

$$0 \leq p_i \leq 1, \quad 1 \leq i \leq m;$$

$$\prod_{1 \leq i \leq m} p_i \geq \underline{R}'.$$

Thus, from a formal viewpoint, this problem is completely equivalent to problem (6.4)-(6.6) of computing the best lower confidence limit for  $R(\mathbf{p})$  under binomial tests with no observed failure. Using the previous results, we can write expressions for the lower  $\square$ -confidence limit for the main system's PFFO, if this system is series, parallel, series-parallel, parallel-series or has a structure belonging to class  $K'$ . For the most general class, namely,  $K'$ , we have from (6.38)

$$\underline{R} = 1 - [1 - (\underline{R}')^{\frac{1}{b}}]^b,$$

where  $\underline{R}'$  is the known lower  $\square$ -confidence limit for the auxiliary system and  $b$  is a number of units in the main cut of the main system. (For a parallel structure  $b$  equals a number of the system's unit; for a parallel-series structure  $b$  equals a number of parallel connected series subsystems; for a series-parallel structure  $b$  equals a number of units in the smallest redundant group, etc.)

## 7.7 APPENDIX

### 7.7.1 Confidence Clopper-Pearson Limits for Parameter of Binomial Distribution

Let us observe  $d$  failures in sequence of  $N$  independent tests. Then the lower  $\square$ -confidence limit  $\underline{p}$  for the PFFO can be found from the following equation:

$$\sum_{0 \leq k \leq d} \binom{N}{k} (1 - \underline{p})^k \underline{p}^{N-k} = 1 - \gamma.$$

The upper  $\square$ -confidence limit  $\bar{p}$  for the same conditions can be found from equation:

$$\sum_{0 \leq k \leq N-d} \binom{N}{k} (1 - \bar{p})^k (\bar{p})^{N-k} = 1 - \gamma.$$

In particular, the lower  $\square$ -confidence limit for case  $d=0$  is found in equation  $\underline{p}^N = 1 - \square$ .

Tables for  $\underline{p}$  and  $\bar{p}$  are given in Appendix of the book.

### 7.7.2 Best Lower Confidence Limit for the PFFO in Case of No-Failure-Test

Let us consider all possible lower limits for PFFO  $R(p)$  with the confidence level of not less than  $\square$ , i.e., all of possible functions of observed data  $\varphi(\mathbf{d}) = \varphi(d_1, \dots, d_m)$  such that

$$P_p \{ \varphi(\mathbf{d}) \leq P(\mathbf{p}) \} \geq \square$$

for all of  $\mathbf{p} \in \square$  where  $\square = \{ \mathbf{p} : 0 \leq p_i \leq 1, 1 \leq i \leq m \}$  is a set of all of the possible values for parameters  $\mathbf{p} = (p_1, \dots, p_m)$ .

**Theorem 6.3** Let the function  $R(\mathbf{p}) = R(p_1, \dots, p_m)$  be a monotone non-decreasing for each parameters and continuous for  $\mathbf{p} \in \square$ . Then any lower confidence limit  $\varphi(\mathbf{d})$  with the confidence level of not less than  $\square$  for  $R(\mathbf{p})$  at the point  $\mathbf{d} = (0, 0, \dots, 0)$  satisfies the inequality

$$\varphi(0, 0, \dots, 0) \leq \underline{R}$$

where  $\underline{R}$  is the lower confidence limit (6.4).

**Proof** Assume the opposite:  $\varphi(\mathbf{0}) > \underline{R}$ . From continuity of function  $R(\mathbf{p})$  in  $H_0$ , the minimum in (6.4) is reached at some point

$$\tilde{\mathbf{p}} = (\tilde{p}_1, \dots, \tilde{p}_m) \in H_0.$$



Because of the monotonicity of  $R(\mathbf{p})$ , the minimum in (6.4) is reached on the border of the area  $H_0$ , and

$$\prod_{1 \leq i \leq m} \tilde{p}_i^{N_i} = 1 - \gamma.$$

Let us consider an interval between points  $\tilde{p}$  and  $\mathbf{p}' = (1, 1, \dots, 1)$  which can be given in a parametrical form as a set of points

$$p = \tilde{p} + t(p' - \tilde{p}) \quad (6.42)$$

where  $t$  is a parameter,  $0 \leq t \leq 1$ . Because of the convexity of region  $H_0$ , this interval belongs to this region. Consider the function  $R(\mathbf{p})$  on interval (6.42). Let us introduce function

$$g(t) = R[\tilde{p} + t(p' - \tilde{p})], \quad 0 \leq t \leq 1.$$

Under our assumptions, function  $g(t)$  is continuous and monotone non-decreasing by  $t$ . Note that from

$$g(0) = R(\tilde{p}) = \underline{R}(0) < \varphi(0) \leq R(1, 1, \dots, 1) = g(1).$$

follows that such point  $t'$ ,  $0 < t' \leq 1$ , exists that

$$\begin{aligned} g(t') &= \varphi(0); \\ g(t) &< \varphi(0, 0, \dots, 0), \quad 0 \leq t < t'. \end{aligned} \quad (6.43)$$

Now consider the probability  $P_p\{\varphi(\mathbf{d}) \leq R(p)\}$  where  $p$  belongs to interval (6.42) and  $0 \leq t < t'$ . Because of the monotonicity of (6.43), we have

$$\begin{aligned} g(0) &= R(\tilde{p}) = \underline{R}(0) < \varphi(0) \leq R(1, 1, \dots, 1) = g(1). \\ P_p\{\varphi(\mathbf{d}) \leq R(p)\} &\leq 1 - P_p\{d_1 = \dots = d_m = 0\} = 1 - \prod_{1 \leq i \leq m} p_i^{N_i}. \end{aligned}$$

From the strong monotone increments of function

$$h(t) = \prod_{1 \leq i \leq m} [\tilde{p}_i + t(1 - \tilde{p}_i)]^{N_i}$$

in  $t$  and  $h(0) = 1 - \gamma$ , it follows that the probability

$$P_p\{\varphi(\mathbf{d}) \leq R(p)\} < \gamma$$

in all of the internal points  $p$  of interval (6.42) for parameter  $0 < t < 1$ . Thus, the confidence limit  $\varphi(\mathbf{d})$  has the confidence level strictly less than  $\alpha$ , which contradicts the theorem's conditions. Consequently,

$$\varphi(0) \leq \underline{R}$$

### 7.7.3 Maximum of Convex Function on a Convex Set

An area  $\Omega$  in  $m$ -dimensional Euclidean space  $R_m$  is convex if for each of its two points  $\mathbf{x}=(x_1, \dots, x_m)$  and  $\mathbf{y}=(y_1, \dots, y_m)$ , this area also contains an entire interval between these points. In other words, any point of a type  $\mathbf{z}=\alpha\mathbf{x}+(1-\alpha)\mathbf{y}$ , where  $0 \leq \alpha \leq 1$  belongs to this area. A function of  $m$  variables  $f(\mathbf{x})=f(x_1, \dots, x_m)$  is called convex (strictly convex) if

$$f[\alpha\mathbf{x}+(1-\alpha)\mathbf{y}] \leq (<) \alpha f(\mathbf{x})+(1-\alpha)f(\mathbf{y})$$

for any  $\mathbf{x}, \mathbf{y}$ ,  $0 \leq \alpha \leq 1$ .

A point  $x$  is called an inner point of a convex area  $\Omega$ , if it belongs to an interval which lies totally inside the area. The "surface" point is any point of the area  $\Omega$  which is not inner.

Let  $\Omega$  be a closed convex region and  $f(x)$  be a continuous and strictly convex function. Then  $\max_{\Gamma} f(x)$  is reached at a "surface" point of the area  $\Omega$  and the point in which maximum is

reached is unique. To prove it, assume the opposite: the maximum is reached at the inner point  $z \in \Omega$ . Then there exists such points  $x$  and  $y$  belonging to  $\Omega$  and such  $0 < \alpha < 1$  that  $z = \alpha x + (1 - \alpha)y$ , and, consequently,  $f(z) < \alpha f(x) + (1 - \alpha)f(y)$  and from this  $f(z) < \max [f(x), f(y)]$ . But the latter contradicts the statement that  $\max f(x)$  is reached at point  $z$ . If function  $f(x)$  is convex (not necessarily strictly convex), then  $\max_{\Gamma} f(x)$  is reached in a "surface" point of  $\Omega$ , but such point might be not unique. (See details in [11].)

Let us consider, for example, the problem of finding the maximum in (6.12) under the restrictions of (6.11). Because function  $f(\mathbf{z})$  monotone increases in each of its variables, its maximum value is reached on such points at which the first inequality in (6.11) turns into equality, and, consequently, we can choose restrictions of the type

$$\sum_{1 \leq i \leq m} N_i z_i = A, \quad (6.44)$$

$$z_i \geq 0, \quad 1 \leq i \leq m.$$

Surface points (corner) of the convex area  $\Omega$  determined by restrictions (6.44) are points of the type:  $z^i=(0, \dots, 0, A/N_i, 0, \dots, 0)$  where all coordinates except a single one are 0's. The simplest way to check this is to turn to Figure 6.8 where (for two-dimensional case) we marked in bold the two unique corner points.

Figure 6.8

#### 7.7.4 Proof of Theorem 6.1

Maximization of function (6.18) is equivalent to maximization of function

$$h(z) = \sum_{1 \leq j \leq n_i} \ln(1 - e^{-z_j}) \quad (6.45)$$

under the same restrictions (6.19). Function (6.45) is monotone increased by each of its variables and is strictly concave. It follows that inside the area

$$\{z_{ij} : \sum_{1 \leq j \leq n_i} N_{ij} z_{ij} = x_i, \quad z_{ij} \geq 0\} \quad (6.46)$$

there is a unique conditional local minimum which, at the same time, is the global one and is determined by a system of Lagrange equations

$$\frac{\partial h}{\partial z_{ij}} = \frac{1}{e^{z_{ij}} - 1} = \alpha N_{ij}, \quad 1 \leq j \leq n_i \quad (6.47)$$

where  $\alpha$  is a Lagrange's multiplier. (It is not necessary here to investigate if the maximum belongs to the border of area (6.46) or not because  $h(z) = -\alpha$  for  $z_{ij} = 0$ .) Expressing  $z_{ij}$  from (6.47) through  $\alpha$  and substituting this expressions into condition (6.46), we obtain the statement of Theorem. (For the sake of convenience, we use the denotation  $t = 1/\alpha$ .)

#### 7.7.5 Proof of Theorem 6.2

From (6.22) and (6.23), after simple transformations, we obtain the following expression for the derivative:

$$\Psi'_i(x_i) = \frac{t^{n_i-1}(x_i)}{\prod_{1 \leq j \leq n_i} [N_{ij} + t(x_i)] - t^{n_i}(x_i)} \quad (6.48)$$

where  $t(x_i)$  is determined by (6.23). Since the denominator of (6.48) is a polynomial of  $t(x_i)$  with the power  $n_i-1$  and with positive coefficients, then (6.48) is monotone increasing in  $x_i$ . Function  $t(x_i)$ , in its turn, is increasing in  $x_i$ , and it follows that  $\Psi'_i(x_i)$  is increasing in  $x_i$  and  $\Psi_i(x_i)$  convex.

## Problems to Chapter 6

### Exercise 6.1

Consider a system depicted in Figure 6.9.

Figure 6.9

This system consists of two groups of redundant units connected in series. The first group consists of two parallel units with parameters  $p_{11}$  and  $p_{12}$ , and the second one consists of three parallel units with parameters  $p_{21}$ ,  $p_{22}$  and  $p_{23}$ . All these parameters are unknown. Units within each group are not assumed identical. The system's probability of failure free operation is expressed as

$$R=[1-(1-p_{11})(1-p_{12})][1-(1-p_{21})(1-p_{22})(1-p_{23})] \quad (6.49)$$

During eight tests of the system ( $N=8$ ) there was no failure.

Construct the lower 90%-confidence limit for the reliability function (6.49).

### Exercise 6.2

The problem almost completely coincides with the previous one. The difference is in the fact that the units within each group are identical:

$$p_{11}=p_{12}=p_1, \text{ and } p_{21}=p_{22}=p_{23}=p_2. \quad (6.50)$$

Construct the lower 90%-confidence limit for the reliability function

$$R=[1-(1-p_1)^2][1-(1-p_2)^3] \quad (6.51)$$

### Exercise 6.3.

Solve the previous problem if it is known that

$$p_1 \geq p_2. \quad (6.52)$$

Construct the lower 90%-confidence limit for the reliability function (6.51).

**Exercise 6.4** Let us say that a series-parallel system has a structure of type  $(n_1, n_2, \dots, n_m)$  if it consists of  $m$  groups of redundant groups and the  $i$ th one consists of  $n_i$  identical units, each with parameter  $p_i$ . In other words, this system has the following reliability function

$$R = \prod_{1 \leq i \leq m} [1 - (1 - p_i)^{n_i}].$$

Let the system with the structure (2, 3) depicted in Figure 6.10, was tested eight times ( $N=8$ ) with no failures.

Construct the lower 90%-confidence limit for the reliability function of the system with structure (3, 2). In other words, we need to estimate the system reliability of one system on the basis of the test results of another system consisting of the similar units.

**Exercise 6.5.** Under the condition of the previous exercise, find the structure  $(n_1, n_2)$  with the maximum lower 90%-confidence limit if the numbers of units are subjected to the following restrictions:

$$n_1+n_2 \leq 5; n_1 \geq 1, n_2 \geq 1. \quad (6.53)$$

This task represents one of variants of optimal redundancy problem where we should choose the optimal structure of structure  $(n_1, n_2)$  with restrictions (6.53) on the basis of test of system  $(n_1=2, n_2=3)$ . Notice that this problem cannot be solved using point estimates because in this case all variants of the system have a trivial estimate equals 1.

**Exercise 6.6.** A "bridge system" (see Figure 6.10) was tested four times ( $N=4$ ) with no failures. All units are not assumed identical.

Figure 6.10

Construct the lower 0.95-confidence limit for the system reliability function.

## 8. System Confidence Limits Based on Unit Test Results

### 8.1 Introduction

#### 8.1.1 Practical Applications

In real world situations we often need to estimate the system reliability *before* it has been designed. We are thus forced to predict system behavior based on statistical information obtained from unit (components or subsystems) testing. The goal is to use this information in such a way that the result will be adequate to what may be obtained from testing the system as a whole. The same situation arises when testing the entire system, for some reason, is very difficult or even impossible at developmental phase. An analogous circumstance appears if we are only able to test a system in a truncated configuration; perhaps on some "pilot model" of the system. We wish to make a realistic confidence prediction of the prospective system on the basis of testing a truncated configuration, and results of testing its separate units. At the other extreme, units and truncated configurations is all we have for a continuously developing system.

Sometimes unit tests are more effective than a test on the entire system. Assume that the system has a high order of redundancy. Such a system can be highly reliable and is protected against the failure of single units. In this case the total test volume measured as

**(testing hours)  $\times$  (number of tested units)**

can be less for units tests rather than for system test. Of course, this situation can be transformed: we can test the system but collect all information about unit failures, and use the full set of results for system reliability estimation.

Notice that in all these cases a new problem of aggregating of statistical data arises: different units can be tested differently, i.e., by different testing plans (different numbers of tested units, different time of testing, truncation of test, etc.). Thus, the problem of constructing of confidence limits for the system on the basis of unit test data is of great practical interest.

### 8.1.2 Formulation of the Problem

We showed in the previous chapter that the problem of finding the best confidence limits for the system from unit data of no-failure test can be easily solved analytically. In other cases, (for the problems listed 7.1.1) solving this problem is rather complicated.

In general, the problem of constructing the confidence limits for the PFFO of a complex system, based on the unit testing can be formulated in the following way. Let  $m$  be the number of different types of units, and  $\theta = (\theta_1, \theta_2, \dots, \theta_m)$  is a vector of reliability parameters of these units. (Here  $\theta_i$  is a reliability parameter of a unit of the  $i$ th type.) A set of all possible values of the vector  $\theta$  we denote  $\Theta$ . Let  $R$  be the system PFFO which depends on the unit parameters:

$$R = R(\theta) = R(\theta_1, \theta_2, \dots, \theta_m). \quad (7.1)$$

We assume that dependence (7.1) is known but the units parameters *are unknown* though we possess the test results  $x_i$  for each unit of the  $i$ th type,  $i=1, \dots, m$ . Test results  $x_i$  for each unit can be obtained by either of the two methods: (1) individual testing of units or (2) individual registration of unit failures during the test of the system as a whole. A set of all available test data is denoted by  $\mathbf{x} = (x_1, x_2, \dots, x_m)$ . This vector is random and its distribution  $P_{\theta}\{\mathbf{x}\}$  depends on the set of unknown parameters  $\theta$ .

Let  $\underline{R} = \underline{R}(x)$  and  $\bar{R} = \bar{R}(x)$  be the lower and upper confidence limits for each test outcome  $x$ . Both  $\underline{R}$  and  $\bar{R}$  depends on random outcomes. Consequently, they are random variables themselves. This interval  $[\underline{R}, \bar{R}]$  is said to have the confidence probability  $\gamma$  for the unknown value  $R = R(\theta)$  if

$$P_{\theta} \{ \underline{R} \leq R(\theta) \leq \bar{R} \} \geq \gamma \quad (7.2)$$

for all  $\theta \in \Theta$ . The maximum possible value of  $\gamma$ , satisfying (7.2) for all  $\theta$ , is usually called the confidence coefficient. In an analogous way, a function  $\varphi(\mathbf{x})$  of test outcomes is called the lower (upper) confidence limit with the confidence coefficient  $\gamma$  for  $R(\theta)$  if for all  $\theta \in \Theta$

$$P_{\theta} \{ \varphi(\mathbf{x}) \leq (\geq) R(\theta) \} \geq \gamma. \quad (7.3)$$

Let us consider the requirement for validation of inequalities (7.2) and (7.3) for all possible  $\theta$  in more detail. As we have mentioned above, the distribution of outcomes  $\mathbf{x}$  depends on  $\theta$ .

Therefore the confidence probability in (7.2) and (7.3), may also depend on  $\alpha$ . Thus, (7.2) guarantees that the confidence interval is valid for any  $\alpha$ . However, if we know that  $\alpha$  belongs to a narrower subset  $\alpha_0, \alpha_0 < \alpha < \alpha_1$ , then the confidence interval will also be narrower.

## 8.2 Calculation by Direct Substitution

### 8.2.1 Point Estimates for Units

Assume that test results  $x_i$  for different units are independent and for each parameter we can find a point estimate (for instance, maximum likelihood),  $\hat{\theta}_i = \hat{\theta}_i(x_i)$ . The point estimate,  $\hat{R}$ , for the system PFFO,  $R$ , is most often obtained by substitution of the point estimates of parameters  $\hat{\theta}_i$  into the function (7.1), that is,

$$\hat{R} = R(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_m) \quad (7.4)$$

In some cases such a procedure delivers an unbiased estimate of  $R$ , if the estimates  $\hat{\theta}_i$  were unbiased themselves (see Example 7.1 below). However, even if the estimate (7.4) is biased, it usually possesses asymptotically optimal properties (unbiased and efficient), if estimates  $\hat{\theta}_i$  possess these properties. The latter statement is illustrated in Example 7.2.

#### Example 7.1 (Series system, Binomial test)

Consider a series system of  $m$  different units. Let us assume that we apply the binomial testing plan. The system PFFO is expressed as

$$R = \prod_{1 \leq i \leq m} p_i$$

where  $p_i$  is the PFFO of the  $i$ th type unit. Each unit is assumed to be tested separately. The number of tested units,  $N_i$ , and the number of failures,  $d_i$  for unit of each type are known.

In this case  $\theta_i = p_i$  and  $x_i = d_i$ . The standard unbiased point estimate for parameter  $p_i$  is

$$\hat{p}_i = 1 - \frac{d_i}{N_i}.$$

Corresponding estimate for the system PFFO like (7.4) is

$$\hat{R} = \prod_{1 \leq i \leq m} \hat{p}_i = \prod_{1 \leq i \leq m} \left(1 - \frac{d_i}{N_i}\right).$$

Since the test results  $d_i$  were assumed to be independent, this estimate is unbiased for  $R$ .  $\square$

#### Example 7.2 (Series-parallel system, binomial test)

Consider a series-parallel system of  $m$  redundant groups. The  $i$ th group consists of  $n_i$  redundant units connected in parallel (loaded regime of redundant units). We again assume that the binomial testing plan is applied.

The system PFFO has the form

$$R = \prod_{1 \leq i \leq m} [1 - (1 - p_i)^{n_i}].$$

Suppose  $N_i$  units of each type were tested  $d_i$  failures have been observed. The number of tested units does not depend on the size of redundant group.

In this case we generally use the following point estimate for the system PFFO:

$$\hat{R} = \prod_{1 \leq i \leq m} [1 - (1 - \hat{p}_i)^{n_i}] = \prod_{1 \leq i \leq m} \left[ 1 - \left( \frac{d_i}{N_i} \right)^{n_i} \right].$$

This estimate is biased for  $n_i > 1$ . This is because

$$E \left( \frac{d_i}{N_i} \right)^{n_i} \neq (1 - p_i)^{n_i}. \quad \square \square \square$$

### 8.2.2 Confidence Limits

Thus, the problem of system point estimator construction can be easily solved by a direct substitution of point estimates of unit parameters into the system reliability function (7.1). Finding the confidence limits this point estimate (7.4) is more complex problem. To illustrate this, let us try to construct the lower limit for the system PFFO defined in (7.1) by direct use the lower confidence limits of unit parameters  $\square_i$ . Assume that the function  $R(\square) = R(\square_1, \square_2, \dots, \square_m)$  is monotonically increasing in each  $\square_i$ . This assumption translates to a natural condition that the system PFFO improves with the growth of units' reliability. Assume further that the lower  $\square$ -confidence limit  $\underline{\theta}_i = \underline{\theta}_i(x_i)$  is constructed for each parameter  $\square_i$  on the basis of test results  $x_i$ , that is,

$$P_{\theta}(\underline{\theta}_i \leq \theta_i) \geq \gamma, \quad i = \overline{1, m}. \quad (7.5)$$

Let us now find the lower confidence limit  $\underline{R}$  for the system PFFO,  $R$ , by direct substitution of confidence limits  $\underline{\square}_i$  into (7.1),

$$\underline{R} = R(\underline{\square}_1, \underline{\square}_2, \dots, \underline{\square}_m). \quad (7.6)$$

Monotonicity of function  $R(\square)$  implies:

$$\bigcap_{1 \leq i \leq m} \{\theta_i \leq \theta_i\} \subset \{R(\underline{\theta}_1, \underline{\theta}_2, \dots, \underline{\theta}_m) \leq R(\theta_1, \theta_2, \dots, \theta_m)\}.$$



From this, equation (7.5), and the independence of test results, we have

$$P_{\theta} \{R \leq R(\theta)\} \geq P_{\theta} \left\{ \bigcap_{1 \leq i \leq m} (\theta_i \leq \theta_i) \right\} = \prod_{1 \leq i \leq m} P_{\theta} (\theta_i \leq \theta_i) \geq \gamma^m.$$

Thus, the only statement we can make is that the lower confidence limit (7.6) possesses the confidence coefficient *not less* than  $\gamma^m$ .

However, we don't know to what real value of  $\gamma$  it corresponds. Moreover, this value decreases very fast with the growth of the number of redundant groups. This confidence interval is too conservative and ineffective even for small  $m$ . For instance, for  $m=10$  and  $\gamma=0.9$ ,  $\gamma^m \approx 0.35$ . It means, in particular, that if we need to construct the confidence limit for the system PFFO with the confidence coefficient not less than 0.9, we should increase individual unit confidence limits up to  $\approx 0.99$ . Thus, the more the number of redundant groups in the system, the lower is the system PFFO estimate from direct substitution.

Thus, an attempt to solve the problem via direct substitution of units' confidence limits is ineffective. The problem attracted the attention of American and Russian researchers in the past two decades and some proper approaches were discovered. Some of these approaches are described in the following section.

### 8.2.3 General Method

The method of constructing the confidence limits for a function of several arguments  $R=R(\mathbf{x})=R(x_1, x_2, \dots, x_m)$  is a natural generalization of the method developed for a function with one argument (see Section 1.4).

#### Belyaev Method

The following simplified method was considered in [Belyaev, 1966, 1968]. Let  $S$  be a system statistic, or some function of test results:  $S=S(\mathbf{x})=S(x_1, \dots, x_m)$ . We can use as such a statistic as a point estimate of  $R$ , that is,  $S = \hat{R}$ . Consider the plane  $(R, S)$  represented on Figure 7.1.

Figure 7.1

As we did in one dimensional case (see Figure 1.2), let us find (for each fixed value  $R$ ) a corresponding  $\gamma$ -zone of  $H_R$  on the  $S$ -axis. Probability that statistic  $S$  will occur in  $H_R$  will be not less than  $\gamma$ .

Unlike an one-dimensional case, a multi-dimensional case may be degenerate in the sense that different values of a vector of parameters might correspond to the given fixed value  $R=R(\mathbf{x})$ . Let us denote

$$A_R = \{ \mathbf{x} : R(\mathbf{x}) = R \}$$

the set of values of parameters-vector  $\theta$ , for which the value of the reliability function,  $R(\theta)$  is equal to  $R$ . The distribution function of r.v.  $S$  for the given value of a vector of parameters  $\theta$  is denoted by

$$F(t, \theta) = P_{\theta}(S \leq t). \quad (7.7)$$

For the sake of simplicity, let us assume that function (7.7) is continuous and strictly monotonic in  $t$ .

For a given fixed  $\theta$ , let us choose  $t_1(\theta)$  and  $t_2(\theta)$  in the same manner as in the one dimensional case (Section 1.4):

$$\begin{cases} F(t_1, \theta) = \alpha, \\ F(t_2, \theta) = 1 - \beta \end{cases} \quad (7.8)$$

that is,  $t_1(\theta)$  and  $t_2(\theta)$  are respective quantiles of levels  $\alpha$  and  $\beta$  of distribution (7.7). By construction, the probability that statistic  $S$  will be covered by the interval  $[t_1(\theta), t_2(\theta)]$  is equal to  $\beta - \alpha$ . For a fixed value of the reliability index  $R$ , we construct the interval  $[K_1(R), K_2(R)]$  by joining all intervals  $[t_1(\theta), t_2(\theta)]$  for different  $\theta$  from  $A_R$ . This interval is considered as a  $\beta$ -zone  $H_R$  of the reliability index,  $R$ . The lower and the upper limits of the  $\beta$ -zone has the form (see Figure 7.1):

$$\begin{aligned} K_1(R) &= \min_{\theta \in A_R} t_1(\theta), \\ K_2(R) &= \min_{\theta \in A_R} t_2(\theta), \end{aligned} \quad (7.9)$$

Thus, the probability that the  $\beta$ -zone  $H_R$  covers statistic  $S$  satisfies the inequality

$$P_{\theta} \{S \in H_R\} = P_{\theta} \{K_1(R) \leq S \leq K_2(R)\} \geq \gamma \quad (7.10)$$

for each  $\theta \in A_R$  and for any values of  $R$ .

Let  $\mathbf{x}^* = (x_1^*, \dots, x_m^*)$  denote the value of a random vector  $\mathbf{x} = (x_1, \dots, x_m)$  obtained as a test result, and  $S^* = S(\mathbf{x}^*)$  denote the corresponding statistic  $S = S(\mathbf{x})$ . We call values  $\mathbf{x}^*$  and  $S^*$  the *observed value of vector  $\mathbf{x}$*  and the *observed value of statistic  $S$*  respectively. We assume further that the limits of  $\beta$ -zone,  $K_1(R)$  and  $K_2(R)$ , are continuous and monotonically increasing in  $R$ . The lower and upper limits for  $R = R(\theta)$  are defined from conditions

$$K_2(\underline{R}) = S^*, \quad K_1(\overline{R}) = S^*, \quad (7.11)$$

where  $S^*$  is the observed value of statistic  $S$  (see Figure 7.1). Then, by construction and taking into account (7.10), we have

$$P_{\theta} \{\underline{R} \leq R(\theta) \leq \overline{R}\} = P_{\theta} \{S^* \in H_R\} = P_{\theta} \{K_1(R) \leq S^* \leq K_2(R)\} \geq \gamma$$

for each  $\theta \in A_R$  and for any allowed value of  $R$ . From here it follows that

$$P_{\theta} \{ \underline{R} \leq R(\theta) \leq \bar{R} \} \geq \gamma$$

for possible values of  $\theta$ . It means that the interval  $[\underline{R}, \bar{R}]$  is the confidence interval for  $R=R(\theta)$  with the confidence coefficient not less than  $1-\alpha-\beta$ .

If the limits of the  $\theta$ -zone monotonically decreases in  $R$ , we can write, repeating above arguments, that the lower and upper limits can be found from the equations:

$$K_1(\underline{R}) = S^*, \quad K_2(\bar{R}) = S^*. \quad (7.11)$$

### Bol'shev-Loginov Method

A closely related approach was suggested in [Bol'shev (1965), Bol'shev and Loginov (1966)]. In Belyaev approach, the main idea was to find the maximum limits of the  $\theta$ -zone for each fixed value of  $R$ . It is also possible to find these limits in another way; namely, for each given test result  $S$ . Roughly speaking, we are finding the limits for a given fixed  $R$  in the vertical direction, (i.e., along  $S$ -axis on Figure 7.1) in the first case. In the second case, for the given fixed  $S^*$ , the limits are found along the  $R$ -axis. Let us introduce functions

$$\begin{aligned} \Phi_1(S^*, R) &= \min_{\theta \in A_R} F(S^*, \theta) \\ \Phi_2(S^*, R) &= \max_{\theta \in A_R} F(S^*, \theta) \end{aligned} \quad (7.12)$$

Assume that these functions are continuous and strictly monotonically decreasing in  $R$ . Then (7.12) gives the lower ( $\underline{R}$ ) and upper ( $\bar{R}$ ) limits of the confidence interval for  $R(\theta)$  obtained

$$\begin{aligned} \Phi_1(\underline{R}, S^*) &= 1 - \beta \\ \Phi_2(\bar{R}, S^*) &= \alpha. \end{aligned} \quad (7.13)$$

from the test results  $S^*$ .

If these functions are monotonically increasing in  $R$ , then the lower and upper limits of the confidence interval can be found from the following equations:

$$\begin{aligned} \Phi_1(\underline{R}, S^*) &= \alpha \\ \Phi_2(\bar{R}, S^*) &= \beta. \end{aligned}$$

### Neumann Method

A general approach to constructing confidence sets, which includes Belyaev and Bol'shev-Loginov methods as particular cases, originates from well known works [Neumann(1935), (1937)]. Let  $S=S(\mathbf{x})$  again be some statistic. The set  $C_\alpha$  of values of statistic  $S$  defined as

$$C_\alpha = \{S: t_1(\theta) \leq S \leq t_2(\theta)\} \quad (7.14)$$

corresponds to some possible value of parameter  $\theta=(\theta_1, \dots, \theta_m)$ . As done previously,  $t_1(\theta)$  and  $t_2(\theta)$  are chosen under condition (7.8). The following equality can be established for each  $\alpha$

$$P_\theta \{S \in C_\alpha\} = \alpha \quad (7.15)$$

(Here, as above,  $\alpha=1-\alpha-\alpha$ .) Let  $S^*$  be an observed value of statistic  $S$  from the test. Each  $S$  has its reflection on set  $H(S^*)$  of parameter  $\theta$  such that value  $S^*$  belongs to the set  $C_\alpha$ :

$$H(S^*) = \{\theta: S^* \in C_\alpha\}.$$

Owing to (7.14), the set  $H(S^*)$  is given by the following conditions:

$$\begin{aligned} t_2(\theta) &\geq S^* & (7.16) \\ t_1(\theta) &\leq S^* \end{aligned}$$

Taking into account the definition of  $t_1(\theta)$  and  $t_2(\theta)$ , the inequalities (7.16) can also be written directly via the distribution function of statistic  $S$  in the following form:

$$\begin{aligned} F(S^*, \theta) &\geq \alpha, \\ 1 - F(S^*, \theta) &\geq \beta, \end{aligned} \quad (7.17)$$

For each fixed  $\alpha$ , events  $\{\theta \in H(S^*)\}$  and  $\{S^* \in C_\alpha\}$  are equivalent by construction. It follows from (7.15), that for any  $\alpha$  equality holds.

$$P_\theta \{\theta \in H(S^*)\} = \alpha \quad (7.18)$$

A collection of sets  $H(S^*)$ , which satisfies (7.18) for all possible  $\alpha$ , is called a collection of  $\alpha$ -confidence sets for  $\theta$ .

Let us now determine the lower and upper limits of  $R=R(\alpha)$ .

$$\begin{aligned} \bar{R} &= \max_{\theta \in H(S^*)} R(\theta), \\ \underline{R} &= \min_{\theta \in H(S^*)} R(\theta), \end{aligned} \quad (7.19)$$

The minimum and maximum are taken in the confidence set  $H(S^*)$ . (We have assume that the minimum and maximum attained on the set  $H(S^*)$ . If it is not the case, one must use **infimum** and **supremum** instead of **min** and **max**, respectively.) For each fixed  $\alpha$  the following relationship is valid:

$$\{\theta \in H(S^*)\} \subset \{\underline{R} \leq R(\theta) \leq \bar{R}\}.$$

Taking (7.17) into account, we have

$$P_\theta \{\underline{R} \leq R(\theta) \leq \bar{R}\} \geq \gamma.$$

It means that the interval  $(\underline{R}, \bar{R})$  thus constructed, is the confidence interval for  $R=R(\theta)$  with the confidence coefficient not less than  $1-\alpha-\beta$ .

**Remark 7.1** Under some additional monotonicity and continuity assumption on  $R(\theta)$ , which almost always hold in reliability problems, all three approaches considered above are shown to be factually equivalent. In other words, expressions (7.11), (7.13) and (7.19) give the same confidence interval  $(\underline{R}, \bar{R})$  if one uses same  $\alpha$ ,  $\beta$  and statistic  $S$  (see Pavlov, 1982, pp.82-83) under these restrictions. From a computational viewpoint, these different methods are dual descriptions of mathematical programming problems (conditional optimization problems). Thus, the function  $F(S^*, \theta)$ , for which extremum in (7.12) is searched, represents restrictions in (7.17) for the last approach. The function  $R(\theta)$ , for which extremum in (7.19) is searched, represents restrictions of the type  $R=R(\theta)$  in the Belyaev and Bol'shev-Loginov approaches (for more details, see [Pavlov, 1982]).

#### 8.2.4 Arbitrary Distribution of Statistic $S$

Previously we assumed in the previous sub-section that statistic  $S$  has continuous distribution. Now we consider a more general case where the distribution of test results  $\mathbf{x}=(x_1, x_2, \dots, x_m)$  and the statistic  $S=S(\mathbf{x})$  are arbitrary (possibly, discrete). In this case let us denote

$$\begin{aligned} t_1(\theta) &= \sup \{t: P_\theta(S < t) \leq \alpha\}, \\ t_2(\theta) &= \inf \{t: P_\theta(S > t) \leq \beta\}, \end{aligned} \quad (7.20)$$

where  $\alpha + \beta < 1$ . Let us introduce the set

$$C_\theta = \{S: t_1(\theta) \leq S \leq t_2(\theta)\}. \quad (7.21)$$

Continuity of the function  $P_\theta(S > t)$  from the right and of the function  $P_\theta(S < t)$  from the left, together with (7.20) imply the inequality

$$P_\theta \{S \in C_\theta\} = P_\theta \{t_1(\theta) \leq S \leq t_2(\theta)\} \geq \gamma \quad (7.22)$$

is valid for each fixed  $\theta$  where  $\alpha + \beta = 1 - \alpha - \beta$ .

Let  $S^*$  be an observed value of statistic  $S$ . Let the set

$$H(S^*) = \{\theta: S^* \in C_\theta\} \quad (7.23)$$

correspond to each value of  $S^*$ . In accordance with (7.21) this set is given by the inequalities

$$\begin{cases} t_1(\theta) \leq S^* \\ t_2(\theta) \geq S^* \end{cases} \quad (7.24)$$

Due to (7.22), the inequality

$$P_\theta \{\theta \in H(S^*)\} = P_\theta \{S^* \in C_\theta\} \geq \gamma \quad (7.25)$$

holds for each fixed  $\theta$ . In other words, sets  $H(S^*)$  form a collection of confidence sets for  $\theta$  with the confidence coefficient not less than  $\gamma$ .

Let us again determine the lower and upper limits  $\underline{R}$  and  $\bar{R}$  as the minimum and maximum values of the function  $R(\theta)$ , respectively, for all values of parameter  $\theta$  which belong to the confidence set  $H(S^*)$ . It can be shown (see p.1 of the Appendix to this chapter) that the final expressions for the confidence limits have the form

$$\underline{R} = \min R(\theta), \quad \bar{R} = \max R(\theta) \quad (7.26)$$

where minimum and maximum are taken for all values of the parameters  $\theta = (\theta_1, \dots, \theta_m)$  which satisfy the following inequalities:

$$\begin{cases} P_\theta(S \leq S^*) \geq \alpha \\ P_\theta(S \geq S^*) \geq \beta \end{cases} \quad (7.27)$$

The interval  $(\underline{R}, \bar{R})$  is the confidence interval for  $R=R(\theta)$  with the confidence coefficient not less than  $1-\alpha-\beta$ . Formulas (7.26) and (7.27) include previously considered confidence limits as particular cases.

From formulas (7.26) and (7.27) we can easily obtain one-side confidence limits for  $R=R(\theta)$ . For instance, let the reliability function  $R(\theta)=R(\theta_1, \dots, \theta_m)$  be monotonically increasing in each argument  $\theta_i$  and the function  $P_\theta(S \leq S^*)$  [ $P_\theta(S \geq S^*)$ ] is monotonically decreasing (increasing) in each  $\theta_i$ . Setting  $\alpha = 1-\beta$  and  $\beta = 0$ , and from (7.26) and (7.27), we obtain the lower confidence limit for  $R$ :

$$\underline{R} = \min R(\theta) \quad (7.28)$$

where minimum is taken for all parameters  $\theta$  satisfying the inequality

$$P_{\theta}(S \geq S^*) \geq 1 - \gamma. \quad (7.29)$$

In correspondence with (7.12) and (7.13) the value  $\underline{R}$  is determined from the following equation in respect to  $R$ :

$$\max_{\theta \in A_R} P_{\theta}(S \geq S^*) = 1 - \gamma. \quad (7.30)$$

where the maximum is taken in the set  $A_R$  of parameters for which  $R(\theta) = R$ .

In an analogous way, setting  $\alpha = 1 - \alpha$  and  $\alpha = 0$ , we obtain from (7.26) and (7.27) the upper  $\alpha$ -confidence limit for  $R$ :

$$\bar{R} = \max R(\theta) \quad (7.31)$$

where the maximum is taken for all parameters  $\theta$  satisfying the inequality

$$P_{\theta}(S \leq S^*) \geq 1 - \gamma. \quad (7.32)$$

The value  $\bar{R}_{505}$  is determined from the following equation:

$$\max_{\theta \in A_R} P_{\theta}(S \leq S^*) = 1 - \gamma. \quad (7.33)$$

For the chosen statistic  $S = S(\mathbf{x})$ , the lower and upper confidence limits of (7.33) can not be improved for more general conditions [see (Pavlov, 1977, a,b)]. Different statistics  $S' = S'(\mathbf{x})$  generate different confidence limits. The problem of choosing the best initial statistic is still open.

### Example 7.3 (Binomial test)

Let us consider a standard binomial test scheme. The system consists of  $m$  types of units, and the reliability index of the  $i$ th type unit equals  $p_i$ . The system PFFO,  $R$ , is the function of the vector of units reliability parameters  $\mathbf{p} = (p_1, \dots, p_m)$ :

$$R = R(\mathbf{p}) = R(p_1, \dots, p_m). \quad (7.34)$$

We assume that this function is increasing in each parameter  $p_i$ , that is, the system PFFO increases if the unit reliability increases. The Bernoulli test is invoked for determining each parameter  $p_i$ . During testing  $N_i$  units were tested and  $d_i$  failures have been observed. All tests are supposed to be independent.

In this case the vector of unknown parameters  $\theta$  is the vector of binomial parameters  $\mathbf{p}$ , and the vector of test results  $\mathbf{x}$  is the vector of the numbers of failures for units of different types,  $\mathbf{d} = (d_1,$

... , $d_m$ ). We need to construct the lower confidence limit with the confidence coefficient not less than the given  $\square\square\square$  on the basis of the test results  $\mathbf{d}$ .  $\square$

Let us take the point estimate of the system PFFO for the initial statistic  $S=S(\mathbf{x})$ , that is,

$$S = \hat{R} = R(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_m) \quad (7.35)$$

where  $\hat{p}_i = 1 - \frac{d_i}{N_i}$  is the standard point estimate of the parameter  $p_i$ ,  $\overline{1, m}$ . Let  $\mathbf{d}^*=(d_1^*, \dots, d_m^*)$  be the observed value of the vector of failures, and  $S^*=S(\mathbf{d}^*)$  be the corresponding observed statistic  $S$ . From (7.28) and (7.29) we obtain the following formula for the lower confidence limit (for  $R$  with confidence coefficient not less than  $\square\square\square$ ):

$$\underline{R} = \max R(p_1, p_2, \dots, p_m) \quad (7.36)$$

where the minimum is taken over all parameters  $\mathbf{p}=(p_1, \dots, p_m)$  which satisfy the following conditions:

$$\sum_{S(\mathbf{d}) \geq S(\mathbf{d}^*)} \prod_{1 \leq i \leq m} \binom{N_i}{d_i} p_i^{N_i - d_i} \cdot (1 - p_i)^{d_i} \geq 1 - \gamma, \quad (7.37)$$

$$0 \leq p_i \leq 1, \quad i = \overline{1, m} \quad (7.38)$$

We notice that inequality (3.37) corresponds to (7.29).

In accordance with (7.30), we want to find maximum of (7.37), that is, the value of  $\underline{R}$  can be found from the equation in respect to  $R$

$$\max_{\mathbf{p} \in A_R} \sum_{S(\mathbf{d}) \geq S(\mathbf{d}^*)} \prod_{1 \leq i \leq m} \binom{N_i}{d_i} p_i^{N_i - d_i} \cdot (1 - p_i)^{d_i} = 1 - \gamma, \quad (7.39)$$

where the maximum is taken over the set  $A_R$  of parameters satisfying restrictions

$$\begin{aligned} R(p_1, p_2, \dots, p_m) &= R, \\ 0 \leq p_i &\leq 1, \quad i = \overline{1, m} \end{aligned} \quad (7.40)$$

Sometimes it is more convenient to rewrite (7.36) in another form. Remember that the statistic  $S(\mathbf{d})=S(d_1, \dots, d_m)$  is any function of test results which is monotone non-decreasing for each  $d_i$ . For instance,  $S(\mathbf{d})$  can be a point estimate of  $R$ . Let us order all possible values of the vector of test results,  $\mathbf{d}=(d_1, \dots, d_m)$  with decreasing value of the statistic  $S(\mathbf{d})$ . In other words, each value of  $m$ -dimensional vector  $\mathbf{d}=(d_1, \dots, d_m)$  with positive integer coordinates corresponds to a number  $n = n(\mathbf{d})$  which is determined in such a way that for any vector  $\mathbf{K}=(K_1, \dots, K_m)$  with  $S(\mathbf{K}) > S(\mathbf{d})$ , we have  $n(\mathbf{d}) \geq n(\mathbf{K})$ . Then the confidence limit (7.36) as



$$\underline{R} = \min R(p_1, \dots, p_m), \quad (7.41)$$

where the minimum is taken under the constrains

$$\sum_{n(\mathbf{d}) \geq n(\mathbf{d}^*)} \prod_{1 \leq i \leq m} \binom{N_i}{d_i} p_i^{N_i - d_i} \cdot (1 - p_i)^{d_i} \geq 1 - \gamma, \quad (7.42)$$

$$0 \leq p_i \leq 1, \quad i = \overline{1, m} \quad (7.43)$$

The difference between (7.42) and (7.37) is in the summation limit. Here  $n(\mathbf{d}) = n(d_1, \dots, d_m)$  is such a ordering of test result vectors  $\mathbf{d}$  that  $n(\mathbf{d})$  monotonically increases (non-decreases) in each  $d_i$ . The first number is assigned to the null-vector  $\mathbf{d} = (0, 0, \dots, 0)$ . The next one to vectors of the type  $(0, \dots, 0, 1, 0, \dots, 0)$ , and so on. This approach, connected with the ordering of test results, was used in [Buehler, 1957] for a parallel system consisting of two units (see Example 7.5 below and also Section 7.4).

Example 7.4 (Test with no failures)

Under conditions of the previous example, let us consider a particular case where no failure was observed of any unit:  $d_1^* = d_2^* = \dots = d_m^* = 0$ . In this case the formula for the lower  $\square$ -confidence limit for the system PFFO follows from (7.36)-(7.38) or from (7.41)-(7.43).

$$\underline{R} = \min R(p_1, p_2, \dots, p_m) \quad (7.44)$$

where the minimum is taken under the restrictions

$$\prod_{1 \leq i \leq m} p_i^{N_i} \geq 1 - \gamma, \quad (7.45)$$

$$0 \leq p_i \leq 1, \quad i = \overline{1, m} \quad (7.46)$$

This is the best lower  $\square$ -confidence limit for the system PFFO in the case of no failure test (see Chapter 6).  $\square$

Example 7.5 (Buehler Problem) Find the PFFO for a parallel system, consisting of two units. The system is tested by a binomial scheme. The index of interest is

$$R(\mathbf{p}) = R(p_1, p_2) = 1 - (1 - p_1)(1 - p_2).$$

Each vector of test results,  $\mathbf{d} = (d_1, d_2)$ , corresponds to some number  $n(\mathbf{d}) = n(d_1, d_2)$  such that  $n(\mathbf{d})$  is monotonically decreasing in each  $d_i, i=1, 2$ . The minimum number, naturally, is assigned to the vector  $(0,0)$ . The next one to the vectors  $(0,1)$  or  $(1,0)$ , and so on. From (7.41)-(7.43) we get the following lower  $\square$ -confidence limit for the system PFFO

$$\underline{R} = \min_{\mathbf{p}} \{1 - (1 - p_1)(1 - p_2)\} \quad (7.47)$$

where minimum is taken in all  $\mathbf{p}=(p_1, p_2)$  which satisfy (7.42). The solution of this problem was given by Buehler(1957).□

Example 7.6 (Series system, Binomial test) Under conditions of Example 7.3 consider a particular case where a series system consists of  $m$  different units. In this case the system PFFO is

$R = \prod_{1 \leq i \leq m} p_i$ . Let the point estimate of the system PFFO be taken as an initial statistic  $S=S(\mathbf{d})$ , that is,

$$S(\mathbf{d}) = \hat{R}(\mathbf{d}) = \prod_{1 \leq i \leq m} \left(1 - \frac{d_i}{N_i}\right).$$

Denote the observed value of the vector  $\mathbf{d}=(d_1, \dots, d_m)$  by  $\mathbf{d}^*=(d_1^*, \dots, d_m^*)$ . Then from (7.36)-(7.38) we obtain the lower  $\alpha$ -confidence limit for the system PFFO

$$\underline{R} = \min \prod_{1 \leq i \leq m} p_i$$

where the minimum is taken over all parameters  $(p_1, \dots, p_m)$  which satisfy the restrictions

$$\sum_{\hat{R}(\mathbf{d}) \geq \hat{R}(\mathbf{d}^*)} \prod_{1 \leq i \leq m} \binom{N_i}{d_i} p_i^{N_i - d_i} \cdot (1 - p_i)^{d_i} \geq 1 - \gamma, \quad (7.48)$$

$$0 \leq p_i \leq 1, \quad i = \overline{1, m} \quad 517.$$

The difference between (7.48) and (7.37) is in the limit of the summation.

If probabilities of failure are small, i.e.,  $q_i=1-p_i \ll 1$ , and the number of tested units,  $N_i$ , is large, we can use the Poisson approximation for solving this problem. In this case we can consider that the number of failures,  $d_i$ , approximately has the Poisson distribution with parameter  $\lambda_i=N_i q_i$ ,  $i = \overline{1, m} \quad 518$ .

For conditions above, we can write an approximation for  $R$ :

$$R = \prod_{1 \leq i \leq m} (1 - q_i) \approx \exp\left(-\sum_{1 \leq i \leq m} q_i\right) = \exp\left(-\sum_{1 \leq i \leq m} \frac{\lambda_i}{N_i}\right).$$

Thus, the problem of finding the lower confidence limit of  $R$  is reduced to the problem of finding the upper confidence limit  $\bar{f} \quad 519$  for the function of the Poisson parameters  $\lambda=(\lambda_1, \dots, \lambda_m)$ :

$$f(\Lambda) = \sum_{1 \leq i \leq m} \frac{\Lambda_i}{N_i}.$$

It follows that the Poisson approximation leads us to an approximate lower  $\square$ -confidence limit for the system PFFO in the form  $\underline{R} = e^{-\bar{f}}$  where  $\bar{f} = \max_{1 \leq i \leq m} \sum \frac{\Lambda_i}{N_i}$ . The maximum here is taken over all parameters  $\square = (\square_1, \dots, \square_m)$  satisfying the conditions

$$\sum_{\hat{R}(\mathbf{d}) \geq \hat{R}(\mathbf{d}^*)} \prod_{1 \leq i \leq m} e^{-\Lambda_i} \left( \frac{\Lambda_i^{d_i}}{d_i!} \right) \geq 1 - \gamma,$$

$$\square_i \geq 0, \quad i = \overline{1, m} \quad (7.48)$$

The problem of constructing of an approximate confidence limit for the PFFO of a series system was considered by Bol'shev and Loginov (1966) for equal  $N_i$ 's; and by Pavlov (1973) and Sudakov (1974) for arbitrary  $N_i$ .

### 8.3 Series Structures

#### 8.3.1 Binomial model

Consider a series system consisting of units of  $m$  different types. The number of units of the  $i$ th type equals  $r_i$ ,  $i = \overline{1, m}$ . All system's units are independent. The system PFFO for some fixed operation time  $t_0$  is

$$R = \prod_{1 \leq i \leq m} p_i^{r_i}, \quad (7.49)$$

where  $p_i$  is the PFFO of a unit of the  $i$ th type. Assume that  $N_i$  units of type  $i$  have been tested during time  $t_0$  and  $d_i$  failures were observed. The test results  $d_1, \dots, d_m$  are supposed to be independent. Let us construct the confidence limits of the system PFFO (7.49) on the basis of the vector of test results  $\mathbf{d} = (d_1, \dots, d_m)$ . Notice that for practical purposes, most important is the lower confidence limit.

Consider the case where the units TTFs are exponentially distributed. The PFFO for the unit of the  $i$ th type during time  $t$  is given by the formula  $p_i(t) = \exp(-\square_i t)$ , where  $\square_i$  is the unit's failure rate. Then the system PFFO defined in (7.49) has the form

$$R = \prod_{1 \leq i \leq m} p_i^{r_i}(t_0) = e^{-t_0 \sum_{1 \leq i \leq m} r_i \lambda_i}. \quad (7.50)$$

Assume that units of the  $i$ th type were tested by plan  $[N_i R T_i]$ . This notation means that initially there were  $N_i$  units, failed units were replaced by new ones, and the test duration was  $T_i$ . The number of observed failures equals  $d_i$ . We need to construct the lower  $\alpha$ -confidence limit for the system PFFO defined by (7.50) on the basis of test results  $\mathbf{d}=(d_1, \dots, d_m)$ . We would like to emphasize, in contrast to the binomial case, values  $T_i$  for different types of units may be different and do not necessarily coincide with the operational time  $t_0$ .

R.v.  $d_i$  has the Poisson distribution with the parameter  $\lambda_i=N_i T_i \lambda_i$ ,  $i = \overline{1, m}$  (5.24). The problem is now reduced to the finding the upper  $\alpha$ -confidence limit,  $\bar{f} = \bar{f}(\mathbf{d})$ , (5.25) of the following function of the Poisson parameters  $\lambda=(\lambda_1, \dots, \lambda_m)$

$$f(\lambda) = \sum_{1 \leq i \leq m} \left( \frac{r_i}{N_i T_i} \right) \lambda_i.$$

Finally the lower  $\alpha$ -confidence limit for the system PFFO can be found as

$$\underline{R} = \exp(-\bar{f}t_0).$$

Notice that we meet the same problem if we use the Poisson approximation for a binomial model. Indeed, in accordance with well known limit theorems of probability theory, binomial distribution of the r.v.  $d_i$  can be approximated by a Poisson distribution with the parameter  $\Lambda_i = N_i q_i$  if  $q_i \ll 1$  and  $N_i \gg 1$ , so that  $N_i q_i$  remains fixed. (Here  $q_i = 1 - p_i$  is the probability of failure of a unit of the  $i$ th type). It follows that for highly reliable units, that is,  $q_i \ll 1$ ,  $i = \overline{1, m}$  526, and large number of tested units,  $N_i$ , the problem of approximate confidence estimate of the system PFFO (7.49) can be reduced to the confidence estimation of value

$$R = \prod_{1 \leq i \leq m} (1 - q_i)^{r_i} \approx \exp\left(-\sum_{1 \leq i \leq m} r_i q_i\right) = \exp\left(-\sum_{1 \leq i \leq m} \frac{r_i}{N_i} \Lambda_i\right).$$

It means that the problem again is reduced to the finding the upper  $\alpha$ -confidence limit  $\bar{f} = \bar{f}(d)$  527 of the following function of Poisson parameters:

$$f(\Lambda) = \sum_{1 \leq i \leq m} \frac{r_i}{N_i} \Lambda_i.$$

Again the lower  $\alpha$ -confidence limit for the system PFFO can be found as  $\underline{R} = e^{-\bar{f}}$ .

### 8.3.2 Lidstrom-Madden Method

The Lidstrom-Madden method is discussed in a well known book by Lloyd and Lipov (1962). At that time, the method was considered heuristic and approximate because there was no proof that this is indeed the lower confidence limit  $\underline{R}$  for the system PFFO (7.49). The correct confirmation of this method was obtained by Pavlov (1973), Sudakov (1974) and others (see below Sections 7.6.1 and 7.6.6).

The main idea of this method is grounded on the hypothetical construction of possible outcomes of a system test based on the results of tests of individual units. For the sake of simplicity, consider a simple case where  $m=2$  and  $r_1=r_2=1$ , that is the system consists of two different units. Without any loss in generality we can assume that

$$\min_{1 \leq i \leq m} N_i = N_1 \quad (7.51)$$

that is, units are ordered by the increasing number of tested units. The unit test results can be given by two sets

$$(x_{11}, x_{12}, \dots, x_{1N_1}) \quad (7.52)$$

$$\text{and } (x_{21}, x_{22}, \dots, x_{2N_2}), \quad (7.53)$$

where  $x_{ij}$  is an indicator of failure of the  $i$ th unit at the  $j$ th test

$$x_{ij} = \begin{cases} 1 & \text{if the unit has failed,} \\ 0 & \text{if the unit has not fail.} \end{cases}$$

Using these notations, let us try to enumerate all possible outcomes of the system tests. For this purpose, for each test result of the first set,  $x_{1j}$  we correspond a randomly chosen test result of the second set,  $x_{2j}$ . Thus we obtain  $N_1$  pairs

$$(x_{1j}, x_{2k}) \quad (7.54)$$

each of which can be interpreted as the result of testing of a system. The number of "tested" systems is equal to  $N_1$  and the random number of the system failures,  $\xi$ , is equal to the number of pair for which at least one unit has failed (at least one of  $x_{1j}$  or  $x_{2j}$  equals 1). For a given set of test results (7.52) and (7.53), the number of such pairs,  $\xi$ , is random since we chose the pairs randomly. Therefore, we take as the number of the system failures the value  $D_1$  (later is called "equivalent number of failures") which is the mathematical expectation of  $\xi$ . It is clear that

$$D_1 = E\xi = N_1(1 - \hat{R})$$

where  $\hat{R}$  is the point estimate of the system PFFO. In turn, this value of  $\hat{R}$  which is calculated as

$$\hat{R} = \hat{p}_1 \hat{p}_2 = \left(1 - \frac{d_1}{N_1}\right) \cdot \left(1 - \frac{d_2}{N_2}\right).$$

Following such a heuristic procedure, we obtain  $N_1$  system tests (remember that  $N_1$  is the minimum number of tested units) and equivalent number of failures,  $D_1$ . The lower  $\square$ -confidence limit  $\underline{R}$  for the system PFFO is defined as

$$\underline{R} = \underline{P}_{\square}(N_1, D_1) \quad (7.55)$$

where  $\underline{P}_{\square}(N, D)$  is the standard Clopper-Pearson lower  $\square$ -confidence limit [Clopper and Pearson, 1934] for the binomial parameter  $p$ . Confidence limit (7.55) corresponds to the Bernoulli test with  $N$  units tested and  $d$  failures observed.

In the case of  $m \geq 2$  and  $r_1 = r_2 = \dots = r_m = 1$  the lower confidence limit  $\underline{R}$  for the system PFFO is found by formula (7.55) in analogous way. Again  $N_1$  is the minimum number of tested units defined

in (7.51) and  $D_1$  is the equivalent number of failures. The latter value is calculated using the formula  $D_1 = N_1(1 - \hat{R})$  533. The point estimate  $\hat{R}$  534 is calculated as

$$\hat{R} = \prod_{1 \leq i \leq m} \left(1 - \frac{d_i}{N_i}\right).$$

Obviously, the equivalent number of failures,  $D_1$ , might be non-integer. In this case  $\underline{P}_{\square}(N_1, D_1)$  can be calculated by interpolation of corresponding values in tables of binomial distribution or with the use of the beta function. In this case the lower  $\square$ -confidence limit  $\underline{P}_{\square}(N_1, D_1)$  can be found by solving the following equation for  $p$

$$B_p(N_1 - D_1, D_1 + 1) = 1 - \square$$

$B_p(a, b)$  is the beta function defined as

$$B_p(a, b) = \frac{\int_0^p x^{a-1} (1-x)^{b-1} dx}{\int_0^1 x^{a-1} (1-x)^{b-1} dx}$$

(The reader can find some details in Section 2 of Appendix to the current chapter.) Corresponding numerical tables of  $P_{\square}(N, d)$  can be found in [Pearson, 1934], [Nation Bureau of Standards, 1950], [U.S. Army Ordnance Corps Pamphlet, 1952], [Romig, 1953], [Bol'shev and Smirnov, 1965], [Sudakov, 1975], [Ushakov, ed., 1994], and others.

Sometimes it is more convenient to write formula(7.55) for the lower  $\gamma$ -confidence limit of the system PFFO as

$$\underline{R} = \min_{1 \leq i \leq m} \underline{P}_{\gamma}(N_i, D_i), \quad (7.56)$$

where  $D_i = N_i(1 - \hat{R})$  536. This expression allows us to give one more interpretation of this method. For each unit  $i$  let us calculate the equivalent number of failures  $D_i$  from the condition

$$\frac{D_i}{N_i} = 1 - \hat{R}, \quad \hat{R} = \prod_{1 \leq i \leq m} \left(1 - \frac{d_i}{N_i}\right). \quad (7.57)$$

In other words,  $D_i$  is defined in such a way that the point estimate of the  $i$ th unit coincides with the point estimate of the system as a whole. For each unit we construct the lower  $\square$ -confidence limit for the PFFO,  $\underline{P}_{\square}(N_i, D_i)$  based on the equivalent number of failures. The minimum of these confidence limits, obtained this way, is taken as the lower confidence limit  $\underline{R}$  for the entire system. This interpretation allows us to expand the method for series systems for  $r_i \gg 1$ ,  $i = \overline{1, m}$  537, and for

series-parallel systems (see below Section 7.6.1). For a series system with  $r_i \gg 1$ ,  $i = \overline{1, m}$  538, the lower  $\square$ -confidence limit of the PFFO (7.49) is calculated by the formula

$$\underline{R} = \min_{1 \leq i \leq m} [P_\gamma(N_i, D_i)]^{r_i}, \quad (7.58)$$

where the equivalent number of failures,  $D_i$ , is found from conditions

$$\left(1 - \frac{D_i}{N_i}\right)^{r_i} = \hat{R},$$

$$\hat{R} = \prod_{1 \leq i \leq m} \left(1 - \frac{d_i}{N_i}\right)^{r_i}. \quad (7.59)$$

It means that the value of  $D_i$  is chosen in such a way that the point estimate of the PFFO of the subsystem consisting of units of the  $i$ th type will coincide with the point estimate of the system as a whole. Formulas (7.56) and (7.58) represent a particular case of the general method of "equivalent tests" for series-parallel systems which will be considered below in Section 7.6.1.

Example 7.7 Consider a case of no failure test where  $d_1 = d_2 = \dots = d_m = 0$ .

In this case the point estimate of the system PFFO is

$$\hat{R} = \prod_{1 \leq i \leq m} \left(1 - \frac{d_i}{N_i}\right)^{r_i} = 1$$

and, correspondingly, all equivalent numbers of failure  $D_i = 0$ ,  $i = \overline{1, m}$  541. The Clopper-Pearson lower  $\square$ -confidence limit for this case is

$$\underline{P}_\gamma(N_i, D_i) = \underline{P}_\gamma(N_i, 0) = (1 - \gamma)^{\frac{1}{N_i}}.$$

To be concrete, assume that  $\min_{1 \leq i \leq m} \left(\frac{N_i}{r_i}\right) = \frac{N_1}{r_1}$ . 542 From (7.58), the Lindstrom-Madden method (method of "equivalent tests") delivers (in the no failure case) the following lower  $\square$ -confidence limit of the system PFFO:

$$\underline{P} = \min_{1 \leq i \leq m} (1 - \gamma)^{\frac{r_i}{N_i}} = (1 - \gamma)^{\frac{r_1}{N_1}}$$

This coincides with the best lower confidence limit obtained by Solov'yev and Mirny for the same case (see Chapter 6).  $\square$



Example 7.8 Consider a system consisting of units of three different types ( $m=3$ ). The number of units of each type,  $r_i$ , and test results,  $N_i$  and  $d_i$ , are presented in Table 7.1.

.....Table 7.1

We need to construct the lower confidence limit with the confidence probability  $\square=0.9$  for the system PFFO,  $R=p_1p_2p_3$ . The point estimate of the system PFFO in this case equals

$$\hat{R} = \hat{p}_1 \hat{p}_2 \hat{p}_3 = \left(1 - \frac{d_1}{N_1}\right) \left(1 - \frac{d_2}{N_2}\right) \left(1 - \frac{d_3}{N_3}\right) = \left(1 - \frac{1}{10}\right) \left(1 - \frac{2}{40}\right) \left(1 - \frac{1}{60}\right) = 0.84.$$

Minimum number of tested units equals 10 ( $N_1=10$ ). Applying (7.56) and (7.57), we obtain that the equivalent number of failures is

$$D_1 = N_1(1 - \hat{R}) = 10(1 - 0.84) = 1.6.$$

The lower  $\square$ -confidence limit  $\underline{R}$  for the system PFFO, obtained from the equation  $B_p(N_1 - D_1, D_1 + 1) = 1 - \square$ , is equal to  $\underline{R} = \underline{P}_{\square}(N_1, D_1) = \underline{P}_{0.9}(10, 1.6) = 0.594$ . ■

Example 7.9 Let  $m=2$ ,  $r_1=r_2=1$ . The test results are  $N_1=100$ ,  $d_1=1$  and  $N_2=200$ ,  $d_2=8$ . We need to construct the lower confidence limit with the confidence probability  $\square=0.9$  for the system PFFO,  $R=p_1p_2$ . The point estimate of the system PFFO in this case equals

$$\hat{R} = \hat{p}_1 \hat{p}_2 = \left(1 - \frac{d_1}{N_1}\right) \left(1 - \frac{d_2}{N_2}\right) = \left(1 - \frac{1}{100}\right) \left(1 - \frac{8}{200}\right) = 0.95.$$

The minimum number of tested units equals 100 ( $N_1=100$ ). Applying (7.56) and (7.57), we obtain that the equivalent number of failures is

$$D_1 = N_1(1 - \hat{R}) = 100(1 - 0.95) = 5.$$

the lower  $\square$ -confidence limit  $\underline{R}$  for the system PFFO equals

$$\underline{R} = \underline{P}_{\square}(N_1, D_1) = \underline{P}_{0.9}(100, 5) = 0.909. \blacksquare$$

Example 7.10 Consider a system consisting of three different types of units ( $m=3$ ). The number of units of each type,  $r_i$ , and test results,  $N_i$  and  $d_i$ , are presented in Table 7.2. The structure of the system is depicted in Figure 7.2

.....Figure 7.2

..... Table 7.2

We need to construct the lower confidence limit with the confidence probability  $\square\square\square=0.95$  for the system PFFO,  $R=p_1^2p_2p_3$ . The point estimate of the system PFFO in this case equals

$$\hat{R} = \hat{p}_1^2 \hat{p}_2 \hat{p}_3 = \left(1 - \frac{d_1}{N_1}\right)^2 \left(1 - \frac{d_2}{N_2}\right) \left(1 - \frac{d_3}{N_3}\right) = \left(1 - \frac{4}{100}\right) \left(1 - \frac{3}{150}\right) = 0.941.$$

The "minimum relative number of tested units" (that is, the number of tested units,  $N_i$ , divided on the number of units,  $r_i$ ) corresponds to the type 1:

$$\min_{1 \leq i \leq m} \left(\frac{N_i}{r_i}\right) = \frac{N_1}{r_1} = \frac{40}{2} = 20.$$

From (7.58) and (7.59) we have

$$\left(1 - \frac{D_i}{N_i}\right)^2 = \hat{R}$$

and from here the equivalent number of failures for the units of the first type is

$$D_1 = N_1 \left(1 - \sqrt{\hat{R}}\right) = 40 \left(1 - \sqrt{0.941}\right) = 1.2.$$

The lower  $\square$ -confidence limit for the system PFFO is

$$\underline{R} = [\underline{P}_\gamma(N_1, D_1)]^2 = [\underline{P}_{0.95}(40, 1.2)]^2 = (0.879)^2 = 0.771. \blacksquare$$

Above method can be applied to the exponential model in analogous way. Denote the volume of the  $i$ th unit test (the total time of testing) by  $S_i = N_i T_i$ . Then the lower  $\square$ -confidence limit for the system PFFO (7.50) can be calculated by the following way:

$$\underline{R} = \min_{1 \leq i \leq m} \exp\{-t_0 r_i \bar{\lambda}_\gamma(S_i D_i)\}. \quad (7.60)$$

$\bar{\lambda}_\gamma(S, a) = \frac{\Delta_{1-\gamma}(a)}{S}$  543 and,  $\square_{1-\square}(a)$  is the standard upper  $\square$ -confidence limit for the parameter of the Poisson distribution built by the test result  $a$ . Therefore,  $\bar{\lambda}_\gamma(S, d)$  544 is the standard upper  $\square$ -confidence limit for the parameter of the exponential distribution built on the basis of test result  $(S, d)$  where  $S$ . is the total test time and  $d$  is the number of failures (see Sections 1.4.8 and 2.2).

The equivalent number of failures  $D_i$  for units of the  $i$ th type is found from the following condition

$$r_i \left( \frac{D_i}{S_i} \right) = \hat{f},$$

$$\hat{f} = \sum_{1 \leq j \leq m} r_j \left( \frac{d_j}{S_j} \right) \quad (7.61)$$

From (7.61) it follows that

$$D_i = \sum_{1 \leq j \leq m} \left( \frac{S_i}{r_i} \cdot \frac{l_j}{S_j} \right) d_j. \quad (7.62)$$

**Example 7.11** Consider a particular case of no failure test where  $d_1=d_2= \dots =d_m=0$ . To be concrete assume that the first type unit has the minimum relative volume of testing:

$$\min_{1 \leq i \leq m} \left( \frac{S_i}{r_i} \right) = \frac{S_1}{r_1}$$

In this case, the equivalent number of failures  $D_i=0$ ,  $i = \overline{1, m}$ . 546 Taking into account that  $\Delta_{1-\gamma}(0) = \ln \left( \frac{1}{1-\gamma} \right) 547$ , from (7.60) we obtain the following lower  $\square$ -confidence limit for the system PFFO:

$$\underline{R} = \min_{1 \leq i \leq m} \exp \left( -t_0 \ln \left( \frac{1}{1-\gamma} \right) \cdot \frac{r_i}{S_i} \right) = \min_{1 \leq i \leq m} (1-\gamma)^{t_0 \frac{r_i}{S_i}} = (1-\gamma)^{t_0 \frac{r_1}{S_1}}.$$

This limit coincides with the limit obtained in Example 7.3 for the binomial model, if the volume of testing there equals  $N_i = \frac{S_i}{t_0} 548$ ,  $i = \overline{1, m}$ . 549.

Lindstrom-Madden method (or, the method of "equivalent testing") considered above is often used in practice for series systems and it delivers rather effective confidence limits for the system PFFO. At the same time, its deficiency lies in its insensitivity to the test volume of units of any type except that of the minimum,  $N_j$ . As a simple example, let us consider two systems consisting of two units. Let us assume the following results for the first and second cases, respectively:  $N_1=10, d_1=0, N_2=20, d_2=2$ , and  $N_1=10, d_1=0, N_2=100, d_2=10$ . For both cases the point estimate of the system PFFO is the same and are equal to

$$\hat{R} = \left( 1 - \frac{d_1}{N_1} \right) \left( 1 - \frac{d_2}{N_2} \right) = 0.9.$$

The lower  $\alpha$ -confidence limit (for instance, for  $\alpha = 0.95$ ) is the same for the both cases :

$$\underline{R} = \underline{P}_{\alpha}(N_1, D_1)$$

where the equivalent number of failures is  $D_1 = N_1(1 - \hat{R})$

$550 = 10(1 - 0.9) = 1$ , therefore  $\underline{R} = \underline{P}_{0.95}(10, 1) = 0.606$ .

Intuitively the confidence limit in the second situation should be better than in the first one, because the number of tested units of the second type is essentially higher (though the point estimate

of the PFFO of the second type unit is the same:  $\hat{p}_2 = 1 - \frac{d_2}{N_2} = 0.9551$ ). ■

For the normal approximation, the improvement of the Lindstrom-Madden method was obtained by Pavlov, 1992a (see also Ushakov, ed., 1994, pp.346-350).

## 8.4 Parallel Structures

### 8.4.1 System of Different Units

Consider a parallel system consisting of  $m$  different units. The system PFFO is

$$R = R(p) = 1 - \prod_{1 \leq i \leq m} (1 - p_i) \quad (7.63)$$

where  $p_i$  is the PFFO of the  $i$ th unit. Each unit was tested by binomial plan:  $N_i$  units of the  $i$ th type were tested and  $d_i$  failures have been observed,  $i = \overline{1, m}$ . The test results  $\mathbf{d} = (d_1, \dots, d_m)$  are assumed to be independent.

Confidence limits (7.63) were obtained in [Buehler, 1957] for the case where  $m=2$ . The Buehler lower  $\alpha$ -confidence limit for the system PFFO has the form

$$\underline{R} = \min \left[ 1 - \prod_{1 \leq i \leq m} (1 - p_i) \right] \quad (7.64)$$

where the minimum is taken over all parameters of units  $\mathbf{p} = (p_1, \dots, p_m)$  which satisfy the following inequalities

$$\left\{ \begin{array}{l} \sum_{n(\mathbf{d}) \leq n(\mathbf{d}^*)} \prod_{1 \leq i \leq m} \binom{N_i}{d_i} p_i^{N_i - d_i} (1 - p_i)^{d_i} \geq 1 - \gamma, \\ 0 \leq p_i \leq 1, \quad i = \overline{1, m}. \end{array} \right. \quad (7.65)$$

Here  $\mathbf{d}^*=(d_1^*, \dots, d_m^*)$  is the observed value of the vector  $\mathbf{d}=(d_1, \dots, d_m)$  obtained in the result of testing, and  $n(\mathbf{d})=n(d_1, \dots, d_m)$  is monotone decreasing in each  $d_i$  and gives some ordering on the set of all possible results of testing.

As we mentioned above, for such an ordering of  $n(\mathbf{d})$ , the lower  $\square$ -confidence limit  $\underline{R}$ , defined in (7.64) and (7.65) can not be improved (see Section 7.2). For different types of ordering of  $n(\mathbf{d})$ , we obtain different confidence limits  $\underline{R}$ . The question of how to choose the best (optimal) ordering is still open. We usually choose the principle of ordering basing on qualitative arguments or convenience of calculation.

The numerical solution of the minimization problem (7.64) under restrictions (7.65) is comparatively not difficult for small  $m$  and  $d_i^*$ . For the case where  $m=2$ , the numerical values of the lower confidence limit  $\underline{R}$  for some test results  $(N_1, d_1^*)$  and  $(N_2, d_2^*)$  (mainly, for  $N_1=N_2$ ) are given in [Owen, 1962], [Steck, 1957], [Lipow 1958, 1959], and [Shick, 1959]. In general case where  $m$ ,  $N_i$  and  $d_i^*$  are arbitrary an approach based on the idea that one can utilize results for series systems for (constructing the corresponding limits for parallel system consisting of the same units) was suggested in [Pavlov, 1982] and [Sudakov, 1986]. Consider this approach in more detail. Let us take a supplementary (imaginary) series system consisting of the same  $m$  units. For this system the PFFO has the form

$$R' = R'(\mathbf{p}) = \prod_{1 \leq i \leq m} p_i. \quad (7.66)$$

Let  $\mathbf{d}^*=(d_1^*, \dots, d_m^*)$  is the observed value of the vector of failures. Construct the lower  $\square$ -confidence limit  $\underline{R}'=\underline{R}'(\mathbf{d}^*)$  for the PFFO of the series system described by (7.66), for instance, on the basis of the Lindstrom-Madden method:

$$\underline{R}'=\underline{P}_{\square}(N_1, D_1) \quad (7.67)$$

where  $\underline{P}_{\square}(N, d)$  is the standard Clopper-Pearson lower  $\square$ -confidence limit,  $N_1$  is the minimum number of tested units,  $D_1$  is the equivalent number of failures determining by the formula (see Section 7.3 for details)

$$D_1 = N_1 \left[ 1 - \prod_{1 \leq i \leq m} \left( 1 - \frac{d_i^*}{N_i} \right) \right]. \quad (7.68)$$

After this, the lower  $\square$ -confidence limit for the PFFO of the initial parallel system, described by (7.63), can be found as follows

$$\underline{R} = \min \left[ 1 - \prod_{1 \leq i \leq m} (1 - p_i) \right] \quad (7.69)$$

Here the minimum is taken over all parameters  $\mathbf{p}=(p_1, \dots, p_m)$  which satisfy the following restrictions:

$$\begin{cases} \prod_{1 \leq i \leq m} p_i \geq \underline{R}', \\ 0 \leq p_i \leq 1, \quad i = 1, m. \end{cases} \quad (7.70)$$

It is easy to find that the obtained value  $\underline{R}$  is the lower  $\square$ -confidence limit for the PFFO of the initial parallel system. Indeed, by construction of  $\underline{R}$  the following relation

$$\{\underline{R}' \leq R(\mathbf{p})\} \subset \{\underline{R} \leq R(\mathbf{p})\}$$

is valid for each fixed  $\mathbf{p}$ . It follows that

$$P_p \{\underline{R} \leq R(\mathbf{p})\} \geq P_p \{\underline{R}' \leq R(\mathbf{p})\} \geq \gamma$$

that is,  $\underline{R}$  is the lower  $\square$ -confidence limit of the PFFO of a parallel structure,  $R(\mathbf{p})$ .

Minimum of (7.69) under restrictions (7.70) can be easily found analytically. (Notice that from the formal viewpoint, this problem coincides with the problem of finding the lower confidence limit for no failure test considered in Chapter 6.) From the solution obtained in Section 6.4, it follows that minimum (7.69) is attained in the point  $p_1 = p_2 = \dots = p_m = \sqrt[m]{\underline{R}'}$  which corresponds to the equally reliable units.

So, the lower  $\square$ -confidence limit (7.69) of the parallel structure PFFO is

$$\underline{R} = 1 - \left(1 - \sqrt[m]{\underline{R}'}\right)^m \quad (7.71)$$

**Example 7.12** Consider a case of no failure test ( $d_1^* = d_2^* = \dots = d_m^* = 0$ ). The numbers of tested units are:  $N_1 = N_2 = \dots = N_m = N$ . From (7.67) and (7.68) we obtain that  $D_1 = 0$  and the lower  $\square$ -confidence limit for the PFFO of a supplementary series structure equals

$$\underline{R}' = \underline{P}_\gamma(N_1, D_1) = \underline{P}_\gamma(N, 0) = (1 - \gamma)^{\frac{1}{N}}.$$

From (7.71) we obtain the following lower  $\square$ -confidence limit of the parallel system PFFO:

$$\underline{R} = 1 - \left[1 - (1 - \gamma)^{\frac{1}{mN}}\right]^m.$$

This limit coincides with the best lower confidence limit (6.27) obtained in Section 6.4 above for the same case.  $\square$

**Example 7.13** Consider a parallel system consisting of two units. Test results are  $N_1 = 20$ ,  $d_1^* = 1$  and  $N_2 = 40$ ,  $d_2^* = 4$ . We need to construct the lower confidence limit of the PFFO with the confidence

coefficient not less than  $\alpha=0.9$ . Let us first construct the lower  $\alpha$ - confidence limit for the PFFO of a series system with the same units, i.e.,  $R'=p_1p_2$ . In this case the minimum number of tested units is  $N_1=20$ . Applying formulas (7.67) and (7.68) for a series system, we have the "equivalent number of failures" equal to

$$D_1 = N_1 \left[ 1 - \left( 1 - \frac{d_1^*}{N_1} \right) \cdot \left( 1 - \frac{d_2^*}{N_2} \right) \right] = 20 \left[ 1 - \left( 1 - \frac{1}{20} \right) \cdot \left( 1 - \frac{4}{40} \right) \right] = 2.9.$$

The lower 0.9-confidence limit for the series system PFFO equals

$$\underline{R}' = \underline{P}_\gamma(N_1, D_1) = \underline{P}_{0.9}(20; 2.9) = 0.701.$$

Applying afterwards formula (7.71), we obtain the lower 0.9-confidence limit for the parallel system PFFO

$$\underline{R} = 1 - \left( 1 - \sqrt[m]{\underline{R}'} \right)^m = 1 - \left( 1 - \sqrt[2]{0.701} \right)^2 = 0.974. \quad \square$$

#### 8.4.2 System with Replicated Units

Consider a parallel system consists of  $m$  different types of units. There are  $n_i$  units of the  $i$ th type in the system. In this case the system PFFO has the form

$$R = R(\mathbf{p}) = 1 - \prod_{1 \leq i \leq m} (1 - p_i)^{n_i} \quad (7.72)$$

where  $\mathbf{p}=(p_1, \dots, p_m)$ , and  $p_i$  is the unit PFFO,  $i = \overline{1, m}$ . 560

For constructing the lower confidence limit for the PFFO of this system, let us again consider first a corresponding supplementary series system consisting of the same units. For this series system, the PFFO is:

$$R' = R'(\mathbf{p}) = \prod_{1 \leq i \leq m} p_i^{n_i}. \quad (7.73)$$

Let  $\mathbf{d}^*=(d_1^*, \dots, d_m^*)$  be an observed value of the vector of unit failures. Applying again the results above to series systems, let us construct the lower confidence limit for the PFFO of a series system,  $\underline{R}'=\underline{R}'(\mathbf{d}^*)$ , described by (7.73). By formulas (7.58) and (7.59) we have

$$\underline{R}' = \min_{1 \leq i \leq m} [\underline{P}_\gamma(N_i, D_i)]^{n_i} \quad (7.74)$$

where the "equivalent number of failures",  $D_i$ , is determined from the condition

$$\left(1 - \frac{D_i}{N_i}\right)^{n_i} = \prod_{1 \leq j \leq m} \left(1 - \frac{d_j^*}{N_j^*}\right)^{n_j} \quad (7.75)$$

Then we find the lower  $\square$ - confidence limit for the PFFO of the initial parallel system described by (7.72):

$$\underline{R} = \min \left[ 1 - \prod_{1 \leq i \leq m} (1 - p_i)^{n_i} \right] \quad (7.76)$$

where minimum is taken over all  $\mathbf{p} = (p_1, \dots, p_m)$  which satisfy the following restrictions

$$\left\{ \begin{array}{l} \prod_{1 \leq i \leq m} p_i^{n_i} \geq \underline{R}', \\ 0 \leq p_i \leq 1, \quad i = \overline{1, m}. \end{array} \right. \quad (7.77)$$

The proof that this limit has a confidence coefficient not less than  $\square$  completely coincides with that given in Section 7.4.1.

For finding minimum (7.76), we introduce new arguments  $\mathbf{z} = (z_1, \dots, z_m)$  where  $z_i = -\ln p_i$ ,  $i = \overline{1, m}$  565. Then the problem (7.76)-(7.77) can be rewritten as follows: to find

$$\underline{R}' = 1 - e^{\overline{f}}$$

where

$$\overline{f} = \max f(\mathbf{z}) \quad (7.78)$$

and

$$f(\mathbf{z}) = \sum_{1 \leq i \leq m} n_i \ln(1 - e^{-z_i}).$$

Maximum in (7.78) is taken under the following restrictions

$$\left\{ \begin{array}{l} \sum_{1 \leq i \leq m} n_i z_i \leq -\ln \underline{R}', \\ z_i \geq 0, \quad i = \overline{1, m}. \end{array} \right. \quad (7.79)$$



We can check by direct differentiating that function  $f(z)$  is monotone decreasing in each  $z_i$  and strictly concave ("convex up") in  $\mathbf{z}=(z_1, \dots, z_m)$ . It follows that maximum (7.78) is attained in the unique point which satisfies the following Lagrange equation system

$$\begin{cases} \frac{\partial f}{\partial z_i} = \frac{n_i}{e^{z_i} - 1} = \lambda n_i, & i = \overline{1, m}. & (7.80) \\ \sum_{1 \leq i \leq m} n_i z_i = -\ln R' & & (7.81) \end{cases}$$

where  $\lambda$  is the Lagrange multiplier. (Notice that in this case we don't need to analyze the boundary points of area (7.81) because  $f(\mathbf{z})=-\lambda$  for any  $z_i=0$ .) This system of equations has an obvious solution

$$z_1 = z_2 = \dots = z_m = \frac{-\ln R'}{\sum_{1 \leq i \leq m} n_i}.$$

In the force of strong concavity of function  $f(\mathbf{z})$ , this solution is unique. From here it follows that minimum (7.76) under restrictions (7.77) is attained at a "symmetrical point" which corresponds equally reliable units, that is,

$$p_1 = p_2 = \dots = p_m = \sqrt[n]{R'}$$

where  $n=n_1+n_2+ \dots +n_m$  is the total number of units within the system. Thus, the lower  $\lambda$ -confidence limit for the PFFO of a parallel system described by (7.72) has the form:

$$\underline{R} = 1 - \left(1 - \sqrt[n]{R'}\right)^n \quad (7.82)$$

**Example 7.14** Consider a parallel system consisting of units of two types. The numbers of units of each type are  $n_1=2$  and  $n_2=1$  (see Figure 7.3). Test results are  $N_1=10$ ,  $d_1^*=1$  and  $N_2=20$ ,  $d_2^*=1$ , respectively.

.....Figure 7.3  
The system PFFO is written as

$$R = 1 - (1 - p_1)^2 (1 - p_2) \quad (7.83)$$

We need to construct the lower confidence limit for the PFFO with the confidence coefficient not less than 0.9.

Let us introduce a supplementary series system consisting of the same units (see Figure 7.4), PFFO of which is  $R'=p_1^2 p_2$ .

First we construct the confidence limit of the series system PFFO,  $R'$ . In this case, the point estimate for  $R'$  is

$$\hat{R}' = \left(1 - \frac{d_1^*}{N_1}\right) \left(1 - \frac{d_2^*}{N_2}\right) = \left(1 - \frac{1}{10}\right) \left(1 - \frac{1}{20}\right) = 0.769.$$

"Equivalent numbers of failures",  $D_i$ , can be found from (7.75):

$$D_1 = N_1 \left(1 - \sqrt{\hat{R}'}\right) = 10 \cdot \left(1 - \sqrt{0.769}\right) = 1.25$$

$$D_2 = N_2 \left(1 - \hat{R}'\right) = 20(1 - 0.769) = 4.62.$$

The lower  $\square$ - confidence limit for  $R'$  in correspondence with (7.74) equals

$$\begin{aligned} \underline{R}' &= \min \{ \underline{P}_{\gamma}^2(N_1, D_1), \underline{P}_{\gamma}(N_2, D_2) \} = \\ &= \min \{ \underline{P}_{0.9}^2(10, 1.25), \underline{P}_{0.9}(20, 4.62) \} = \min \{ (0.634)^2, 0.605 \} = 0.401. \end{aligned}$$

Applying formula (7.82), we find the lower 0.9- confidence limit for the PFFO of the considering parallel system (7.83), taking into account that the total number of units within the system equals  $n = n_1 + n_2 = 2 + 1 = 3$ , can be written as

$$\underline{R} = 1 - \left(1 - \sqrt[3]{\underline{R}'}\right)^n = 1 - \left(1 - \sqrt[3]{0.401}\right)^3 = 0.982. \blacksquare$$

### 8.5 Series-Parallel System

Consider a system which consists of  $m$  subsystems (redundant groups). In turn, each subsystem consists of  $n_i$  identical redundant units in parallel, that is, redundant units are working in loading regime. The PFFO of this system can be written as

$$R = \prod_{1 \leq i \leq m} \left[1 - (1 - p_i)^{n_i}\right] \quad (7.84)$$

where  $p_i$  is the PFFO of a unit of the  $i$ th subsystem,  $i = \overline{1, m}$ . Assume that each unit was tested in accordance with a binomial plan:  $N_i$  units were tested and  $d_i$  failures were registered. We need to construct the lower  $\square$ - confidence limit for the PFFO (7.84) on the basis of these test results. A simple solution of the problem exists only for a no-failure-test:  $d_1^* = d_2^* = \dots = d_m^* = 0$  (see Section 6.3). In other cases the solution of the problem is rather difficult.

### 8.5.1 "Super Reliable" System

Assume that all numbers of observed failures,  $d_i^*$ , satisfy inequalities

$$d_i^* \leq n_i - 1 \quad \text{for all } i = \overline{1, m}. \quad (7.85)$$

It means that the number of failures does not exceed the number of redundant units for each subsystem. Such a system we will conditionally call "super reliable".

We apply the general method described above in Section 7.1 (see also Example 7.3). Let us take an unbiased point estimate  $\hat{R}$  of the system PFFO as an initial statistic  $S$ . This unbiased estimate for the binomial scheme of testing, is

$$\frac{d_i(d_i - 1) \cdot \dots \cdot (d_i - n_i + 1)}{N_i(N_i - 1) \cdot \dots \cdot (N_i - n_i + 1)}.$$

We further assume that inequality  $N_i \geq n_i$  is valid for each type of units, that is, the number of tested units of each type is not less than the number of these units within the  $i$ th subsystem. If unit tests are independent, then the unbiased estimate for the system PFFO can be written as

$$\hat{R} = \hat{R}(\mathbf{d}) = \prod_{1 \leq i \leq m} \left[ 1 - \frac{d_i(d_i - 1) \cdot \dots \cdot (d_i - n_i + 1)}{N_i(N_i - 1) \cdot \dots \cdot (N_i - n_i + 1)} \right]. \quad (7.86)$$

Assuming  $S(\mathbf{d}) = \hat{R}(\mathbf{d})$  in (7.36) - (7.38), we obtain the following lower confidence limit for the system PFFO

$$\underline{R} = \min \prod_{1 \leq i \leq m} [1 - (1 - p_i)^{n_i}], \quad (7.87)$$

where minimum is taken over all values of unit parameters  $\mathbf{p} = (p_1, \dots, p_m)$  which satisfy inequalities

$$\left\{ \begin{array}{l} \sum_{\hat{R}(\mathbf{d}) \geq \hat{R}(\mathbf{d}^*)} \prod_{1 \leq i \leq m} \binom{N_i}{d_i} p_i^{N_i - d_i} (1 - p_i)^{d_i} \geq 1 - \gamma \quad (7.88) \\ 0 \leq p_i \leq 1, \quad i = \overline{1, m}. \quad (7.89) \end{array} \right.$$

From (7.85) and (7.86) it follows that  $\hat{R}(\mathbf{d}^*) = 1$ . So, the sum in (7.88) is taken over all  $\mathbf{d} = (d_1, \dots, d_m)$  which satisfy inequality

$$\hat{R}(\mathbf{d}) \geq 1. \quad (7.90)$$

From the definition of estimate  $\hat{R}(\mathbf{d})$  (7.88), it follows that inequality (7.90) is equivalent to the following inequalities:

$$0 \leq d_i \leq n_i - 1 \quad \text{for all } i = \overline{1, m} \quad (7.91)$$

Thus the sum in (7.88) is taken over the set of all vectors  $\mathbf{d} = (d_1, \dots, d_m)$  which satisfy inequalities (7.91). This set represents an  $m$ -dimensional hyper cube in the space of test results  $\mathbf{d}$ . In this situation, inequality (7.88) takes the following form:

$$\prod_{1 \leq i \leq m} f_i(p_i) \geq 1 - \gamma \quad (7.92)$$

where

$$f_i(p_i) = \sum_{0 \leq d_i \leq n_i - 1} \binom{N_i}{d_i} p_i^{N_i - d_i} (1 - p_i)^{d_i}.$$

We introduce new arguments

$$z_i = -\ln f_i(p_i), \quad i = \overline{1, m} \quad (7.93)$$

for a more convenient calculation of  $\underline{R}$ .

After some simple transformations, the problem of calculation of minimum of (7.87) can be represented in the following form:

$$\underline{R} = \exp\left(-\max_{1 \leq i \leq m} \sum \varphi_i(z_i)\right), \quad (7.94)$$

where

$$\varphi_i(z_i) = -\ln[1 - [1 - p_i(z_i)]^{n_i}]$$

and  $p_i(z_i)$  is an inverse function corresponding to (7.93). Maximum in (7.94) is taken under the restrictions

$$\left\{ \begin{array}{l} \sum_{1 \leq i \leq m} z_i \leq -\ln(1 - \gamma) \\ z_i \geq 0, \quad i = \overline{1, m}. \end{array} \right. \quad (7.95)$$

It is easy to show by direct differentiation that  $\varphi_i'(z_i) \geq 0$ ,  $\varphi_i''(z_i) \leq 0$ ,  $i = \overline{1, m}$ . So, the function

$$\varphi(z) = \sum_{1 \leq i \leq m} \varphi_i(z_i)$$

is monotone increasing in each  $z_i$  and convex ("convex down") in  $z = (z_1, \dots, z_m)$ . In accordance with well known results in the theory of convex programming (see Section 3 of Appendix to Chapter 6), it follows that maximum in (7.94) is attained at one of  $m$  "corner" points of area (7.95). This point has the form

$$(0_1, \dots, 0_{i-1}, -\ln(1-\alpha), 0_{i+1}, \dots, 0_m)$$

where  $0_i$  means zero on the  $i$ th position.

After some simple transformations, we can obtain the lower confidence limit of the system PFFO in the form

$$R = \min_{1 \leq i \leq m} \left\{ 1 - [1 - \underline{P}_\gamma(N_i, r_i)]^{n_i} \right\}, \quad (7.96)$$

where  $\underline{P}_\alpha(N, d)$  is the standard Clopper-Pearson lower  $\alpha$ -confidence limit of a binomial parameter  $p$ . Here  $N$  is the number of units tested,  $d$  is the number of failures,  $r_i = n_i - 1$  is the number of redundant units within the  $i$ th subsystem. (For more details see Section 2 in Appendix to the current chapter.) If  $n_1 = n_2 = \dots = n_m = 1$ , formula (7.96) includes for the Mirny-Solovyev result for a no-failure-test as a particular case (see Sections 6.1 and 6.3).

### 8.5.2 Method of Hyperplane

Consider a solution of the problem for the case where the Poisson approximation is valid. Let we have  $N_i$  Bernoulli trials (the number of tested units) and the probability that an event has been occurred equals  $q_i$ . As is well known in the probability theory, the binomial distribution converges to the Poisson distribution with parameter  $\lambda_i = N_i q_i$ , if  $q_i \rightarrow 0$  and  $N_i \rightarrow \infty$  in such a way that  $N_i q_i = \text{const}$ . (In our case, a binomial r.v. is represented by the number of failures,  $d_i$ .) This theorem above allows us to use the Poisson approximation for a distribution of r.v.  $d_i$  if units, consisting a system, are highly reliable,  $q_i \ll 1$ , and the numbers of tested units,  $N_i$ , are large. In this case, we can use the following approximate formula for the system PFFO

$$R = \prod_{1 \leq i \leq m} (1 - q_i^{n_i}) \approx \exp\left(-\sum_{1 \leq i \leq m} q_i^{n_i}\right) = \exp\left(-\sum_{1 \leq i \leq m} c_i \Lambda_i^{n_i}\right), \quad (7.97)$$

where the coefficients are  $c_i = \left(\frac{1}{N_i}\right)^{n_i}$  586. So, the problem is reduced to constructing the upper confidence limit of the following function of Poisson parameters  $\Lambda = (\Lambda_1, \dots, \Lambda_m)$ :

$$f(\Lambda) = \sum_{1 \leq i \leq m} c_i \Lambda_i^{n_i} \quad (7.98)$$

on the basis of test results  $\mathbf{d} = (d_1, \dots, d_m)$ . Here  $d_i$  is a r.v. with the Poisson distribution with unknown parameter  $\Lambda_i$ ,  $i = \overline{1, m}$ . 588 All these r.v.'s are mutually independent.

Notice that the construction of confidence limit for the system PFFO (7.84) can be reduced to the analogous problem if system units have exponential distribution of TTF,  $p_i(t) = e^{-\lambda_i t}$  589, where  $\lambda_i$  is the failure rate,  $i = \overline{1, m}$  590. Assume that the  $i$ th type units were tested in accordance with plan  $[N_i R T_i]$  (see Chapter 2) and  $d_i$  failures have been registered. Then r.v.  $d_i$  approximately has a Poisson distribution with parameter  $\Lambda_i = n_i T_i \lambda_i$ . The system PFFO (7.84) can be presented in the form

$$R = \prod_{1 \leq i \leq m} \{1 - [1 - p_i(t)]^{n_i}\} = \prod_{1 \leq i \leq m} \{1 - [1 - e^{-\lambda_i t}]^{n_i}\} = \exp\left(-\sum_{1 \leq i \leq m} f_i(\Lambda_i)\right)$$

where

$$f_i(\Lambda_i) = -\ln\left\{1 - \left[1 - e^{-\frac{t \Lambda_i}{N_i T_i}}\right]^{n_i}\right\}.$$

The problem is again reduced to construction of the upper confidence limit for function

$$f(\Lambda) = \sum_{1 \leq i \leq m} f_i(\Lambda_i) \quad (7.99)$$

on the basis of independence of Poisson r.v.'s  $\mathbf{d} = (d_1, \dots, d_m)$ . In the case of highly reliable units ( $\Lambda_i t \ll 1$ ) function (7.99) approximately has the form analogous to (7.98):

$$f(\Lambda) \approx \sum_{1 \leq i \leq m} c_i \Lambda_i^{n_i},$$

where the coefficients are  $c_i = \left(\frac{t}{N_i T_i}\right)^{n_i}$  592,  $i = \overline{1, m}$  593.

Further, let us take the total number of failures (for all types of units)

$$S = S(\mathbf{d}) = d_1 + d_2 + \dots + d_m$$

as an initial statistic for constructing the confidence limit. Let  $d_1^*, d_2^*, \dots, d_m^*$  be the registered numbers of failures and  $S=d_1^*+d_2^*+ \dots +d_m^*$  is the corresponding value of observed statistic  $S$ . Applying (7.31) and (7.32), we find that the upper  $\square$ -confidence limit for  $f(\square)$  has the form:

$$\bar{f} = \max f(\Lambda). \quad (7.100)$$

Here the maximum is taken over all parameters  $\square=(\square_1, \dots, \square_m)$  in such a way that inequality

$$P_{\square}(S \leq S^*) \geq 1 - \square \quad (7.101)$$

holds.

Statistic  $S$  is the sum of independent Poisson r.v.'s. Thus,  $S$  itself has the Poisson distribution with parameter  $\square_1 + \square_2 + \dots + \square_m$ . After simple transformations, inequality (7.101) can be rewritten as

$$\sum_{1 \leq i \leq m} \Lambda_i \leq \Delta_{1-\gamma} (d_1^* + d_2^* + \dots + d_m^*) \quad (7.102)$$

where  $\square_{1-\square}(\mathbf{d})$  is the standard upper  $\square$ -confidence limit of parameter of the Poisson distribution based on test results  $\mathbf{d}$ . Besides, parameters  $\square_i$  must satisfy the obvious restrictions

$$\Lambda_i \geq 0, \quad i = \overline{1, m}. \quad (7.103)$$

Thus, maximum in (7.100) is taken under restrictions (7.102) and (7.103). Function  $f(\square)$  is monotone increasing in each variable and Convex ("convex down") in vector  $\square$ . From here it follows (see Section 3 of Appendix to Chapter 6) that the maximum mentioned above is attained at one of "corner points" of the area described by (7.102) and (7.103). These points have the form of the following type

$$\underbrace{(0, \dots, 0)}_{i-1}, \Delta_{1-\gamma} \left( \sum_{1 \leq i \leq m} d_i^* \right), 0_{i+1}, \dots, 0_m), \quad i = \overline{1, m}$$

where  $0_i$  means that 0 is at the  $i$ th position. From here we conclude that the upper  $\square$ -confidence limit of  $f(\square)$  has the form

$$\bar{f} = \max_{1 \leq i \leq m} f \left[ 0_1, \dots, 0_{i-1}, \Delta_{1-\gamma} \left( \sum_{1 \leq i \leq m} d_i^* \right), 0_{i+1}, \dots, 0_m \right]. \quad (7.104)$$

After this the lower  $\square$ -confidence limit for the system PFFO is calculated as  $\underline{R} = e^{-\bar{f}}$  598.

This solution has the following meaning. First, we calculate a lower  $\square$ -confidence limit for each individual  $i$ th subsystem under the assumption that there were observed number of failures

equal to  $d_1^* + d_2^* + \dots + d_m^*$ . Then the minimum of such confidence limits is considered as the lower confidence limit of the system PFFO. Notice that for a series system and a no-failure-test,  $n_1 = n_2 = \dots = n_m = 1$  and  $d_1^* = d_2^* = \dots = d_m^* = 0$ , the obtained solution coincides with the result obtained by Lindstrom-Madden method (see Section 7.3) and by Mirny-Solovyev (Sections 6.1 and 6.3).

The area given by (7.102) and (7.103) in the space of parameters  $\square_1, \dots, \square_m$  is a simplex restricted by  $m$ -dimensional plane:

$$\Lambda_1 + \Lambda_2 + \dots + \Lambda_m = \Delta_{1-\gamma} \left( \sum_{1 \leq i \leq m} d_i^* \right).$$

This is the reason why this method is often called as "method of hyperplane". This method was considered in (Lipow, 1958, 1959), Mirny and Solovyev (1964), (Belyaev, 1966a, b), (Bol'shev and Loginov, 1966), (Belyaev et al., 1967), and others.

This method effectively works in the following cases:

1. Series systems with equal (or very close) volumes of unit test, that is, for  $n_1 = n_2 = \dots = n_m = 1$  and  $N_1 = N_2 = \dots = N_m$  (or for the exponential model  $N_1 T_1 = N_2 T_2 = \dots = N_m T_m$ ). In this case the method of hyperplane produces confidence limits which coincide with the ones obtained by the Lindstrom-Madden method (see Section 7.3).

2. Series-parallel systems with equal numbers of units within different subsystems,  $n_1 = n_2 = \dots = n_m$ , and equal volumes of tests  $N_1 = N_2 = \dots = N_m$  in the case where all (or almost all) failures have occurred within a single subsystem (remaining subsystems are highly reliable).

3. Series-parallel systems with equal volumes of tests,  $N_1 = N_2 = \dots = N_m$ , in the case where the system PFFO coincides with (or is close to) the PFFO of a subsystem with the minimum number of redundant units

$$n_j = \min(n_1, \dots, n_m)$$

(other subsystems are highly reliable).

4. Series-parallel systems with equal numbers of units within different subsystems,  $n_1 = n_2 = \dots = n_m$ , in the case where the system PFFO coincides (or is close to) the PFFO of a subsystem with the minimum number of redundant units

$$N_j = \min(N_1, \dots, N_m)$$

(other subsystems are highly reliable).

In other cases, the efficiency of the method of hyperplane is significantly worse.

### 8.5.3 Method of Hypercube



Let  $d_1^* = d_2^* = \dots = d_m^*$  be observed number of failures of tested units. We calculate the upper  $\gamma$ -confidence limit of each parameter  $\lambda_i$  of the  $i$ th unit by standard formula  $\bar{\lambda}_i = \Delta_{1-\gamma}(d_i^*)$  (7.99). Let us consider the  $m$ -dimensional cube

$$H(d^*) = \{\Lambda: 0 \leq \Lambda_i \leq \Delta_{1-\gamma}(d_i^*), \quad i = \overline{1, m}\} \quad (7.105)$$

in the space of parameters  $\Lambda = (\lambda_1, \dots, \lambda_m)$ . Then, taking into account the test results for different types of units, we can write the following relations:

$$P\{\Lambda \in H(d^*)\} = P\left\{\bigcap_{1 \leq i \leq m} [\Lambda_i \leq \Delta_{1-\gamma}(d_i^*)]\right\} = \prod_{1 \leq i \leq m} P[\Lambda_i \leq \Delta_{1-\gamma}(d_i^*)] \geq \gamma^m.$$

Thus, sets  $H(d^*)$  form a collection of confidence sets for  $\Lambda$  with the confidence coefficient not less than  $\gamma^m$ . It follows that the value of

$$\bar{f} = \max_{\Lambda \in H(d^*)} f(\Lambda) \quad (7.106)$$

produces the upper  $\gamma$ -confidence limit for  $f(\Lambda)$  with the confidence coefficient not less than  $\gamma^m$ .

Since function  $f(\Lambda) = f(\lambda_1, \dots, \lambda_m)$  defined by (7.98) and (7.99) is monotone increasing in each parameter  $\lambda_i$ , the maximum in (7.106) is calculated as

$$\bar{f} = f(\bar{\lambda}_1, \dots, \bar{\lambda}_m)$$

Corresponding lower confidence limit for the system PFFO,  $\underline{R} = e^{-\bar{f}}$  (7.102), is calculated by simple substitution of the  $\gamma$ -upper confidence limit for individual parameters  $\lambda_i$  into a function which gives the dependence of system PFFO on unit parameters. An obvious deficiency of the method is in fast decreasing of confidence coefficient  $\gamma^m$  with increasing of  $m$  (number of different unit types). It normally leads to a very conservative confidence limits of the system PFFO.

#### 8.5.4 Method of Truncated Hypercube

This method, suggested in [Belyaev, 1966,b] and [Belyaev, et al., 1967], represents a combination of the two methods: hyperplane and hypercube. Let  $\lambda_0 = \lambda_0$  in (7.102) and  $\lambda_i = \lambda_i$  in (7.105). Form a confidence set in the space of unit parameters  $\Lambda = (\lambda_1, \dots, \lambda_m)$  by the means of intersection of corresponding confidence sets obtained by the methods of hyperplane and hypercube. The resulting set is given by restrictions

$$\sum_{1 \leq i \leq m} \Lambda_i \leq \Delta_{1-\gamma_0} \left( \sum_{1 \leq i \leq m} d_i^* \right) \quad (7.107)$$

$$0 \leq \Lambda_i \leq \Delta_{1-\gamma_i}(d_i^*), \quad i = \overline{1, m}. \quad (7.108)$$

Confidence coefficient  $\alpha$  for sets given by (7.107) and (7.108) satisfies inequality

$$\alpha \alpha \geq \alpha_0 + \alpha_1^m - 1.$$

This inequality follows from the well known formula for intersection of two events

$$P(AB) = P(A) + P(B) - P(A \cap B) \geq P(A) + P(B) - 1.$$

Thus, the lower  $\alpha$ -confidence limit for the system PFFO with the confidence coefficient not less than value of  $\alpha_0 + \alpha_1^m - 1$  is calculated as

$$\underline{R} = e^{-\bar{f}}$$

where

$$\bar{f} = \max f(\Lambda) \quad (7.109)$$

and the maximum is taken over the area given by constrains (7.107) and (7.108). This area in the  $m$ -dimensional space of parameters  $\Lambda$  is a hypercube truncated by a hyperplane. For series-parallel systems function  $f(\Lambda)$  defined in (7.98) or (7.99) is monotone increasing in each  $\Lambda_i$  and convex ("convex down"). Thus, maximum in (109) is attained at one of "corner" points. This allows one to locate the maximum easily with the help of a computer. Using examples from [Belyaev, et al., 1967], we show below that this method gives better results in comparison with the methods of hyperplane or hypercube.

**Example 7.15** A system consists of ten redundant groups (subsystems) connected in series ( $m=10$ ). The  $i$ th subsystem consists of  $n_i$  parallel identical units. Each unit was tested by the binomial plan:  $n_i$  is the number of tested units and  $d_i$  is the number of registered failures. Corresponding values are given in Table 7.3.

..... Table 7.3  
 Values of the lower confidence limit with confidence coefficient  $\alpha=0.9$  for the system PFFO for this example, calculated with the help of different methods, are given in the left column of Table 7.4.  $\square$

..... Table 7.4  
**Example 7.16** A system consists of ten redundant groups (subsystems) connected in series ( $m=10$ ). Each unit was tested by the binomial plan analogously to Example 7.5.1. Corresponding values are given in Table 7.5

..... Table 7.5  
 Values of the lower confidence limit with confidence coefficient  $\alpha=0.98$  for the system PFFO, calculated with the help of different methods for this example, are given in the second column of Table 7.4.  $\square$

Example 7.5.3 A system consists of twenty redundant groups (subsystems) connected in series ( $m=20$ ). Each group consists of two parallel units,  $n_i=2$ . Equal number of each unit were tested,  $N_1=N_2= \dots =N_{20}=100$ . Corresponding numbers of failures are  $d_1=d_2= \dots =d_5=0$  and  $d_6=d_6= \dots =d_{20}=1$ .

Values of the lower confidence limit with confidence coefficient  $\alpha=0.9$  for the system PFFO, calculated with the help of different methods for this example, are given in the third column of Table 7.4. □

Notice that the method of truncated hypercube, nevertheless, still gives too conservative confidence limit of the system PFFO.

### 8.5.5 Modified Hyperplane Method

The method of hyperplane, as it was shown in (Pavlov, 1972) can be improved for series-parallel systems if  $n_i \geq 2$  for all  $i$ ,  $i = \overline{1, m}$ , that is, if each subsystem has at least one redundant unit. Then the confidence limit defined in (7.104) we can use value

$$\tilde{S} = \max \left[ \max_{1 \leq i \leq m} d_i^*, \sum_{1 \leq i \leq m} d_i^* - r \right] \quad (7.110)$$

instead of the total number of failures,  $\sum_{1 \leq i \leq m} d_i^*$ . Above we used notation  $r = \min_{1 \leq i \leq m} n_i - 1$  which denotes the minimum number of redundant units among all subsystems.

Example 7.5.4 Consider a series system consisting of three subsystems (see Figure 7.5). All data related to the example are given in Table 7.6.

.....Figure 7.5  
 ..... Table 7.6

We need to construct the lower confidence limit for the system PFFO with the confidence coefficient  $\alpha=0.9$ .

In this case  $r=2$  and  $\tilde{S} = \max(2; 4-2)=2.609$  The method of hyperplane gives the following lower  $\alpha$ - confidence limit for the system PFFO

$$\underline{R} = \exp \left\{ - \max_{1 \leq i \leq m} \left[ \frac{1}{N_i} \Delta_{1-\alpha} \left( \sum_{1 \leq i \leq m} d_i^* \right)^{n_i} \right] \right\} = \exp \left[ \frac{\Delta_{0.1}(4)}{25} \right]^3 = 0.965.$$

The modified method of hyperplane produces the lower confidence limit (with the same confidence coefficient  $\alpha=0.9$ ) equal to

$$\underline{R} = \exp \left\{ - \max_{1 \leq i \leq m} \left[ \frac{\Delta_{1-\gamma}(\tilde{S})}{N_i} \right]^{n_i} \right\} = \exp \left[ - \frac{\Delta_{0.1}(2)}{25} \right]^3 = 0.991.$$

The latter estimate is significantly better than previous one. □

**Example 7.19** Consider a system consisting of ten subsystems,  $m=10$ . All data concerning unit tests result are presented in Table 7.7. In this case the total number of failures is  $d_1^* + \dots + d_{10}^* = 6$ . The minimum number of redundant units equals

$$r = \min(n_1, \dots, n_m) - 1 = 2.$$

..... Table 7.7

The lower  $\square$ -confidence limit with the confidence coefficient  $\square=0.9$  calculated with the help of the hyperplane method equals

$$\underline{R} = \exp \left[ - \frac{\Delta_{0.1}(6)}{40} \right]^3 = 0.982.$$

Now from (7.110) we find that

$$\tilde{S} = \max(1; 6-2) = 4$$

Thus the modified method of hyperplane gives

$$\underline{R} = \exp \left[ - \frac{\Delta_{0.1}(4)}{40} \right]^3 = 0.992. \quad \square$$

**Example 7.20** Consider again Example 7.17. For that example, the minimum number of redundant units among all subsystems is  $r = \min(n_1, \dots, n_m) - 1 = 1$ , the total number of failures is  $d_1^* + \dots + d_m^* = 15$ , and the value of  $\tilde{S}$  is  $\max(1, 15-1) = 14$ .

For the confidence coefficient  $\square=0.9$  the method of hyperplane (see Table 7.4 above) gives the  $\square$ - of the system PFFO  $\underline{R}=0.955$ . The modified method of hyperplane produces

$$\underline{R} = \exp \left[ - \frac{\Delta_{0.1}(14)}{100} \right]^3 = 0.960. \quad \square$$

All methods considered in this section are correct and give the guaranteed confidence coefficient for series-parallel systems. Nevertheless, in many cases they produce a too

conservative estimate of system reliability to be of practical interest. It leads one to create heuristic or approximate methods which give better confidence limits. We consider these approaches in the next section. Besides, in Section 7.7 we suggest more effective method though it needs more sophisticated computations.

## **9.**

***9.1 see before***

***9.2 see before***

***9.3 see before***

***9.4 see before***

***9.5 see before***

## 9.6 Approximate Methods

All algorithms considered previously deliver guaranteed value of the confidence coefficient for series-parallel systems, that is, they satisfy the following condition

$$P_p\{\underline{R} \leq R_p\} \geq \square \quad (7.111)$$

for all possible values of unit parameters  $\mathbf{p}=(p_1, \dots, p_m)$ . At the same time, these strong methods give conservative estimates  $\underline{R}$  of the system PFFO  $R=R(\mathbf{p})$ . There are heuristic and approximate methods which can be described by condition

$$P_p\{\underline{R} \leq R_p\} \square \square$$

which might lead to an opposite effect. As we have discussed overestimating the reliability indices is often unacceptable, in practice. Thus, these methods must be used with care.

### 9.6.1 Method of "Equivalent Tests"

This method is a natural extension of the Lindstrom-Madden method (7.56) for the series-parallel systems. Taking into account the vector of test results  $\mathbf{d}^*=(d_1^*, d_2^*, \dots, d_m^*)$ , let us find point estimates of unit parameters  $\hat{p}_i = 1 - \frac{d_i^*}{N_i}$  and a point estimate of the system PFFO (7.84):

$$\hat{R} = R(\hat{p}_1, \dots, \hat{p}_m) = \prod_{1 \leq i \leq m} \left[ 1 - \left( \frac{d_i^*}{N_i} \right)^{n_i} \right].$$

Determine the "equivalent number of failures",  $D_i$ , for the  $i$ th subsystem units from the condition

$$R(\underbrace{1, \dots, 1}_{i-1}, 1 - \frac{D_i}{N_i}, 1, \dots, 1) = \hat{R} \quad (7.112)$$

where  $1_i$  means that the  $i$ th component of the vector equals 1. From (7.112), we obtain

$$1 - \left( \frac{D_i}{N_i} \right)^{n_i} = \prod_{1 \leq i \leq m} \left[ 1 - \left( \frac{d_i^*}{N_i} \right)^{n_i} \right]. \quad (7.113)$$

Now we determine the lower  $\square$ -confidence limit for the system PFFO (7.84) as

$$\underline{R} = \min_{1 \leq i \leq m} R(\underbrace{1, \dots, 1}_{i-1}, \underline{P}_\gamma(N_i, D_i), 1, \dots, 1) = \min_{1 \leq i \leq m} \left\{ 1 - [1 - \underline{P}_\gamma(N_i, D_i)]^{n_i} \right\} \quad (7.114)$$

where  $\underline{P}_\square(N, d)$  is the Clopper-Pearson lower  $\square$ -confidence limit calculated on the basis of the binomial test ( $N$  units were tested and  $d$  failures have been observed). If the value of  $D_i$  is not integer, the value of  $\underline{P}_\square(N_i, D_i)$  is found from solution of the equation

$$B_p(N_i - D_i, D_i + 1) = 1 - \gamma$$

where  $B_p(a, b)$  is beta function (see Section 2 of Appendix to the current chapter).

Thus, we first find the "equivalent number of failures",  $D_i$ , for each  $i$ th subsystem in such a way that the point estimate of each subsystem, calculated on the basis of such value, coincides with the point estimate  $\hat{R}$  of the system PFFO. Then on the basis of this "equivalent number of failures", we construct the  $\square$ -lower confidence limit of the PFFO of a unit of the  $i$ th type and for the  $i$ th subsystem itself. The minimum confidence limit among the lower  $\square$ -confidence limit for the subsystems is taken as the lower  $\square$ -confidence limit for the entire system. This method includes the Lindstrom-Madden method (7.56) for series systems as a particular case. In an analogous way, it is applied to the exponential model.

Explanations of this method can be obtained for the normal approximation in the case of a large test volume (see below Section 7.6.5 and [Krol, 1974, 1975] and [Pavlov, 1982]).

This method relates to the general strong method considered in Section 7.2. Let us take the point estimate of the system PFFO

$$S(\mathbf{d}) = \hat{R} = \prod_{1 \leq i \leq m} \left[ 1 - \left( \frac{d_i}{N_i} \right)^{n_i} \right]$$

as an initial statistic  $S=S(\mathbf{d})$  for constructing the confidence limit. Then in accordance with (7.36)-(7.38) the lower  $\square$ -confidence limit for the system PFFO has the form

$$\underline{R} = \min \prod_{1 \leq i \leq m} [1 - (1 - p_i)^{n_i}] \quad (7.115)$$

where minimum is taken over all values of parameter  $\mathbf{p}=(p_1, \dots, p_m)$  which satisfy the following inequalities

$$\sum_{S(\mathbf{d}) \geq S(\mathbf{d}^*)} \prod_{1 \leq i \leq m} \binom{N_i}{d_i} p_i^{N_i - d_i} (1 - p_i)^{d_i} \geq 1 - \gamma, \quad (7.116)$$

$$0 \leq p_i \leq 1, \quad i = \overline{1, m} \quad (7.117)$$

Now consider  $m$  "corner points" of the area (7.116)-(7.117) which has the following form

$$p^{(i)} = (\underbrace{1, \dots, 1}_{i-1}, \pi_i, 1, \dots, 1), \quad i = \overline{1, m}.$$

Substituting the  $i$ th "corner point"  $p^{(i)}$  into inequality (7.116), the value  $\square_i$  can be found from equation

$$\sum_{0 \leq d_i \leq L_i} \binom{N_i}{d_i} \pi_i^{N_i - d_i} (1 - \pi_i)^{d_i} = 1 - \gamma \quad (7.118)$$

where  $L_i$  is the maximum integer quantity among  $r$  ones satisfying the inequality

$$S(\underbrace{0, \dots, 0}_{i-1}, r, 0, \dots, 0) \geq S(d^*)$$

from where we have

$$L_i = \max \left\{ r: \left(1 - \frac{1}{N_i}\right)^{n_i} \geq \prod_{1 \leq i \leq m} \left(1 - \frac{d_i^*}{N_i}\right) \right\}.$$

It shows that the "equivalent number of failures",  $D_i$ , connected with the quantity  $L_i$  by the following relation

$$L_i = [D_i], \quad i = \overline{1, m} \quad (7.119)$$

where  $[D_i]$  is the integer part of  $D_i$ .

From (7.118) follows that value of  $\square_i$  coincides with the standard Clopper-Pearson lower  $\square$ -confidence limit calculated on the basis of  $L_i$  failures in  $N_i$  tests:

$$\square_i = \underline{P}_{\square}(N_i, L_i).$$

All "corner points" belong to the area (7.115) in which minimum is searched, so from here the following inequality for the strong lower confidence limit,  $\underline{R}$ , follows:

$$\begin{aligned} \underline{R} &= \min_{1 \leq i \leq m} R(p^{(i)}) = \min_{1 \leq i \leq m} R(\underbrace{1, \dots, 1}_{i-1}, \pi_i, 1, \dots, 1) = \\ &= \min_{1 \leq i \leq m} [1 - (1 - \pi_i)^{n_i}] = \min_{1 \leq i \leq m} \{1 - [1 - \underline{P}_{\gamma}(N_i, L_i)]^{n_i}\}. \end{aligned}$$



The right part of this inequality coincides with the lower confidence limit calculated by the method of "equivalent tests" if the "equivalent numbers of failures",  $D_i$ , are integer. (If  $D_i$  are non-integer, the difference is not significant since  $D_i - L_i = D_i - [D_i]$  from (7.119).)

Thus, the lower confidence limit calculated by the method of "equivalent tests" is always as good as the strong lower confidence limit (7.115). The above argument shows that from the computational viewpoint this heuristic procedure factually reduces the finding minimum in (7.115) among the "corner points" of the area determined by (7.116)-(7.117) instead of searching within the entire area. Correspondingly, the proof of correctness of the method is reduced to the question: how much does the absolute minimum value within the entire area differ from the minimum value found in  $m$  "corner points" (see Section 7.6.6).

Example 7.21 Consider Example 7.18. Let us construct the lower confidence limit for the system PFFO by the method of "equivalent tests". The point estimate of the system PFFO in this case is

$$\hat{R} = \prod_{1 \leq i \leq m} [1 - (1 - \hat{p}_i)^{n_i}] = \prod_{1 \leq i \leq m} \left[ 1 - \left( \frac{d_i^*}{N_i} \right)^{n_i} \right]$$

$$= \left[ 1 - \left( \frac{1}{25} \right)^3 \right] \left[ 1 - \left( \frac{1}{25} \right)^3 \right] \left[ 1 - \left( \frac{2}{25} \right)^3 \right] = 0.99936.$$

From condition (7.113) we find that the "equivalent number of failures",  $D_1$ , for the first subsystem is

$$1 - \left( \frac{D_1}{N_1} \right)^{n_1} = \hat{R}$$

from where we get

$$D_1 = N_1 \left( 1 - \hat{R} \right)^{\frac{1}{n_1}} = 25 \sqrt[3]{64 \cdot 10^{-5}} = 2.16.$$

For the second and third subsystems, the "equivalent number of failures",  $D_2$  and  $D_3$ , coincide with  $D_1$ . Further, from (7.114) we find the lower  $\square$ -confidence limit (with the confidence coefficient  $\square=0.9$ ) for the system PFFO

$$\underline{R} = 1 - [1 - \underline{P}_\gamma(N_1, D_1)]^3 = 1 - [1 - \underline{P}_\gamma(25; 2.160)]^3 = 1 - (1 - 0.795)^3 = 0.991.$$

In this case the confidence limit coincides with that found in Example 7.18 by the modified method of hyperplane. ■

Example 7.22 Consider again the same system as in the previous example but with different test results (see Table 7.8).

..... Table 7.8

In this case, the point estimate of the system PFFO is

$$\hat{R} = \left[ 1 - \left( \frac{1}{25} \right)^3 \right] \cdot \left[ 1 - \left( \frac{1}{50} \right)^3 \right] \cdot \left[ 1 - \left( \frac{2}{100} \right)^3 \right] = 0.99992.$$

The minimum number of tested units is  $n_1=25$ . From (7.113) we find the "equivalent number of failures"  $D_1$  for the first subsystem:

$$D_1 = N_1(1 - \hat{R})_{n_1}^{\frac{1}{3}} = 25 \cdot \sqrt[3]{0.00008} = 1.08.$$

Thus, the lower  $\square$ - confidence limit with the confidence coefficient  $\square=0.9$  for the system PFFO calculated by the method of "equivalent tests" is

$$\begin{aligned} \underline{R} &= 1 - [1 - \underline{P}_{\square}(N_1, D_1)]^3 = 1 - [1 - \underline{P}_{0.9}(25, 1.08)]^3 = \\ &= 1 - (1 - 0.848)^3 = 0.996. \end{aligned}$$

Notice that in this case, the minimum number of tested units,  $N_i=25$ , and observed numbers of failures,  $d_i^*$ , are the same as in Examples 7.6.1 and 7.5.4. Thus the methods of hyperplane and modified hyperplane method give in this case the previous quantities equal to 0.966 and 0.991, respectively. It shows that they are not sensitive to increasing  $N_2$  and  $N_3$  if the minimum number  $N_1$  is fixed. The method of "equivalent tests" delivers higher values of lower  $\square$ - confidence limit of the system PFFO. ■

Example 7.23 Apply the method of "equivalent tests" to the system considered above in Example 7.17.

In this case the point estimate of the system PFFO is

$$\hat{R} = \prod_{1 \leq i \leq m} [1 - (1 - \hat{p}_i)^{n_i}] = \left[ 1 - \left( \frac{1}{100} \right)^2 \right]^{15} = 0.9985.$$

From condition (7.113) we find the "equivalent number of failures" for the first subsystem:

$$D_1 = N_1(1 - \hat{R})_{n_1}^{\frac{1}{2}} = 100 \cdot \sqrt{0.0015} = 3.88.$$

In this case all values  $n_i$  and  $N_i$  are equal, therefore all "equivalent number of failures",  $D_i$ , for different subsystems coincides with  $D_1$ . The lower  $\square$ -confidence limit of the system PFFO calculated with the help of the method of "equivalent tests" equals

$$\begin{aligned} \underline{R} &= 1 - [1 - \underline{P}_{\square}(N_1, D_1)]^2 = 1 - [1 - \underline{P}_{0.9}(100, 3.88)]^2 \\ &= 1 - (1 - 0.923)^2 = 0.994. \end{aligned}$$

In this case the method of "equivalent tests" produces the  $\square$ -lower confidence limit much higher than other methods considered above (see Table 7.4 and Examples 7.17 and 7.20).■

### 9.6.2 Method of Reduction

This method was considered in [Martz and Duran, 1985] and [Tyoskin and Kursky, 1986]. Let us begin with a series-parallel system described by reliability index (7.84). The idea of the method is in the following. First we construct the point estimate,  $\hat{R}_i$  621, and lower  $\square$ -confidence limit,  $\underline{R}_i$ , for each subsystem PFFO  $R_i = 1 - (1 - p_i)^{n_i}$ . 622 These values are constructed in a standard way

$$\begin{aligned} \hat{R}_i &= 1 - (1 - \hat{p}_i)^{n_i} = 1 - \left( \frac{d_i^*}{N_i} \right)^{n_i}, \\ \underline{R}_i &= 1 - [1 - \underline{P}_{\square}(N_i, d_i^*)]^{n_i} \end{aligned}$$

where  $\underline{P}_{\square}(N_i, d_i^*)$  is the standard Clopper-Pearson lower  $\square$ -confidence limit. Then we replace the  $i$ th subsystem (redundant group) by an "equivalent unit" with the "equivalent number of tested units",  $M_i$ , and "equivalent number of failures",  $r_i$ , which are chosen from the conditions

$$\left\{ \begin{array}{l} 1 - \frac{r_i}{M_i} = \hat{R}_i, \\ \underline{P}_{\square}(M_i, r_i) = \underline{R}_i. \end{array} \right. \quad (7.120)$$

In other words, values of  $M_i$  and  $r_i$  are chosen in such a way that the point estimate and lower  $\square$ -confidence limit for an "equivalent unit" coincides with the corresponding quantities of the  $i$ th subsystem.

Thus, an initial series-parallel system is replaced by some supplementary (imaginary) series system consisting of  $m$  "equivalent units" for each of which we have test results  $M_i$  and  $r_i$ ,  $i = \overline{1, m}$  624. After this, the lower  $\square$ -confidence limit can be constructed with the help of any known method for a series system (for instance, by the Lindstrom-Madden method).

Notice that in contrast to the previous method of "equivalent tests" (see Section 7.6.1) where the number of tests  $N_i$  was kept constant, in this case both  $r_i$  and  $M_i$ , determined from (7.120), are varied. Both these quantities might be non-integer. In this case the lower  $\square$ -confidence limit,  $P_{\square}(M_i, r_i)$ , is found as the solution of equation

$$B_p(M_i - r_i, r_i + 1) = 1 - \square$$

where  $B_p(a, b)$  is beta function (see Section 2 of Appendix to the current chapter).

### 9.6.3 Method of Reduction for Complex Systems

In contrast to the previous method, the method of "equivalent tests" can be easily extended to systems with more complex structure than series-parallel.

#### Series-Parallel Systems with Different Units

Consider a system consisting of  $m$  redundant groups (subsystems). The  $i$ th subsystem might consist of  $n_i$  different units in parallel. In this case, the system PFFO has the form

$$R = \prod_{1 \leq i \leq m} \left[ 1 - \prod_{1 \leq j \leq n_i} (1 - p_{ij}) \right]$$

where  $p_{ij}$  is the PFFO of the  $j$ th unit of the  $i$ th subsystem. For each parameter  $p_{ij}$ , we have results of independent binomial tests with the following results:  $N_{ij}$  units were tested and  $d_{ij}^*$  of them have failed. The PFFO of the  $i$ th subsystem is denoted by

$$R_i = 1 - \prod_{1 \leq j \leq n_i} (1 - p_{ij}).$$

As before, we construct a corresponding point estimate  $\hat{R}_i$  for this index:

$$\hat{R}_i = 1 - \prod_{1 \leq j \leq n_i} (1 - \hat{p}_{ij}) = 1 - \prod_{1 \leq j \leq n_i} \left( \frac{d_{ij}^*}{N_{ij}} \right)$$

and the lower  $\square$ -confidence limit,  $R_i$ , which can be constructed with the help of any known method for a parallel system, for instance, that considered in Section 7.4.1.

After this, the entire  $i$ th subsystem is replaced by the supplementary "equivalent unit" with "equivalent number of tests",  $M_i$ , and "equivalent number of failures",  $r_i$ . The numbers  $M_i$  and  $r_i$  are again found from (7.120). Thus, we replace an initial series-parallel system by a series system

consisting of  $m$  equivalent units. After this, the lower  $\alpha$ -confidence limit can be constructed by the Lindstrom-Madden method.

### Systems with Reducible Structures

The method of reduction can be extended on the so-called reducible structures. Remember that a reducible structure is such a structure which can be obtained from an initial simple series (or parallel) system by replacing units of this structure by series and parallel substructures. Such procedure of replacement can be recurrently continued. It is obvious that a reducible structure can be "converted" up to a single unit by the inverse procedure (replacement simple series and parallel fragments by a single "equivalent unit"). In such a manner an entire initial reducible system can be transformed into a single "equivalent unit". An example of a structure of such type is depicted in Figure 8.1 of the next chapter.

The method of reduction is applied for statistical problems as follows. Each  $k$ th parallel structure (fragment of a system) is replaced by "equivalent unit" with "equivalent number of tests",  $N_k$  and "equivalent number of failures",  $r_k$ . (These values are found as above.)

In an analogous manner, we replace each  $i$ th series structure (fragment of a system) by "equivalent unit" with "equivalent number of tests",  $N_k$ , and "equivalent number of failures",  $D_k$ , found as

$$D_k = N_k (1 - \hat{R}_k)$$

Here  $\hat{R}_k$  is the point estimate of the PFFO of the fragment replaced. In this case the point estimate  $1 - \frac{D_k}{N_k}$  and the lower  $\alpha$ -confidence limit  $\underline{P}_\alpha(N_k, D_k)$  of the PFFO of the "equivalent unit" coincide with the corresponding characteristics of the replaced fragment.

As we mentioned above, the procedure of recursive reduction allows one to represent an initial reducible system as a single "equivalent unit".

Example 7.24 Consider a series-parallel system analogous to that in Examples 7.18 and 7.21 (each redundant group consists of identical units). The structure of the system is depicted in Figure 7.5 and input data are presented in Table 7.6.

The reduction method gives for this system the lower  $\alpha$ -confidence limit of the system PFFO 0.987 with the confidence coefficient  $\alpha=0.9$ . Methods of hyperplane, modified method of hyperplane and method of "equivalent tests" give for this case quantities 0.966, 0.991 and 0.991, respectively. □

#### 9.6.4 Method of Fiducial Probabilities

The idea of the method was suggested in [Fisher, 1935] and developed by many authors (for instance, see [Rao, 1965]; [Fraser, 1961]). Consider this approach on a simple example. Let  $\theta_i$  be a reliability parameter of unit  $i$  and random variable  $x_i$  be the result of testing this unit. Denote the distribution function of random variable  $x_i$  for a fixed values of parameter  $\theta_i$  by

$$F_i(t, \theta_i) = P_{\theta_i}(x_i \leq t) \quad (7.121)$$

For the sake of simplicity, assume that function (7.121) is continuous in  $t$ . Assume also that this function is continuous and monotone increasing in parameter  $\theta_i$ .

Let  $x_i^*$  be a value of random variable  $x_i$  observed in the result of test. For fixed  $x_i^*$ , we consider parameter  $\theta_i$  as a random variable with the distribution  $F_i(x_i^*, \theta_i)$ . The distribution of parameter  $\theta_i$  defined in such a way is called a fiducial distribution. The upper and lower  $\theta_i$ -fiducial limits  $\underline{\theta}_i$  and  $\bar{\theta}_i$  for parameter  $\theta_i$  for given fixed test result  $x_i^*$  are determined from the following conditions

$$\begin{aligned} F_i(x_i^*, \underline{\theta}_i) &= 1 - \gamma, & (7.122) \\ F_i(x_i^*, \bar{\theta}_i) &= \gamma. \end{aligned}$$

It means that they are corresponding quantiles of fiducial distribution  $F_i(x_i^*, \theta_i)$ . The  $\theta_i$ -fiducial limits coincides with corresponding  $\theta_i$ -lower confidence limit for parameter  $\theta_i$  as it can be seen from equations (1.31) for confidence limit.

Notice that in this approach parameter  $\theta_i$  is not a random variable, but some unknown constant, and function  $F_i(x_i^*, \theta_i)$  is a distribution function of the result of observations,  $x_i$ , for given fixed value of parameter  $\theta_i$ . So, we interpret  $F_i(x_i^*, \theta_i)$  as a distribution function of parameter  $\theta_i$  for given fixed  $x_i$ . Correctness of the procedure is not obvious and even more -- doubtful. The probabilistic ("physical") sense of such transform also is unclear. (For details see, for instance, [Rao, 1965, Section 5b]; [Zacks, 1971, Section 10.6]; [Depster, 1964]; [Stein, 1959]; in application to reliability tests - [Pavlov, 1982, Sections 2.4, 4.5, and 4.6].) Nevertheless, independently of the interpretation and validation of the method, we might consider it a convenient formal approach which gives good practical results in many cases.

Let  $R$  be a reliability index of a system, consisting of  $m$  units of different types. This index is a function  $R=R(\theta)=R(\theta_1, \dots, \theta_m)$  of parameters  $\theta=(\theta_1, \dots, \theta_m)$  of the units. Let  $\mathbf{x}^*=(x_1^*, \dots, x_m^*)$  be observed values of the vector of test results  $\mathbf{x}=(x_1, \dots, x_m)$  where  $x_i$  is the test result for the  $i$ th type units. Test results for different units are assumed independent. For each parameter  $\theta_i$  let us construct fiducial distribution  $F_i(x_i^*, \theta_i)$  in a manner proposed above. Now the reliability index,  $R=R(\theta_1, \dots, \theta_m)$ , can be considered as a function of fiducial random variable  $\theta_i$  with corresponding distributions. The fiducial distribution function for  $R$  for a given vector of test results  $\mathbf{x}^*$  is determined by the following formula:

$$\Phi(\mathbf{x}^*, R) = \int \dots \int_{R(\theta_1, \dots, \theta_m) \leq R} \prod_{1 \leq i \leq m} f_i(x_i^*, \theta_i) d\theta_i \quad (7.123)$$

where  $f_i(x_i^*, \theta_i) = \frac{\partial F_i(x_i^*, \theta_i)}{\partial \theta_i}$  632 is the density function of the fiducial distribution of parameter  $\theta_i$ ,  $i = \overline{1, m}$  633.

The lower and upper  $\gamma$ -fiducial limits  $\underline{R}$  and  $\overline{R}$  634 for the system PFFO,  $R$ , are determined from conditions

$$\begin{aligned} \Phi(x^*, \underline{R}) &= 1 - \gamma, & (7.124) \\ \Phi(x^*, \overline{R}) &= \gamma, \end{aligned}$$

Analytical calculation of distribution (7.123) and limits  $\underline{R}$  and  $\overline{R}$  635 in (7.124) is usually too complicated. Nevertheless, in many practical cases these values can be found by Monte Carlo simulation.

The fiducial approach discussed above was used for reliability problems (mostly for binomial and exponential models) in [Springer and Thompson, 1964], [Senetsky and Shishonok, 1967], [Farkhad-Zadeh, 1979], [Groisberg, 1980], [Pavlov, 1980, 1981 a,b]. More available sources are [Gnedenko, ed., 1983] and [Ushakov, ed., 1994].

**Example 7.25** (Binomial model) Let  $\theta_i = p_i$  be the unit PFFO,  $x_i = d_i$  be the number of failures during the test, and  $N_i$  be the number of tested units of type  $i$ . In this case, distribution function (7.121) has the form

$$F_i(d_i, p_i) = \sum_{0 \leq j \leq d_i} \binom{N_i}{j} p_i^{N_i - j} (1 - p_i)^j \quad (7.125)$$

Function (7.125) is a distribution function of the test result  $d_i$  for a given fixed value of parameter  $p_i$  and, at the same time, is a fiducial distribution function of parameter  $p_i$  for a given fixed test result  $d_i$ . Further, let the system consist of  $m$  different types of units with parameters  $\mathbf{p} = (p_1, \dots, p_m)$  and function  $R = R(\mathbf{p}) = R(p_1, \dots, p_m)$  reflect the dependence of the system PFFO on parameters of units. For instance, for a series-parallel systems considered above

$$R = \prod_{1 \leq i \leq m} [1 - (1 - p_i)^{n_i}].$$

The value of  $R$  is considered below as a fiducial random variable with a distribution which is determined by distributions of parameters (7.125). Afterwards, the  $\gamma$ -fiducial limit for  $R$  is constructed as mentioned above.  $\square$

**Example 7.26** (Exponential model) A system consists of  $m$  different units each of which has the exponential distribution of TTF. The failure rate of the  $i$ th unit is  $\theta_i$ ,  $i = \overline{1, m}$  637. In this case  $\theta_i = \lambda_i$  and  $x_i = S_i$  where  $S_i$  is the total time of testing of all units of the  $i$ th type until occurrence of  $r_i$  failures. In this case function

$$F_i(S_i, \lambda_i) = 1 - e^{-\lambda_i S_i} \sum_{0 \leq j \leq r_i - 1} \frac{(\lambda_i S_i)^j}{j!} \quad (7.126)$$

is a function of test results  $S_i$  for a given fixed value of parameter  $\square_i$  and simultaneously the same function is a fiducial distribution of parameter  $\square_i$  for a given fixed value of the test result  $S_i$ . Let  $R=R(\square)=R(\square_1, \dots, \square_m)$  be a function that gives the dependence of the system PFFO on unit parameters. Then  $\square$ -fiducial limit for  $R$  can be constructed on the basis of fiducial distribution (7.126) for parameters  $\square_i, i = \overline{1, m}$  639, as it was described above.  $\square$

Sometimes the fiducial approach produces obviously ineffective limits (which is illustrated by the example below).

**Example 7.27** (Binomial model, No-failure-test) Take a series system considered in Example 7.25 for particular case:  $n_1=n_2= \dots =n_m=1$ . All units are tested in equal numbers:  $N_1=N_2= \dots =N_m=N$  and no failures has been observed:  $d_1=d_2= \dots =d_m=0$ .

In this case the lower  $\square$ -fiducial limit for the system PFFO can be easily found analytically. Indeed, from (7.125) it follows that parameter  $p_i$  is a fiducial random variable with distribution function

$$F_i(d_i, p_i) = F_i(0, p_i) = p_i^{N_i} = p_i^N, \quad i = \overline{1, m}. \quad (7.127)$$

The system PFFO in this case equals

$$R = p_1 p_2 \dots p_m. \quad (7.128)$$

Thus, the lower  $\square$ -fiducial limit  $\underline{R}$  for  $R$  equals a quantile of the level of  $(1-\square)$  for random variable (7.128) under the condition that each parameter  $p_i$  has distribution (7.127). This random variable can be found, for instance, by the following way. Introduce random variable's  $\xi_i = p_i^N, i = \overline{1, m}$ . Each of random variable  $\square_i$  has the uniform distribution on interval  $[0, 1]$ . Introduce also random variable's  $\eta_i = -\ln \xi_i, i = \overline{1, m}$ .

Since  $\square_i$  has the uniform distribution on interval  $[0, 1]$ , random variable  $\eta_i$  has the exponential distribution,  $1-e^{-x}$ . Thus,  $R$  can be written in the following form

$$R = p_1 \cdot p_2 \cdot \dots \cdot p_m = \prod_{1 \leq i \leq m} \xi_i^{\frac{1}{N}} = \exp\left(-\frac{1}{N} \sum_{1 \leq i \leq m} \eta_i\right).$$

Then use the well known fact that random variable  $2(\square_1 + \square_2 + \dots + \square_m)$  has the standard  $\square^2$  distribution with  $2m$  degrees of freedom (see Section 1.2). Inequality  $R \geq \underline{R}$  is equivalent to inequality



$$2 \sum_{1 \leq i \leq m} \eta_i \leq -2N \cdot \ln \underline{R}.$$

After simple transformations we obtain that the lower  $\square$ -fiducial limit  $\underline{R}$  for the system PFFO equals

$$\underline{R} = e^{-\frac{\chi_{\gamma}^2(2m)}{2N}} \quad (7.129)$$

where  $\chi_{\gamma}^2(2m)$  is the quantile of the level  $\square$  for the  $\square^2$  distribution with  $2m$  degrees of freedom. The lower limit (7.129) rapidly decreases with increasing of the number of system units,  $m$ . On the first glance, this fact seems natural, since the system has a series structure. From another hand, the best lower  $\square$ - confidence limit obtained in [Mirny and Solovyev, 1964] for the same situation (see Sections 6.1 through 6.3) is

$$\underline{R} = (1 - \gamma)^{\frac{1}{N}} = e^{-\frac{\chi_{\gamma}^2(2)}{2N}} \quad (7.130)$$

and it does not depend on  $m$  and for  $m > 1$  is always better than (7.129). The more  $m$ , the worse fiducial limit (7.129). (Notice incidentally that Bayesian approach possesses the same deficiency.)

Nevertheless, if we do not deal with no-failure-tests, the fiducial method (as well as Bayesian one) can deliver good results (see Chapter 8). Besides, this method is rather universal and can be applied for systems with various types of structures. It makes this method very popular in engineering applications.

Notice also that the fiducial method is approximate in the sense that for the  $\square$ -fiducial limit (for instance, lower) does not guarantee the inequality for the confidence coefficient

$$P_{\theta} \{ \underline{R} \leq R(\theta) \} \geq \gamma \quad (7.131)$$

for all possible values of parameters  $\square = (\square_1, \dots, \square_m)$ . Although for  $m=1$ , that is for one unknown parameter,  $\square$ -fiducial limit simultaneously represents  $\square$ -confidence limit, for  $m > 1$  justification of (7.131) does not follow from anywhere (for all  $\square$ ). In Chapter 8 it will be shown that there are examples where the fiducial method does not work. However, at the same time one can find many examples where this method is valid. For instance, for a wide class of "exponential models" for systems with complex structures, the fiducial approach delivers strong lower  $\square$ - confidence limit, that is, its application is correct and effective.

### 9.6.5 Bootstrap Method

The idea of this method [Efron, 1979, 1982] will be illustrated on the example of the binomial model. Let  $R=R(\mathbf{p})=R(p_1, \dots, p_m)$  be a function expressing the dependence of the system

PFFO on binomial parameters of its units  $\mathbf{p}=(p_1, \dots, p_m)$ . The point estimate  $\hat{R}$  645 of reliability index  $R$  is written as

$$\hat{R} = R(\hat{\mathbf{p}}) = R(\hat{p}_1, \dots, \hat{p}_m)$$

where  $\hat{p}_i = 1 - \frac{d_i}{N_i}$  646 is the point estimate of parameter  $p_i$ .

Denote the distribution function of the point estimate,  $\hat{R}$  647, for a given vector of parameters  $\mathbf{p}=(p_1, \dots, p_m)$  by  $\Phi(t, \mathbf{p}) = P_p(\hat{R} \leq t)$ .

We need to construct the lower  $\square$ - confidence limit  $\underline{R}$  for the system PFFO,  $R=R(\mathbf{p})$ . Let us define  $\underline{R}$  as the quantile of the level  $(1-\square)$  of the distribution function of estimate  $\hat{R}$  648, that is, from the following condition

$$\Phi(\underline{R}, \mathbf{p}) = 1 - \gamma. \quad (7.132)$$

The distribution of estimate  $\hat{R}$  650 depends on parameters  $\mathbf{p}=(p_1, \dots, p_m)$  which are unknown by the formulation of the problem. Therefore a direct determination of the value of  $\underline{R}$  from (7.132) is impossible. To pass side this obstacle, let us set  $\mathbf{p}=\mathbf{p}^*$  where  $\mathbf{p}^*=(p_1^*, \dots, p_m^*)$  and

$p_i^* = 1 - \frac{d_i^*}{N_i}$ ,  $i = \overline{1, m}$ , 651 where  $d_i^*$  is the number of observed failures. Function  $\square(t, \mathbf{p}^*)$  is called

a bootstrap distribution of estimate  $\hat{R}$  652. So, value  $\underline{R}$  is determined from the condition

$$\square(\underline{R}, \mathbf{p}^*)=1-\square. \quad (7.133)$$

In other words, for finding  $\underline{R}$  we use an estimate of distribution  $\square(t, \mathbf{p}^*)$  instead of the distribution of estimate,  $\square(t, \mathbf{p})$ .

Finding an exact analytical expression for a distribution function in the left side of (7.133) is usually a difficult task. Therefore most often Monte Carlo simulation is used for estimating  $\square(t, \mathbf{p}^*)$ .

In this case for simulation we use estimates  $p_i^* = 1 - \frac{d_i^*}{N_i}$  653 which are taken from real tests (see Section 2.6) instead of unknown values of parameters  $p_i$ .

The main benefit of this method is its universality, simplicity and clearness. However, this method sometimes produces (for lower limits) too optimistic estimates (see, for instance, [Martz and Duran, 1985]).

### 9.6.6 Normal Approximation

Consider a binomial testing plan for a series system. For the case of highly reliable units the system PFFO can be approximately written in the form

$$R = \prod_{1 \leq i \leq m} (1 - q_i) \approx \exp(-\sum_{1 \leq i \leq m} q_i)$$

where  $q_i = 1 - p_i$  is the probability of the  $i$ th unit PFFO,  $q_i \ll \frac{1}{m}$ ,  $i = 1, \dots, m$ . The construction of the lower confidence level for the system PFFO reduced to the construction of the UCL for the value

$$f = f(q) = \sum_{1 \leq i \leq m} q_i$$

As an initial statistic  $S$  which is used for the construction of the confidence limit, let us take the unbiased point estimate  $\hat{f}$  for  $f$  that is,

$$S = \hat{f} = \sum_{1 \leq i \leq m} \hat{q}_i \quad (7.134)$$

where  $\hat{q}_i = \frac{d_i}{N_i}$  is the standard point estimate for parameter  $q_i$ ,  $i = 1, \dots, m$ . Let us denote the mathematical expectation and variance of statistics  $S$  for a given vector of parameters  $\mathbf{q} = (q_1, \dots, q_m)$  by  $\mathbf{E}(\mathbf{q})$  and  $\mathbf{V}(\mathbf{q})$ :

$$\mathbf{E}(\mathbf{q}) = ES = E\left(\sum_{1 \leq i \leq m} \hat{q}_i\right) = \sum_{1 \leq i \leq m} q_i \quad (7.135)$$

$$\mathbf{V}(\mathbf{q}) = \text{Var}\{S\} = \sum_{1 \leq i \leq m} \text{Var}\{\hat{q}_i\} = \sum_{1 \leq i \leq m} \frac{q_i(1 - q_i)}{N_i} \approx \sum_{1 \leq i \leq m} \frac{q_i}{N_i}$$

Distribution function of statistic  $S$  for given  $\mathbf{q}$  is denoted by  $F(t, \mathbf{q}) = P_{\mathbf{q}}(S \leq t)$ . Random variable  $S$  represents the sum of  $i$  independent random variables with finite moments. Therefore for large enough  $m$  and  $N_i$  the distribution of statistic  $S$  can be approximated by the normal distribution with the mean  $\mathbf{E}(\mathbf{q})$  and variance  $\mathbf{V}(\mathbf{q})$

$$F(t, q) \approx \Phi\left(\frac{t - \mathbf{E}(\mathbf{q})}{\sqrt{\mathbf{V}(\mathbf{q})}}\right) \quad (7.136)$$

where  $\Phi(u)$  is the standard normal distribution.

Applying general functions (7.9) and (7.11), we obtain that the upper  $\gamma$ -confidence limit  $\bar{f}$  for  $f = f(q)$  is defined from the equation

$$K_1(\bar{f}) = S^* \quad (7.137)$$

where  $S^* = \sum_{1 \leq i \leq m} \frac{d_i^*}{N_i}$ ,  $d_i^*$  is the number of observed failures, and function  $K_1(f)$  is defined as

$$K_1(f) = \min_{q \in A_f} t_1(\mathbf{q})$$

where minimum is taken over the set  $A_f = \{q: f(q) = f\}$ , and the value  $f_1(q)$  is the quantile of level  $(1-\gamma)$  of the distribution function of statistic  $S$ . In accordance to (7.136)  $t_1(q)$  is defined from the equation

$$\Phi\left(\frac{t_1 - E(\mathbf{q})}{\sqrt{V(\mathbf{q})}}\right) = 1 - \gamma$$

from where

$$t_1(\mathbf{q}) = E(\mathbf{q}) - u_\gamma \sqrt{V(\mathbf{q})}$$

where  $U_\gamma$  is the quantile of the level  $\gamma$  of the standard normal distribution. (Here the difference is understood as operations over vector components.)

Thus, for the normal approximation, the upper  $\gamma$ -confidence limit  $\bar{f}$  is defined from (7.137) where

$$K_1(f) = \min_{q \in A_f} E(\mathbf{q}) - u_\gamma \sqrt{V(\mathbf{q})} \quad (7.138)$$

are minimum is taken over all  $\mathbf{q} = (q_1, \dots, q_m)$  that satisfy the constrain

$$\sum_{1 \leq i \leq m} q_i = f \quad (7.139)$$

$$0 \leq q_i \leq 1, i=1, \dots, m$$

The value of  $U_\gamma > 0$  if  $\gamma > 1/2$ . Taking into account inequality (7.135), minimum (7.138) is written in the form

$$K(f) = E(f) - u_\gamma \sqrt{V^+(f)}$$

where

$$V^+(f) = \max_{q \in A_f} V(\mathbf{q}) = \max_{q \in A_f} \sum_{1 \leq i \leq m} \frac{q_i}{N_i} \quad (7.140)$$

is the maximum value of the variance  $V(\mathbf{q})$  within the area of parameter  $\mathbf{q}$  satisfying constrains (7.139). Maximum (7.140) belongs to one of the ‘‘corner’’ points of the form:  $(\underbrace{0, \dots, 0}_{i-1}, f, 0, \dots, 0)$ .

It produces  $V^+(f) = f/N_1$  where  $N_1 = \min_{1 \leq i \leq m} N_i$  is the minimum volume of tests among all units.

Thus, the upper  $\gamma$ -confidence limit  $\bar{f}$  for the normal approximation can be defined from the equation

$$\bar{f} - u_\gamma \sqrt{\frac{\bar{f}}{N_1}} = \sum_{1 \leq i \leq m} \frac{d_i^*}{N_i} \quad (7.141)$$

Let us introduce an ‘‘equivalent number of failures’’ as follows

$$\tilde{D}_1 = N_1 S^* = \sum_{1 \leq i \leq m} \left( \frac{N_1}{N_i} \right) d_i^*$$

For small numbers of failures  $d_i \ll N_i$ ,  $i = 1, \dots, m$ , the value  $\tilde{D}_1$  coincides with the “equivalent number of failures”  $D_1$  given by formula (7.57), differing only in the infinitesimally small values of the second and higher orders:

$$\sum_{1 \leq i \leq m} \left( \frac{N_1}{N_i} \right) d_i^* \approx N_1 \left[ 1 - \prod_{1 \leq i \leq m} \left( 1 - \frac{d_i^*}{N_i} \right) \right]$$

Equation (7.141) for the confidence limit can also be written in the form

$$\bar{f} - u_\gamma \sqrt{\frac{\bar{f}}{N_1}} = \frac{\tilde{D}_1}{N_1} \quad (7.142)$$

Finally, the lower confidence level for the system PFFO can be computed as  $\underline{R} = \exp(-\bar{f})$ .

From (7.142) it follows that the lower confidence level for the system PFFO is computed in the same way as the lower confidence level for a single unit of type 1 with the minimum test volume  $N_1$  and the “equivalent number of failures”  $\tilde{D}_1$ . Strictly speaking, the idea of “equivalent tests” for the series system under present consideration. Thus, for the construction of the lower confidence level of the PFFO of a series system, if we take the point statistic (7.134) as the initial statistic  $S$ , then the normal approximation of this confidence limit corresponds to that obtained by the method of “equivalent tests” (for  $\gamma > 1/2$ ). Analogous fact takes place for series-parallel systems (see [Krol', 1974, 1975] and [Pavlov, 1982]).

### 9.7 Method of the Use of Basic Structures

Accurate methods of confidence estimation considered in Sections 7.2 and 7.5 (that is, the methods for which a given value of  $\gamma$  is guaranteed by inequality 7.111) in many cases deliver very conservative estimates for the system reliability. This is the reason for the use of different heuristic or approximate methods, based in some sense, on asymptotic results (see Section 7.6). Notice that main existing methods of confidence estimation of complex systems reliability do not take into account of optimization approaches due to added complications. In addition notice that asymptotic methods for large samples, for instance, the maximum likelihood method (see [Wilks, 1938], [Madansky, 1965] and others) do not appear to be a good approximation for small samples.

We suggest an approach based on the construction of an algorithm of system confidence limit estimation based on the methods existing for systems with simpler structures below. Actually, this approach is the development of a simple idea (already used in Sections 4.5 and 7.4) of constructing the system confidence limits on the basis of a simpler system consisting of the same units.

The construction of the algorithm is realized in two stages. At first, an ensemble of simple auxiliary structures is introduced, each of which consists of the same units as the investigated system. For each of these structures, the confidence limits are constructed with the help of “trivial algorithms” used for developing the algorithm for the complex structure confidence estimation. Then some operations are introduced (see Theorem 7.7.1) which allows us to combine these “trivial algorithms”. Then the best algorithm is searched among the class of these combined algorithms. It is shown that for some general conditions there is an optimal algorithm (Theorem 7.7.2) which gives the confidence limits that are better for arbitrary test results than any “trivial algorithm” of the basic ensemble (Theorem 7.7.3).

### 9.7.1 Basic Ensemble of “Trivial Algorithms”.

Let a system consist of  $m$  units of different types, and  $\mathbf{z}=(z_1, z_2, \dots, z_m)$  be the vector of reliability parameters of units, where  $z_i$  is the parameter of the  $i$ th type unit,  $i=1, \dots, m$ . Vector  $\mathbf{z}$  takes its values from the area  $Z=\{\mathbf{z}: z_i \geq 0, i=1, \dots, m\}$  of  $m$ -dimensional Euclidean space. Let  $\mathbf{x}=(x_1, \dots, x_m) \in X$  be the vector of test results and  $P_z\{\cdot\}$  be the distribution of test results  $\mathbf{x} \in X$  for given value of vector  $\mathbf{z} \in Z$ . Let  $f=f(\mathbf{z})$  denote the function expressing the dependence between the reliability index and unit reliability parameters  $\mathbf{z}$ . This function is called system reliability function. We are interested in construction of the confidence limit (say, upper) with the confidence coefficient not less than given  $\gamma$  for  $f=f(\mathbf{z})$ , that is, we are searching a function of the test results  $\bar{f} = \bar{f}(x)$  such that the inequality

$$P_z\{\bar{f} \geq f(\mathbf{z})\} \geq \gamma \quad (7.143)$$

holds for all  $\mathbf{z} \in Z$ .

In correspondence with the general method of confidence sets (see Sections 1.4 and 7.2), the limit  $\bar{f} = \bar{f}(x)$  is found as the maximum value of the reliability function  $f(\mathbf{z})$  over entire  $\gamma$ -confidence set  $H_x \subset Z$ . A collection of sets in the space of parameters

$$H_x \subset Z, \quad \mathbf{x} \in X \quad (7.144)$$

is called a collection of  $\gamma$ -confidence sets for  $\mathbf{z} \in Z$ , if inequality

$$P_z\{\mathbf{z} \in H_x\} \geq \gamma \quad (7.145)$$

holds for all  $\mathbf{z} \in Z$ . The upper  $\gamma$ -confidence limit  $\bar{f} = \bar{f}(x)$  for  $f=f(\mathbf{z})$  is computed as

$$\bar{f} = \bar{f}(x) = \max_{\mathbf{z} \in H_x} f(\mathbf{z}) \quad (7.146)$$

It follows directly from (7.145) that  $\bar{f}$  constructed in such way satisfies inequality (7.143) for all  $\mathbf{z} \in Z$ , that is,  $\bar{f}$  is the  $\gamma$ -UCL for  $f=f(\mathbf{z})$ .

Thus, an algorithm for computation of the confidence limit  $\bar{f} = \bar{f}(x)$  is defined by a given collection of confidence sets (7.144) and by the operation of taking the optimum in

(7.146). Different collections of  $\gamma$ -confidence sets correspond to different algorithms of computing of the confidence set.

Further assume that there are several different collections of  $\gamma$ -confidence sets

$$\{H_x^\mu \subset Z, x \in X\}, \mu \in M \quad (7.147)$$

such that

$$P_z\{\mathbf{z} \in H_x^m\} \geq \gamma$$

for all  $\mathbf{z} \in Z$  and for all  $\mu \in M$ . The value of  $m$  is the “mark” of a collection of confidence sets in (7.147). Each collection of confidence sets  $H_x^m, \mathbf{x} \in X$ , corresponds to its own algorithm of computation of the following kind of the confidence limit:

$$\bar{f}_\mu = \bar{f}_\mu(x) = \max_{z \in H_x^\mu} f(z), \mu \in M \quad (7.148)$$

Thus, the value of  $\mu$  is the “mark” of the algorithm of computation of the confidence limit (7.148), and  $M$  defines a collection of such algorithms. An aggregation of all algorithms (7.139) with  $\mu \in M$  is called basis collection of algorithms for computation of the confidence limit for the system reliability function,  $f(\mathbf{z})$ .

Examples of such basis collections are collections of algorithms for systems with the simplest structures as series and parallel. Let us consider, for instance, a binomial model. Let  $z_i = -\ln p_i$  where  $p_i$  is the binomial parameter equal to the PFFO of a unit of the  $i$ th type, and  $x_i = d_i$  be the number of failures observed during a test of  $N_i$  units of the  $i$ th type,  $i = 1, \dots, m$ ,  $\mathbf{x} = (d_1, \dots, d_m)$ . Assume that we use these units of an initial complex system for constructing some series system in which a unit of the  $i$ th type is used  $\mu_i$  times,  $i = 1, \dots, m$ . The PFFO of this series system is

$$R_\mu = \prod_{1 \leq i \leq m} p_i^{\mu_i} = \exp\left(-\sum_{1 \leq i \leq m} \mu_i z_i\right) \quad (7.149)$$

where  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$  is the vector defining the number of different units in the system. Let us construct the lower confidence level on the basis of one of known methods, for instance, the Lindstrom-Madden, (7.58), (7.59) for the system reliability (7.140):

$$\underline{R}_\mu = \underline{R}_\mu(x) = \min_{1 \leq i \leq m} [P_\gamma(N_i, D_i)]^{\mu_i} \quad (7.150)$$

where values  $D_i$  are found from the conditions

$$\left(1 - \frac{D_i}{N_i}\right)^{\mu_i} = \hat{R},$$

$$\hat{R} = \prod_{1 \leq j \leq m} \left(1 - \frac{d_j}{N_j}\right)^{\mu_j}$$

By definition of the lower confidence level  $\underline{R}_\mu$  we have

$$P_z\{\underline{R}_\mu \leq R_\mu\} \geq \gamma.$$

The above inequality can be written for all  $\mathbf{z} \in Z$  in the following equivalent form:

$$P_z\left\{\sum_{1 \leq i \leq m} \mu_i z_i \leq \bar{g}_\mu(x)\right\} \geq \gamma \quad (7.151)$$

where  $\bar{g}_\mu(x) = -\ln \underline{R}_\mu(x)$ .

Consider a collection of sets in the space  $Z$  of the unit parameters  $\mathbf{z}=(z_1, \dots, z_m)$ :

$$H_x^\mu = \{z: \sum_{1 \leq i \leq m} \mu_i z_i \leq \bar{g}_\mu(x)\}, \quad x \in X \quad (7.152)$$

where  $\mathbf{x}=(d_1, \dots, d_m)$  is the vector of the numbers of failures observed at the different unit tests,  $X$  is the set of all vectors  $\mathbf{x}$  with positive integer coordinates. By (7.151) the collection of sets of the type (7.152) forms the collection of  $\gamma$ -confidence sets for  $\mathbf{z} \in Z$ . It means that the value

$$\bar{f}_\mu = \bar{f}_\mu(x) = \max_{z \in H_x^\mu} f(z) \quad (7.144)$$

is the upper  $\gamma$ -confidence limit for the reliability function  $f=f(\mathbf{z})$ .

Further, let us consider an ensemble of all such series structures with various numbers of units of different types,  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$ . For any fixed  $\boldsymbol{\mu}$ , each structure of such type is characterized by its own collection of confidence sets (7.152) and its own algorithm for confidence limit computation (7.153) for the reliability function of the initial system. We thus obtain a basis set of computational algorithms for the confidence limit of the form

$$\bar{f}_\mu(x) = \max_{z \in H_x^\mu} f(z), \quad \boldsymbol{\mu} \in M \quad (7.154)$$

where  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$  is the “mark” of an algorithm (and simultaneously it is the vector of numbers of units in the corresponding auxiliary series system). The set  $M$  represents the set of all  $m$ -dimensional vectors  $\boldsymbol{\mu}$  with positive integer coordinates:

$$M = \{\boldsymbol{\mu}: \mu_i = 0, 1, \dots; i=1, \dots, m\}.$$

Notice that the confidence limit  $\underline{R}_m$  is not just defined for integer, but for any positive value  $\mu_i$ . Therefore below we will use  $M$  defined by the following way:

$$M = \{\boldsymbol{\mu}: \mu_i \geq 0; i=1, \dots, m\}.$$

A collection of algorithms of the kind (7.154) we will call a basis collection of “trivial” algorithms. Further the basis collection of confidence sets (7.152) and corresponding collection of “trivial” algorithms (7.154) will be significantly extended, and within this new extended class of algorithms we will find the optimal algorithm (Theorem 7.7.1). This algorithm delivers for all test results  $\mathbf{x}$  the best confidence limit for  $f(\mathbf{z})$  in comparison with any “trivial” algorithm of the basis set (7.154).

## 9.7.2 Constructing New Algorithms from the Basis Collection of “Trivial” Algorithms.

Denote the boundary hyper-plane for the confidence set in (7.152) by



$$\Gamma_x^\mu = \{z \in Z: \sum_{1 \leq i \leq m} \mu_i z_i = g_\mu(x)\} \quad (7.155)$$

Assume that the reliability function of a system,  $f(\mathbf{z})=f(z_1, \dots, z_m)$  is monotone increasing in each variable, has continuous partial derivatives of the first order, and convex in  $\mathbf{z} \in Z$ . Assume also that the following conditions hold:

- A. Function  $g_m(\mathbf{x})$  is continuous in  $\mu \in M$ .
- B. For any  $\mu$  and  $\nu$  in  $M$ , sets  $\Gamma_x^\mu$  and  $\Gamma_x^\nu$  have non-empty intersection for any  $\mathbf{x} \in X$ .
- C.  $H_x^\mu = H_x^{c\mu}$  for any  $c > 0$ ,  $\mathbf{x} \in X$ ,  $\mu \in M$ .

The justness of these conditions can be proved directly on the basis of (7.150), (7.152), and (7.155). Notice that the latter condition shows that the set

$$M = \{ \mu: \|\mu\|=1; \mu_i \geq 0; i=1, \dots, m \}$$

can be considered as the set  $M$ . In other words, a set of all “directing” unitary vectors  $\mu$  with positive coordinates where

$$\|\mu\| = \sqrt{\sum_{1 \leq i \leq m} \mu_i^2}$$

is the standard norm in  $m$ -dimensional Euclid space .

Now let  $s$  be a curve in  $Z$  given in a parametrical form:

$$s = \{ \mathbf{z}: z_i = j_i(t), i=1, \dots, m; t \geq 0 \} \quad (7.156)$$

where  $j_i(t)$  are continuous, monotone functions increasing in  $t$ .

Let any ensemble  $S$  of non-intersected curves of type (7.156) and covering the entire space  $Z$

$$S = \{s_v: \mathbf{v} \in V\} \quad (7.148)$$

be called  $S$ -division of the parameter space  $Z$ . Arbitrary curves  $s_v$  and  $s_u$  above do not intersect if  $v \neq u$  [may be with the exception for a unique point of the origin,  $\mathbf{z}=(0, 0, \dots, 0)$  ] and

$$\bigcup_{v \in V} s_v = Z .$$

An example of such a division is  $S=\{s_v: \mathbf{v} \in V\}$  of the space  $Z$  by various rays (semi-lines) initiating at the origin:

$$s_v = \{ \mathbf{z}: z_i = v_i t; 1 \leq i \leq m; t \geq 0 \}$$

where  $\mathbf{v}=(v_1, \dots, v_m)$  is a “directing” unitary vector,  $\|\mathbf{v}\|=1$  which determines the direction of ray  $s_v$  and  $V$  is the set of all “directing” unitary vectors with positive coordinates

$$V = \{ \mathbf{v}: \|\mathbf{v}\|=1; v_i \geq 0; i=1, \dots, m \}.$$

Let some  $S$ -division (7.157) of the parameter space  $Z$  exist and  $a=a(\mathbf{v})$  be a one-to-one reflection

$$a: V \Rightarrow M.$$

Then the following theorem holds.

**Theorem 7.7.1.** The collection of sets

$$H_x(S, \alpha) = \bigcup_{\mathbf{v} \in V} \{s_{\mathbf{v}} \cap H_x^{a(\mathbf{v})}\}, \mathbf{x} \in X \quad (7.158)$$

represents a collection of  $\gamma$ -confidence sets for  $\mathbf{z} \in Z$ .

**Proof.** Let  $\mathbf{z}$  be any fixed point in  $Z$ . Then by the definition of  $S$ -division there is a unique  $\mathbf{v} \in V$  such that  $\mathbf{z} \in s_{\mathbf{v}}$ . For these fixed  $\mathbf{z}$  and  $\mathbf{v}$ , we have

$$P_z\{\mathbf{z} \in H_x(S, a)\} = P_z\{\mathbf{z} \in (s_{\mathbf{v}} \cap H_x^{a(\mathbf{v})})\} = P_z\{\mathbf{z} \in H_x^{a(\mathbf{v})}\} \geq \gamma.$$

The proof of the theorem follows from this statement. ■

Thus each pair  $S$  and  $a$

$$S = \{s_{\mathbf{v}}, \mathbf{v} \in V\}, \quad a: V \Rightarrow M$$

produces a new collection of  $\gamma$ -confidence sets formed in accordance to operation (7.149) on the basis of initial confidence limits  $H_{\mathbf{x}}, \mathbf{x} \in X, \mu \in M$ . For instance a trivial (identically equal to a constant) reflection

$$a(\mathbf{v}) \equiv \mathbf{v} \in M$$

in accordance with (7.154) gives the collection of confidence sets  $H_x(S, a), \mathbf{x} \in X$ , belonging to the basis collection of sets (7.147). However, an arbitrary reflection  $a=a(\mathbf{v})$ , not identically equal to a constant, produces a new limit for the system reliability uncton,  $f=f(\mathbf{z})$ .

Pair  $A=(S, a)$  is called an algorithm of constructing the upper confidence limits (7.149) and corresponding confidence limit  $\bar{f}$  for  $f=f(\mathbf{z})$ . This upper confidence limits  $\bar{f} = \bar{f}_A(\mathbf{x})$ , computing by the algorithm  $A$ , has the form

$$\bar{f}_A(x) = \sup_{z \in H_x(A)} f(z) = \sup_{\mathbf{v} \in V} \sup_{z \in s_{\mathbf{v}} \cap H_x^{a(\mathbf{v})}} f(z) \quad (7.159)$$

where  $H_x(A) = H_x(S, a)$  is the confidence limit formed in accordance with (7.158).

The sense of the construction for computing a new confidence limit given above can be explained in the following way. Assume that we a priori know that parameter  $\mathbf{z}$  belongs to some subset  $s_\nu \subset Z$ . Then the confidence limit (7.154) for  $\hat{f} = f(\mathbf{z})$  can be improved by the formula

$$\bar{f}'_\mu = \sup_{z \in s_\nu \cap H_x^\mu} f(z) \quad (7.160)$$

Notice that a collection of confidence limits  $H_x^m$  with an arbitrary mark  $\mu$ ,  $\mu \in M$ , can be used in the formula (7.160). If there is no prior information concerning parameter  $\mathbf{z}$ , then for computing the UCL we need to find the maximum of the value (7.151) over various subsets  $s_\nu$ , which cover the entire parametrical space  $Z$ . The mark  $m$ , generally speaking, can change dependently on  $\nu$  as some function  $\mu = a(\nu)$ . In other words, we use a “best” collection of confidence limits from the basis ensemble (7.147) for estimation of the function  $f(\mathbf{z})$  on each subset  $s_\nu$ . This leads us to constructions (7.158) and (7.159) where the choice of the function  $a(\nu)$  dividing the space  $Z$  onto subsets  $s_\nu$  can be arbitrary. Thus the problem is in finding the function  $a(\nu)$  and optimal division of  $L$ , that is, in choosing the optimal algorithm  $A = (S, a)$  among various pairs  $(S, a)$ .

### 9.7.3 Constructing the optimal algorithm

Let  $S^*$  be a class of all S-divisions of the parameter space  $Z$ , and  $W$  be a class of all reflections  $a = a(\nu) \in M$  such that

$$\{\mu: \mu = a(\nu), \nu \in V\} = M. \quad (7.161)$$

This condition means that all collections of confidence limits from the basis ensemble (7.147) are used for constructing the confidence limit (7.150). Let  $A = S^* \times W$  be a class of corresponding algorithms  $A = (S, a)$  of computation of the confidence limit (7.150) for the system reliability function  $f = f(\mathbf{z})$ .

Denote the gradient of function  $f(\mathbf{z})$  by

$$\nabla f(z) = \left( \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_m} \right)$$

and introduce set  $V_f, V_f \subset M$ , of all normed gradients of function  $f(\mathbf{z})$

$$V_f = \left\{ \nu: \nu = \frac{\nabla f(z)}{\|\nabla f(z)\|}, z \in Z \right\}.$$

Let  $A^* = (S^*, \alpha^*)$ , where division  $S^* = \{s_\nu^*, \nu \in V_f\}$  of space  $Z$  and reflection  $\alpha^* = \alpha^*(\nu) \in M$  are defined as follows

$$s_\nu^* = \{ \mathbf{z}: \nabla f(z) = t\nu, t \geq 0 \}, \nu \in V_f \quad (7.162)$$

$$\alpha^*(\mathbf{v})=\mathbf{v}, \quad \mathbf{v} \in V_f. \quad (7.163)$$

The following theorem shows that the confidence limit  $\bar{f}_{A^*}(x)$ , that is calculated in respect to the algorithm  $A^*=(S^*, \alpha^*)$  from (7.159), is the best one for  $\mathbf{x} \in X$  in comparison with the limit  $\bar{f}_A(\mathbf{x})$ , that is calculated by any other algorithm  $A \in A^*$ .

Theorem 7.7.2. For any algorithm  $A \in A^*$  the inequality

$$\bar{f}_A(x) \geq \bar{f}_{A^*}(x) \quad (7.164)$$

holds for all  $\mathbf{x} \in X$ .

Proof. Let  $\psi=\psi(\mu)$ ,  $\mu \in M$  is the reflection inverse to  $\alpha=\alpha(\mathbf{v})$ ,  $\mathbf{v} \in V$ , that is

$$\psi(\mu)=\{\mathbf{v}: \alpha(\mathbf{v})=\mu\} \subset V.$$

If the reflection  $\alpha: V \Rightarrow M$  is not unique, then subsets  $\psi(\mu) \subset V$  might include more than one point. The formula (7.150) can be rewritten as follows

$$\bar{f}_A(x) = \sup_{\mathbf{v} \in V} \max_{z \in I_v \cap H_x^{\alpha(\mathbf{v})}} f(z) = \sup_{\mu \in M} \sup_{\mathbf{v} \in \psi(\mu)} \max_{z \in I_v \cap H_x^\mu} f(z). \quad (7.165)$$

On the other hand, for the limit corresponding to algorithm  $A^*$ , we have

$$\bar{f}_{A^*}(x) = \sup_{\mathbf{v} \in V_f} \max_{z \in I_v^* \cap H_x^{\alpha^*(\mathbf{v})}} f(z) = \sup_{\mathbf{v} \in V_f} \max_{z \in I_v^* \cap H_x^\mu} f(z).$$

Let  $\mathbf{z}^*$  be the crossing point of subset  $s_{\mathbf{v}^*}$  with the border  $\Gamma_{\mathbf{v}^*}^*$  of the set  $H_{\mathbf{v}^*}^*$ . At this point by the definition of division  $S^*$  condition

$$\nabla f(\mathbf{z}^*) = \mathbf{v}$$

holds due to the convexity of function  $f(\mathbf{z})$ , and at this point the minimum

$$\min_{z \in \Gamma_x^{\mathbf{v}}} f(z) = f(\mathbf{z}^*). \quad (7.166)$$

is attained. Taking into account the monotone increase of the function  $f(z)$  in each variable  $z_i$ , it follows that

$$\max_{z \in I_v^* \cap H_x^{\mathbf{v}}} f(z) = \max_{z \in I_v^* \cap \Gamma_x^{\mathbf{v}}} f(z) = f(\mathbf{z}^*) = \min_{z \in \Gamma_x^{\mathbf{v}}} f(z).$$

From here, with condition  $V_f \in M$ , we have

$$\bar{f}_{A^*}(x) = \sup_{\mathbf{v} \in V_f} \min_{z \in \Gamma_x^{\mathbf{v}}} f(z) \leq \sup_{\mu \in M} \min_{z \in \Gamma_x^{\mathbf{v}}} f(z). \quad (7.167)$$

For any  $\mathbf{v} \in V$  and  $\mu \in M$  the inequality

$$\max_{z \in \mathcal{E}_v \cap H_x^{\mathbf{v}}} f(z) = \max_{z \in \mathcal{E}_v \cap \Gamma_x^{\mathbf{v}}} f(z) \geq \min_{z \in \Gamma_x^{\mathbf{v}}} f(z)$$

and, moreover, the inequality

$$\sup_{\mathbf{v} \in \mathcal{V}(\mu)} \max_{z \in \mathcal{E}_v \cap H_x^{\mathbf{v}}} f(z) \geq \min_{z \in \Gamma_x^{\mathbf{v}}} f(z) \quad (7.168)$$

hold. The proof now follows from (7.165), (7.167), and (7.168). ■

Notice that by the condition (7.161) the class  $A^*$  does not contain algorithms  $A=(S,a)$  with trivial reflections  $a=a(\mathbf{v})$  of the form  $a(\mathbf{v}) \equiv \mu$ . Therefore from theorem 7.7.2 does not follow directly that the limit  $\bar{f}_{A^*}(x)$  is optimal in comparison with any of limits  $\bar{f}_{\mu}(x)$  of the form (7.154) which corresponds to the initial collection of “trivial” algorithms. This fact follows from the next theorem.

**Theorem 7.7.3** For any  $\mu \in M$  the limit  $\bar{f}_{A^*}(x)$  satisfies the inequality

$$\bar{f}_{A^*}(x) \leq \bar{f}_{\mu}(x)$$

for all  $\mathbf{x} \in X$ .

*Proof* In correspondence with (7.167), the value  $\bar{f}_{A^*}(x)$  can be written in the form

$$\bar{f}_{A^*}(x) = \sup_{\mu \in V_f} \min_{z \in \Gamma_x^{\mathbf{v}}} f(z) \quad (7.169)$$

Therefore for any  $\varepsilon > 0$  there exists such  $\mu(\varepsilon) \in V_f \subset M$  that

$$\min_{z \in \Gamma_x^{\mathbf{v}(\varepsilon)}} f(z) > \sup_{\mu \in V_f} \min_{z \in \Gamma_x^{\mathbf{v}}} f(z) - \varepsilon = \bar{f}_{A^*}(x) - \varepsilon.$$

For any  $\mu \in M$ , the set  $\Gamma_x^{\mathbf{m}}$  due to the condition  $B$  has at least one common point  $\tilde{z}$  with the set  $\Gamma_x^{\mathbf{m}(\varepsilon)}$ . Thus,

$$\min_{z \in \Gamma_x^{\mathbf{v}(\varepsilon)}} f(z) \leq f(\tilde{z}) \leq \max_{z \in \Gamma_x^{\mathbf{v}}} f(z) = \max_{z \in H_x^{\mu}} f(z) = \bar{f}_{\mu}(x), \quad \mu \in M.$$

and taking into account the previous inequality, we have

$$\bar{f}_{A^*}(x) < \bar{f}_{\mu}(x) + \varepsilon, \quad \mu \in M.$$

Due to the arbitrariness of  $\varepsilon$ , the needed inequality is proved. ■

As an example, consider an application of the optimal algorithm  $A^*$  for computation off the lower confidence level of the PFFO of a series-parallel system for a binomial model. The PFFO of this system (7.84) can be written via parameters  $z_i = -\ln p_i$  in the following form

$$R = \prod_{1 \leq i \leq m} \{1 - (1 - p_i)^{n_i}\} = \exp\{-f(\mathbf{z})\}$$

where

$$f(\mathbf{z}) = \sum_{1 \leq i \leq m} f_i(z_i),$$

$$f_i(z_i) = -\ln\{1 - (1 - e^{-z_i})^{n_i}\}$$

The confidence estimation of the system reliability index  $R$  from below is reduced to the constructing of the UCL for  $f(\mathbf{z})$ . Function  $f(\mathbf{z})$  is monotonically increasing in each  $z_i$  and convex in  $\mathbf{z} \in Z$ .  $S^*$ -division of the space of parameters  $Z$  in correspondence with (7.162) has the form

$$S_v^* = \{\mathbf{z}: f_i(z_i) = tv_i, i=1, \dots, m; t=0\}$$

where  $\mathbf{v}=(v_1, \dots, v_m) \in V_f \subset M$ . The upper confidence limit (7.159) for  $f(\mathbf{z})$  corresponding to the optimal algorithm  $A^*$  has the form

$$\bar{f}_{A^*}(x) = \max_{\mu \in V_f} \max_{z \in \Gamma_\mu^* \cap H_x^\mu} f(z) \quad (7.170)$$

or in correspondence to (7.167)

$$\bar{f}_{A^*}(x) = \max_{\mu \in V_f} \min_{z \in \Gamma_x^\mu} f(z). \quad (7.171)$$

Notice that if the number of units of different subsystems  $n_i > 1, i=1, \dots, m$ , then the set  $V_f = M$ , that is the external maximum in (7.171) or (7.170) is taken over set  $M$ .

**Example 7.28** Consider a series-parallel system from Example 7.15. The system consists of  $m=10$  parallel subsystems connected in series. The numbers of units,  $n_i$ , in the subsystems and the test results,  $N_i, d_i$ , are given in Table 7.2 above. In this case the lower  $\gamma$ -confidence level (for  $\gamma=0.9$ ) for the system reliability computing with the help of algorithm  $A^*$  equals  $\underline{R}=0.9998$ . This value coincides with the limit found for this example by the heuristic method of “equivalent tests” (see Table 7.4 above).■

## 9.8 Bayes Method

Let us demonstrate this method applied to a binomial testing plan. Consider a system consisting of units of  $m$  different types. The system reliability index,  $R=R(\mathbf{p})=R(p_1, \dots, p_m)$ , is a function of vector  $\mathbf{p}=(p_1, \dots, p_m)$  representing the results of tests of individual units (here  $p_i$  is the PFFO of a unit of the  $i$ th type). For each unit of each type we have the results of tests: number of failures  $d_i$  in  $N_i$  tests. Unit test results are assumed to be independent.

Assume also that for each parameter  $p_i$  a prior density function  $h_i(p_i)$  is given and parameters  $p_i$  are also assumed independent. Thus the prior density of the distribution  $h(\mathbf{p})$  has the form

$$h(\mathbf{p}) = \prod_{1 \leq i \leq m} h_i(p_i).$$

In correspondence with the standard Bayes procedure (see Chapter 4 above, formula 4.1) the posterior density of the distribution of vector  $\mathbf{p}=(p_1, \dots, p_m)$  for given test results  $\mathbf{d}=(d_1, \dots, d_m)$  has the form

$$h(\mathbf{p}|\mathbf{d}) = \frac{h(\mathbf{p})L(\mathbf{d}|\mathbf{p})}{\varphi(\mathbf{d})} \quad (7.172)$$

where  $L(\mathbf{d}|\mathbf{p})$  is the likelihood function, the probability of the test result  $\mathbf{d}$  for given vector of parameters  $\mathbf{p}$ :

$$L(\mathbf{d}|\mathbf{p}) = \prod_{1 \leq i \leq m} \binom{N_i}{d_i} (1-p_i)^{d_i} p_i^{N_i-d_i}$$

and

$$\varphi(\mathbf{d}) = \int_0^1 \dots \int_0^1 h(\mathbf{p})L(\mathbf{d}|\mathbf{p}) dp_1 \dots dp_m.$$

It follows that the posterior density of distribution (7.172) is expressed by formula

$$h(\mathbf{p}|\mathbf{d}) = \prod_{1 \leq i \leq m} h_i(p_i|d_i) \quad (7.173)$$

where  $h_i(p_i|d_i)$  is the posterior density of the distribution of parameter  $p_i$ :

$$h_i(p_i|d_i) = \frac{h_i(p_i)(1-p_i)^{d_i} p_i^{N_i-d_i}}{\int_0^1 h_i(u)(1-u)^{d_i} u^{N_i-d_i} du}. \quad (7.174)$$

In Binomial scheme of testing considered here, one usually assumes that the prior distribution  $h_i(p_i)$  is a standard Beta distribution with parameters  $(a_i, b_i)$ , that is

$$h_i(p_i) = \frac{(1-p_i)^{a_i-1} p_i^{b_i-1}}{B(a_i, b_i)} \quad (7.175)$$

where  $B(a_i, b_i)$  is the beta function. In this case the posterior distribution (7.174) is also beta, but with different parameters (see Section 4.3 above)

$$h_i(p_i|d_i) = \frac{(1-p_i)^{a_i+d_i-1} p_i^{b_i+N_i-d_i-1}}{B(a_i+d_i, b_i+N_i-d_i)}.$$

Bayesian  $\gamma$ -confidence limit (for instance, lower)  $\underline{R}=\underline{R}(\mathbf{d})$  of the system PFFO,  $R=R(\mathbf{p})$ , can be found from the equation

$$\iiint_{R(p_1, \dots, p_m) \geq \underline{R}} \prod_{1 \leq i \leq m} h_i(p_i | d_i) dp_i = \gamma, \quad (7.176)$$

that is  $\underline{R}$  is the quantile of level  $(1-\gamma)$  of prior distribution of  $R(\mathbf{p})=R(p_1, \dots, p_m)$ . Analytical evaluation of limit  $\underline{R}$  on the basis of (7.176) might be too complicated, although it can be easily found with the Monte Carlo simulation. For a Monte Carlo simulation, independent random values of parameters  $p_i^{(j)}$ ,  $i=1, \dots, m$ , are generated on the basis of the posterior distributions (7.174). After this the value of the system reliability  $R^{(j)}=R(p_1^{(j)}, \dots, p_m^{(j)})$  is calculated, and the computational process goes to the next  $(j+1)$ th step. On the basis of  $n$  realizations one can construct a corresponding empirical distribution function, and afterwards the lower Bayesian  $\gamma$ -confidence limit,  $\underline{R}$ , is taken equal to the quantile of the level  $(1-\gamma)$  of this empirical distribution.

The Bayesian approach might deliver too conservative confidence limit  $\underline{R}$  for the reliability of series systems consisting of large number of subsystems and units if there were few failures, that is the case is close to the non-failure tests almost for all types of units (see Section 7.6.4, Example 7.27 above). Nevertheless, for “medium” numbers of failures, this approach gives admissible results. Besides, the merit of the Bayesian approach for this problem is that it is universal and visual. The Bayesian approach was used for analysis of the reliability of complex systems in many works, among them [Barlow, 1985], [Cole, 1975], [Dostal and Iannuzzelli, 1977], [Mann, Schafer, and Singpurwalla, 1974], [Martz and Waller, 1982, 1990], [Martz, Waller and Fickas, 1988], [Mastran, 1976], [Mastran and Singpurwalla, 1978], [Natvig and Eide, 1987], [Savchuk, 1989], [Springer and Thompson, 1966, 1967, 1968], [Smith and Springer, 1976], and other.

An additional merit of the Bayesian approach lays in a possibility to use a mixed information of testing of subsystems and units of the same system. Consider, for instance, a situation where in addition to the unit test results  $d_1, \dots, d_m$  we also know the results of test of  $K$  different subsystems of the system. These subsystems could be of series-parallel type. During the tests only subsystems failures were registered, not its units. Let  $R_j=R_j(\mathbf{p})$  is the PFFO of the  $l$ th subsystem,  $1 \leq j \leq K$ . If the  $j$ th subsystem has a series structure then

$$R_j(p) = \prod_{1 \leq i \leq m} p_i^{v_{ij}}$$

where  $v_{ij}$  is the number of units of the  $i$ th type comprising the  $j$ th subsystem. If the  $j$ th subsystem has a parallel structure then

$$R_j(p) = 1 - \prod_{1 \leq i \leq m} (1 - p_i)^{v_{ij}}.$$

In addition, suppose we have information about test of  $M_j$  subsystems of the  $j$ th type and it is known that there were  $D_j$  failures. The vector of the test results has the form



$$z=(\mathbf{d},\mathbf{D})=(d_1, \dots, d_m; D_1, \dots, D_K)$$

where  $\mathbf{d}=(d_1, \dots, d_m)$  is the vector of test results for units, and  $\mathbf{D}=(D_1, \dots, D_K)$  is the vector of test results for the subsystems. As usual, all test results are assumed to be independent. We need to construct, for instance, the lower confidence limit  $\underline{R}$  for the system PFFO,  $R=R(\mathbf{p})$ .

In the frame of the Bayesian method such problems are solved in the same manner as for more simple problem where only unit test results were known. However, pure calculational difficulties can arise. Indeed, the posterior density in this case is again determined by the formula equivalent to (7.172):

$$h(\mathbf{p}|\mathbf{d},\mathbf{D}) = \frac{h(\mathbf{p})L(\mathbf{d},\mathbf{D}|\mathbf{p})}{\varphi(\mathbf{d},\mathbf{D})} \quad (7.177)$$

where  $L(\mathbf{d},\mathbf{D}|\mathbf{p})$  is the probability of test results  $(\mathbf{d}, \mathbf{D})$  for the given vector of parameters,  $\mathbf{p}$ :

$$L(\mathbf{d},\mathbf{D}|\mathbf{p}) = \prod_{1 \leq i \leq m} \binom{d_i}{N_i} (1-p_i)^{d_i} p_i^{N_i-d_i} \prod_{1 \leq l \leq K} \binom{D_l}{M_l} [1-R_l(\mathbf{p})]^{D_l} [R_l(\mathbf{p})]^{M_l-D_l} \quad (7.178)$$

$$\varphi(\mathbf{d},\mathbf{D}) = \int_0^1 \dots \int_0^1 h(\mathbf{p})L(\mathbf{d},\mathbf{D}|\mathbf{p}) dp_1 \dots dp_m .$$

Now the lower Bayesian  $\gamma$ -confidence limit  $\underline{R}$  for the system PFFO,  $R(\mathbf{p})=R(p_1, \dots, p_m)$  is determined similarly to (7.176):

$$\int_{R(p_1, \dots, p_m) \geq \underline{R}} \dots \int h(\mathbf{p})L(\mathbf{d},\mathbf{D}|\mathbf{p}) dp_1 \dots dp_m = \gamma .$$

## 9.9 Appendices

9.9.1 Derivation of formulas (7.26) and (7.27) for the confidence limits in the case of arbitrary distribution of test results  $x$  and statistic  $S=S(\mathbf{x})$ .

Let

$$\underline{R} = \inf_{\theta \in H(S^*)} R(\theta), \quad \bar{R} = \sup_{\theta \in H(S^*)} R(\theta),$$

where set  $H(S^*)$  is defined by inequalities

$$\begin{aligned} t_1(\boldsymbol{\theta}) &\leq S^* \\ t_2(\boldsymbol{\theta}) &\geq S^*. \end{aligned}$$

In correspondence with (7.20) and (7.25), values  $\underline{R}$  and  $\bar{R}$  give the confidence interval for  $R=R(\boldsymbol{\theta})$  with the confidence coefficient not less than  $\gamma=1-\alpha-\beta$ .

Further, let us assume that the following conditions hold

Condition A. Function  $R(\boldsymbol{\theta})$  is continuous in  $\boldsymbol{\theta}$ .

Condition B. Functions of the form  $P_{\boldsymbol{\theta}}(S \leq S^*)$  and  $P_{\boldsymbol{\theta}}(S \geq S^*)$  are continuous in  $\boldsymbol{\theta}$ .

Directly from the definition of  $t_1(\boldsymbol{\theta})$  and  $t_2(\boldsymbol{\theta})$  in (7.20) the following applications follow:

$$\begin{aligned} t_1(\boldsymbol{\theta}) \leq S^* &\Rightarrow P_{\boldsymbol{\theta}}(S \leq S^*) \geq \alpha \quad (7.179) \\ t_2(\boldsymbol{\theta}) \geq S^* &\Rightarrow P_{\boldsymbol{\theta}}(S \geq S^*) \geq \beta \end{aligned}$$

and

$$\begin{aligned} P_{\boldsymbol{\theta}}(S \leq S^*) > \alpha &\Rightarrow t_1(\boldsymbol{\theta}) \leq S^* \quad (7.180) \\ P_{\boldsymbol{\theta}}(S \geq S^*) > \beta &\Rightarrow t_2(\boldsymbol{\theta}) \geq S^* \end{aligned}$$

Let us denote the set of parameters  $\boldsymbol{\theta}$  that satisfy the conditions  $P_{\boldsymbol{\theta}}(S \leq S^*) \geq \alpha$  and  $P_{\boldsymbol{\theta}}(S \geq S^*) \geq \beta$  by  $G(S^*)$ . Introduce also the set  $L(S^*)$  of parameters  $\boldsymbol{\theta}$  that satisfy the conditions  $P_{\boldsymbol{\theta}}(S \leq S^*) > \alpha$  and  $P_{\boldsymbol{\theta}}(S \geq S^*) > \beta$ . Due to (7.170) and (7.171) the following relations are valid

$$L(S^*) \subset H(S^*) \subset G(S^*) \quad (7.172)$$

By continuity of function  $R(\boldsymbol{\theta})$  the set  $L(S^*)$  is open and the set  $G(S^*)$  is closed.

Now assume that together with conditions  $A$  and  $B$  above, the following conditions also hold:

Condition C. Minimum and maximum of function  $R(\boldsymbol{\theta})$  are attained on the set  $G(S^*)$ . (This condition is surely holds if, for instance, the set  $G(S^*)$  is restricted.)

Condition D. Closure  $L^*(S^*)$  of set  $L(S^*)$  coincides with set  $G(S^*)$ .

All conditions from  $A$  through  $D$  given above are not practically very restrictive for the common reliability problems. From (7.172) due to condition  $C$  the following inequalities follow:

$$\min_{\theta \in G(S^*)} R(\theta) \leq \underline{R} \leq \inf_{\theta \in L(S^*)} R(\theta)$$

$$\sup_{\theta \in L(S^*)} R(\theta) \leq \bar{R} \leq \max_{\theta \in G(S^*)} R(\theta).$$

It gives the following lower and upper confidence limits due to the continuity of  $R(\theta)$  and condition  $D$ :

$$\underline{R} = \min R(\theta)$$

$$\bar{R} = \max R(\theta)$$

where minimum and maximum are taken over set  $G(S^*)$ . It delivers the proof of the statement.

Notice in conclusion that formulas (7.26) and (7.27) for the confidence limits  $\underline{R}$  and  $\bar{R}$  are valid for more general case where statistic  $S$  depends on the test results and a parameter  $\theta$ , that is,  $S=S(\mathbf{x}, \theta)$ . In this case inequalities (7.27) can be written in the form

$$\begin{aligned} P_{\theta}\{S(\mathbf{x}, \theta) \leq S(\mathbf{x}^*, \theta)\} &\geq \alpha \\ P_{\theta}\{S(\mathbf{x}, \theta) \geq S(\mathbf{x}^*, \theta)\} &\geq b\beta \end{aligned}$$

where  $\mathbf{x}^*$  is the observed value of random vector  $\mathbf{x}$ .

### 9.9.2 Computation of confidence limits for binomial testing plan.

The lower  $\gamma$ -confidence Clopper-Pearson limit for the binomial parameter  $p$  (the PFFO) is defined from the following equation:

$$\sum_{0 \leq j \leq d} \binom{N}{j} (1-p)^j p^{N-j} = 1-\gamma$$

where  $N$  is the number of tests and  $d$  is the observed number of failures. For integer  $N$  and  $d$  the left part of this equation can be written also in the form

$$\sum_{0 \leq j \leq d} \binom{N}{j} (1-p)^j p^{N-j} = B_p(N-d, d+1)$$

where

$$B_p(a, b) = \frac{\int_0^p x^{a-1} (1-x)^{b-1} dx}{\int_0^1 x^{a-1} (1-x)^{b-1} dx}$$

is the beta function. Thus, the equation for finding the lower confidence level can be written in the form

$$B_p(N-d, d+1) = 1-\gamma$$

The left part of this equation is defined for all positive values  $d < N$  (not necessarily integer). The solution of this equation relative to  $p$  is denoted by  $\underline{P}_\gamma(N, d)$ . Thus the value of  $\underline{P}_\gamma(N, d)$  is the lower  $\gamma$ -confidence Clopper-Pearson limit for parameter  $p$  obtained on the basis of  $N$  tests with  $d$  failures. This limit is valid for all positive  $d < N$ , that allows one to apply it for construction of confidence limits for the system reliability using various methods of “equivalent tests” (see Sections 7.6.1 and 7.6.2 above).

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## Problems to Chapter 7

7.1 A series system consists of two different units. Test of  $N_1=100$  units of the first type has no failures,  $d_1=0$ . Test of  $N_2=200$  units of the second type found  $d_2=4$ . Construct the lower  $\gamma$ -confidence level with the confidence coefficient not less than  $\gamma=0.9$  for the system PFFO  $R=p_1p_2$ .

7.2 Consider a series system consisting of units of different types,  $m=3$ . The number of units of each type,  $r_i$ , and results of tests,  $N_i$  and  $d_i$ , are given in Table 7.9. Construct the lower  $\gamma$ -confidence level with the confidence coefficient  $\gamma=0.95$  for the system PFFO  $R=p_1^2 p_2 p_3$ .

7.3 Consider a parallel system consisting of two units,  $m=2$ . Test results are  $N_1=10$ ,  $d_1=0$ , and  $N_2=20$ ,  $d_2=1$ , respectively. Construct the lower  $\gamma$ -confidence level with the confidence coefficient  $\gamma=0.9$  for the system PFFO

$$R=1-(1-p_1)(1-p_2).$$

7.4 Consider a parallel system consisting of units of two types,  $m=2$ . There are  $n_1=2$  units of the first type and a single unit of the second type,  $n_2=1$ . Test results are  $N_1=12$ ,  $d_1=0$ , and  $N_2=6$ ,  $d_2=1$ , respectively. Construct the lower  $\gamma$ -confidence level with the confidence coefficient  $\gamma=0.95$  for the system PFFO  $R=1-(1-p_1)^2(1-p_2)$ .

7.5 Consider a series-parallel system consisting of  $m=10$  subsystems that was considered in Example 7.5.1 above. Find lower  $\gamma$ -confidence level for the system PFFO by the method of “equivalent tests” (for  $\gamma=0.9$ ).

7.6 Find the lower  $\gamma$ -confidence level for the PFFO of the system considered in Example 7.5.2 above (for  $\gamma=0.9$ ) by the method of “equivalent tests”.

## 10. Confidence Limits for Systems Consisting of Units with Exponential Distribution of Time to Failure.

### 10.1 Introduction

The methods considered in previous chapters are valid for constructing confidence limits of simple series or series-parallel structures. For systems with more complex structures one uses usually different heuristic or approximate methods. However the correctness of application of these methods remains unclear, in other words, it is not known if this method produces confidence limits with guaranteed confidence coefficient for a complex system. The answer can be obtained by Monte Carlo simulation of the test process and constructing the confidence limit of a particular system for particular set of parameters. But a specific solution for this particular case does not deliver any information about results for other parameters. Moreover, there is no information for other types of system structures. Enumerating all the specter of structures of interest is practically impossible. Thus, Monte Carlo simulation is not the best way of validation of one method or another, though it is frequently used for engineering purposes in reliability analysis.

In this chapter we suggest methods of constructing confidence limits that are accurate in the sense that for them we guarantee the confidence limit not less some given value  $\gamma$  for all realizations of parameters. Besides, these methods work for a wide class of complex systems, particularly, for systems with repair if an additional suggestion is made, namely: units have exponential distribution of time to failure.

## 10.2 Method of “Replacement”

### 10.2.1 Description of the method

Let a system consist of  $m$  different types, and the distribution of TTF of the  $i$ th unit is exponential:  $F_i(t) = 1 - \exp(-\lambda_i t)$  with unknown parameter, FR,  $\lambda_i$ ,  $1 \leq i \leq m$ . Assume that a test of the unit  $i$  was performed by the standard plan of the type  $[N_i, U, r_i]$ , that is, without replacement of failed units, or by the plan  $[N_i, R, r_i]$ , that is, with replacement of failed units. (See details in Chapter 2, Section 2.1.) The test results in the form of summarized unit testing time are  $S_i$ ,  $1 \leq i \leq m$ . The test results are assumed independent.

Let  $R$  be some system reliability index and

$$R(\boldsymbol{\lambda}) = R(\lambda_1, \dots, \lambda_m) \quad (8.1)$$

be a function expressing the dependence of this reliability index on element parameters  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ . It is necessary to construct the confidence limit (for instance, for lower) for the system reliability  $R = R(\boldsymbol{\lambda})$  on the basis of unit test results.

Let us introduce the notation

$$\bar{\lambda}_i = \frac{\chi_\gamma^2(2r_i)}{2S_i} \quad (8.2)$$

where  $\chi_\gamma^2(2r_i)$  is a quantile of level  $\gamma$  for the  $\chi^2$ -distribution with  $2r_i$  degrees of freedom,  $r_i$  is the number of failures of the units of the  $i$ th type,  $1 \leq i \leq m$ . Let us also introduce the vector

$\bar{\boldsymbol{\lambda}} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)$  of upper  $\gamma$ -confidence limits for the separate unit parameters. The function (8.1), as a rule, is monotone decreasing for each its parameter  $\lambda_i$ . That is, they satisfy the natural condition: the system reliability decreases with decreasing unit reliability.

Take  $\underline{R}$  as a lower confidence limit of the reliability index  $R$ : a value that is computed by a direct substitution of upper  $\gamma$ -confidence limit into function (8.1), that is

$$\underline{R} = R(\bar{\boldsymbol{\lambda}}) = R(\bar{\lambda}_1, \dots, \bar{\lambda}_m) \quad (8.3)$$

Since  $R(\boldsymbol{\lambda})$  is monotonically decreasing in each parameter, the following relation is valid:

$$\bigcap_{1 \leq i \leq m} \{\bar{\lambda}_i \geq \lambda_i\} \subset \{R(\bar{\lambda}_1, \dots, \bar{\lambda}_m) \leq R(\lambda_1, \dots, \lambda_m)\}$$

Taking into account the unit test independence, we have

$$P\{R(\bar{\boldsymbol{\lambda}}) \leq R(\boldsymbol{\lambda})\} \geq P\left\{\bigcap_{1 \leq i \leq m} (\bar{\lambda}_i \geq \lambda_i)\right\} = \prod_{1 \leq i \leq m} P(\bar{\lambda}_i \geq \lambda_i) = \gamma^m.$$

Therefore the lower confidence limit (8.3) has the confidence coefficient not less than the value of  $\gamma^m$ . This procedure of the confidence limit construction, obviously, corresponds to the previously considered method of rectangular (Chapter 7, Section 7.5.3).

This simple procedure is universal enough and can be applied always if the function  $R(\lambda)$  is monotone. Notice that this condition is almost always valid. Nevertheless, confidence coefficient for this procedure decreases very fast with growing number of system unit types  $m$ . Due to this fact, this simple approach gives too conservative a confidence estimate of the system reliability.

In Pavlov (1979, 1980a, 1982) and others, it is shown that for many of the models of complex systems considered below (including systems with repairable units), the lower confidence limit of the system reliability (8.3) can be used with the preservation of the initial confidence coefficient

for  $\gamma \geq 1 - e^{-\frac{3}{2}} \cong 0.778$ ). In other words, the lower confidence limit of the system reliability with the given confidence coefficient can be done by a simple substitution of confidence limits  $\bar{\lambda}_i$  for unit parameters (with the same confidence coefficient) into the function (8.1).

This procedure will be called the method of substitution. In many practical cases this method allows us to obtain a simple and effective solution immediately. Although, as we will show below, sometimes this method still produces very conservative estimates of the system reliability (see Sections 8.5 and 8.6).

## 10.2.2 Conditions of Method Application

We will further assume that function  $R(\lambda) = R(\lambda_1, \dots, \lambda_m)$  is monotone decreasing in each of its parameter  $\lambda_i$ ,  $1 \leq i \leq m$ . Besides for application of the replacement method we need some conditions of convexity of function  $R(\lambda)$ .

Function  $R(\lambda) = R(\lambda_1, \dots, \lambda_m)$  is called quasi-concave (convex) if the region of parameters  $\lambda$  of the form

$$\{\lambda: R(\lambda) \geq (\leq) C\}$$

is concave (convex) for any constant  $C$ . It is easy to see that convex (concave) function is simultaneously quasi-convex (quasi-concave) although the inverse statement is not correct. (One can find details in Appendix 1 to the Chapter.)

In Sections 8.2 through 8.4 we will assume that the confidence coefficient  $\gamma$  satisfies the inequality

$\gamma \geq 1 - \exp(-\frac{3}{2}) \approx 0.778$ . We also assume that function  $R(\lambda) = R(\lambda_1, \dots, \lambda_m)$  is monotonically

increasing in each parameter  $\lambda_i$  and quasi-concave. Then the lower  $\gamma$ -confidence limit  $R$  for  $R(\lambda)$  can be calculated by the method of replacement, that is by formula (8.3). (See Theorem 8.6 in Appendix to the Chapter.)

### 10.2.3 Systems with Series-Parallel Structure

#### Series structure

Let a system consists of  $m$  different units connected in series. The PFFO of such system with independent units is defined as

$$R = \prod_{1 \leq i \leq m} p_i(t)$$

where  $p_i(t) = \exp(-\lambda_i t)$  is the PFFO of the  $i$ th unit,  $1 \leq i \leq m$ . This expression can be rewritten via parameters  $\lambda = (\lambda_1, \dots, \lambda_m)$  as

$$R(\lambda) = \exp\left(-t \sum_{1 \leq i \leq m} \lambda_i\right) \quad (8.5)$$

Function (8.5) monotone decreases in each  $\lambda_i$  and quasi-concave in  $\lambda$ , since the region of parameters

$$\{\lambda: R(\lambda) \geq C\} = \{\lambda: \sum_{1 \leq i \leq m} \lambda_i \leq -\ln \frac{C}{t}\}$$

is convex. So, by (8.4) the lower  $\gamma$ -confidence limit for  $R$  can be calculated as

$$\underline{R} = \exp\left(-t \sum_{1 \leq i \leq m} \bar{\lambda}_i\right)$$

#### Series Structure with replicated units

In an analogous manner we can consider a series structure where units of some type are replicated several times. Let  $r_i$  be the number of units of the  $i$ th type. Then the system PFFO equals

$$R = \prod_{1 \leq i \leq m} p_i^{r_i}(t) = \exp\left(-t \sum_{1 \leq i \leq m} r_i \lambda_i\right)$$

The lower  $\gamma$ -confidence limit for the PFFO can again be computed with the help of the substitution method as

$$\underline{R} = \exp\left(-t \sum_{1 \leq i \leq m} r_i \bar{\lambda}_i\right).$$

#### Series-Parallel Structure (Loaded Redundancy)



Consider a system consisting of  $m$  parallel subsystems in series. Each of subsystems consists of  $n_i$  parallel identical and independent units. Each unit has exponential d.f. of TTF with parameter  $\lambda_i$ ,  $1 \leq i \leq m$ . In this case the system PFFO can be written as

$$R = \prod_{1 \leq i \leq m} \{1 - [1 - p_i(t)]^{n_i}\}$$

where  $p_i(t) = \exp(-\lambda_i t)$ . Using vector parameter  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ , we can write

$$R(\boldsymbol{\lambda}) = \exp\{-f(\boldsymbol{\lambda})\}$$

where  $f(\boldsymbol{\lambda}) = \sum_{1 \leq i \leq m} f_i(\lambda_i)$ ,  $f_i(\lambda_i) = -\ln\{1 - [1 - \exp(-\lambda_i t)]^{n_i}\}$ .

It is easy to show, for instance, by direct differentiation, that functions  $f_i(\lambda_i)$  are monotone increasing and convex. It follows that the area of the form

$$\{\boldsymbol{\lambda}: R(\boldsymbol{\lambda}) \geq C\} = \{\boldsymbol{\lambda}: f(\boldsymbol{\lambda}) \leq -\ln C\}$$

is convex. Thus, in accordance with (8.4), the lower  $\gamma$ -confidence limit for  $R(\boldsymbol{\lambda})$  can be calculated by the method of substitution as

$$\underline{R} = \exp\left\{\sum_{1 \leq i \leq m} f_i(\bar{\lambda}_i)\right\}.$$

### Series Connection of Systems of Type “k out of n”

Consider a system consisting of  $m$  parallel subsystems in series. Each of subsystems consists of  $n_i$  parallel identical and independent units. The  $i$ th subsystem failure occurs if  $k_i$  or more units of this fails,  $1 \leq k_i \leq n_i$ . Each unit again has exponential d.f. of TTF with parameter  $\lambda_i$ ,  $1 \leq i \leq m$ . System described above is a particular case of this general model. In this case the system PFFO can be written as

$$R = \prod_{1 \leq i \leq m} R_i(p_i)$$

where  $p_i = p_i(t) = \exp(-\lambda_i t)$  and

$$R_i(p_i) = \sum_{0 \leq d \leq k_i - 1} \binom{n_i}{d} (1 - p_i)^d p_i^{n_i - d}.$$

The system PFFO expressed via parameters  $\lambda_i$  can be written as

$$R(\boldsymbol{\lambda}) = \exp\left\{-\sum_{1 \leq i \leq m} f_i(\lambda_i t)\right\} \quad (8.6)$$

where

$$f_i(\lambda_i t) = -\ln R_i[\exp(-\lambda_i t)] \quad (8.7)$$

It is easy to show by direct differentiation that each in functions (8.7) is monotone increasing and convex in  $\lambda_i$ . Notice that convexity of these functions follows from the known fact that a system

of type “*k out of n*”, consisting of identical units with exponential d.f. of TTF, has IFR distribution of TTF. Moreover, this fact is also correct even for systems of such type consisting of identical units with IFR d.f. of TTF (see Barlow and Proschan, 1965). From here by definition of IFR distribution, it follows that each function in (8.7) is convex in  $t$  and, consequently, in  $\lambda_i$  for fixed  $t$ .

It follows that (8.6) is monotone decreasing in each  $\lambda_i$  and quasi-concave in  $\lambda$ . Thus, the lower  $\gamma$ -confidence limit for  $R(\lambda)$  can be calculated with the help of the method of substitution as

$$\underline{R} = \exp\left\{-\sum_{1 \leq i \leq m} f_i(\bar{\lambda}_i t)\right\}$$

### Series-Parallel Structure (Unloaded Redundancy)

Consider a system consisting of  $m$  parallel subsystems in series. Each of subsystems consists of  $n_i$  identical and independent units, one of them main and other spare (unloaded redundancy). TTF of the  $i$ th subsystem represents the sum of i.i.d. exponential r.v.'s with parameter  $\lambda_i$ ,  $1 \leq i \leq m$ :

$$\tau_i = \sum_{1 \leq j \leq n_i} \xi_j$$

where  $\xi_j$  is TTF of unit  $j$  of subsystem  $i$ . The PFFO for the  $i$ th subsystem has the form:

$$R_i(\lambda_i t) = P(\tau_i > t) = e^{-\lambda_i t} \sum_{0 \leq d \leq n_i - 1} \frac{(\lambda_i t)^d}{d!}.$$

The system PFFO is determined as

$$R(\lambda) = \prod_{1 \leq i \leq m} R_i(\lambda_i t) = \exp\left\{-\sum_{1 \leq i \leq m} \varphi_i(\lambda_i t)\right\}$$

where

$$\varphi_i(\lambda_i t) = -\ln R_i(\lambda_i t). \quad (8.8)$$

It is easy again verify that each of functions (8.8) is monotone increasing and convex in  $\lambda_i$ . Notice that the convexity is follows from the fact that the sum of i.i.d IFR distributed r.v.'s has IFR distribution (see Barlow and Proschan, 1965). Thus we again obtained that the lower confidence limit for  $R(\lambda)$  can be calculated with the help of the method of substitution as

$$\underline{R} = \exp\left\{-\sum_{1 \leq i \leq m} \varphi_i(\bar{\lambda}_i t)\right\}$$

### 10.3 Systems with Complex Structure

#### 10.3.1 "Recurrent" Structures

For all previously considered systems, we assumed that units within a redundant group are identical to the main ones. Now consider a now more general case where the number of redundancy levels is arbitrary and redundant units might differ from main ones.

Consider, at first, a separate redundant group consisting of  $n$  units. If  $\lambda_1, \dots, \lambda_n$  are the unit failure rates, then the PFFO for loaded redundancy for this redundant group is

$$R = 1 - \prod_{1 \leq i \leq n} (1 - e^{-\lambda_i t}) \quad (8.9)$$

Further instead of (8.9) we will use an approximate formula for highly reliable groups, that is for  $t^n \prod_{1 \leq i \leq n} \lambda_i \ll 1, 1 \leq i \leq n$ :

$$R \approx \exp(-t^n \prod_{1 \leq i \leq n} \lambda_i) \quad (8.10)$$

This formula gives the lower estimate for index  $R$  for all  $\lambda_1, \dots, \lambda_n$  (Gnedenko, et al., 1965).

In an analogous way, one can obtain an expression for unloaded redundancy. Instead of the clumsy and practically useless formula

$$R = \int \dots \int_{y_1 + \dots + y_n \geq t} \prod_{1 \leq i \leq n} \lambda_i \exp(-\lambda_i y_i) dy_1 \dots dy_n$$

we will use an approximation

$$R \approx \exp(-\frac{t^n}{n!} \prod_{1 \leq i \leq n} \lambda_i) \quad (8.11)$$

which gives the lower estimate for index  $R$  for all  $\lambda_1, \dots, \lambda_n$  if  $\frac{t^n}{n!} \prod_{1 \leq i \leq n} \lambda_i \ll 1$  (see Gnedenko, et al., 1965) and delivers the lower limit for PFFO.

Let a system consist of  $n$  units. The system PFFO for some given time  $t$  has the form

$$R = H(p_1, \dots, p_n)$$

where  $p_i = \exp(-\lambda_i t)$ ,  $\lambda_i$  is the FR of the  $i$ th unit. The system PFFO, expressed via parameter  $\lambda = (\lambda_1, \dots, \lambda_n)$  can be written in the form

$$R=R(\boldsymbol{\lambda})=R(\lambda_1, \dots, \lambda_n) \quad (8.12)$$

where  $R(\lambda_1, \dots, \lambda_n)=H[\exp(-\lambda_1 t) \dots \exp(-\lambda_n t)]$ .

Introduce operation  $\mathbf{S}(i, n)$  of substituting the subsystem of  $n$  units with FRs  $\alpha_1, \dots, \alpha_n$  instead of the  $i$ th unit. PFFO After this substitution can be written in the form

$$R=R[\lambda_1, \dots, \lambda_{i-1}, \Lambda(\boldsymbol{\alpha}), \lambda_{i+1}, \dots, \lambda_m] \quad (8.13)$$

where  $\Lambda(\boldsymbol{\alpha})=-(1/t)\ln\pi(\boldsymbol{\alpha})$ ,  $\pi(\boldsymbol{\alpha})$  is the PFFO of the new subsystem,  $\boldsymbol{\alpha}=(\alpha_1, \dots, \alpha_n)$ .

Let  $\mathbf{S}_1=\mathbf{S}_1(i, n)$  denote the operation where the subsystem is series. For this operation

$$\pi(\boldsymbol{\alpha}) = \prod_{1 \leq j \leq n} e^{-\alpha_j t} = \exp(-t \sum_{1 \leq j \leq n} \alpha_j)$$

and transformation (8.13) has the form:

$$\Lambda(\boldsymbol{\alpha}) = \sum_{1 \leq j \leq n} \alpha_j \quad (8.14)$$

Let now  $\mathbf{S}_2=\mathbf{S}_2(i, n)$  denote the operation where the subsystem is parallel. For this case, if redundancy is loaded, due to (8.10) we have

$$\pi(\boldsymbol{\alpha}) \approx \exp(-t^n \prod_{1 \leq j \leq n} \alpha_j).$$

If the redundancy is unloaded then by (8.11)

$$\pi(\boldsymbol{\alpha}) \approx \exp(-\frac{t^n}{n!} \prod_{1 \leq j \leq n} \alpha_j).$$

Thus, transformation (8.13) for operation  $\mathbf{S}_2$  has the form:

$$\begin{aligned} \Lambda(\boldsymbol{\alpha}) &\approx t^{n-1} \prod_{1 \leq j \leq n} \alpha_j \\ \Lambda(\boldsymbol{\alpha}) &\approx \frac{t^{n-1}}{n!} \prod_{1 \leq j \leq n} \alpha_j \end{aligned} \quad (8.15)$$

for loaded and unloaded redundancy, respectively.

Let us say that a system has recurrent, or reducible structure if it can be obtained from some series structure by sequential application of procedure  $\mathbf{S}_1$  and  $\mathbf{S}_2$  in finite number of iterations. Structures of such type were considered in Section 7.6.3. Obviously the class recurrent structures includes series, parallel, series-parallel and parallel-series structures as

particular cases. More sophisticated recurrent structure is depicted in Figure ???1. where numbers denote unit types. There are no restrictions on the number of levels of “recurrence” or on the identity of main and redundant units. Let us introduce a class of functions of the following type:

$$R(\lambda) = \exp\left(- \sum_{1 \leq i \leq M} A_i \prod_{j \in G_i} \lambda_j^{n_{ij}}\right)$$

where  $A_i \geq 0, i=1, \dots, M, n_{ij}$  are arbitrary integer positive numbers,  $G_i, i=1, \dots, M$  are arbitrary (possibly intersecting) subsets of subscripts from set  $\{1, 2, \dots, m\}$ .

Assume that some recurrent structure has reliability function of type (8.16). Then application of operations  $S_1$  or  $S_2$  to each of its units corresponds to the substitution of some parameters  $\lambda_j$  in the initial expression by sums of type (8.14) or products of type (8.15) that again produces formula of type (8.16) though from larger number of variables. Notice that the reliability function of a series system

$$R(\lambda) = \exp\left(- \sum_{1 \leq i \leq M} \lambda_i t\right)$$

obviously is function of type (8.16). It follows that for any recurrent structure the reliability function has the form (8.16).

Thus the lower confidence limit for the PFFO of the system with recurrent structure of type (8.16) is reduced to the construction of the upper confidence limit for the function of type

$$f(\lambda) = \sum_{1 \leq i \leq M} A_i \prod_{j \in G_i} \lambda_j^{n_{ij}} \quad (8.17)$$

where  $A_i$  are positive coefficients,  $G_i$  are arbitrary subsets of unit’s subscripts. Notice that if we have prior information about identity of some units within the system, function (8.17) again transforms to another function of the form of (8.17) but with smaller number of parameters. Function (8.17) of general type is not quasi-convex or quasi-concave in  $\lambda=(\lambda_1, \dots, \lambda_m)$ . Nevertheless, as Pavlov (1982) showed (see p.2 of Appendix to the chapter, Theorem 8.7), the upper  $\gamma$ -confidence limit for function of type (8.17) can be constructed by the method of substitution, that is, with the help of formula

$$\bar{f} = f(\bar{\lambda}) = \sum_{1 \leq i \leq M} A_i \prod_{j \in G_i} \bar{\lambda}_j^{n_{ij}} \quad (8.18)$$

where  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)$  is vector of standard upper  $\gamma$ -confidence limits for unit parameter [here  $\gamma \geq 1 - \exp(-3/2)$ ]. It means that  $\gamma$ - lower confidence limit for the PFFO of recurrent structure, that is the index of type (8.16) can be also constructed with the help of the method of substitution as

$$\underline{R} = R(\bar{\lambda}) = R(\bar{\lambda}_1, \dots, \bar{\lambda}_m)$$

Example 8.3.1. Consider a system with a recurrent structure depicted in Figure 8.1.

Figure 8.1

The number of different unit types is equal  $m=16$ . All redundant units are in loaded regime. Test results for different unit types,  $r_i, S_i, 1 \leq i \leq m$ , are given in Table 8.1.

Table 8.1

This table also contains upper 0.9-confidence limit,  $\bar{\lambda}_i$  for parameters  $\lambda_i$ . Application of the method of substitution in this case gives the lower confidence limit with confidence coefficient not less than 0.9 for the system PFFO (for  $t=1$ ) equal to  $\underline{R}=0.953$ . ■

Example 8.3.2. Consider a system with the same input data as above but for the condition that all redundant units are in unloaded regime. In this case the method of substitution gives  $\underline{R}=0.963$ . ■

### 10.3.2 Monotone Structures

A wide class of real systems can be described with the help of model of monotone structure (see Section 4.2 above, and Barlow & Proschan, 1965). Let a system consists of  $m$  units each of which is characterized by the failure probability during time  $t$ :  $q_i = 1 - p_i = 1 - \exp(-\lambda_i t)$ ,  $1 \leq i \leq m$ . Let us introduce the following notation:

$H_i \subset \{1, 2, \dots, m\}$  = a set of unit's subscripts belonging to a system's minimum cut  $i$ ,

$C_i$  = an event that the  $i$ th units has failed,

$B_i = \bigcap_{j \in H_i} C_j$  = an event that all units belonging to a system's minimum cut  $i$  have failed,

$B$  = an event that the system has failed.

In this notation the probability of system failure,  $Q$ , can be written in the form

$$Q = P(B) = P\left(\bigcup_{1 \leq i \leq M} B_i\right) \quad (8.19)$$

where  $M$  is the total number of system's minimum cuts.

For the case of highly reliable systems ( $\lambda_i t \ll 1$ ) the following approximation can be used:  $q_i \approx \lambda_i t$ . Notice that  $\lambda_i t$  gives an over estimate of  $q_i$ . From (8.19) in the case of independent units, we have

$$Q \approx \sum_{1 \leq i \leq M} P(B_i) = \sum_{1 \leq i \leq M} P\left(\bigcap_{j \in H_i} C_j\right) = \sum_{1 \leq i \leq M} \prod_{j \in H_i} q_j \approx \sum_{1 \leq i \leq M} \prod_{j \in H_i} (\lambda_j t) \quad (8.20)$$

This approximation is valid under assumption that the probability of two or more failures of cuts is negligibly small. Notice that (8.20) gives an over estimate for  $Q$ :

$$Q \leq \sum_{1 \leq i \leq M} \prod_{j \in H_i} q_i \leq \sum_{1 \leq i \leq M} \prod_{j \in H_i} (\lambda_i t).$$

Thus, for highly reliable systems the confidence estimation of  $Q$  from above is reduced to the construction of the upper confidence limit for

$$Q \leq \sum_{1 \leq i \leq M} \prod_{j \in H_i} (\lambda_i t) \quad (8.21)$$

where  $H_i$  are some (in general case intersecting) subsets of unit's subscripts. For series-parallel system each of parameters  $\lambda_i$  belongs to only one of product in (8.21). In other words, each unit belongs to only one system's cut. For complex systems, each unit might belong to different intersecting cuts.

Sometimes we can assume in advance that some units within the system are identical, i.e., have same reliability parameters. In this case (8.21) for the system failure probability has the form

$$Q \approx \sum_{1 \leq i \leq M} \prod_{j \in H_i} (\lambda_j t)^{n_{ij}} \quad (8.22)$$

where  $n_{ij}$  is the number of identical units with equal parameter  $\lambda_i$  in cut  $H_i$ . (8.21) is a particular case of (8.22).

Thus, for the case of highly reliable systems the problem (with the approximation made above) is reduced to the construction of the upper confidence limit of function of type (8.22). This function has the same form as (8.17) considered in the previous section. For the latter we showed that (8.18) can be used for the upper confidence limit calculation. So, for arbitrary complex monotone structures upper  $\gamma$ -confidence limit for the probability of system failure can be calculated (if  $\gamma \geq 1 - \exp(-3/2)$ ) by the method of substitution with the help of formula

$$\bar{Q} = Q(\bar{\lambda}) = \sum_{1 \leq i \leq M} \prod_{j \in H_i} (\bar{\lambda}_j t)^{n_{ij}}$$

#### 10.4 Systems with Repairable Units

For repairable systems we often meet a situation where the system MTTF is much larger than mean time of renewal. Usually it is easier to estimate renewal (repair) time than MTTF, because it can be done by special and simple experiments. We will assume that the mean renewal time is known with accuracy.

### 10.4.1 System with a Series Structure

Let a system consists of  $n$  units in series. The d.f. of the  $i$ th unit's TTF is assumed exponential with unknown parameter  $\lambda_i$ , that is, MTTF of this unit is equal to  $T_i=1/\lambda_i$ . The mean renewal time,  $a_i$ , is assumed known. All units are assumed independent in sense of failures and repair.

Consider the construction of upper confidence limit for main system reliability indices for this case. The most widely used reliability index in this case is stationary availability coefficient, the probability that the system is in up state at the arbitrary stationary moment of time  $t$  ( $t \rightarrow \infty$ ). The availability coefficient of each unit  $i$  is determined by well known formula:

$$K_i(T_i) = \frac{T_i}{T_i + a_i}.$$

For independent units the system's availability coefficient has the form

$$K(T) = \prod_{1 \leq i \leq m} K_i(T_i) = \prod_{1 \leq i \leq m} \left( \frac{T_i}{T_i + a_i} \right). \quad 8.23)$$

where  $\mathbf{T}=(T_1, \dots, T_m)$ .

Since repair parameters are assumed known, the problem is reduced to the confidence estimation of (8.23) depending on unknown parameters  $\mathbf{T}=(T_1, \dots, T_m)$ . Introduce now a vector of the lower confidence limits for unit parameters :  $\underline{\mathbf{T}} = (\underline{T}_1, \dots, \underline{T}_m)$  where

$$\underline{T}_i = \frac{1}{\lambda_i} = \frac{\lambda S_i}{\chi_\gamma^2(2r_i)}$$

is the lower  $\gamma$ -confidence limit for parameter  $T_i$ ,  $1 \leq i \leq m$ . As it is shown below (see p.1 of Appendix to the chapter , Theorem 8.4), lower  $\gamma$ -confidence limit for reliability index  $R=R(\mathbf{T})$  can be calculated by the method of substitution (for  $\gamma \geq 1 - \exp[-3/2]$ ), that is by formula

$$\underline{R} = R(\underline{\mathbf{T}}) = R(\underline{T}_1, \dots, \underline{T}_m)$$

if function  $R(\underline{T}_1, \dots, \underline{T}_m)$  is monotone increasing in each  $T_i$  and quasi-concave in vector  $\mathbf{T}$ . It is easy to see that reliability index (8.23) satisfies all this conditions. Indeed, monotone increasing of  $K(\mathbf{T})$  in each  $T_i$  is obvious. Besides, by direct differentiating we can prove that function

$$\ln K_i(T_i) = \ln \frac{T_i}{T_i + a_i}$$

is concave in  $T_i$ . Thus, the area of parameters



$$\{\mathbf{T}: K(\mathbf{T}) \geq C\} = \{\mathbf{T}: \sum_{1 \leq i \leq m} \ln K_i(T_i) \geq \ln C\}$$

is convex, and, consequently, it follows that function  $K(\mathbf{T})$  is quasi-concave. So, lower  $\gamma$ -confidence limit

for the system availability coefficient (8,23) can be calculated by the method of substitution as follows

$$\underline{K} = K(\underline{\mathbf{T}}) = \prod_{1 \leq i \leq m} \frac{T_i}{T_i + a_i}.$$

Analogously, we can prove that this method is valid for confidence estimation from above for other main reliability indices. Consider the coefficient of operative availability probability of failure free operation on time interval  $(t, t + \tau)$  in stationary regime for  $t \rightarrow \infty$ . This reliability index has the form

$$K_\tau = K_\tau(\mathbf{T}) = K(\mathbf{T}) \cdot \exp\left(-\tau \sum_{1 \leq i \leq m} \frac{1}{T_i}\right).$$

Another standard reliability index, MTBF, has the form

$$L = L(\mathbf{T}) = \left(\sum_{1 \leq i \leq m} \frac{1}{T_i}\right)^{-1}.$$

It is easy to see that functions  $K_\tau(\mathbf{T})$  and  $L(\mathbf{T})$  are monotone increasing in each  $T_i$  and quasi-concave in  $\mathbf{T}$ . So, lower  $\gamma$ -confidence limit for each of these reliability indices can be calculated by the method of substitution as follows

$$\underline{K}_\tau = K_\tau(\underline{\mathbf{T}}) = \prod_{1 \leq i \leq m} \left(\frac{T_i}{T_i + a_i}\right) \cdot \exp\left(-\tau \sum_{1 \leq i \leq m} \frac{1}{T_i}\right),$$

$$\underline{L} = L(\underline{\mathbf{T}}) = \left(\sum_{1 \leq i \leq m} \frac{1}{T_i}\right)^{-1}.$$

#### 10.4.2 Series-parallel system with loaded redundancy.

Consider a system consisting of  $m$  subsystems (redundant groups) in series. Each group consists of  $n_i+1$  identical units in parallel. All failures and renewals are independent. TTF of unit  $i$  has exponential d.f. with unknown mean  $T_i$ . The mean repair time for each group is known

and equal  $a_i$ . Random repair time has an arbitrary d.f. The repair process for all units is independent, that is, repair(replacement) on all units begins immediately after failure.

In the frame of assumption made above availability coefficient of each unit of type  $i$  equals  $\frac{T_i}{T_i + a_i}$ ,  $1 \leq i \leq m$ . For the  $i$ th group we can write

$$K_i(T_i) = 1 - \left[1 - \frac{T_i}{T_i + a_i}\right]^{n_i+1}$$

and for the entire system

$$K(\mathbf{T}) = \prod_{1 \leq i \leq m} K_i(T_i) = \prod_{1 \leq i \leq m} \left[1 - \left(\frac{a_i}{a_i + T_i}\right)^{n_i+1}\right] \quad (8.24)$$

In this case (see Gnedenko, et al., 1965, Section 2,4 and 6.2) the MTBF of each redundant group can be found as

$$l_i(T_i) = \frac{a_i}{n_i + 1} \left[ \left(1 + \frac{T_i}{a_i}\right)^{n_i+1} - 1 \right] \quad (8.25)$$

It follows (see *ibid.*, Section 2.4)

$$L(\mathbf{T}) = \frac{1}{\sum_{1 \leq i \leq m} \frac{1}{l_i(T_i)}} = \frac{1}{\sum_{1 \leq i \leq m} \frac{n_i + 1}{a_i \left[ \left(1 + \frac{T_i}{a_i}\right)^{n_i+1} - 1 \right]}}$$

Consider the PFFO on a time interval of the length  $\tau$ . Let us denote random TTF (in stationary regime) of the  $i$ th redundant group by  $\xi_i$ . The d.f. of this r.v. in general differs from exponential if  $(n_i+1) \geq 2$ . Nevertheless (*ibid.*, Section 6.2), if the group MTBF is much larger than mean repair time, that is  $l_i(T_i) \gg a_i$ , then d.f. of r.v.  $\xi_i$  is approximately exponential :

$$P(\xi_i > \tau) \approx \exp\left[-\frac{\tau}{l_i(T_i)}\right]$$

therefore stationary PFFO for the system as a whole can be calculated by formula

$$P_{\tau}(\mathbf{T}) \approx \exp\left[-\sum_{1 \leq i \leq m} \frac{\tau}{l_i(T_i)}\right]$$

where  $l_i(T_i)$  is calculated by (8.25).

Coefficient of operative availability (unconditional stationary probability of failure-free operation on interval of length  $\tau$ ) can be found from

$$P_{\tau}(\mathbf{T}) \approx \prod_{1 \leq i \leq m} \left[1 - \left(\frac{a_i}{a_i + T_i}\right)^{n_i+1}\right] \exp\left[-\sum_{1 \leq i \leq m} \frac{\tau}{l_i(T_i)}\right]$$

Main reliability indices, such as  $K(\mathbf{T})$ ,  $L(\mathbf{T})$ ,  $K_{\tau}(\mathbf{T})$  and  $P_{\tau}(\mathbf{T})$  are monotone increasing in  $T_i$  can be seen directly from the above equations. In p.3 of the Appendix to this chapter, we show that all these functions are quasi-concave in vector  $\mathbf{T}=(T_1, \dots, T_m)$ . Therefore the lower confidence limits for them (for  $\gamma \geq \exp\{-3/2\}$ ) can be calculated by the method of substitution as  $\underline{K}=K(\underline{\mathbf{T}})$ ,  $\underline{L}=L(\underline{\mathbf{T}})$ ,  $\underline{K}_{\tau}=K_{\tau}(\underline{\mathbf{T}})$  and  $\underline{P}_{\tau}=P_{\tau}(\underline{\mathbf{T}})$ .

#### 10.4.3 General model of series connection of renewal subsystems

Results obtained above can be expanded on a wider class of systems. Consider a system consisting of  $m$  subsystems in series. All failures and renewals are independent. Let subsystem  $i$  consists of  $A_i+B_i+C_i$  identical units with same unknown parameter - MTBF  $T_i$ . Among these units,  $A_i$  of them are main,  $B_i$  are loaded redundant units, and  $C_i$  are unloaded redundant units. There are  $D_i$  repair facilities each of which can repair a single failed unit in a time. The mean time of repair of unit  $i$  equals  $a_i$ ,  $1 \leq i \leq m$ . For this model we assume that d.f. of random TTF as well as that for repair time are exponential.

The subsystem  $i$  is in operational state if at least  $A_i$  its units are in up state. Stochastic process  $N_i(t)$ , describing changing the number of non-failed units within  $i$ th subsystem, is a Birth-and-Death process. The  $i$ th subsystem failure corresponds to the moment when process  $N_i(t)$  crosses the threshold  $n_i=B_i+C_i$  from below. Processes  $N_i(t)$ ,  $1 \leq i \leq m$ , are independent. The system has failed if at least one subsystem has failed.

Consider the process for  $N_i(t)$ . Transitive intensity from state  $k$  to state  $k+1$  equals  $\frac{\alpha_{ki}}{T_i}$

and from state  $k$  to state  $k-1$  equals  $\frac{\beta_{ki}}{T_i}$  where coefficients  $\alpha_{ki}$  and  $\beta_{ki}$  are defines by values  $A_i$ ,

$B_i$ ,  $C_i$  and  $D_i$  and also by the regime of repair and replacement within the  $i$ th subsystem.

Assume, for example, that subsystem  $i$  operates in the following manner. If a main unit has failed, it is replaced by a unit from unloaded redundant group. If loaded redundant unit has

failed or is directed for replacement of failed main unit, an unloaded unit takes its place. Repair of failed units begins immediately. Coefficients  $\alpha_{ki}$  and  $\beta_{ki}$  for such a model are defined as:

$$\alpha_{ki} = \begin{cases} A_i + B_i, & k \leq C_i \\ A_i + B_i + C_i, & k > C_i \end{cases}$$

$$\beta_{ki} = \begin{cases} k_i, & k \leq D_i \\ D_i, & k > D_i \end{cases}$$

It is easy to write down transitive intensities for other possible regimes.

Let us evaluate main reliability indices, using known results from theory of stochastic processes. Let  $\pi_{ki}$  be the stationary probability that  $N_i(t) = k$ . This value is determined (see, for instance *ibid*, Section 6.3):

$$\pi_{ki} = \frac{c_{ki} \left( \frac{a_i}{T_i} \right)^k}{\sum_{0 \leq l \leq Z_i} c_{li} \left( \frac{a_i}{T_i} \right)^l}$$

where  $Z_i = A_i + B_i + C_i$  is the total number of units in the  $i$ th subsystem, and coefficients  $c_{ki}$  are defined via  $\alpha_{ki}$  and  $\beta_{ki}$  as

$$c_{ki} = \frac{\alpha_{0i} \cdot \alpha_{1i} \cdot \dots \cdot \alpha_{k-1,i}}{\beta_{1i} \cdot \beta_{2i} \cdot \dots \cdot \beta_{ki}}, \quad 1 \leq k \leq Z_i.$$

Availability coefficient of subsystem  $i$  is the stationary probability that the number of failed units has not exceeded the system failure level  $n_i$ :

$$K_i = \lim_{t \rightarrow \infty} P\{N_i(t) \leq n_i\} = \sum_{0 \leq k \leq n_i} \pi_{ki}.$$

Thus, availability coefficient of subsystem  $i$  represents a ratio of polynomials with positive coefficients

$$K_i(T_i) = \frac{\sum_{0 \leq k \leq n_i} c_{ki} \left(\frac{a_i}{T_i}\right)^k}{\sum_{0 \leq k \leq z_i} c_{ki} \left(\frac{a_i}{T_i}\right)^k} \quad (8.26)$$

where  $c_{ki} \geq 0$ . The system availability coefficient is defined as

$$K(\mathbf{T}) = \prod_{1 \leq i \leq m} K_i(T_i) \quad (8.27)$$

Formulae (8.23)-(8.24) are particular cases of (8.26)-(8.27).

The MTBF of subsystem  $i$ , that is process  $N_i(t)$  transition time from state  $n_i$  to state  $n_i+1$  is given by (see *ibid.*, Section 6.3):

$$l_i(T_i) = \frac{T_i}{\alpha_{n_i,i} c_{n_i,i}} \left(\frac{T_i}{a_i}\right)^{n_i} \sum_{0 \leq k \leq n_i} c_{ki} \left(\frac{a_i}{T_i}\right)^k \quad (8.28)$$

For the system as a whole, the MTBF (for stationary regime) is determined as

$$L(\mathbf{T}) = \left( \sum_{1 \leq i \leq m} \frac{1}{l_i(T_i)} \right)^{-1}.$$

Distribution of TTF of the system or its individual subsystems is not, in general, exponential, though the exponent d.f. gives a good approximation. For instance, system's availability coefficient for time interval  $\tau$  has the form of the following approximation (see Gnedenko and Ushakov, 1995)

$$K_\tau(\mathbf{T}) \approx \prod_{1 \leq i \leq m} K_i(T_i) \cdot \exp\left(-\sum_{1 \leq i \leq m} \frac{\tau}{l_i(T_i)}\right)$$

where characteristics of individual subsystems are defined by (8.26) and (8.28).

Stationary system PFFO on time interval  $\tau$  is equal to

$$P_\tau \approx \exp\left[-\sum_{1 \leq i \leq m} \frac{\tau}{l_i(T_i)}\right].$$

Consider the moment  $t=0$  where all system's units are operable. The mean time of the transition of process  $N_i(t)$  from state 0 to failure state  $n_i+1$  is determined by

$$l_i^0(T_i) = \sum_{0 \leq k \leq n_i} \frac{T_i}{\alpha_{ki} c_{ki}} \left(\frac{T_i}{a_i}\right)^k \sum_{0 \leq r \leq k} c_{ri} \left(\frac{a_i}{T_i}\right)^r$$

(see Gnedenko, et al., 1965, Ushakov, ed., 1994) which after changing the order of summation gives

$$l_i^0(T_i) = T_i \sum_{0 \leq r \leq n_i} E_{ri} \left(\frac{T_i}{a_i}\right)^r \quad (8.29)$$

where coefficients  $E_{ri}$  are defined as

$$E_{ri} = \sum_{r \leq k \leq n_i} \frac{c_{k-r,i}}{\alpha_{ki} c_{ki}}$$

The system PFFO on time interval  $(0, \tau)$  can be calculated by approximate formula

$$P_\tau^0(\mathbf{T}) \approx \exp \left[ - \sum_{1 \leq i \leq m} \frac{\tau}{l_i^0(T_i)} \right]$$

where characteristics of individual subsystems  $l_i^0(T_i)$  are found from (8.29).

All the main reliability indices:  $K(\mathbf{T})$ ,  $K_\tau(\mathbf{T})$ ,  $P_\tau(\mathbf{T})$ ,  $L(\mathbf{T})$  and  $P_{0\tau}(\mathbf{T})$  are monotone increasing in each  $T_i$  as it can be seen directly from corresponding formulas. In addition (see p.3 to Appendix to the chapter) all of these functions are concave in vector  $\mathbf{T}=(T_1, \dots, T_m)$ . Thus, lower  $\gamma$ -confidence limit (for  $\gamma > 1-\exp(-3/2)$ ) for those reliability indices can be calculated by substitution method as  $K=K(\mathbf{T})$ ,  $K_\tau=K_\tau(\mathbf{T})$ ,  $P_\tau=P_\tau(\mathbf{T})$ ,  $L=L(\mathbf{T})$  and  $P_{0\tau}=P_{0\tau}(\mathbf{T})$ . It allows one to calculate the lower confidence limits of main reliability indices for sufficiently general models of renewable systems.

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## 11.5 Method of Fiducial Probabilities

### 11.5.1 Introduction and method description

As before in this chapter, we assume that system units have exponential d.f of TTF and the test of units of type  $i$  continues up until  $r_i$  failures. Test is assumed to be performed in accordance with plans  $[N_i U r_i]$  or  $[N_i R r_i]$ ,  $1 \leq i \leq m$ . Let  $S_i$  be the total test time of units of type  $i$ . R.v.  $S_i$  has gamma d.f. of the form

$$F_i(S_i, \lambda_i) = 1 - e^{-\lambda_i S_i} \sum_{0 \leq j \leq r_i - 1} \frac{(\lambda_i S_i)^j}{j!} \quad (8.30)$$

where  $\square_i$  is unknown parameter of failure rate of unit of type  $i$ .

For each fixed  $S_i$ , (8.30) considering as a function of parameter  $\square_i$  possesses all formal properties of d.f. in respect to parameter  $\square_i$ . Let  $S_i^*$  be observed value of r.v.  $S_i$  obtained in the test. For given fixed value  $S_i^*$  we consider parameter  $\square_i$  as a r.v. with d.f.

$$F_i(S_i^*, \lambda_i) = 1 - e^{-\lambda_i S_i^*} \sum_{1 \leq j \leq r_i - 1} \frac{(\lambda_i S_i^*)^j}{j!} \quad (8.31)$$

Defined in such a way d.f. (8.31) of parameter  $\square_i$  is called fiducial (see also Section 7.6.4 above). Corresponding density of fiducial distribution of parameter  $\square_i$  has the form

$$\varphi_i(S_i^*, \lambda_i) = \frac{\partial}{\partial \lambda_i} F_i(S_i^*, \lambda_i) = e^{-\lambda_i S_i^*} \frac{(S_i^*)^{r_i}}{(r_i - 1)!} \lambda_i^{r_i - 1},$$

that is, it is a density of standard gamma distribution with parameters  $(S_i^*, r_i)$ .

Let, further,  $R$  be a reliability index which is a function

$$R = R(\square) = R(\square_1, \dots, \square_m) \quad (8.32)$$

of unit parameters  $\square = (\square_1, \dots, \square_m)$  where  $m$  is the number of different system unit types. Let also  $\mathbf{S}^* = (S_1^*, \dots, S_m^*)$  be a vector of test results for different types of units. As above, test results for different unit types are assumed independent.

For given fixed  $\mathbf{S}^*$  let us consider reliability indices (8.32) as a function of independent r.v.'s  $\square_i$  with fiducial d.f.'s (8.31) mentioned above. Corresponding fiducial d.f. of  $R$  for given  $\mathbf{S}^*$  has the form

$$\Phi(S^*, R) = \int_{R(\lambda) \leq R} \dots \int \prod_{1 \leq i \leq m} \varphi_i(S_i^*, \lambda_i) d\lambda_i$$

where  $\varphi_i(S_i^*, \square_i)$  is mentioned above fiducial densities of parameters  $\square_i$ . The lower and upper (one-sided) fiducial limits with confidence coefficient  $\square$ ,  $\underline{R}$  and  $\overline{R}$  for system reliability index  $R=R(\square)$  is defined from conditions

$$\begin{aligned} \Phi(S^*, \underline{R}) &= 1 - \gamma, \\ \Phi(S^*, \overline{R}) &= \gamma, \end{aligned}$$

that is, as corresponding quantiles of fiducial d.f. of r.v.  $R$ . D.f.  $\square(S^*, R)$  and limits  $\underline{R}$  and  $\overline{R}$  can be easily found with the help of Monte Carlo simulation.

As we mention in Section 7.6.4, in multi-dimensional case, where  $m > 1$ , the fiducial method is approximate in the sense that the fiducial limit constructed by this method (for instance, upper  $\overline{R}$ ) may not be with confidence coefficient equal to  $\square$ . Its real confidence coefficient might be essentially smaller than  $\square$ . As a simple example, illustrating this fact, coincide the following.

Example 8.5.1 Let reliability index has the form  $R=R(\square)$

$$R = \min(\square_1, \dots, \square_m) \quad (8.33)$$

and we need to construct the UCL for  $R$ . This task can be interpreted as follows. Reliability index (8.33) has the sense of the best (minimum) parameter of failure rate among the unit failure rates of different types. Thus, we need on the basis of results of independent unit tests of  $m$  different types to construct UCL of failure rate for the best (most reliable) unit type. (We do not know in advance which unit type is the best.)

Let us apply for solving this problem the fiducial method considered above. In accordance with this method, parameters  $\lambda_i$  are assumed independent r.v.'s with d.f.'s (8.31). Corresponding fiducial d.f. reliability index (8.33) has the form

$$\begin{aligned} \Phi(S^*, R) &= P\{\min_i \lambda_i \leq R\} = 1 - P\{\min_i \lambda_i > R\} = 1 - \prod_{1 \leq i \leq m} P\{\lambda_i > R\} = \\ &= 1 - \prod_{1 \leq i \leq m} [1 - F_i(S_i^*, R)]. \end{aligned}$$

So, the upper fiducial limit,  $\overline{R}$ , with confidence coefficient  $\square$ , for reliability index  $R$  is determined from



$$1 - \prod_{1 \leq i \leq m} [1 - F_i(S^*, \bar{R})] = \gamma. \quad (8.34)$$

Notice that standard  $\square$ -UCL  $\hat{\lambda}_i$  (coinciding with the upper fiducial limit) for each individual parameter  $\square_i$  is determined from

$$F_i(S^*, \bar{\lambda}_i) = \gamma. \quad (8.35)$$

Left sides of (8.34) and (8.35) satisfy inequality

$$1 - \prod_{1 \leq i \leq m} [1 - F_i(S^*, t)] > F_i(S^*, t)$$

for all  $t > 0$ . From here inequality

$$\bar{R} < \bar{\lambda}_i$$

for all  $1 \leq i \leq m$ . So,

$$\bar{R} < \min_i \bar{\lambda}_i$$

and it follows the following inequality for the confidence probability

$$P_\lambda \{\bar{R} \geq R(\lambda)\} = P_\lambda \{\bar{R} \geq \min_i \lambda_i\} \leq P_\lambda \{\min_i \bar{\lambda}_i \geq \min_i \lambda_i\}.$$

Take in this inequality  $\square_1 = \dots = \square_m$ , we obtain that in this case confidence probability satisfies inequality

$$P_{\lambda} \{ \bar{R} \geq R(\lambda) \} \leq P_{\lambda} \{ \min_i \bar{\lambda}_i \geq \lambda_i \} = P_{\lambda} \left\{ \bigcap_{1 \leq i \leq m} (\bar{\lambda}_i \geq \lambda_i) \right\} = \prod_{1 \leq i \leq m} P_{\lambda} (\bar{\lambda}_i \geq \lambda_i) = \gamma^m.$$

from here it follows that then upper fiducial limit has confidence coefficient not larger than  $\gamma^m$ . Form instance, for  $m=10$  and  $\gamma=0.9$  factual value of confidence coefficient is not larger than value  $\gamma^m \approx 0.35$ . ■

Thus, in this example using the fiducial method for confidence estimation of reliability index is incorrect because it leads to essential increasing of reliability (the larger, the larger the problem dimension is, that is, the larger is the number of types of units).

Remark 8.5.1. In example considered above, value  $\min_i \bar{\lambda}_i$  is equal to UCL for  $R$ , calculated in accordance with the method of "substitution". Due to this, from analogous arguing we can show incorrectness of application for this example the method of substituting. □

Thus, application of fiducial method for confidence estimation of complex system might lead, in general case, to essential errors. Therefore the application of this method to one or another concrete situation is needed to be verified. Further we will show, using results (Pavlov, 1980b, 1981a,b), that the application of the fiducial method is correct for sufficiently wide class of complex systems for confidence estimation of system reliability index from below (that is most important for practice).

### 11.5.2 Conditions of the method correctness

Let

$$R=R(\square)=R(\square_1, \dots, \square_m) \quad (8.36)$$

is a function expressing the dependence of system reliability index  $R$  on parameters of units  $\square=(\square_1, \dots, \square_m)$ . Consider the question for each reliability index (8.36) it is possible to state that the application of the fiducial method for confidence estimation is correct (or, in other words, that the fiducial limit for  $R$  is simultaneously  $\square$ -confidence limit for this reliability index).

It is convenient to write (8.36) in the form

$$R= \exp \{ -f(\square) \}. \quad (8.37)$$

Estimate  $R$  from below is reduced to the estimation of function  $f(\square)=f(\square_1, \dots, \square_m)$  from above. Let us introduce new parameters  $\mathbf{z}=(z_1, \dots, z_m)$  where  $z_i= \ln \square_i, 1 \leq i \leq m$ .

Assume that the following main condition holds.

Condition A. Function  $f(\square)$  expressed via parameters  $\mathbf{z}=f(z_1, \dots, z_m)$ , that is, function

$$\tilde{f}(\mathbf{z}) = f(e^{z_1}, \dots, e^{z_m}) \quad (8.38)$$

is convex in  $\mathbf{z}$ .

Then the upper fiducial limit,  $\bar{f}$ , for  $f(\square)$  is simultaneously  $\square$ -UCL, that is, has the confidence coefficient not smaller than  $\square$  (see the proof in p.4 of App). In other words, if condition A holds, the use of the method of fiducial probabilities is correct for confidence estimation of  $f(\square)$  from above. (It corresponds to estimation of reliability index (8.37) from below.)

Let us now show that condition A (function (8.38) convexity) is valid for satisfactory wide class of complex systems, including systems with recurrent structures and with renewal.

### 11.5.3 Systems with series-parallel system

Consider a system consisting of  $m$  subsystems in series. Each subsystem  $i$  consists of  $n_i$  identical units in parallel, each unit is characterized by failure rate  $\square_i$ ,  $1 \leq i \leq m$ . The redundant units regime might be any: loaded or unloaded. Let us assume that within subsystems with subscripts  $1 \leq i \leq l$  redundant units are unloaded, and within subsystems with subscripts  $l+1 \leq i \leq m$  redundant units are loaded. The system PFFO during time  $t_0$  can be written in the form

$$R = R(\lambda) = \prod_{1 \leq i \leq m} h_i(\lambda_i) \quad (8.39)$$

where  $h_i(\square_i)$  is the PFFO of the  $i$ th subsystem which depends on the redundant unit regime. For a subsystem with unloaded redundant units, the PFFO can be written as

$$h_i(\lambda_i) = e^{-\lambda_i t_0} \sum_{0 \leq d \leq n_i - 1} \frac{(\lambda_i t_0)^d}{d!}, \quad 1 \leq i \leq l,$$

and for loaded redundant units as

$$h_i(\lambda_i) = 1 - (1 - e^{-\lambda_i t_0})^{n_i}, \quad l + 1 \leq i \leq m.$$

Obviously, (8.39) includes as particular cases systems with all loaded or all unloaded redundant units. It is easy to check by a direct differentiation that for both types of redundancy function

$$f_i(\lambda_i) = -\ln h_i(\lambda_i)$$

is monotone increasing and convex in  $\lambda_i$ ,  $1 \leq i \leq m$ . Consequently, system reliability index (8.39) can be written as

$$R(\lambda) = \exp \{-f(\lambda)\}$$

where

$$f(\lambda) = \sum_{1 \leq i \leq m} f_i(\lambda_i)$$

is the sum of monotone increasing convex functions. It follows that function

$$f(\mathbf{z}) = f(e^{z_1}, \dots, e^{z_m}) = \sum_{1 \leq i \leq m} f_i(e^{z_i})$$

is convex in  $\mathbf{z}$ , that is, Condition A holds. It means (see Section 8.5.2) that the fiducial method can be used for confidence estimation of  $f(\lambda)$  from above, or for confidence estimation of reliability index (8.39) from below.

Example 8.5.2. Consider series-parallel system consisting of  $m=10$  subsystem in series. Within subsystems from 1 through 4, redundant units are unloaded, and within subsystems from 5 through 10, redundant units are loaded. The numbers of units in subsystems,  $n_i$ , and the test results (the number of failures,  $r_i$ , and the total test time,  $S_i$ ) are given in Table 8.2.

We need to construct the LCL with confidence coefficient  $\alpha=0.95$  for the system reliability index (8.39) for  $t_0=1$ .

Table 8.2

The use of the method of fiducial probabilities in this case gives  $\underline{R}_{0.95}=0.856$ . Notice that the method of substitution, considered above, gives for the same case essentially worse lower limit:  $\underline{R}_{0.95}=0.786$ . ■

Example 8.5.3. Consider the same system as in the previous example with the difference that all subsystems are consisted of unloaded redundant units. In this case the method of fiducial probabilities gives  $\underline{R}_{0.95}=0.918$ , and the method of substitution  $\underline{R}_{0.95}=0.876$ . ■

#### 11.5.4 Systems with complex structure

As we mentioned above (see Section 8.3), the confidence estimate of a system with complex structure can be reduced to the construction of UCL of function

$$f(\lambda) = \sum_{1 \leq i \leq M} A_i \prod_{j \in G_i} \lambda_j^{n_{ij}} \quad (8.40)$$

where  $A_i$  are positive coefficients,  $G_i$  is some (in general case, intersecting) subsets of subscripts,  $G_i \subseteq \{1, 2, \dots, m\}$  where  $m$  is the number of different types of units within the system,  $\lambda_j$  is failure rate of unit  $j$  (see formulas (8.17) and (8.22) above).

Function (8.40), written via parameters  $z_j = \ln \lambda_j$ ,  $1 \leq j \leq m$ , has the form

$$\tilde{f}(z) = f(e^{z_1}, \dots, e^{z_m}) = \sum_{1 \leq i \leq M} A_i \exp\left(\sum_{j \in G_i} n_{ij} z_j\right).$$

This function is convex in  $\mathbf{z} = (z_1, \dots, z_m)$  because it represents the sum of convex functions, that is, Condition A above holds. Thus, the UCL for reliability index of type (8.40) can be constructed by the method of fiducial probabilities. It means that this method is correct for confidence estimation of complex system PFFO from below.

#### 11.5.5 Systems with renewal

Consider at the beginning a system with independent renewal, i.e., a failed unit begins to be repaired immediately after its failure. Let a system consist of  $m$  subsystems in series. Each subsystem  $i$  consists of  $n_i$  identical redundant units in parallel. Processes of failure and renewal in all subsystems are independent.

Unit's TTF and unit's intensity of repair have exponential d.f. with parameters  $\lambda_i$  and  $\mu_i$ , respectively,  $1 \leq i < m$ .

Parameters  $\lambda = (\lambda_1, \dots, \lambda_m)$  and  $\mu = (\mu_1, \dots, \mu_m)$  are unknown but there are test results: the total test time for unit  $i$ ,  $S_i$ , up to the occurrence of  $r_i$  failures, and the total time of repair  $V_i$  of  $l_i$  units of this type. In other words, each  $S_i$  and  $V_i$  represent the sum of corresponding r.v.'s. Notice that these test results might be obtained in the result of unit tests, test of a system as a whole, or as a combination of data of both types of testing.

System's reliability index is a function of mentioned parameters

$$R = R(\lambda, \mu) = R(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_m) \quad (8.41)$$

We need to construct the confidence limit (lower or upper) for reliability index (8.41) based on the tr. Systems with renewal are quite different from systems without renewal. They are characterized

by different reliability indices. Nevertheless, from formal viewpoint, parameters  $\lambda_i$  are same "exponential" parameters like  $\lambda_i$ . Therefore all results obtained above for systems without renewal can be extended to systems with renewal.

We again write reliability index (8.41) in the form

$$R = \exp\{-f(\lambda, \mu)\}.$$

Estimation of  $R$  from below is reduced to estimate of function  $f(\lambda, \mu) = f(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_m)$  from above.

Introduce parameters:  $z_i = \ln \lambda_i$  and  $y_i = \ln \mu_i$ ,  $1 \leq i \leq m$ . As above, denote  $\mathbf{z} = (z_1, \dots, z_m)$  and  $\mathbf{y} = (y_1, \dots, y_m)$ . Condition A given in Section 8.5.2 in this case transforms into the following.

Condition A'. Function  $f(\lambda, \mu)$  expressed via parameters  $(\mathbf{z}, \mathbf{y})$ , that is

$$\tilde{f}(\mathbf{z}, \mathbf{y}) = f(e^{z_1}, \dots, e^{z_m}, e^{y_1}, \dots, e^{y_m})$$

is convex in  $(\mathbf{z}, \mathbf{y})$ .

The condition of applicability of the method of fiducial probabilities for confidence estimation of renewal system reliability indices is formulated in a way similar to above. If Condition A' holds, then the upper fiducial limit for  $f(\lambda, \mu)$  is simultaneously  $\lambda$ -UCL (see the proof in p.4 of App). In other words, if condition A' holds, the use of the method of fiducial probabilities is correct for confidence estimation of  $f(\lambda, \mu)$  from above. (It corresponds to estimation of reliability index (8.41) from below.)

Condition A' is valid for many practical cases. For instance, one of the most frequently used reliability index for renewal systems is availability coefficient. For the case of independent unit repair and failures

$$K = \prod_{1 \leq i \leq m} K_i(\lambda_i, \mu_i) \quad (8.42)$$

where  $K_i(\lambda_i, \mu_i)$  is availability coefficient of subsystem  $i$  which is defined by (see Section 8.4.2 above)

$$K_i(\lambda_i, \mu_i) = 1 - \left( \frac{\lambda_i}{\lambda_i + \mu_i} \right)^{n_i}.$$

reliability index (8.42) can be presented in the form

$$K = e^{f(\lambda, \mu)}$$

where

$$f(\lambda, \mu) = \sum_{1 \leq i \leq m} f_i(\lambda_i, \mu_i), \quad (8.43)$$

and in turn

$$f_i(\lambda_i, \mu_i) = -\ln \left[ 1 - \left( \frac{\lambda_i}{\lambda_i + \mu_i} \right)^{n_i} \right].$$

confidence estimation of availability coefficient from below is reduced to the construction of the UCL for function (8.43). This function can be written via parameters  $(\mathbf{z}, \mathbf{y})$  in the form

$$\tilde{f}(\mathbf{z}, \mathbf{y}) = f(e^{z_1}, \dots, e^{z_m}, e^{y_1}, \dots, e^{y_m}) = \sum_{1 \leq i \leq m} \varphi_i(z_i - y_i) \quad (8.44)$$

where

$$\varphi_i(u) = -\ln \left[ 1 - \left( \frac{e^u}{1 + e^u} \right)^{n_i} \right].$$

By direct differentiation we can show that functions  $\varphi_i(u)$  are convex in  $u$ ,  $1 \leq i \leq m$ , and the convexity of function (8.44) follows. Thus, Condition A' holds and, consequently,  $\square$ -UCL  $\bar{f}$  for  $f(\square, \square)$  can be calculated by the fiducial method. After this, the  $\square$ -LCL for the system availability coefficients calculated by

$$\underline{K} = e^{-\bar{f}}.$$

**Example 8.5.4.** Let a system consists of  $m=10$  subsystems in series. The number of units within subsystems, and test results (that is, values of  $S_b$ ,  $r_i$ ,  $V_b$ ,  $l_i$ ) are given in Table 8.4.

We need to construct the LCL for the system availability coefficient (8.42) with confidence coefficient  $\square=0.9$ .

Table 8.4

In this case, the use of fiducial method gives the LCL  $\underline{K}_{0.9}=0.678$ .■

Consider now construction of the LCL for another standard system reliability index with renewal, the MTBF (in stationary regime). For the model with independent renewal, this reliability index is defined by formula (see Section 8.4.2 above, and Gnedenko et al., 1965, Ushakov, ed.(1994):

$$L = \left( \sum_{1 \leq i \leq m} \frac{n_i \mu_i}{\left(1 + \frac{\mu_i}{\lambda_i}\right)^{n_i} - 1} \right)^{-1}.$$

Confidence estimation of  $L$  from below is reduced to construction of UCL for

$$f(\lambda, \mu) = \sum_{1 \leq i \leq m} \frac{n_i \mu_i}{\left(1 + \frac{\mu_i}{\lambda_i}\right)^{n_i} - 1}. \quad (8.45)$$

In the most interesting for practice cases of highly reliable systems where  $\lambda_i \ll \mu_i$  for all  $i$ , from (8.45) it follows the following approximation

$$f(\lambda, \mu) \approx \sum_{1 \leq i \leq m} n_i \left( \frac{\lambda_i^{n_i}}{\mu_i^{n_i-1}} \right).$$

Expressing this formula via parameters  $(z, y)$ , we have

$$\tilde{f}(z, y) \approx \sum_{1 \leq i \leq m} n_i \exp[n_i z_i - (n_i - 1) y_i].$$

This function is convex in  $(z, y)$ . Thus, Condition A' holds and the UCL,  $\bar{f}$ , for  $f(\lambda, \mu)$  can be calculated by fiducial method. After this the LCL for the system MTBF,  $L$ , can be found as  $\underline{L} = \frac{1}{\bar{f}}$ .



In analogous way, we can show the correctness of fiducial method for confidence estimate from below another standard reliability index, operative availability coefficient,  $K_{\tau}$ . Remember that this reliability index presents the probability of successful operation during interval  $\tau$  in stationary regime. For highly reliable systems ( $\lambda_i \ll \mu_i$ ) the following approximation is valid:

$$K_{\tau} \approx K e^{-\frac{\tau}{L}}$$

where  $K$  and  $L$  are availability coefficient and MTBF of the system. Using previous formulas, we can rewrite the latter formula:

$$K_{\tau} \approx e^{-f(\lambda, \mu)}$$

where  $f(\lambda, \mu)$  being expressed via parameters  $(z, y)$  takes the form

$$\tilde{f}(z, y) = \sum_{1 \leq i \leq m} \varphi_i(z_i - y_i) + \tau \sum_{1 \leq i \leq m} e^{n_i z_i - (n_i - 1) y_i}$$

This function is convex in  $(z, y)$ , so it follows that the LCL for  $K_{\tau}$  can be constructed by the fiducial method.

In analogous way, on the basis of Condition A', we can show the correctness of the fiducial method for confidence estimation from below standard reliability indices like the availability coefficient,  $K$ , operative availability coefficient,  $K_{\tau}$ , MTBF,  $L$ , for a general model of series-parallel system with renewal (see Section 8.4.3).

### 11.6 Method of "Tangent" Functions

Consider one more approach, which is more complex in sense of computations but allows, in particular, to improve the fiducial method. Besides this approach is sufficiently general and can be applied not only to considered in this chapter "exponential" case, that is, where a system consisting of units with exponential d.f. of TTF. (See Remark 8.6.1 below.)

Let  $f(\lambda) = f(\lambda_1, \dots, \lambda_m)$  be a system reliability index where  $\lambda_i$  is failure rate of a unit  $i$ . Again introduce new parameters  $z_i = \ln \lambda_i$ ,  $1 \leq i \leq m$  and assume that function

$$\tilde{f}(z) = f(e^{z_1}, \dots, e^{z_m}) \quad (8.46)$$

is convex in  $\mathbf{z}=(z_1, \dots, z_m)$  . As we show above, this condition holds of many standard reliability index and models of complex systems.

Consider a class of all linear functions of type

$$h_b(z) = b_0 + \sum_{1 \leq i \leq m} b_i z_i, \quad \mathbf{b} \in B \quad (8.47)$$

where  $\mathbf{b}=(b_0, b_1, \dots, b_m)$  is vector of coefficients and

$B=\{\mathbf{b}: -\infty < b_i < \infty, 1 \leq i \leq m\}$ .

Since function (8.46) is convex, it can be presented as maximum in some subset of linear functions of the type (8.47), that is,

$$\tilde{f}(z) = \max_{\mathbf{b} \in B'} h_b(z) \quad (8.48)$$

where  $B' \subseteq B$ . Obviously, that function  $h_b(z)$  has the sense of tangent "planes" for function  $\tilde{f}(z)$  .

We call functions  $h_b(z)$  "tangent functions" for function  $\tilde{f}(z)$  .

Denote by  $\mathbf{b}(\mathbf{z}) \in B'$  vector of coefficients for which (for given fixed  $\mathbf{z}$ ) maximum in (8.48) attains, that is,

$$\tilde{f}(z) = \max_{\mathbf{b} \in B'} h_b(z) \quad (8.49)$$

Let  $\bar{h}_b = \bar{h}_b(S)$  is the upper fiducial limit with confidence coefficient  $\alpha$  for function  $h_b(z)$  for given vector of test results  $\mathbf{S}=(S_1, \dots, S_m)$  . Value  $\bar{h}_b$  simultaneously represents the  $\alpha$ -UCL for  $h_b(z)$  (see p.4 of App). Denote also by  $\bar{f} = \bar{f}(S)$  the upper fiducial limit for  $f$ . From (8.48) follows inequality

$$\tilde{f}(z) \geq h_b(z), \quad \mathbf{b} \in B'$$

for all  $\mathbf{z}$ . It follows inequalities for fiducial limits

$$\bar{f}(S) \geq \bar{h}_b(S), \quad \mathbf{b} \in B'$$

for all  $\mathbf{S}$ . So, Inequality

$$\bar{f}(\mathbf{S}) \geq \sup_{b \in B'} \bar{h}_b(\mathbf{S}), \quad (8.50)$$

holds for all  $\mathbf{S}$ . Introduce

$$H(\mathbf{S}) = \sup_{b \in B'} \bar{h}_b(\mathbf{S}), \quad (8.51)$$

which we call a "confidence majorant". Taking into account (8.49), we have for arbitrary fixed  $\mathbf{z}$

$$P\{H(\mathbf{S}) \geq \tilde{f}(\mathbf{z})\} = P\{\sup_{b \in B'} \bar{h}_b(\mathbf{S}) \geq h_{b(\mathbf{z})}(\mathbf{z})\} \geq P\{\bar{h}_{b(\mathbf{z})}(\mathbf{S}) \geq h_{b(\mathbf{z})}(\mathbf{z})\} \geq \gamma$$

that is,  $H(\mathbf{S})$  is  $\square$ -UCL for function  $f$ . Due to (8.50) Inequality

$$\bar{f}(\mathbf{S}) \geq H(\mathbf{S}) \quad (8.52)$$

holds for all  $\mathbf{S}$ .

The latter Inequality shows, first, that the upper fiducial limit  $\bar{f}(\mathbf{S})$  presents the  $\square$ -UCL for  $f$  because of  $H(\mathbf{S})$  is such a limit. It proves the correctness of the use of the fiducial method for confidence estimation from above (if condition of convexity of function (8.46) holds). Besides, from (8.52) follows that  $\square$ -UCL  $H(\mathbf{S})$  is better (for all test results  $\mathbf{S}$ ) in comparison with the upper fiducial limit  $\bar{f}(\mathbf{S})$  with confidence coefficient  $\square$ .

The precise calculation of a "confidence majorant"  $H(\mathbf{S})$  is sufficiently complicated if maximum in (8.48) and (8.51) is taken over infinite set  $B'$ . Nevertheless, convex function  $\tilde{f}(\mathbf{z})$  can be always presented approximately with the given in advance accuracy in the form of maximum over a finite set of "tangent" linear functions:

$$\tilde{f}(\mathbf{z}) = \max\{h_1(\mathbf{z}), \dots, h_N(\mathbf{z})\}.$$

After this "confidence majorant" is calculated by formula

$$H(S) = \max \{ \bar{h}_1(S), \dots, \bar{h}_N(S) \}.$$

where each of confidence limits  $\bar{h}_j(S)$  for individual "tangent" linear functions  $h_j(z)$ ,  $1 \leq j \leq N$ , is calculated (if it is impossible to do analytically), for example, by a common fiducial method with the help of Monte Carlo simulation.

This procedure of confidence limits construction for  $H(S)$  is called method of "tangent functions", or method of "confidence majorant". Thus, if the condition of function (8.46) convexity holds, then method of tangent functions allows to improve confidence estimation of function  $f$  from above in comparison with fiducial method. (See examples below.) Remember that it corresponds to the confidence estimation of the system PFFO from below (see Section 8.5).

Example 8.6.1. Consider the system from Example 8.5.2. Fiducial method gives  $\underline{R}_{0.95}=0.856$ . Method of "tangent functions" gives  $\underline{R}_{0.95}=0.914$ . ■

Example 8.6.2. Consider a series-parallel system with renewal, similar to that in Example 8.5.5. Let the system consists of  $m=4$  subsystems in series. Each subsystem consists of  $n_i$  parallel identical units. A failed unit is repaired immediately after its failure independently on the state of other units. All needed data for this example are in Table 8.5.

We need to find LCL with confidence coefficient not less than  $\square=0.95$  for the system availability coefficient

$$K = \prod_{1 \leq i \leq m} \left[ 1 - \left( \frac{\lambda_i}{\lambda_i + \mu_i} \right)^{n_i} \right].$$

Fiducial method gives  $\underline{K}_{0.95}=0.533$ , and method of "tangent functions" gives  $\underline{K}_{0.95}=0.686$ . ■

Remark 8.6.1. It is easy to see that method of "tangent functions", suggested by Pavlov (1981a,b, 1982, pp.157-159), can be applied not only for "exponential case", investigated in this chapter. Let us consider a general function

$$f(\theta) = f(\theta_1, \dots, \theta_m) \quad (8.53)$$

reflecting the dependence of reliability index  $f$  from vector of parameters  $\square=(\square_1, \dots, \square_m)$ , where  $\square_i$  is parameter of unit reliability. We need to construct  $\square$ -UCL for  $f$  on the basis of test results  $\mathbf{x}$ .

Assume that there is a basis set of the "tangent functions"

$$h_{\mathbf{b}}(\square) = h_{\mathbf{b}}(\square_1, \dots, \square_m), \quad \mathbf{b} \in B \quad (8.54)$$

such that function (8.53) can be represented in the form of maximum over some subset of "tangent functions". In other words, for arbitrary  $\square$  inequality

$$f(\theta) = \max_{b \in B'} h_b(\theta) \quad (8.55)$$

where  $B' \subseteq B$ , and maximum (8.55) is attained on some  $b(\square) \in B'$  for arbitrary  $\mathbf{b}$ . Assume also that for each of "tangent functions"  $h_b(\square) \square$ -UCL  $\bar{h}_b = \bar{h}_b(x)$ ,  $b \in B$ , can be constructed. Then value

$$H(x) = \sup_{b \in B'} \bar{h}_b(x)$$

gives  $\square$ -UCL for function (8.53).

The proof of this fact is completely similar to that for "exponential" case. The problem is in "optimal" (more precisely, rational) and choice of a sufficiently constructive class of "tangent functions" (8.54) for a concrete situation.  $\square$

In conclusion note that all three main methods (substitution, fiducial and "tangent functions") represents a set of methods each of which improves a previous one but consumes more calculating time. Method of substitution needs calculation of function  $f(\square)$  only in a single point  $\square = \bar{\lambda}$  in the space of parameters  $\square$ . Fiducial method requires multiple calculations of function  $f(\square)$  in different points of the parametrical space. These pints might be calculated with the help of Monte Carlo simulation. Finally, method of "tangent functions", roughly speaking, needs multiple repetition of fiducial method for different "tangent functions".

**12.**

**12.1**

**12.2 aaa**

**12.3 aaa**

**12.4 aaa**

**12.5 aaa**

**12.6 aaa**

## 12.7 Appendix to Chapter

### 12.7.1 Confidence Limits for Quasi-Convex and Quasi-Concave Functions

Let  $\Omega$  is a convex area in  $m$ -dimensional Euclid space. Function of  $m$  variables

$$g(\mathbf{x})=g(x_1, \dots, x_m)$$

determined in area  $\Omega$  is called quasi-concave if for any  $\alpha > 0$  and  $\beta > 0$ ,  $\alpha + \beta = 1$ ,  $\mathbf{x} \in \Omega$  and  $\mathbf{y} \in \Omega$ ,  
Inequality

$$g(\alpha \mathbf{x} + \beta \mathbf{y}) \geq \min [g(\mathbf{x}), g(\mathbf{y})]$$

or, equivalently, if area

$$\{\mathbf{x} \in \Omega: g(\mathbf{x}) \geq C\}$$

is convex for arbitrary constant  $C$ . Function  $g(\mathbf{x})$  is called quasi-convex if function  $-g(\mathbf{x})$  is concave. It is easy to see that concave (convex) function is simultaneously quasi-concave (quasi-convex), although an inverse statement is, generally speaking, not true. Notice also that if function  $g(\mathbf{x})$  is convex (concave) and function of a single variable  $h(u)$  is monotone decreasing in  $u$ , then function  $h[g(\mathbf{x})]$  is quasi-concave (convex).

Let  $F_{\Omega}(t) = P(\Omega < t)$  is d.f. of r.v.  $\Omega$ . Denote by  $\Omega$  class of d.f.'s  $F(t)$  such that  $-\ln [1-F(t)]$  is convex in  $t$  for such  $t$  that  $F(t) < 1$ , and  $\tilde{\Phi}$  class of d.f.'s  $F(t)$  such that  $-\ln F(t)$  is convex in  $t$  for such  $t$  that  $F(t) > 0$ . If r.v.  $\Omega$  has d.f.  $F_{\Omega}(t)$  belonging to  $\Omega$  ( $\tilde{\Phi}$ ), we will denote it by  $\Omega \in \Omega$  (or by  $\Omega \in \tilde{\Phi}$ ).

From definition of the classes, defined above, it follows that if  $\Omega \in \Omega$  then  $-\Omega \in \tilde{\Phi}$ , and, on the contrary, if  $\Omega \in \tilde{\Phi}$  then  $-\Omega \in \Omega$ . If  $\Omega \in \Omega$  ( $\Omega \in \tilde{\Phi}$ ) and  $C > 0$  then  $C\Omega \in \Omega$  ( $C\Omega \in \tilde{\Phi}$ ). Classes  $\Omega$  and  $\tilde{\Phi}$  include such d.f.'s like normal, exponential, Weibull-Gnedenko (with shape parameter  $\alpha \geq 1$ ), and some others. Notice that under an additional condition:  $F(0) = 0$ , class  $\Omega$  coincides with class of IFR- distributions.

**Lemma 8.1.** Let  $\Omega_1, \dots, \Omega_m$  are independent r.v.'s, and  $u_1, \dots, u_m$  such constants that  $P\{\Omega_k \leq (\geq) u_k\} \geq \gamma$ ,  $1 \leq k \leq m$ . Then if  $\Omega_k \in \Omega$  ( $\Omega_k \in \tilde{\Phi}$ ) and  $\gamma \geq 1 - \exp(-3/2)$ , then

$$P\left\{ \sum_{1 \leq k \leq m} \xi_k \leq (\geq) \sum_{1 \leq k \leq m} u_k \right\} \geq \gamma. \quad (8.56)$$

**Proof.** It is enough to consider the case where  $\Omega_k \in \Omega$ ,  $1 \leq k \leq m$ , since the second half of the proof is follows from a simple transition to r.v.'s  $-\Omega_k$ . Since class  $\Omega$  is closed in respect to convolution of independent r.v.'s, it is enough to prove (8.56) for  $m=2$ . Let  $F_k(t) = P(\Omega_k < t)$  and  $F_k(u_k) = \gamma$ ,  $1 \leq k \leq 2$ .

Due to convexity of  $\square'_k(t) = -\ln[1-F_k(t)]$ , it lies above the tangent at point  $[u_k, \square'_k(u_k)]$ , and it follows that for all  $t$

$$\square_k(t) \geq \max[0, \square_k(u_k) + \square'_k(u_k)\square(t-u_k)]$$

or, taking into account that  $\square_k(u_k) = -\ln(1-\square)$ , we obtain

$$\square_k(t) \geq \square^*_k(t) = \max[0, \square(t-u_k)]$$

where  $\square_k = u_k + \ln(1-\square)[\square'_k(u_k)]^{-1}$ ,  $\square_k = \square'_k(u_k)$ ,  $1 \leq k \leq 2$ . Denote  $F^*_k(t) = 1 - \exp[-\square^*_k(t)]$  we obtain  $F^*_k(t) \leq F_k(t)$  for all  $t$ . It gives

$$\begin{aligned} \int_{t_1+t_2 \leq u_1+u_2} \dots \int dF_1(t_1) dF_2(t_2) &\geq J^* = \int_{t_1+t_2 \leq u_1+u_2} \dots \int dF^*_1(t_1) dF^*_2(t_2) \\ &= \int_{t_1+t_2 \leq c} \dots \int \beta_1 e^{-\beta_1 t_1} \beta_2 e^{-\beta_2 t_2} dt_1 dt_2 \end{aligned}$$

where

$$c = -\ln(1-\gamma) \frac{\beta_1 + \beta_2}{\beta_1 \beta_2}.$$

Inequality  $J^* \geq \square$  after integration can be written in the form

$$\frac{\beta_1 e^{-\beta_1 c} - \beta_2 e^{-\beta_2 c}}{\beta_1 - \beta_2} \leq 1 - \gamma.$$

Setting  $\square_1 \geq \square_2$  and denoting  $\square = \square_2 / \square_1$ , we can write

$$h(\mu) = \mu + e^{-B\mu} - \mu e^{-\frac{B}{\mu}} \leq 1$$

where  $B = -\ln(1-\square)$ . It is easy to show that for  $B \geq 3/2$  and  $0 < \square \leq 1$

$$h''(\mu) = B^2 \left( e^{-B\mu} - \frac{1}{\mu^3} e^{-\frac{B}{\mu}} \right) \geq 0.$$

holds. From here follows that if  $B \geq 3/2$ , function  $h(\square)$  is convex on interval  $(0, 1)$ . Since  $h(+0)=h(1)=1$ , it follows that for  $B \geq 3/2$  and  $0 < \square \leq 1$  Inequality  $h(\square) \leq 1$  holds, and inequality  $J^* \geq \square$  and (8,56) follow. ■

Let us have  $m$  independent test results where  $X_k = \{x_k\}$  is the results of the  $k$ th test,  $P_{a_k}$  is a family of d.f.'s on  $X_k$  where

$a_k \in A_k, 1 \leq k \leq m$ . Let  $\theta_k(a_k)$  is given on  $A_k$  real function and  $\underline{\theta}_k(x_k)$  and  $\bar{\theta}_k(x_k)$  are r.v.'s defined on  $X_k$  such that

$$\inf_{a_k \in A_k} P_{a_k} \{ \underline{\theta}_k(x_k) \leq \theta_k(a_k) \} \geq \gamma$$

$$\inf_{a_k \in A_k} P_{a_k} \{ \bar{\theta}_k(x_k) \geq \theta_k(a_k) \} \geq \gamma$$

$1 \leq k \leq m, 0 < \gamma < 1$ . Further we use an abbreviate notation:  $\theta_k = \theta_k(a_k), \underline{\theta}_k = \underline{\theta}_k(x_k), \bar{\theta}_k = \bar{\theta}_k(x_k), \theta = (\theta_1, \dots, \theta_m), \underline{\theta} = (\underline{\theta}_1, \dots, \underline{\theta}_m), \bar{\theta} = (\bar{\theta}_1, \dots, \bar{\theta}_m)$ .

**Theorem 8.1** Let  $\theta_k \in \Phi, 1 \leq k \leq m$ , and function  $R(\theta) = R(\theta_1, \dots, \theta_m)$  is strictly increasing (decreasing) monotone in each variable and quasi-concave (quasi-convex). Then if  $\gamma \geq 1 - \exp(-3/2)$ , then

$$\inf_{a \in A} P_a \{ R(\underline{\theta}) \leq (\geq) R(\theta) \} \geq \gamma. \quad (8.57)$$

■

**Theorem 8.2.** Let  $\bar{\theta}_k \in \tilde{\Phi}, 1 \leq k \leq m$ , and function  $R(\theta) = R(\theta_1, \dots, \theta_m)$  is strictly increasing (decreasing) monotone in each variable and quasi-convex (quasi-concave). Then if  $\gamma \geq 1 - \exp(-3/2)$ , then



$$\inf_{a \in A} P_a \{R(\bar{\theta}) \geq (\leq) R(\theta)\} \geq \gamma.$$

Proof. Consider at the beginning the proof of Theorem 8.1 if  $R(\theta)$  is increasing in each variable and quasi-concave function. Let  $a \in A$  and  $\theta = \theta(a)$ . Since set  $H = \{\theta: R(\theta) \geq R(\theta)\}$  is convex, then there exists a plane in  $m$ -dimensional space of the form

$$L = \left\{ \theta: \sum_{1 \leq k \leq m} c_k (\theta_k - \theta_k) = 0 \right\}$$

which comes via a border point  $\theta \in H$  and the area  $H$  locates on one side of this plane, that is,

$$H \subset \left\{ \theta: \sum_{1 \leq k \leq m} c_k \theta_k \geq \sum_{1 \leq k \leq m} c_k \theta_k \right\}$$

From here follows that

$$\begin{aligned} & \left\{ \theta: \sum_{1 \leq k \leq m} C_k \theta_k < \sum_{1 \leq k \leq m} C_k \theta_k \right\} \subset \\ & \subset \left\{ \theta: R(\theta) < R(\theta) \right\} \subset \left\{ \theta: R(\theta) \leq R(\theta) \right\} \end{aligned}$$

Due to the monotonicity of  $R(\theta)$  coefficients  $c_k \geq 0, 1 \leq k \leq m$ . If function  $R(\theta)$  has continuous partial derivatives then coefficients  $c_k$  are defined in a unique way:

$$c_k = \alpha \frac{\partial R(\theta)}{\partial \theta_k}, \quad 1 \leq k \leq m,$$

with the accuracy of a positive multiplier  $\alpha$ . Using Lemma 8.1 we obtain

$$P_a \{R(\theta) \leq R(\theta)\} \geq P_a \left\{ \sum_{1 \leq k \leq m} C_k \theta_k < \sum_{1 \leq k \leq m} C_k \theta_k \right\} = P_a \left\{ \sum_{1 \leq k \leq m} C_k \theta_k \leq \sum_{1 \leq k \leq m} C_k \theta_k \right\} \geq \gamma$$

that proves (8.57) for the case where  $R(\theta)$  is increasing in each variable and quasi-concave function. The proof the statement in braces follows from the previous after transition to function  $(-R)$ . Notice also that function  $R(-\theta)$  is monotone decreasing (increasing) in each variable and quasi-

convex (quasi-concave) if function  $R(\theta)$  is monotone increasing (decreasing) in each variable and quasi-convex (quasi-concave). Since, additionally, r.v.'s  $\theta_k$  and  $\bar{\theta}_k \in \tilde{\Phi}$ , so the proof Theorem 8.2 follows from the previous after transition to variables  $\theta_1, \theta_2, \dots, \theta_m$ . ■

The generalization of previous statements for the case where function  $R(\theta)$  is monotone increasing in some of the variables and decreasing in others is given by the following Theorem.

**Theorem 8.3.** Let function  $R(\theta)$  is strictly monotone increasing in  $\theta_1, \dots, \theta_n$  and decreasing in  $\theta_{n+1}, \dots, \theta_m$ , and  $\theta_i \geq 1 - e^{-\frac{\gamma}{2}}$ . Then:

(1) If  $(\theta_1, \dots, \theta_n) \in \tilde{\Phi}$  and  $(\bar{\theta}_{n+1}, \dots, \bar{\theta}_m) \in \tilde{\Phi}$ , and function  $R(\theta)$  is quasi-concave, then

$$\inf_{a \in A} P_a \{R(\theta_1, \dots, \theta_n, \bar{\theta}_{n+1}, \dots, \bar{\theta}_m) \leq R(\theta)\} \geq \gamma.$$

(2) If  $(\bar{\theta}_1, \dots, \bar{\theta}_n) \in \tilde{\Phi}$  and  $(\theta_{n+1}, \dots, \theta_m) \in \tilde{\Phi}$ , and function  $R(\theta)$  is quasi-convex, then

$$\inf_{a \in A} P_a \{R(\bar{\theta}_1, \dots, \bar{\theta}_n, \theta_{n+1}, \dots, \theta_m) \geq R(\theta)\} \geq \gamma.$$

The proof is analogous to the given above.

Consider now an application of the results above for an "exponential" case. Let  $\lambda_k$  is parameter (failure rate) for unit of type  $k$  and

$$\bar{\lambda}_k = \frac{\chi_\gamma^2(2r_k)}{2S_k}$$

is standard  $\lambda$ -UCL for  $\lambda_k$ . (See Section 8.2.1 above).

Denote also by  $\lambda = (\lambda_1, \dots, \lambda_m)$  vector of unit parameters and by  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)$  vector of corresponding UCLs. In some cases, it is convenient to use parameters  $\mathbf{T} = (T_1, \dots, T_m)$  where  $T_k = 1/\lambda_k$  is MTBF of a unit of type  $k$ ,  $1 \leq k \leq m$ . Vectors of corresponding confidence limits for individual parameters are denoted by  $\underline{\mathbf{T}} = (\underline{T}_1, \dots, \underline{T}_m)$  and  $\bar{\mathbf{T}} = (\bar{T}_1, \dots, \bar{T}_m)$  where

$$\underline{T}_k = \frac{2S_k}{\chi_\gamma^2(2r_k)},$$

$$\bar{T}_k = \frac{2S_k}{\chi_{1-\gamma}^2(2r_k)},$$

are standard lower and upper  $\alpha$ -confident limits for  $T_k$ .

R.v.  $S_k$  (the total test time of units of type  $k$ ) has gamma d.f. with density

$$f_k(t) = \lambda_k \frac{(\lambda_k t)^{r_k-1}}{(r_k-1)!} e^{-\lambda_k t}. \blacksquare$$

On the basis of this it easy to show that  $S_k \square \square$  and  $S_k \square \tilde{\Phi}$ . From here, it follows that confident limits  $\underline{T}_k \square \square$  and  $\underline{T}_k \square \tilde{\Phi}$ , and  $\bar{T}_k \square \square$  and  $\bar{T}_k \square, \square \tilde{\Phi}$ ,  $1 \leq k \leq m$ . From previous results the following theorem follows.

Theorem 8.4. If function  $R(\mathbf{T})$  is monotone increasing (decreasing) in each variable and quasi-concave (quasi-convex) then for  $\alpha \geq 1 - e^{-\frac{3}{2}}$

$$\inf_{\mathbf{T}} P_T \{R(\mathbf{T}) \leq (\geq) R(\mathbf{T}) \geq \gamma\}.$$

If in addition one of the following conditions holds:

- (a) for some  $C$  set  $\varphi_C = \{\mathbf{T}: R(\mathbf{T}) \leq (\geq) C\}$  is restricted
- (b) for some  $C$  set

$$\Psi_C = \{\lambda: R\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_m}\right) \leq (\geq) C\} \quad (8.58)$$

is restricted, then

$$\inf_{\mathbf{T}} P_T \{R(\mathbf{T}) \leq (\geq) R(\mathbf{T})\} = \gamma. \quad (8.59)$$

Proof. Since  $T_k \in \Phi, 1 \leq k \leq m$ , and function  $R(\mathbf{T})$  is monotone increasing (decreasing) in each variable and quasi-concave (quasi-convex), the proof Inequality (8.58) follows from Theorem 8.1. Let us show (8.59). Let, for instance, condition (a) holds. Then

$$0 < \sup_{\mathbf{T} \in \varphi_C} T_k = \eta_k < \infty, \quad 1 \leq k \leq m.$$

exists. From here the relation

$$\{\mathbf{T}: R(\mathbf{T}) \leq (\geq) c\} \subset \{\mathbf{T}: T_k \leq \eta_k\}$$

follows. Due to it for any  $T \in \varphi_C$

$$P_T \{R(\mathbf{T}) \leq (\geq) R(\mathbf{T})\} \leq P_T \{R(\mathbf{T}) \leq (\geq) c\} \leq P_T \{T_k \leq \eta_k\} = P_{T_k} \{T_k \leq \eta_k\}$$

For any  $\varepsilon > 0$  such  $T_\varepsilon = (T_{\varepsilon_1}, \dots, T_{\varepsilon_m}) \in \varphi_C$  exists that  $T_k - \varepsilon < T_k < T_k + \varepsilon$ . Since  $P_{T_k} \{T_k \leq \eta_k\}$  is continuous in  $T_k$ , then

]

$$P_{T_k} \{R(\mathbf{T}) \leq (\geq) R(T_\varepsilon)\} \leq P_{T_k} \{T_k \leq \eta_k\} + \delta_\varepsilon = \gamma + \delta_\varepsilon$$

where  $\delta_\varepsilon > 0$  and  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = 0$ . From here, (8.59) follows. The proof for condition (b) is similar. ■

Theorem 8.5. If function  $R(\mathbf{T})$  is monotone increasing (decreasing) in each variable and quasi-convex (quasi-concave), then for  $\gamma \geq 1 - e^{-\frac{3}{2}}$

$$\inf_{\mathbf{T}} P_{\mathbf{T}} \{R(\mathbf{T}) \geq (\leq) R(T)\} = \gamma.$$

Proof. Since  $\bar{T}_k \in \tilde{\Phi}, 1 \leq k \leq m$ , in accordance with Theorem 8.2,

$$\inf_{\mathbf{T}} P_{\mathbf{T}} \{R(\bar{\mathbf{T}}) \geq (\leq) R(T)\} \geq \gamma.$$

Let further  $\mathbf{u}=(u_1, \dots, u_m)$  be some fixed point  $u_k>0, 1\leq k\leq m$ , and  $c=R(\mathbf{u})$ . Due to the quasi-convexity of  $R(\mathbf{T})$  there exists a plane in  $m$ -dimensional space of type

$$\sum_{1\leq k\leq m} b_k(T_k - u_k) = 0, \quad b_k > 0, \quad T_k > 0, \quad 1 \leq k \leq m,$$

which passes via point  $\mathbf{T}=\mathbf{u}$  and such that

$$\varphi_c = \{\mathbf{T}: R(\mathbf{T}) \leq (\geq) c\} \subset \left\{ \mathbf{T}: \sum_{1\leq k\leq m} b_k(T_k - u_k) \leq 0, \quad T_k > 0 \right\} = G.$$

Since area  $G$  is restricted, area  $\varphi_c$  is also restricted. After this the proof is analogous to that for Theorem 8.4. ■

Let  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_m)$  be vector of  $\square$ -UCLs. The following theorem can be formulated.

**Theorem 8.6.** If function  $f(\square)$  is monotone increasing (decreasing) in each variable and quasi-concave (quasi-convex), then for  $\square \geq 1 - e^{-\frac{3}{2}}$

$$\inf_{\lambda} P_{\lambda} \{f(\bar{\lambda}) \leq (\geq) f(\lambda)\} = \gamma.$$

**Proof.** It is enough to consider the case where  $f(\square)$  is quasi-concave and monotone decreasing in each variable function. Introduce function  $g_f(\mathbf{T})=f\left(\frac{1}{T_1}, \dots, \frac{1}{T_m}\right)$ . For any  $\mathbf{T}, \mathbf{u}$  and  $\alpha>0, \beta>0$ ,  $\alpha+\beta=1$ , we have

$$\begin{aligned} g_f(\alpha\mathbf{T} + \beta\mathbf{u}) &= f\left(\frac{1}{\alpha T_1 + \beta u_1}, \dots, \frac{1}{\alpha T_m + \beta u_m}\right) \geq \\ &\geq f\left(\frac{\alpha}{T_1} + \frac{\beta}{u_1}, \dots, \frac{\alpha}{T_m} + \frac{\beta}{u_m}\right) \geq \min[g_f(\mathbf{T}), g_f(\mathbf{u})] \end{aligned}$$

From here follows that  $g_f(\mathbf{T})$  is quasi-concave and monotone increasing in each variable function. Besides, set

$$\left\{ \lambda: g_f\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_m}\right) \geq g_f\left(\frac{1}{u_1}, \dots, \frac{1}{u_m}\right) \right\} = \{ \lambda: f(\lambda) \geq f(\mathbf{u}) \}$$

is restricted for any  $\mathbf{u}$  and it follows that the function satisfies condition (b) of Theorem 8.4, and the proof follows from this theorem. ■

### 12.7.2 Construction of confident limits by the substitution method for function of type (8.17)

We need further additional knowledge about some distribution families.

Continuous distribution density  $f(t)$  of r.v.  $\square$  is called Polya second order density (we use below abbreviation P2-density), if one of the following equivalent condition holds:

(1) Function  $\frac{f(t-\Delta)}{f(t)}$  is monotone increasing in  $t$  for  $\square > 0$  and  $t$  such that  $f(t) > 0$ .

(2) Function  $\varphi(t) = -\ln f(t)$  is convex in  $t$  such that  $f(t) > 0$ .

It is easy to see that such d.f. like normal, exponential, Gnedenko-Weibull (with shape parameter  $a \geq 1$ ) and some other frequently used distributions have P2-density (for more details, see Barlow and Proschan, 1965). It is easy to show that if r.v.  $\square$  has P2-density, then  $\square \square \square$  and  $\xi \in \tilde{\Phi}$  where  $\square$  and  $\tilde{\Phi}$  are classes of logarithmically convex distributions. Indeed, let  $\square$  has

P2-density and  $F(t) = \int_{-\infty}^t f(u) du$  is corresponding d.f. Then function

$$-\frac{d}{dt} \ln[1 - F(t)] = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{\int_t^{\infty} f(x) dx} = \frac{1}{\int_t^{\infty} \frac{f(x)}{f(t)} dx} = \frac{1}{\int_0^{\infty} \frac{f(t+u)}{f(t)} du}$$

is monotone increasing in  $t$  for such  $t$  that  $F(t) < 1$ , and it follows that  $\square \square \square$ . (In particular, from here follows the well-known fact that if r.v.  $\square$  is non-negative and has P2-density, then it has IFR distribution.) Besides, from the definition above follows that if r.v.  $\square$  has P2-density, then  $(-\square)$  also has P2-density. Consequently,  $(-\square) \square \square$  and it follows that  $\square = (-\square) \square \tilde{\Phi}$ . Thus, if r.v.  $\square$  has P2-density, then  $\square \square \square$  and  $\square \square \tilde{\Phi}$ .

Consider, further, function  $f(\square)$  of type

$$f(\lambda) = \sum_{1 \leq i \leq M} A_i \prod_{j \in G_i} \lambda_j^{n_{ij}} \quad (8.60)$$

where  $A_i$  are positive coefficients and  $G_i$  are arbitrary subsets of set of subscripts,  $G_i \subseteq \{1, 2, \dots, m\}$ . Let us show that for arbitrary function of type (8.60)  $\square$ -UCL can be calculated (for  $\square \geq 1 - \exp(-3/2)$ ) by the method of substitution, that is by formula  $\bar{f} = f(\bar{\lambda})$ . For the sake of convenience, introduce new parameters  $z_i = \ln \square_i$ ,  $1 \leq i \leq m$ . For parameter  $z_i$  value

$$\bar{z}_i = \ln \bar{\lambda}_i = \ln \left[ \frac{\chi_\gamma^2(2r_i)}{2S_i} \right]$$

is  $\square$ -UCL. It is not difficult to find that  $\bar{z}_i$  has P2-density. For this purpose, we need to show that r.v.  $\square_i = \ln S_i$  has P2-density. R.v.  $S_i$  has gamma d.f. (or Erlang d.f. of order  $r_i - 1$ ) with density

$$f_i(t) = \frac{\lambda_i^{r_i} t^{r_i-1}}{(r_i - 1)!} e^{-\lambda_i t}.$$

R.v.  $\square_i$  has d.f.

$$H_i(t) = P(\square_i \leq t) = P(\ln S_i \leq t) = P(S_i \leq e^t) = F_i(e^t)$$

where  $F(\square)$  is d.f. of  $S_i$ . After simple transforms, it gives the following density of r.v.  $\square_i$ :

$$h_i(t) = H'_i(t) = f_i(e^t) e^t = \frac{\lambda_i^{r_i}}{(r_i - 1)!} e^{-\varphi(t)}$$

where  $\varphi(t) = \square e^t - r_i t$  is a convex function. So, r.v.  $\square_i$  (and, consequently, confident limit  $\bar{z}_i$ ) has P2-density.

Introduce vector of parameters  $\mathbf{z} = (z_1, \dots, z_m)$  and vector of  $\square$ -UCLs  $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_m)$ . From the previous results follows

**Theorem 8.7** Let function  $g(\mathbf{z}) = g(z_1, \dots, z_m)$  is monotone increasing in each  $z_i$  and quasi-convex, then for  $\square \geq 1 - e^{-\frac{3}{2}}$

$$P\{g(\bar{\mathbf{z}}) \geq g(\mathbf{z})\} \geq \gamma.$$

Proof. As we showed above, UCL  $\bar{z}_i$  has P2-density. Then  $\bar{z}_i \in \tilde{\Phi}$ ,  $1 \leq i \leq m$ , and the needed proof follows from Theorem 8.2.

Let us express function (8.60) via parameters  $\mathbf{z}$ :

$$g(\mathbf{z}) = f(e^{z_1}, \dots, e^{z_m}) = \sum_{1 \leq i \leq M} A_i \exp\left(\sum_{j \in G_i} h_{ij} z_j\right) \quad (8.61)$$

This function is convex in  $\mathbf{z}$  being the sum of convex functions. Thus, using Theorem 8.7, we show that for function of type (8.61) for parameters  $\mathbf{z}$ , or for function of type (8.60) of parameters of type  $\square$ ,  $\square$ -UCL can be calculated by the method of substitution. ■

### 12.7.3 Proof of quasi-convexity of reliability index for renewable systems

Let  $g(\mathbf{T})$  be a function of type (8.27) where  $K_i(T_i)$  are given in (8.26). Notice that (8.23) and (8.24) are particular cases of (8.26) and (8.27). For proof of quasi-concavity of functions (8.27), it is sufficient to show convexity in  $\mathbf{z}$  of function

$$\varphi_{nm}(\mathbf{z}) = -\ln \left( \frac{\sum_{0 \leq k \leq m} \frac{A_k}{z_k}}{\sum_{0 \leq k \leq n} \frac{A_k}{z_k}} \right)$$

where  $m < n$ ,  $A_k > 0$ ,  $0 \leq k \leq n$ . Denote

$$\varphi_l(\mathbf{z}) = \frac{\sum_{0 \leq k \leq l} \frac{k A_k}{z_k}}{\sum_{0 \leq k \leq l} \frac{A_k}{z_k}}.$$

After differentiation of  $\varphi_{nm}(\mathbf{z})$ , we obtain



$$\varphi'_{nm}(z) = \frac{1}{z} [\psi_m(z) - \psi_n(z)] = \frac{1}{z} \sum_{m+1 \leq l \leq n} [\psi_{l-1}(z) - \psi_l(z)] = \frac{1}{z} \sum_{m+1 \leq l \leq n} \varphi'_{l,l-1}(z).$$

It follows that it is sufficient to show the convexity of function  $\varphi_{l,l-1}(z)$ ,  $m+1 \leq l \leq n$ . We can write

$$\varphi_{l,l-1}(z) = -\ln [1-h(z)],$$

$$\varphi'_{l,l-1}(z) = \frac{h'(z)}{1-h(z)}$$

where

$$h(z) = \frac{A_l}{A_l + A_{l-1}z + \dots + A_0 z^l}.$$

Since  $A_k$ ,  $0 \leq k \leq m$ , function  $f(z)$  is convex for  $z \geq 0$  that gives the convexity of functions  $\varphi_{l,l-1}(z)$  and  $\varphi_{nm}(z)$ . Consequently, function (8.27) can be presented as a monotone decreasing function of the sum of monotone decreasing convex functions and is quasi-concave.

Let  $g(\mathbf{T})$  be the function of the following form:

$$g(\mathbf{T}) = \kappa \left\{ \sum_{1 \leq i \leq m} \left( \frac{1}{\sum_{1 \leq k \leq n_i} A_{ki} T_i^k} \right) \right\} \quad (8.62)$$

where  $\square(u)$  is monotone decreasing in  $u$  and  $A_{ki} > 0$ . (Functions of type (8.62) are  $L(T)$  and  $P_{\square}(T)$ .)  
Function

$$\frac{1}{\sum_{1 \leq k \leq n_i} A_{ki} T_i^k}$$

is monotone decreasing and convex in  $T_i$ ,  $1 \leq i \leq m$ , from where follows that (8.62) is quasi-concave function.

Now let  $g(\mathbf{T})$  be the function

$$g(T) = g_1(T) \exp \left\{ - \sum_{1 \leq i \leq m} \frac{1}{\sum_{1 \leq k \leq n_i} A_{ki} T_i^k} \right\} \quad (8.63)$$

where  $g_1(T)$  is function of type (8.27). Function  $K_{\square}(T)$  belongs to type (8.63).

Formula (8.63) can be also presented in the form of a monotone decreasing function of the sum of monotone decreasing convex functions and, consequently, is quasi-concave.

#### 12.7.4 Conditions of correctness of the method of fiducial probabilities for calculation of confidence limits for the system PFFO

Let us investigate the question: for what functions  $f(\square) = f(\square_1, \dots, \square_m)$  fiducial limit with confidence coefficient  $\square$  coincides with  $\square$ -confidence limit? It is not difficult problem to show that such functions are, for instance,

$$\varphi_b(\lambda) = b_0 \prod_{1 \leq i \leq m} \lambda_i^{b_i}, \quad \mathbf{b} \in B \quad (8.64)$$

where  $\mathbf{b} = (b_0, \dots, b_m)$  is vector of constant coefficients, and  $B = \{\mathbf{b}: b_0 > 0, -\square < b_i < \square, 1 \leq i \leq m\}$ . For arbitrary function of type (8.64), fiducial approach is correct for the construction of UCL and LCL of  $\varphi_b(\square)$ . Indeed, for each fixed  $\square_i$  r.v.  $\square_i = \square_i S_i$  has distribution independent on  $\square_i$ . For given

fixed test results  $S_i^*$ , fiducial distribution of parameter  $\square_i$  coincides with distribution of r.v.  $\frac{\xi_i}{S_i^*}$ .

Thus, upper fiducial limit  $\bar{\varphi}_b$  with confidence coefficient  $\square$ , for  $\varphi_b(\square)$  is equal to quantile of level  $\square$  for r.v.

$$\eta_b = b_0 \prod_{1 \leq i \leq m} \left( \frac{\xi_i}{S_i^*} \right)^{b_i},$$

and from here follows

$$\bar{\varphi}_b = b_0 \prod_{1 \leq i \leq m} \left( \frac{1}{S_i^*} \right)^{b_i} t_b(\gamma)$$

where  $t_b(\square)$  is quantile of level  $\square$  for r.v.

$$\tau_b = \prod_{1 \leq i \leq m} \xi_i^{b_i}.$$

It follows that for arbitrary fixed  $\square = (\square_1, \dots, \square_m)$ , the confidence probability corresponding to upper limit  $\bar{\varphi}_b$  is equal to

$$P_\lambda \{ \bar{\varphi}_b \geq \varphi_b(\lambda) \} = P_\lambda \left\{ b_0 \prod_{1 \leq i \leq m} \left( \frac{1}{S_i^*} \right)^{b_i} t_b(\gamma) \geq b_0 \prod_{1 \leq i \leq m} \lambda_i^{b_i} \right\} = P_\lambda \{ \tau_b \leq t_b(\gamma) \} = \gamma$$

that is, upper fiducial limit with confidence coefficient  $\square$  for  $\bar{\varphi}_b(\square)$  coincides with  $\square$ -ICL. Analogously, we can prove a similar relations for lower fiducial limit.

A set of all functions (8.64) for different  $\mathbf{b} \in B$  we call a basis set of functions. For any function from this set, as it was shown above, fiducial method is correct for constructing UCL and LCL. Obviously, the basis set is somewhat narrow. For instance, it does not contain simplest linear functions of type

$$f(\lambda) = \sum_{1 \leq i \leq m} C_i \lambda_i$$

which are related, for instance, to the problem of reliability estimation of series systems. Nevertheless, of basis set of functions (8.64) can be used for expanding the class of functions for which fiducial method can be applied for constructing upper confidence limits.

For the sake of convenience, introduce new parameters  $\mathbf{z} = (z_1, \dots, z_m)$  where  $z_i = \ln \square_i$ ,  $1 \leq i \leq m$ . This vector of parameters,  $\mathbf{z}$ , takes its values from  $Z = \{ \mathbf{z}: -\square < z_i < \square, 1 \leq i \leq m \}$ . In these new variables, a set of functions (8.64) transforms into a set of linear functions

$$h_b(\mathbf{z}) = b_0 + \sum_{1 \leq i \leq m} b_i z_i, \quad \mathbf{b} \in B \quad (8.65)$$

where  $B = \{ \mathbf{b}: -\square < b_i < \square, 1 \leq i \leq m \}$ .

Upper fiducial limit for  $h_b(\mathbf{z})$  is denoted by  $\bar{h}_b = \bar{h}_b(\mathbf{S})$ . Function  $f(\square)$  can be written via parameters  $\mathbf{z}$  as follows

$$\tilde{f}(z) = f(e^{z_1}, \dots, e^{z_m}) \quad (8.66)$$

Now let  $\bar{f} = \bar{f}(S)$  is the upper fiducial limit with confidence coefficient  $\gamma$  for  $f=f(\square)$ . The theorem below shows that the upper fiducial limit coincides with the  $\gamma$ -UCL for  $f$ , if function (8.66) is represented in the form of maximum over a set of functions of type (8.65). Let

$$\tilde{f}(z) = \sup_{b \in B'} h_b(z) \quad (8.67)$$

where  $B' \subseteq B$ . Then the following theorem takes place.

Theorem 8.8. Let supremum in (8.67) is attained for any  $z \in Z$  at some point  $b(z) \in B'$ , that is,

$$\sup_{b \in B'} h_b(z) = h_{b(z)}(z). \quad (8.68)$$

Then for any  $z \in Z$  inequality

$$P\{\bar{f}(x) \geq \tilde{f}(z)\} \geq \gamma.$$

Proof. Due to (8.67), inequality

$$\tilde{f}(z) \geq h_b(z), \quad b \in B'$$

for all  $z \in Z$ . From the construction of fiducial limits  $\bar{f}$  and  $\bar{h}_b$ , analogous inequality for these limits follows

$$\bar{f}(S) \geq \bar{h}_b(S), \quad b \in B'$$

for all  $S$ . It follows that

$$\bar{f}(S) \geq \sup_{b \in B'} \bar{h}_b(S) = H(S) \quad (8.69)$$

for all  $S$ . Thus, it is sufficient to show that

$$P\{H(S) \geq \tilde{f}(z)\} \geq \gamma, \quad z \in Z.$$

Further, for arbitrary fixed  $z \in Z$ , we have

$$P\{H(S) \geq \tilde{f}(z)\} = P\left\{\sup_{b \in B'} \bar{h}_b(S) \geq h_{b(z)}(z)\right\} \geq P\{\bar{h}_b(S) \geq h_{b(z)}(z)\} = \gamma.$$

This completes the proof. ■

Thus, the fiducial method is correct for confidence estimation of  $f$  from above, if function (8.66) can be represented in the form

$$\tilde{f}(z) = \max_{b \in B'} h_b(z). \quad (8.70)$$

This condition holds if function (8.66) is convex in  $z$ . Functions  $f_b(z)$  in (8.70) represent tangent "planes". Thus, Theorem 8.8 establishes that fiducial method is correct for confidence estimation of function  $f(\square)$  from above, if function (8.66) is convex in  $z$ . This result can be, in natural way, expanded to the case where functions are concave. For this purpose, one must consider function  $(-f)$ . From the proof of Theorem 8.8 follows that the fiducial method is not correct for strictly convex (concave) functions  $\tilde{f}(\lambda)$  for confidence estimation from below (above). In this case we obtain confidence limit with confidence coefficient smaller than  $\square$  used for this method. Moreover, this difference between a real confidence coefficient and given  $\square$  can not be evaluated.

As one can see from the proof of Theorem 8.8,

$$H(S) = \sup_{b \in B'} \bar{h}_b(S), \quad (8.71)$$

is also  $\square$ -UCL for  $f$ , and, moreover, better (lower) in comparison with upper fiducial limit  $\bar{f}(S)$  with the same  $\square$  for all test results  $S$ , as it follows from (8.69). So, confidence limit (8.71) improves the fiducial method although needs more complicated calculations (see also Section 8.6 above).

### 13. Sequential Criteria of Hypotheses Test and Confidence Limits for Reliability Indices

#### 13.1 Introduction

Different objects (like units, subsystems or systems) are tested sequentially in many practical cases. In other words, statistical information about objects of interest is collected in sequential bits, not at once. Moment of test termination is not determined in advance. It is determined during the test, depending on the data obtained thus far.

The first result in this direction was obtained for two simple hypotheses in Wald (1947). The sequential criterion he formulated is known as Wald criterion. This method gives an opportunity to decrease the necessary average test volume in comparison with the case where the test volume is fixed in advance. However the classical Wald analysis does not "work" for some problems arising in reliability (test hypotheses and construction of confidence limits for reliability indices of complex systems). It leads to the necessity of modifications and development of new methods.

##### 13.1.1 Sequential Confidence Limits and Hypotheses Test Criteria

The problem of sequential test of hypotheses for reliability indices are close to the problems of construction of sequential confidence limits. Assume that at each test step  $n$ ,  $n=1,2, \dots$ , we can construct the confidence limit (say, the lower)

$$\underline{R}_n = \underline{R}_n(x^{(n)}) = \underline{R}_n(x_1, \dots, x_n) \quad (9.1)$$

for the reliability index of interest  $R$  by some known method. Here  $x_n$  is the test result at the  $n$ th step,  $x^{(n)}$  is a set of all test results,  $x^{(n)} = (x_1, \dots, x_n)$  obtained at first  $n$  steps,  $n=1, 2, \dots$ . Thus, for each fixed test step  $n$  inequality

$$P(\underline{R}_n \leq R) \geq \alpha \quad \square \square \square \square \square$$

holds. Here  $\alpha$  is the confidence coefficient of the lower limit  $\underline{R}_n$ ,  $n=1, 2, \dots$ .

Assume that the test is continued until some (generally speaking, random) step  $\tau$ , and we observe the set of test results  $x^{(\tau)} = (x_1, \dots, x_\tau)$ , and the test is stopped. The value  $\tau$  is called the moment of test stop. Consider a situation where the stop moment  $\tau$  is not fixed in advance but determined during the test in accordance with some stopping rule. At a first glance, it seems natural to use already known confidence limit (9.1) for the estimation of the reliability index  $R$  in this case. In other words, if the test is terminated at  $\tau = n$  then we take value of  $\underline{R}_n$  as the  $\alpha$ -LCL for  $R$ . Nevertheless, that simple solution is not correct in general. Indeed, the confidence probability corresponding to this procedure is equal to

$$P(\underline{R}_v \leq R) = \sum_{1 \leq v < \infty} P(v=n) P(\underline{R}_n \leq R | v=n)$$

where  $\underline{R}_\square = \underline{R}_\square(x_1, \dots, x_\square)$  is the confidence limit (9.1) calculated at the random moment  $\square$ ,  $P(\underline{R}_n \leq R | \square=n)$  is the conditional probability of the event  $\{\underline{R}_n \leq R\}$  under the condition that  $\square=n$ . Conditional and unconditional probabilities,  $P(\underline{R}_n \leq R | \square=n)$  and  $P(\underline{R}_n \leq R)$  do not coincide (except for the case where the random stop moment  $\square$  does not depend on the process of test). Therefore, analogous inequalities

$$P(\underline{R}_{\square} \leq R) \geq \square. \quad (9.3)$$

for  $\underline{R}_\square$  does not follow from the inequalities (9.2) in general.

It means that the confidence coefficient of the confidence limit  $\underline{R}_\square$  can be smaller than  $\gamma$  on an unknown value. Therefore the use of above procedure can lead to errors (except for the situation where the stop moment is determined in advance, or  $\square$  is a r.v. which does not depend on the test results  $x_1, \dots, x_n, \dots$ ). Let us consider the following simple example for illustrative purposes. Assume that we need to confirm that reliability index  $R$  exceed some specified level  $b$  from the test results  $x_1, \dots, x_n, \dots$ . Standard method of solving such a problem consists in the proof that the  $\gamma$ -confidence limit  $\underline{R}$  constructed from the test satisfies the inequality  $\underline{R} \geq b$ . This can be proved, for instance, by the following way. At the current  $k$ th step we construct by some known method the  $\gamma$ -LCL,  $\underline{R}_k = \underline{R}_k(x_k)$ . Here  $x_n$  is the test result at the  $n$ th step. We continue the test until at some  $n$ th step inequality

$$\underline{R}_n \geq b \quad (9.4)$$

holds. The moment of the test stop is determined as

$$\square = \min \{ n: \underline{R}_n \geq b \}.$$

After this step the test is terminated and due to the inequality (9.4) we declare that the reliability requirement  $R \geq b$  is confirmed with the confidence level  $\gamma$ . However, it is clear that if, for instance, the test results  $x_1, \dots, x_n, \dots$  at different steps are independent then using such a way may lead to confusion: we can always confirm an any reliability level although the real level of reliability is arbitrary low! This follows from the fact that inequality (9.4) will be satisfied sooner or later for some random step  $\square$  due to randomness and independence of values  $\underline{R}_n$ ,  $n=1, 2, \dots$ . Thus, a direct transition from usual confidence limits (9.4) found for the fixed test stop to the confidence limit  $\underline{R}_\square$  in the sequential test can lead to erroneous results. Therefore the confidence limits for sequential tests must be constructed by special procedure such that confidence coefficient  $\square$  will be preserved for any stop rule.

**Remark 9.1.1.** The question whether the confidence limit (9.1) found for random moment  $\square$  satisfies inequality (9.3), that is qualified as a  $\square$ -confidence limit, is a particular case of more general problem. Assume we have a sequence of functions depending on test results

$$g_n = g_n(x_1, \dots, x_n), \quad n=1, 2, \dots \quad (9.5)$$

such that at each step  $n$  inequality

$$E\{g_n\} \geq C, \quad n=1, 2, \dots \quad (9.6)$$

holds. Here  $C$  is a constant which does not depend on  $n$ . Let  $\tau$  be some test stop moment defined as the moment of the first entry into the "stop area"  $D$ , that is

$$\tau = \min \{n: (n, x_n) \in D\}.$$

The question is: when can we say that inequality

$$E\{g_{\tau}\} \geq C \quad (9.7)$$

follows from inequalities (9.6). Here  $g_{\tau} = g_{\tau}(x_1, \dots, x_{\tau})$  is function (9.5) depending on test results and computed at random moment  $\tau$ . This problem is discussed in detail, in particular, in Pavlov (1996).

### 13.1.2 Sequential Independent and Identical Tests

Assume that the test result for some object is a sequence of independent and identically distributed random values (vectors)

$$x_1, x_2, \dots, x_n, \dots \quad (9.8)$$

where  $\mathbf{x}_n = (x_{n_1}, \dots, x_{n_l})$  is vector r.v. observed at the  $n$ th step of the test. The distribution of  $x_n$  is defined by density function  $f(x, \theta)$  depending on some unknown vector parameter  $\theta = (\theta_1, \dots, \theta_m)$ . We need to construct the confidence limits for some function of parameter  $\theta$  on the basis of data (9.8):

$$R = R(\theta) = R(\theta_1, \dots, \theta_m) \quad (9.9)$$

Function  $R$  is a reliability index as above. Sometimes we need to test hypotheses about  $R(\theta)$  on the basis (9.8):

$$H_0: R(\theta) \leq R_0, \quad H_1: R(\theta) \geq R_1 \quad (9.10)$$

where  $R_0 < R_1$  are some specified critical levels of index  $R$ .



Example 9.1.1. Let a unit have TTF  $\square$  with d.f.  $F(x, \square)$  and density  $f(x, \square)$  which depend on parameter  $\square$  (in general case vector). During the test the unit is immediately replaced by a new identical one after a failure. As a result we have the following sequence of i.i.d. r.v.'s

$$x_1, x_2, \dots, x_n, \dots \quad (9.11)$$

where  $x_n$  is the TTF (random value of  $\square$ ) at the  $n$ th test step,  $n=1, 2, \dots$ . We need to construct the confidential limits on the basis of (9.11) and check hypotheses (9.10) for unit reliability index  $R=R(\square)$ . The reliability index can be one of the following:

(1) The PFFO for time interval  $t_0$ , that is

$$R(\square)=1-F(t_0, \square) \quad (9.12)$$

(2) The unit life time guarantee,  $t_q(\square)$  with the level  $q$ , which is defined by the equation

$$1-F(t, \square)=q$$

(3) The unit MTTF

$$R(\square)=\int_0^{\infty} x f(x, \theta) dx. \quad \blacksquare$$

Example 9.1.2. ("Exponential" tests)

In condition of the previous example, let us consider a particular case where a unit has exponential d.f. of TTF with density  $f(x, \square)=\square \exp(-\square x)$  with unknown parameter  $\square=\theta$ . In this case any reliability index can be expressed via parameter  $\square$ . It means that the problem is reduced to confidence estimation and test of hypotheses for parameter  $\square$  on the basis (9.11) where i.i.d. r.v.'s have exponential d.f.

Notice that this case is the most investigated in literature.  $\blacksquare$

Example 9.1.3. ("Binomial" test)

In condition of Example 9.1.1 consider a particular case where the index of interest is (9.12). Let us denote the unit PFFO by  $q$ ,  $q=1-R=F(t_0, \square)$ . Assume that each test step a unit is tested for time  $t_0$ .

Introduce an indicator function,  $\square_n$ , which equals 1 if a failure has been observed at the  $n$ th step, and 0 otherwise:  $\delta_n=I(x_n \leq t_0)$ . The moment of failure occurrence,  $x_n$ , is not registered.

In this case we are observing a sequence of i.i.d. r.v.'s

$$\square_1, \square_2, \dots, \square_n, \dots \quad (9.13)$$

where  $\square_n$  takes value 0 or 1 with probabilities  $P(\square_n=0)=1-q$  and  $P(\square_n=1)=q$ ,  $n=1, 2, \dots$ . Thus, in this case the problem is reduced to confidence estimation and test of hypotheses for parameter  $q$  on the basis of (9.13). This "binomial" scheme represents the well-known classical scheme of

sequential independent trials (Bernoulli trials). This is also one of the best investigated cases in the probability theory. ■

Example 9.1.4. (Continuous time)

Let us again refer to Example 9.1.1. Let

$$t_n = x_1 + \dots + x_n \quad (9.14)$$

denote the moment of the  $n$ th failure and  $d(u)$  the number of failures up to moment  $u$ . (In other words,  $n$  satisfies the condition:  $t_n \leq u$ .) We consider a situation where the failure moments (9.14) are continuous in time. Actually, we observe a standard renewal process  $d(u)$ ,  $u \geq 0$ . At each moment  $u$  we have the complete information about previous failure moments

$$t_1 < t_2 < \dots < t_{d(u)} < u$$

(see Figure 9.1). Notice that if d.f. of TTF is exponential, the renewal process represents a Poisson process. In that case, due to the memoryless property, this model can be reduced to the Bernoulli scheme considered above by dividing the time axis into intervals of the length  $h$ ,  $h \rightarrow 0$ .

Figure 9.1

In general case the process behavior after moment  $u$  depends on the prehistory, namely, on the residual time  $z_u = u - t_{d(u)}$ . It means that this scheme cannot be reduced to the Bernoulli scheme and must be studied in the frame of more general models. (This case will be discussed in later sections.) ■

Example 9.1.5. (System with loaded redundant units. Exponential distribution)

Consider a system consisting of  $m$  different types of units. There are  $N_i$  units of type  $i$  within the system,  $1 \leq i \leq m$ . Distribution function of TTF for each unit is exponential:

$$F_i(x, \lambda_i) = 1 - \exp(-\lambda_i x) \quad (9.15)$$

with unknown parameter  $\lambda_i$ ,  $1 \leq i \leq m$ . All units are in operational state, i.e. under load. All unit failures are independent. Any failed unit is immediately replaced by a new identical one. In other words, we observe  $m$  independent tests of different units, and for each type  $i$  there are  $N_i$  independent renewal processes, that is units of type  $i$  are tested by plan  $[N_i R]$  (see Section 2.1). Due to exponentiality of d.f. (9.15) each unit is characterized by the standard Poisson process of failures with the FR  $\lambda_i$ ,  $1 \leq i \leq m$ .

We need to construct confidence limits or/and to test some hypotheses for system reliability index  $R = R(\lambda) = R(\lambda_1, \dots, \lambda_m)$  on the basis of this test. For instance, for a series-parallel system consisting of  $m$  parallel subsystems in series, PFFO for time  $t_0$  has the form

$$R = \prod_{1 \leq i \leq m} [1 - (1 - e^{-\lambda_i t_0})^{N_i}]$$

where  $N_i$  is the number of parallel units in subsystem  $i$ ,  $1 \leq i \leq m$ .

Although we have the model with continuous time, this model can be again reduced to the discrete model (9.8) by dividing the time axis into intervals of length  $h$ . Random value  $x_n$  in (9.8) will be presented by the set of all test results on intervals  $[(n-1)h, nh]$ , or in other words, by the random vector

$$x_n = (d_{1n}, d_{2n}, \dots, d_{mn}) \quad (9.16)$$

where  $d_{in}$  is the number of failures of units of type  $i$  on the interval of length  $h$ ,  $1 \leq i \leq m$ ,  $n=1, 2, \dots$ . Random value  $d_{in}$  has the Poisson d.f. with parameter  $N_i h$ ,  $1 \leq i \leq m$ . The vector of parameters,  $\theta$  is represented by  $\theta = (\theta_1, \dots, \theta_m)$ . Due to the memoryless property for Poisson processes in (9.18), random vectors  $x_1, \dots, x_n, \dots$  are independent and identically distributed. Thus, we have reduced the initial continuous model to the discrete version (9.16). Results for the discrete model can be transferred back to the continuous time case with  $h \rightarrow 0$ . ■

Example 9.1.6. (System with loaded redundancy. General case.)

Consider a test scheme analogous to the previous one but without the assumption of exponentiality of the d.f. of TTF (9.15). In this case we observe  $m$  independent processes of unit tests, each unit is tested by plan  $[N_i, R]$ . In other words, we have  $N_1 + N_2 + \dots + N_m$  independent renewal processes. Each group of  $N_i$  processes is characterized by the same d.f.  $F_i(x, \theta)$ .

We need to construct confidence limits or/and to test some hypotheses for system reliability index  $R = R(\theta) = R(\theta_1, \dots, \theta_m)$ .

It is obvious that this model is a multi-dimensional analogue of Example 9.1.5. This model can not be reduced to (9.8) and must be studied with the help of more general methods. ■

Example 9.1.7. (Binomial test)

Consider the PFFO in conditions of the previous example. Let the test be presented by cycles of length  $t_0$ , in other words, units of type  $i$  are tested by plan  $[N_i, U, t_0]$ ,  $1 \leq i \leq m$ . Result at the  $n$ th test step is presented by the random vector

$$x_n = (d_{1n}, d_{2n}, \dots, d_{mn}) \quad (9.17)$$

where  $d_{in}$  is the number of failed units of type  $i$  within  $n$ th cycle,  $1 \leq i \leq m$ ,  $n=1, 2, \dots$ . Random value  $d_{in}$  has the binomial distribution with parameters  $(N_i, q_i)$  where  $q_i = F_i(t_0, \theta_i)$  is the unit PFFO during time  $t_0$ . All failed units are replaced at the end of each cycle  $t_0$ . Thus we have a sequence of i.i.d. random vectors (9.17) which is used for constructing confidence limits or/and testing some hypotheses for reliability index  $R = R(\mathbf{q}) = R(q_1, \dots, q_m)$  where  $\mathbf{q} = (q_1, \dots, q_m)$  is vector of binomial parameters of units and  $R = R(\mathbf{q})$  is the system PFFO for time  $t_0$ . For instance, for a system consisting of  $m$  parallel subsystem in series, the reliability index has the form

$$R = \prod_{1 \leq i \leq m} (1 - q_i^{N_i})$$

where  $N_i, 1 \leq i \leq m$ , is the number of parallel units in subsystem  $i$ .

This model is a multi-dimensional analogue of the model in Example 9.1.3. ■

Example 9.1.8. (System with spare units)

Consider a system consisting of  $m$  subsystems each of which comprise  $r_i$  identical units, one main and  $r_i-1$  spares (unloaded redundancy). Distribution function of TTF of the  $i$ th unit is denoted by  $F_i(x, \theta_i)$  where  $\theta_i$  is unknown parameter (in general case vector). All failures are independent.

Let  $d_i$  denote the number of units of type  $i$  having failed up to the moment  $t=0$ . The process  $d_i(t)$  is, obviously, the standard renewal process with d.f.  $F_i(x, \theta_i)$ . Thus we observe  $m$ -dimensional stochastic process

$$d(t) = \{d_1(t), \dots, d_m(t)\} \quad (9.18)$$

Each cycle of testing is terminated at the moment of the system failure, i.e. at the moment  $\tau$ :

$$\tau = \min \{t: d_i(t) \geq r_i \text{ at least for one } i, 1 \leq i \leq m\}.$$

Moment  $\tau$  is the moment of the first entering the "stop area"  $D$  by the process (9.18). Area  $D$  is defined in the space of  $m$ -dimensional vectors  $\mathbf{d} = (d_1, \dots, d_m)$  with integer coordinates:

$$D = \bigcup_{1 \leq i \leq m} \{\mathbf{d}: d_i \geq r_i\} \quad (9.19)$$

(for  $m=2$  see also Figure 9.2). Thus we observe a sequence of i.i.d. test cycles of the form:

$$x_n = \{d^{(n)}(t), 0 \leq t \leq \tau_n\}, \quad n=1, 2, \dots \quad (9.20)$$

where  $d^{(n)}(t)$  is a realization of process  $d(t)$  at the  $n$ th test cycle, and  $\tau_n$  is the test duration at this cycle:

$$\tau_n = \min \{t: d^{(n)}(t) \in D\}$$

(see Figure 9.2 where two different realization of reaching stop area  $D$  by process  $d(t)$  are depicted).

Using these information, we need to construct confidence limits or/and to test some hypotheses for system reliability index  $R=R(\theta)$  where  $\theta = (\theta_1, \dots, \theta_m)$  is the vector of unit parameters. For instance,

$$R(\theta) = \prod_{1 \leq i \leq m} [1 - F_i^{(r_i)}(t_0, \theta_i)]$$

where  $F_i^{(r_i)}(t, \theta_i)$  is the d.f. of the  $i$ th subsystem TTF which is  $r_i$ -times convolution of d.f.  $F_i(t, \theta_i)$ . Tests of systems of such type were considered in Ushakov and Gordienko (1978), Gordienko(1979) and some others. ■

### 13.2 Sequential Wald Criterion

#### 13.2.1 Introduction

One of the frequently encountered problem in reliability practice is: to test hypotheses:

$$H_0: R(\square) \leq R_0, \quad H_1: R(\square) \geq R_1 \quad (9.21)$$

on the basis of test results of type (9.8). Here  $\square$  is unknown parameter (in general, a vector),  $R(\square)$  is a reliability index,  $R_0$  and  $R_1$  are some critical levels.  $R_0$  and  $R_1$  play role of rejection and acceptance levels of the tested object by reliability index  $R$ . (Of course, if  $R$  is index characterizing failure, for instance, FR, then  $H_0$  corresponds to acceptance and  $H_1$  to rejection.) The probabilities of errors of type I ( $\square$ ) and of type II ( $\square$ ) corresponds to the manufacturer risk and consumer risk. Criteria of test are defined by the stop rule and decision making rule at the moment of the test stopping.

Hypotheses of type (9.18) are composite. Let us first consider more simple task, namely, test of simple, hypotheses of type:

$$h_0: \square = \square_0, \quad h_1: \square = \square_1$$

where  $\square_0$  and  $\square_1$  are two points of the parametric space. The moment of test stop is denoted by  $\square$ . It is clear that the moment of stop determines the test volume. If this moment is defined in advance, that is, it is a constant  $\square = n$ , the optimal decision rule is determined by well-known Neyman-Pearson criterion (see Section 1.5 and Lehman, 1959). In correspondence with this criterion hypothesis  $h_1$  is accepted if the test results  $x_1, \dots, x_n$  satisfy the inequality

$$\varphi_n(x_1, \dots, x_n) \geq C \quad (9.22)$$

where the function of test result (statistic)  $\varphi_n(x_1, \dots, x_n) \geq C$  is determined as the likelihood ratio

$$\varphi_n = \frac{f(x_1, \theta_1) \cdot f(x_2, \theta_1) \cdot \dots \cdot f(x_n, \theta_1)}{f(x_1, \theta_0) \cdot f(x_2, \theta_0) \cdot \dots \cdot f(x_n, \theta_0)} = \alpha \quad (9.23)$$

and constant  $C$  is found so that the probability of error of type I equals given value of  $\alpha$ , that is, satisfies condition

$$P_0(\varphi_n \geq C) = \alpha \quad (9.24)$$

(Probability  $P_0$  corresponds the value of parameter  $\theta = \theta_0$ .)

For the sake of simplicity we consider continuous case and assume that such a constant  $C$  can be found satisfying equality (9.24) for any  $0 < \alpha < 1$ .

If inequality (9.22) does not hold, that is

$$\varphi_n(x_1, \dots, x_n) < C \quad (9.25)$$

then hypothesis  $H_0$  is accepted.

The Neyman-Pearson criteria (9.22), (9.25) possesses a known optimal property, namely, it has minimum probability of error of type II

$$\alpha = P_1(\varphi_n < C)$$

among all criteria with fixed constant test volume  $n$  and fixed value of the probability of error of type I,  $\alpha$ . (Here  $P_1$  is the probability corresponding the value of parameter  $\theta = \theta_1$ .)

Let  $n^*(\alpha, \beta)$  denote the test volume which is necessary for constructing the Neyman-Pearson criterion with given values of probability of errors of types I and II ( $\alpha$  and  $\beta$ ,  $\alpha + \beta < 1$ ). The value of  $n^*$  is determined as the minimum integer value  $n$ , for which two following inequalities hold

$$P_0(\varphi_n \geq C) \leq \beta$$

$$P_1(\varphi_n < C) \leq \alpha.$$

Notice that due to the optimum property of the Neyman-Pearson criterion, the necessary test volume  $n^*(\alpha, \beta)$  can not be decreased if one takes a criterion with deterministic test volume  $n$ .

Outstanding result obtained by Wald (1947) was that he constructed a sequential criterion significantly decreasing the *average volume* of the necessary test volume in comparison with the value above  $n^*(\alpha, \beta)$  for the same values of probabilities of errors  $\alpha$  and  $\beta$ . The Wald criterion is defined as follows. At current test step  $n$ , taking into account the results already obtained,  $x_1, \dots, x_n$ , one calculates  $\varphi_n(x_1, \dots, x_n)$  considered in (9.23). This value has a meaning of likelihood ratio for hypotheses  $h_0$  and  $h_1$  on the  $n$ th test step. At each  $n$ th step, one checks the following two inequalities

$$B < \varphi_n(x_1, \dots, x_n) < A \quad (9.26)$$

where  $B$  and  $A$  are some given constants such that  $0 < B < 1 < A$ . If both inequalities (9.26) hold then the test continues, that is, the next r.v.  $x_{n+1}$  is subject of observation. In other words, inequality (9.26) determines the "area of test continuation" for the Wald criterion.

Test terminates at the first violation of at least one of inequalities in (9.26). If the left inequality is violated, hypothesis  $h_0$  is accepted. The violation of the right inequality leads to

acceptance of hypothesis  $h_1$ . Thus the moment of test stop  $\nu$  for the Wald criterion is determined from the conditions

$$\varphi_n(x_1, \dots, x_n) \geq (A, B) \text{ for all } 1 \leq n \leq \nu - 1,$$

$$\varphi_\nu(x_1, \dots, x_\nu) \geq (A, B)$$

or in more compact notation

$$\nu = \min \{n: \varphi_n(x_1, \dots, x_n) \geq (A, B)\} \quad (9.27)$$

Vector of test results for any sequential criterion (including the Wald criterion) has the form  $(\nu, x_1, \dots, x_\nu)$ , where  $\nu$  is the stop moment and  $(x_1, \dots, x_\nu)$  is the set of all test results obtained up to this step. Let

$$\varphi_\nu = \varphi_\nu(x_1, \dots, x_\nu) = \frac{f(x_1, \theta_1) \cdot f(x_2, \theta_1) \cdot \dots \cdot f(x_\nu, \theta_1)}{f(x_1, \theta_0) \cdot f(x_2, \theta_0) \cdot \dots \cdot f(x_\nu, \theta_0)}$$

denote the value of function  $\varphi_n$  (likelihood ratio) at the test stop moment  $\nu$ . For the Wald criterion the decision making rule at the stop moment  $\nu$  on the basis of test results  $(\nu, x_1, \dots, x_\nu)$  has the following form

$$\begin{aligned} &\text{if } \varphi_\nu \leq B \text{ hypothesis } h_0 \text{ is accepted} \\ &\text{if } \varphi_\nu \geq A \text{ hypothesis } h_1 \text{ is accepted.} \end{aligned} \quad (9.28)$$

The probabilities of error (risks) of type I and II for any criterion equal  $\alpha = P_0(h_1)$  and  $\beta = P_1(h_0)$  respectively, where the probabilities  $P_j(h_j)$  are taken for  $\nu = \nu_i, i=0,1; j=0,1$ . For the Wald criterion, these values in correspondence with (9.28) are equal to  $\alpha = P_0(\varphi_\nu \geq A)$  and  $\beta = P_1(\varphi_\nu \leq B)$ .

The precise values of risks  $\alpha$  and  $\beta$  for the Wald criterion can be approximately estimated with the help of the following known Wald formulas (Wald, 1947):

$$\begin{aligned} \frac{\beta}{1-\alpha} &= E_0(\varphi_\nu | h_0) \\ \frac{1-\beta}{\alpha} &= E_0(\varphi_\nu | h_1) \end{aligned} \quad (9.29)$$

where  $E_0$  denotes the mathematical expectation for  $\varphi = \varphi_0$  and  $E_0(\varphi_\square | h_j)$  is conditional mathematical expectation of r.v.  $\varphi_\square$  under condition that at the stop moment  $\square$  by the test results  $(\varphi_\square, x_1, \dots, x_n)$  hypothesis  $h_j$  is accepted,  $j=0,1$ . Notice that formulas (9.29) are valid for any criterion with the stop moment ("not depending on the future") such that  $P_i(\varphi_\square < \varphi) = 1, i=1$  or  $2$ .

It is easy to obtain corresponding inequalities and approximate estimates for  $\alpha$  and  $\beta$  of the Wald criterion from (9.29). Indeed from (9.29) for the Wald criterion, it follows by construction that the following conditions are valid:

$$\varphi_\square \leq B \text{ if hypothesis } h_0 \text{ is accepted}$$

$$\varphi_\square \geq A \text{ if hypothesis } h_1 \text{ is accepted.}$$

Thus, corresponding inequalities are valid for mathematical expectations:

$$E_0(\varphi_\square | h_0) \leq B \text{ and } E_0(\varphi_\square | h_1) \geq A.$$

Taking into account (9.29), it follows well known inequalities for precise values of risks  $\alpha$  and  $\beta$  for the Wald criterion:

$$\frac{\beta}{1-\alpha} \leq B, \quad \frac{1-\beta}{\alpha} \geq A \quad (9.30)$$

The area of  $(\alpha, \beta)$  satisfying inequalities (9.30) are shown in Figure 9.3.

Figure 9.3

From these inequalities, also follows:

$$\beta \leq B, \quad \alpha \leq \frac{1}{A} \quad (9.31)$$

They are often used in practice for approximate risk estimation.

Notice that to construct the Wald criterion, test is stopped at such a step  $\varphi = n$  when the process  $\varphi_n$  first time exits interval  $(A, B)$ , that is when r.v.  $\varphi_n$  "jumps" over level  $A$  from below or under level  $B$  from above. Let us neglect the process of that "jump" and assume that at the stop moment one of the following two approximate conditions are fulfilled:

$$\varphi_\square \leq B \text{ if hypothesis } h_0 \text{ is accepted} \quad (9.32)$$

$$\varphi_\square \geq A \text{ if hypothesis } h_1 \text{ is accepted.}$$

Then from (9.29) we obtain known approximate Wald's equalities



$$\frac{\beta}{1-\alpha} \approx B, \quad \frac{1-\beta}{\alpha} \approx A \quad (9.33)$$

These approximate solutions are often used in practice for risk estimation. Let us assume that precise values of risks  $\alpha$  and  $\beta$  are close to approximate values  $\alpha^*$  and  $\beta^*$ , i.e.,

$$\alpha \approx \alpha^* \quad \text{and} \quad \beta \approx \beta^*$$

where approximate risks are found from equations

$$\frac{\beta^*}{1-\alpha^*} = B, \quad \frac{1-\beta^*}{\alpha^*} = A \quad (9.34)$$

from where

$$\alpha^* = \frac{1-B}{A-B}, \quad \beta^* = \frac{B(A-1)}{A-B}$$

(See Figure 9.3 where point  $(\alpha^*, \beta^*)$  is found as cross of lines  $\beta = B - B\alpha$  and  $\alpha = 1 - A\beta$ .) From Figure 9.3 one can see that the precise values of risks  $\alpha$  and  $\beta$  are laying always within the shadowed area, that is,

$$\alpha + \beta \leq \alpha^* + \beta^*.$$

Besides, taking into account (9.34), inequalities (9.31) can be written in the form

$$\beta \leq \frac{\beta^*}{1-\alpha^*}, \quad \alpha \leq \frac{\alpha^*}{1-\beta^*}.$$

The precise values of risks  $\alpha$ ,  $\beta$  can be found with the help of computer programs. (For more details, see p.1 of Appendix to the chapter.)

## 14. nnnn

### 14.1 mmmm

### 14.2 nnnnn

#### 14.2.1 nnnnn

### Average test volume

In accordance with (9.27) the stopping moment  $\nu$  for Wald's criterion can be written as

$$\nu = \min\{n: Z_n \notin (-b, a)\}$$

where  $a = \ln A > 0$ ,  $b = -\ln B > 0$ , and  $Z_n$  is the logarithm of likelihood ratio at the  $n$ th step:

$$Z_n = \ln \varphi_n = \ln \frac{f(\theta_1, x_1) \dots f(\theta_1, x_n)}{f(\theta_0, x_1) \dots f(\theta_0, x_n)}$$

Let us introduce the notation:

$$z = z(x) = \ln \frac{f(\theta_1, x)}{f(\theta_0, x)} \quad (9.35)$$

$$z_n = z(x_n) = \ln \frac{f(\theta_1, x_n)}{f(\theta_0, x_n)}, \quad n = 1, 2, \dots$$

random values  $z_1, \dots, z_n, \dots$  are i.i.d. and  $z_n$  is the  $n$ th observation of r.v.  $z$ .  $Z_n$  is defined as

$$Z_n = z_1 + \dots + z_n. \quad (9.36)$$

Let us find the mathematical expectation  $E_0 \nu$  for the case where hypothesis  $h_0$  is true, that is  $\theta = \theta_0$ . In accordance with (9.36), the equality

$$E_0 Z_\nu = E_0 (z_1 + \dots + z_\nu). \quad (9.37)$$

where  $E_0$  denotes the mathematical expectation under condition  $\theta = \theta_0$ . Applying a well known Wald's equivalence (Wald, 1944) to the right part of (9.37) for the mathematical expectation of the sum of random number of i.i.d r.v.'s, we obtain

$$E_0 (z_1 + \dots + z_\nu) = E_0 \nu E_0 z \quad (9.38)$$

(For a proof of Wald's equivalence, see p.2 of Appendix to the chapter.) For the left side of the equality (9.37,) we have

$$E_0 Z_\nu = P_0(h_0) E_0(Z_\nu | h_0) + P_0(h_1) E_0(Z_\nu | h_1), \quad (9.39)$$

from where

$$E_0 Z_v = (1-\alpha) E_0 (\ln \varphi_v | h_0) + \alpha E_0 (\ln \varphi_v | h_1). \quad (9.40)$$

Neglecting again a "jump" mentioned above and using approximate equalities (9.32) and (9.33), we obtain from the latter equality

$$E_0 Z_v \approx (1-\alpha) \ln \frac{\beta}{1-\alpha} + \alpha \ln \frac{1-\beta}{\alpha} \quad (9.41)$$

After this, the known approximate equality (Wald, 1947) for the average test volume under condition  $\theta = \theta_0$  follows from (9.38) and (9.41) :

$$E_0 v \approx \frac{\omega(\alpha, \beta)}{E_0(-z)} \quad (9.42)$$

where

$$\omega(\alpha, \beta) = (1-\alpha) \ln \frac{1-\alpha}{\beta} + \alpha \ln \frac{\alpha}{1-\beta}.$$

In the same way an analogous approximate equality for the average test volume for hypothesis  $h_1$ , that is for  $\theta = \theta_1$ :

$$E_1 v \approx \frac{\omega(\beta, \alpha)}{E_1(z)} \quad (9.43)$$

Taking into account the definition of r.v.  $z$  in (9.35) , formulae for the average test volume can be also represented in the following form:

$$E_0 v \approx \frac{\omega(\alpha, \beta)}{\rho(\theta_0, \theta_1)} \quad (9.44)$$

$$E_1 v \approx \frac{\omega(\beta, \alpha)}{\rho(\theta_1, \theta_0)}$$

where value

$$\rho(\theta_1, \theta_0) = E_1 z = E_1 \ln \frac{f(\theta_1, x)}{f(\theta_0, x)} \quad (9.45)$$

represents "Kullback-Leibler information distance" between  $\theta_1$  and  $\theta_0$  (or more exactly, between d.f.'s with densities  $f(\theta_1, x)$  and  $f(\theta_0, x)$ ). Notice that in general (Kullback-Leibler, 1951; Kullback, 1959)  $\rho(\theta_1, \theta_0) \neq \rho(\theta_0, \theta_1)$ . Formulae (9.42)-(9.44) for the average test volumes are approximate (without accounting for "jumps"). See p.2 of Appendix to the chapter for the precise calculation of the test volumes with the help of computer programs.

**The lower limit for the average test volume**

From (9.40) one can easily obtain the lower limit of the average test volume for any criterion with fixed risk values  $\alpha$  and  $\beta$ . Indeed, this equality is valid for any criterion, not only for Wald's. Taking into account that function  $\ln u$  is concave in  $u$  and applying Jensen's inequality for the mathematical expectation of concave function we obtain

$$\begin{aligned} E_0(\ln \varphi_v | h_0) &\leq \ln E_0(\varphi_v | h_0) \\ E_0(\ln \varphi_v | h_1) &\leq \ln E_0(\varphi_v | h_1) \end{aligned} \quad (9.46)$$

Taking into account (9.40) and (9.29), we obtain from (9.46) the following inequality

$$E_0 Z_v \leq (1-\alpha) \ln \frac{\beta}{1-\alpha} + \alpha \ln \frac{1-\beta}{\alpha}$$

Then taking into account (9.37) and (9.38), we obtain

$$E_0 v \geq \frac{\omega(\alpha, \beta)}{E_0(-z)} \quad (9.47)$$

In the same way, we can obtain an analogous inequality for the average test volume under condition  $\theta = \theta_1$ :

$$E_1 v \geq \frac{\omega(\beta, \alpha)}{E_1(z)} \quad (9.48)$$

Inequalities (9.47) and (9.48) give us the lower limit for the average test volume under conditions  $\theta = \theta_0$  and  $\theta = \theta_1$  for any ("not depending on the future") criterion for hypotheses  $h_0$  and  $h_1$  for given risks  $\alpha$  and  $\beta$ . These inequalities were obtained by Wald (1947) and Hoeffding (1960).

As one can see from approximate equalities (9.42) and (9.43), the average test volume for Wald's criterion reaches the lower limit in (9.47) and (9.48) at least approximately (without taking into account the "jump" mentioned above). The accurate proof of Wald's criterion optimality was obtained by Wald (1947) and Wald and Wolfowitz (1948) where the following optimum property of this criterion was shown. Let us adopt some Wald's criterion for hypotheses  $h_0$  and  $h_1$  with the stop moment  $v$  and with risks  $\alpha$  and  $\beta$ . Then for any other criterion with the stop moment  $v'$  and with risks  $\alpha'$  and  $\beta'$  such that  $\alpha' \leq \alpha$  and  $\beta' \leq \beta$  the following inequality is valid:

$$E_0 v' \geq E_0 v, \quad E_1 v' \geq E_1 v.$$

The significant property of Wald's criterion is that it simultaneously minimizes two quality indices of the criteria  $E_0 v$  and  $E_1 v$ . among all criteria with the risks not larger than  $\alpha$  and  $\beta$ . The generalization of Wald's results for the case of continuous time was obtained afterwards in Dvoretzky, Kiefer, and Wolfowitz (1953), Epstein and Sobel (1955), and others.

#### 14.2.2 Standard Sequential Wald's Plans of Reliability Indices Control

Suppose we need to test two composite hypotheses about some one-dimensional parameter  $\theta$

$$H_0: \theta \leq \theta_0, \quad H_1: \theta \geq \theta_1, \quad (9.49)$$

on the basis of the test results (9.8). Here  $\theta_0 \leq \theta_1$  are specified levels of parameter  $\theta$ . Levels  $\theta_0$  and  $\theta_1$  mean, respectively, levels of "acceptance" and "rejection" of a tested object by parameter  $\theta$ , if  $\theta$  is the index of type of failure probability,  $q$ , or FR,  $\lambda$ , (see examples below), or if  $\theta$  is the index of type of PFFO.

Standard approach for criterion construction (or in other terms, plans of control of parameter  $\theta$ ) for the case of composite hypotheses of type (9.49) consists in the following. Let us construct Wald's criterion for two simple hypotheses

$$h_0: \theta = \theta_0, \quad h_1: \theta = \theta_1 \quad (9.50)$$

in the same way as it was considered above. Apply this criterion to the test of initial composite hypotheses (9.49). In other words, let us continue the test up to the moment of stop  $\nu$  which was determined above in (9.27). At the stop moment we make the following decision:

$$\begin{aligned} &\text{if } \varphi_\nu \leq B \text{ hypothesis } H_0 \text{ is accepted} \\ &\text{if } \varphi_\nu \geq A \text{ hypothesis } H_1 \text{ is accepted.} \end{aligned} \quad (9.51)$$

Introduce function

$$L(\theta) = P_\theta(H_0) \quad (9.52)$$

equal to the probability of acceptance of hypothesis  $H_0$  under the condition that the true value of parameter is  $\theta$ . Function (9.52) is called the *operative characteristic* of a criterion (plan of control). Since the rule of decision making for Wald's plan has the form of (9.51), the operative characteristic for this plan is determined by

$$L(\theta) = P_\theta(\varphi_\nu \leq B) \quad (9.53)$$

or, in other words, the operative characteristic  $L(\theta)$  for Wald's plan is equal to the probability that function of likelihood ratio  $\varphi_\nu$  will exit the lower limit  $B$  (limit of the "acceptance" area) for a specified values of parameter  $\theta$ . (See Figure 9.4.)

Figure 9.4

In many cases (see examples below) operative characteristic (9.53) is monotone decreasing in parameter  $\theta$ . (See Figure 9.5.)

Figure 9.5

Since hypotheses (9.49) are composite, the probabilities of errors of type I and II are some functions of parameter  $\theta$  which are expressed via the operative characteristic as follows

$$\begin{aligned}\alpha(\theta) &= 1 - L(\theta) \text{ for } \theta \leq \theta_0 \\ \beta(\theta) &= L(\theta) \text{ for } \theta \geq \theta_1.\end{aligned}\tag{9.54}$$

The maximum values of probability of error, that is values

$$\alpha = \max_{\theta \leq \theta_0} \alpha(\theta), \quad \beta = \max_{\theta \geq \theta_1} \beta(\theta)\tag{9.55}$$

are called, respectively, risks of types I and II. If the operative characteristic  $L(\theta)$  is monotone decreasing in  $\theta$ , then

$$\alpha = 1 - L(\theta_0), \quad \beta = L(\theta_1)$$

(See also Figure 9.5.) Notice, that the values  $\alpha$  and  $\beta$  coincide with risks for Wald's criterion for simple hypotheses (9.50). Due to (9.54) and (9.55) the following inequalities are valid

$$\begin{aligned}\alpha(\theta) &\leq \alpha \text{ for } \theta \leq \theta_0 \\ \beta(\theta) &\leq \beta \text{ for } \theta \geq \theta_1.\end{aligned}$$

Thus, Wald's criterion applied for simple hypotheses  $h_0$  and  $h_1$  (corresponding to the boundary point of the acceptance and rejection levels  $\theta_0$  and  $\theta_1$ ) simultaneously gives criterion for composite hypotheses of type (9.49) with the errors of types I and II not larger than values  $\alpha$  and  $\beta$ .

#### Average test volume

Consider calculation of the average test volume for Wald's plan. Denote this plan by

$$N(\theta) = E_0 v\tag{9.56}$$

where  $E_\theta$  is the mathematical expectation under given value of parameter  $\theta$ . For estimating (9.56), let us use formulae of type (9.39)

$$E_\theta Z_v = P_\theta(H_0) E_\theta(Z_v|H_0) + P_\theta(H_1) E_\theta(Z_v|H_1) = \\ L(\theta) E_\theta(\ln \varphi_v|H_0) + [1-L(\theta)] E_\theta(\ln \varphi_v|H_1).$$

Applying approximate formulae (9.32) we obtain

$$E_\theta Z_v \approx [1-L(\theta)]a - L(\theta)b \quad (9.57)$$

where  $a = \ln A$ ,  $b = -\ln B$ . On the other hand, applying Wald's equivalency to the left side of (9.57) (see p.2 of Appendix to the chapter) we have

$$E_\theta Z_v = E_\theta(z_1 + \dots + z_v) = E_\theta v E_\theta z$$

Well known Wald's formula for the average, test volume follows from here

$$N(\theta) \approx \frac{[1-L(\theta)]a - L(\theta)b}{E_\theta z} \quad (9.58)$$

for  $E_\theta z \neq 0$ . Taking into account

$$E_\theta z = E_\theta \ln \frac{f(\theta_1, x)}{f(\theta_0, x)} \\ = E_\theta \ln \frac{f(\theta, x)}{f(\theta_0, x)} - E_\theta \ln \frac{f(\theta, x)}{f(\theta_1, x)} = \rho(\theta, \theta_0) - \rho(\theta, \theta_1)$$

(9.58) can be rewritten in the form

$$N(\theta) \approx \frac{[1-L(\theta)]a - L(\theta)b}{\rho(\theta, \theta_0) - \rho(\theta, \theta_1)} \quad (9.59)$$

Formulae (9.58) and (9.59) are true for such  $\theta$  that  $E_\theta z \neq 0$ , or in other words, for  $\theta \neq \theta'$ , where  $\theta'$  is determined from the condition:  $E_{\theta'} z = 0$ , or from  $\rho(\theta', \theta_0) = \rho(\theta', \theta_1)$ .



Point  $\theta'$  is called "equally distant" (in sense of Kullback-Leibler distance) from points  $\theta_0$  and  $\theta_1$ . In most standard situations, point  $\theta'$  is located between points  $\theta_0$  and  $\theta_1$ . For  $\theta=\theta'$  for the average test volume the approximate formula found by Wald (1947)

$$N(\theta') \approx \frac{ab}{E_{\theta'} z^2} \quad (9.60)$$

is valid.

The precise values of the operative characteristic  $L(\theta)$  and average test volume  $N(\theta)$  for Wald's plan for given fixed boundaries  $A$  and  $B$  and for other sequential plans (with other stop areas) can be calculated with the help of special computer programs. Analogously, with the help of computer one can solve an inverse problem of finding the boundaries for stopping ( $A$  and  $B$ ) in such a way that the precise values of risks coincide with the given values  $\alpha$  and  $\beta$ . (For details see p.1 of Appendix to the chapter.)

A typical graph of dependence of the average test volume,  $N(\theta)$ , of parameter  $\theta$  for sequential Wald's plan is depicted in Figure 9.6

Figure 9.6

As one can see, sequential Wald's plan is the most effective for  $\theta < \theta_0$  and  $\theta > \theta_1$ . In other words, Wald's plan for statistical control allows fast (on average) to accept "good" objects (with parameter  $\theta < \theta_0$ ) and reject "bad" objects (with parameter  $\theta > \theta_1$ ). In areas of values  $\theta < \theta_0$  and  $\theta > \theta_1$  Wald's criterion gives an essential gain in the average test volume in comparison with the best Neyman-Pearson criterion for the case of fixed test volume. However, Wald's criterion becomes less effective in the intermediate area of parameter's values, i.e. if  $\theta_0 < \theta < \theta_1$ . This area is also called "area of uncertainty" or "area of indifference".

Example 9.2.1. (Binomial test. Control of failure probability)

Consider the binomial test considered above in Example 9.1.1. A sequence of i.i.d. r.v.'s is observed

$$\delta_1, \dots, \delta_n, \dots \quad (9.61)$$

where  $\delta_n$  is indicator of unit failure at the  $n$ th step. As was mentioned above this value takes 0 or 1 with probabilities  $P(\delta_n=0)=1-q$  and  $P(\delta_n=1)=q$  where  $q$  is the failure probability. On the basis of test results (9.61) we need to test hypotheses

$$H_0: q \leq q_0, \quad H_1: q \geq q_1$$

where  $q_0 \leq q_1$  is given critical levels of index  $q$ . These values mean "acceptance" and "rejection" levels of this index.

In this case parameter is  $\theta=q$ , observed r.v. at the  $n$ th step is  $x_n=\delta_n$ , and function  $f(x, \theta)=f(\delta, q)=q^\delta(1-q)^{1-\delta}$ ,  $\delta=0, 1$ . From here we obtain that the function of likelihood ratio (9.23) at the  $n$ th test step has the form

$$\varphi_n = \frac{f(\delta_1, q_1) \cdot f(\delta_2, q_1) \cdot \dots \cdot f(\delta_n, q_1)}{f(\delta_1, q_0) \cdot f(\delta_2, q_0) \cdot \dots \cdot f(\delta_n, q_0)} = \left(\frac{q_1}{q_0}\right)^{d_n} \left(\frac{1-q_1}{1-q_0}\right)^{n-d_n}$$

where  $d_n=\delta_1+ \dots +\delta_n$  is the total number of failures during  $n$  steps. The area of test continuation for Wald's plan (9.26) is given by inequalities

$$B < \left(\frac{q_1}{q_0}\right)^{d_n} \left(\frac{1-q_1}{1-q_0}\right)^{n-d_n} < A$$

or after simple transformations

$$C_1 n - C_2 b < d_n < C_1 n + C_2 a, \quad (9.62)$$

where  $a = \ln A > 0$ ,  $b = -\ln B > 0$ ,

$$C_2 = \frac{1}{\ln \left[ \frac{q_1(1-q_0)}{q_0(1-q_1)} \right]},$$

$$C_1 = C_2 \cdot \ln \left( \frac{1-q_0}{1-q_1} \right).$$

The test continues if both inequalities (9.62) are true and stops at such a step  $v=n$  at which at least one of these inequalities is violated. The violation of the left inequality means that hypothesis  $H_0: q \leq q_0$  ("acceptance"), and the violation of the right inequality the decision is to accept hypothesis  $H_1: q \geq q_1$  ("rejection"). Thus, the boundaries of the stopping area (see Figure 9.7) have the form of lines on the plane  $(n, d_n)$ . The equations of these lines are

$d_n = C_1 n - C_2 b$  is the boundary of "acceptance" area

$d_n = C_1 n + C_2 b$  is the boundary of "rejection" area.

Figure 9.7

In this test scheme r.v. (9.35) has the form

$$z = \ln \frac{f(\delta, q_1)}{f(\delta, q_0)} = \delta \ln \left( \frac{q_1}{q_0} \right) + (1 - \delta) \ln \left( \frac{1 - q_1}{1 - q_0} \right)$$

Taking into account that  $E_0 \delta = q_0$ ,  $E_1 \delta = q_1$ , we obtain that (9.42) - (9.44) for the average, test volume for  $q = q_0$  and  $q_1$  has the form

$$E_0 v \approx \frac{\omega(\alpha, \beta)}{\rho(q_0, q_1)}$$

$$E_1 v \approx \frac{\omega(\beta, \alpha)}{\rho(q_1, q_0)}$$
(9.63)

where

$$\rho(q_0, q_1) = q_0 \ln \left( \frac{q_0}{q_1} \right) + (1 - q_0) \ln \left( \frac{1 - q_0}{1 - q_1} \right).$$

Example 9.2.2 ("Exponential" test. Control of FR.)

Suppose we observe a sequence of i.i.d. r.v.'s

$$x_1, \dots, x_n, \dots \quad (9.64)$$

each of which has the exp. d.f. with the density  $f(x, \lambda) = \lambda \exp(-\lambda x)$ . (See also Examples 9.1.1 and 9.1.2.) We need to test the following hypotheses on the basis of test results (9.64)

$$H_0: \lambda \leq \lambda_0, \quad H_1: \lambda \geq \lambda_1 \quad (9.65)$$

where  $\lambda_0 < \lambda_1$  are given critical levels of "acceptance" and "rejection".

In this case the likelihood ratio function (9.23) at the  $n$ th test step has the form

$$\varphi_n = \frac{f(x_1, \lambda_1) \cdot f(x_2, \lambda_1) \cdot \dots \cdot f(x_n, \lambda_1)}{f(x_1, \lambda_0) \cdot f(x_2, \lambda_0) \cdot \dots \cdot f(x_n, \lambda_0)} = \left( \frac{\lambda_1}{\lambda_0} \right)^n e^{-(\lambda_1 - \lambda_0) S_n}$$

where  $S_n = x_1 + \dots + x_n$  is the total test time during  $n$  steps. The area of test continuation for Wald's criterion in this case is given by inequalities

$$B < \left( \frac{\lambda_1}{\lambda_0} \right)^n e^{-(\lambda_1 - \lambda_0) S_n} < A$$

or inequalities

$$C_1 n - C_2 a < S_n < C_1 n + C_2 b \quad (9.66)$$

where  $a = \ln A$ ,  $b = -\ln B$ ,

$$C_2 = \frac{1}{\lambda_1 - \lambda_0}, \quad C_1 = C_2 \cdot \ln \left( \frac{\lambda_1}{\lambda_0} \right).$$

The test continues if both inequalities are true (9.66) and test stops when at least one of these inequalities is violated the first time. If the right inequality in (9.66) does not hold, hypothesis  $H_0$  is accepted. If the right inequality in (9.66) is violated, hypothesis  $H_1$  is accepted ("rejection"). The boundaries of the stopping area (see Figure 9.8) in this case have the form of lines on the plane  $(n, S_n)$ :

$S_n = C_1 n + C_2 b$  is the boundary of "area of acceptance"

$S_n = C_1 n - C_2 a$  is the boundary of "area of rejection"

Figure 9.8

Let us also estimate the average test volume in this case. R.v. (9.35) in this case has the form

$$z = \ln \frac{f(x, \lambda_1)}{f(x, \lambda_0)} = \ln \left( \frac{\lambda_1}{\lambda_0} \right) - x(\lambda_1 - \lambda_0),$$

taking into account that  $E_0 x = 1/\lambda_0$  and  $E_1 x = 1/\lambda_1$ , we obtain that formulae (9.42)-(9.44) for  $\lambda = \lambda_0$  and  $\lambda = \lambda_1$  in this case has the form:

$$E_0 v \approx \frac{\omega(\alpha, \beta)}{\rho(\lambda_0, \lambda_1)} \tag{9.67}$$

$$E_1 v \approx \frac{\omega(\beta, \alpha)}{\rho(\lambda_1, \lambda_0)}$$

where

$$\rho(\lambda_0, \lambda_1) = \frac{\lambda_0 - \lambda_1}{\lambda_0} - \ln \left( \frac{\lambda_1}{\lambda_0} \right).$$

Example 9.2.3. (Poisson process. Control FR.)

Assume that we observe a Poisson process of failures occurrence  $d_t$ ,  $t > 0$ , with unknown parameter  $\lambda$  where  $d_t$  is the number of failures observed up to the moment  $t$ . We need to accept one of two hypotheses on the basis of the results of observation

$$H_0: \lambda \leq \lambda_0, \quad H_1: \lambda \geq \lambda_1$$

where  $\lambda_0 < \lambda_1$  is the given critical levels of "acceptance" and "rejection". The likelihood ratio  $\varphi_t$  at the moment  $t$  (an analogue of value  $\varphi_n$  in the discrete scheme considered above  $n$ ) has the form

$$\varphi_t = \left( \frac{\lambda_1}{\lambda_0} \right)^{d_t} e^{-(\lambda_1 - \lambda_0)t}$$

Thus the area of the test continuation for Wald's criterion at the moment  $t$  is given by inequalities

$$B < \left( \frac{\lambda_1}{\lambda_0} \right)^{d_t} e^{-(\lambda_1 - \lambda_0)t} < A$$

or, after simple transformations,

$$C_1 t - C_2 b < d_t < C_1 t + C_2 a \quad (9.68)$$

where  $a = \ln A$ ,  $b = -\ln B$ ,

$$C_2 = \frac{1}{\ln \left( \frac{\lambda_1}{\lambda_0} \right)}, \quad C_1 = C_2 (\lambda_1 - \lambda_0) \quad (9.69)$$

The test continues until both inequalities (9.68) are true, and test stops at moment  $\tau = t$  when at least one of these inequalities is violated for the first time. If the right inequality in (9.69) does not hold, hypothesis  $H_0$  is accepted. If the right inequality in (9.66) does not hold, hypothesis  $H_1$  is accepted ("rejection"). The boundaries of the stopping area (see Figure 9.9) in this case have the form of lines on the plane  $(t, d_t)$ :

$d_t = C_1 t - C_2 b$  is the boundary of "area of acceptance"

$d_t = C_1 t + C_2 a$  is the boundary of "area of rejection".

Figure 9.9

The moment of the test stop,  $\tau$ , is the moment of exit of the process  $d_t$  from one of the boundaries mentioned above. From (9.42)–(9.44) in the scheme with discrete time, one can easily

obtain analogous formulae for the average test time if  $\lambda=\lambda_0$  and  $\lambda=\lambda_1$  in this scheme with the continuous time:

$$E_0 \tau \approx \frac{\omega(\alpha, \beta)}{\rho(\lambda_0, \lambda_1)}$$

$$E_1 \tau \approx \frac{\omega(\beta, \alpha)}{\rho(\lambda_1, \lambda_0)}$$

where

$$\rho(\lambda_0, \lambda_1) = (\lambda_0 - \lambda_1) - \lambda_0 \ln \left( \frac{\lambda_1}{\lambda_0} \right).$$

Let us consider a numerical example that illustrates a gain in the average test volume obtained by sequential Wald's criterion in comparison with the best Clopper- Pearson criterion with fixed test volume.

Example 9.2.4 (Gain in average test volume)

In the scheme of Example 9.2.2 we need to test hypothesis that FR of type (9.65) where critical levels : "acceptance" and "rejection" are equal to  $\lambda_0=0.1$  and  $\lambda_1=0.2$ . Given values of risk of type I and II equal  $\alpha=0.1$  and  $\beta=0.1$ .

If the test stopping moment is determined in advance and is a constant  $v=n$ , then the best Neyman-Pearson criterion has the following form:

if  $S_n < C$ , hypothesis  $H_1$  is accepted,

if  $S_n \geq C$ , hypothesis  $H_0$  is accepted,

where  $S_n = x_1 + \dots + x_n$  is the total time during  $n$  steps. Risks of type I and II for this criterion equal  $P_0(S_n < C)$  and  $P_1(S_n \geq C)$  where  $P_j$  denotes the probability for the parameter value equal to  $\lambda = \lambda_j$ ,  $j=0$  or  $1$ . Thus, the test volume  $n^* = n^*(\alpha, \beta)$  needed for delivering given risk levels  $\alpha$  and  $\beta$  is determined as a minimum integer number  $n$  which satisfies to the following two inequalities

$$P_0(S_n < C) \leq \alpha \text{ and } P_1(S_n \geq C) \leq \beta.$$

These inequalities can be written in the form

$$P_0(2\lambda_0 S_n < 2\lambda_0 C) \leq \alpha$$

$$P_1(2\lambda_1 S_n \geq 2\lambda_1 C) \leq \beta$$

or, taking into account that r.v.  $2\lambda S_n$  has the standard  $\chi^2$ -distribution with  $2n$  degrees of freedom, we have

$$\chi^2(2\lambda_0 C, 2n) \leq \alpha$$

$$1 - \chi^2(2\lambda_1 C, 2n) \leq \beta.$$

where  $\chi^2(\cdot, 2n)$  is function of the  $\chi^2$ -distribution with  $2n$  degrees of freedom. After simple transformations it gives the needed test volume  $n^*$ , equal to the minimum integer number  $n$  which satisfies inequality

$$\chi^2_{1-\beta}(2n) \leq (\lambda_1/\lambda_0) \chi^2_{\alpha}(2n)$$

where  $\chi^2_{\gamma}(2n)$  is the quantile of level  $\gamma$  of the  $\chi^2$ -distribution with  $2n$  degrees of freedom. Using Table E.16 ( $\chi^2$ -distribution), we find

$$n^* = \min \{n: \chi^2_{0.9}(2n) \leq 2\chi^2_{0.1}(2n)\} = 15.$$

Applying (9.67), we find the average test volume for  $\lambda = \lambda_0$  and  $\lambda = \lambda_1$  for the sequential Wald's criterion (for the same values of  $\lambda_0$ ,  $\lambda_1$ ,  $\alpha$ , and  $\beta$ ):

$$E_0 v \approx \frac{(1-\alpha) \ln\left(\frac{1-\alpha}{\beta}\right) + \alpha \ln\left(\frac{\alpha}{1-\beta}\right)}{\left(\frac{\lambda_1}{\lambda_0}\right) - 1 - \ln\left(\frac{\lambda_1}{\lambda_0}\right)} = \frac{0.9 \cdot \ln 9 - 0.1 \cdot \ln 9}{1 - \ln 2} = 5.7,$$



$$E_1 v \approx \frac{(1-\beta) \ln\left(\frac{1-\beta}{\alpha}\right) + \beta \ln\left(\frac{\beta}{1-\alpha}\right)}{\left(\frac{\lambda_0}{\lambda_1}\right) - 1 - \ln\left(\frac{\lambda_0}{\lambda_1}\right)} = \frac{0.9 \cdot \ln 9 - 0.1 \cdot \ln 9}{\ln 2 - 0.5} = 9.2.$$

Thus, for  $\lambda=\lambda_0$  and  $\lambda=\lambda_1$  the average gain in test volume obtained with Wald's criterion is equal, respectively, to:

$$\frac{n^*}{E_0 v} \approx \frac{15}{5.7} \approx 2.63,$$

$$\frac{n^*}{E_1 v} \approx \frac{15}{9.2} \approx 1.63.$$

**15.**

**15.1**

**15.2**

15.2.1

15.2.2

### 15.2.3 Truncated Sequential Wald's Plans

The area of test continuation for Wald's plan is infinite. Therefore in practice one often uses truncated Wald's plans with finite area of test continuation. Consider, for instance the scheme of Poisson process observation considered in Example 9.2.3. In this case a typical area of test continuation for a truncated Wald's plan is depicted in Figure 9.10.

Figure 9.10

The boundary of the "acceptance area" (if the process  $d_t$  reaches this area then hypothesis  $H_0$  is accepted) is given by conditions:

$$d_t = C_1 t - C_2 b \text{ for } t \leq T$$

$$d_t \leq r \text{ for } t = T$$

The boundary of the "rejection area" (if the process  $d_t$  reaches this area then hypothesis  $H_1$  is accepted) is given by conditions:

$$d_t = C_1 t + C_2 a \text{ for } d_t \leq r$$

$$t \leq T \text{ for } d_t = r$$

where  $(T, r)$  is the truncating point (see Figure 9.10) and coefficients  $C_1$  and  $C_2$  are determined by (9.69).

Truncated Wald's plans lose, generally speaking, their property of minimum of average test volume. If the truncation is not "too severe", that is, if the coordinates of the truncating point  $(T, r)$  are large enough, then we believe that that property is preserved approximately. Analogous arguments are applied to the risks  $\alpha$  and  $\beta$ .

The truncated Wald's plan is specified by four parameters  $A, B, T$  and  $r$ . Precise values of risks  $\alpha$  and  $\beta$ , operative characteristic  $L(\theta)$ , and average test volume  $N(\theta)$  for the truncated Wald's plan can be calculated with the help of computer. (For details, see p.1 of Appendix to this chapter.)

### 15.3 Sequential Test of Composite Hypotheses for Multi-Dimensional Parameters

The sequential Wald's criterion, discussed above, has minimum average test volume  $E_0 \alpha$  and  $E_1 \beta$  for values of parameter  $\theta = \theta_0$  and  $\theta = \theta_1$  (among all criteria with given risks  $\alpha$  and  $\beta$ ). On the other hand, the Wald's criterion's deficiency is its low efficiency in the "area of indifference", that is where values of parameter are  $\theta_0 < \theta < \theta_1$ . This led to the appearance of a set of works by Kiefer and Weiss (1957), Chernoff (1959), Albert (1961), Weiss (1962), Schwarz (1962), Kiefer and Sacks (1963), Ayvazyan (1965), Shiryaev (1965, 1976), Lai (1973), Robbins and Siegmund (1973, 1974), Lorden (1976), Liptser and Shiryaev (1981), Huffman (1983), Pavlov (1985, 1987a, 1990), Draglin and Novikov (1987, 1996), and others in which sequential criteria

with other properties were constructed (for instance, minimization of maximum average in  $\square$  test volume,  $E_{\square}\square$ ).

One important direction of generalization of sequential Wald's criterion for applications (including reliability) is in test of composite hypotheses for multi-dimensional parameter, for instance, of the following type:

$$H_0: R(\square) \leq R_0, H_1: R(\square) \geq R_1 \quad (9.70)$$

where  $R(\square) = R(\square_1, \dots, \square_m)$  is a function (reliability index) depending on multi-dimensional parameter  $\square = (\square_1, \dots, \square_m)$ , and  $R_0 < R_1$  are specified levels of index  $R(\square)$ . Such problems are typical for complex system analysis (see Section 9.1 above and Examples 9.1.5-9.1.8). At the same time such a problem arises for a single unit if its d.f. of TTF,  $F(x, \square)$  depends on vector parameter  $\square$  (see Examples 9.1.1 and 9.1.4). Construction of approximately optimal sequential criteria for composite hypotheses of type (9.70) and close to this problems of construction of approximate optimal sequential confidence limits for  $R(\square)$  were studied in general form by Pavlov (1983a, 1985, 1987a, 1990, 1993), in application to reliability problems by Pavlov (1982, 1983c, 1984a and b, 1986, 1987b), and, in particular, for renewal systems by Pavlov (1988), Pavlov and Ushakov (1989).

Another direction of generalization of Wald's criterion is the study more general test plans, differing from (9.8), including dependent tests. This direction is extremely important for reliability problems because the test scheme of identical and independent objects applies to very restricted class of real tasks (see Examples 9.1.1-9.1.8 above). Problems of sequential test of composite hypotheses of type (9.70) and construction of confidence limits for reliability indices for dependent tests (for Markov type of dependence) were considered by Pavlov (1982, 1983b, 1985).

At first, let us consider sequential test of composite hypotheses of type (9.70) for the simple case of independent identical tests.

### 15.3.1 Sequential Rules of Composite Hypotheses "Exclusion"

At first, consider sequential test of composite hypotheses for multi-dimensional parameter  $\square = (\square_1, \dots, \square_m)$  for a simple scheme of independent and identical sequential tests (9.8). Let there be a composite hypothesis of the type

$$H_0: \square \in D \quad (9.71)$$

where  $D$  is an area of in space parameter  $\square$ . Let us determine for any hypothesis of type (9.71) some random moment (step)  $\square$  which is called the moment (or the rule) of exclusion of this hypothesis. In other words, at moment  $\square$  one makes a decision that hypothesis  $H$  is not true. This moment  $\square$  is determined during the test process

$$x_1, x_2, \dots, x_n, \dots \quad (9.72)$$

in dependence on the current test results. It means that  $\square$  is a random Markovian moment, or, in other terminology, a random moment "independent of the future". (For more details see p.1 of Appendix to Chapter 3.)

For the moment  $\square$  of excluding hypothesis (9.71) we require that the following condition holds

$$P_{\square}(\square < \square) \leq \alpha \text{ for any } \square \in D, \quad (9.73)$$

where  $\alpha$  is given small constant (probability of error). Event  $\{\square < \square\}$  means that hypothesis  $H$  is excluded at some step  $\square$ . So, condition (9.73) has the following meaning: if hypothesis  $H$  is true (the true value of parameter  $\square$  belongs to area  $D$ ), then the probability to exclude this hypothesis by test results (9.72) in some future does not exceed constant  $\alpha$ .

Condition (9.73) does not guarantee usefulness of the moment  $\square$ . Indeed, a trivial exclusion moment  $\square \square \square$ , for instance, satisfies this condition: hypothesis  $H$  never excludes, that is, is considered as true for any test results. Therefore in addition to (9.73) we require that condition:

$$\text{for } \square \in D, \quad E_{\square} \square \rightarrow \min \quad (9.74)$$

must be held. In other words, we wish mathematical expectation  $E_{\square} \square$  of the moment of hypothesis  $H$  exclusion would be small, if this hypothesis is not true ( $\square \square \square D$ ).

It can be shown (see p.3 of Appendix to the chapter) that from (9.73) it follows that mathematical expectation  $E_{\square} \square$  of the moment of hypothesis (9.71) must satisfy inequality

$$E_{\theta} \nu \geq \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, D)} \quad \text{for all } \theta \notin D \quad (9.75)$$

where

$$\rho(\theta, D) = \min_{\theta' \in D} \rho(\theta, \theta') \quad (9.76)$$

and  $\rho(\square, \square')$  is Kullback-Leibler "information distance" between  $\square$  and  $\square'$ :

$$\rho(\theta, \theta') = E_{\theta} \ln \left[ \frac{f(x, \theta)}{f(x, \theta')} \right].$$

Value of  $\rho(\square, D)$  represents Kullback-Leibler "information distance" between  $\square$  and  $D$ . Notice that  $\rho(\square, \square') = 0$  if  $\square = \square'$  and  $\rho(\square, \square') > 0$  if  $\square \neq \square'$ . Therefore,  $\rho(\square, D) = 0$  if  $\square \in D$  and  $\rho(\square, D) \geq 0$  if  $\square \notin D$ . So, (9.75) delivers the lower limit for the mathematical expectation  $E_{\square} \square$  for any (Markovian) moment  $\square$  which satisfies (9.74). Following Pavlov (1985, 1990), consider the rule of construction of  $\square$  where the lower limit of the right side of (9.75) is asymptotically reached for  $\square \square \rightarrow \square \square$ . It allows one to obtain asymptotic solution of the problem of optimization (9.74).

The moment of hypothesis  $H: \square \square D$  exclusion is determined as

$$v(\alpha, D) = \min \left\{ n: \prod_{1 \leq r \leq n} f(x_r, \hat{\theta}_{r-1}) \geq \frac{1}{\alpha} \max_{\theta \in D} \prod_{1 \leq r \leq n} f(x_r, \theta) \right\}, \quad (9.77)$$

where  $\hat{\theta} = \hat{\theta}(x_1, \dots, x_n)$  is the point estimate of parameter  $\theta$  on the basis of test results  $x_1, \dots, x_n$  on first  $n$  steps. The moment determined in such a way, it satisfies the inequality

$$P_{\theta} \{v(\alpha, D) < \infty\} \leq \alpha \quad (9.78)$$

for any  $H: \theta \in D$ .

(See the proof in Appendix to the chapter.) Besides the mathematical expectation of this moment,  $E_{\theta} v(\alpha, D)$  for some general conditions reaches the lower limit in (9.75) asymptotically for  $\alpha \rightarrow 0$ , namely,

$$E_{\theta} v(\alpha, D) = \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, D)} (1 + \varepsilon_{\alpha})$$

for any  $\theta \notin D$  where  $\varepsilon_{\alpha} \rightarrow 0$  if  $\alpha \rightarrow 0$ . (For the proof see *ibid.*) Thus for small probability of error,  $\alpha$ , approximate formula for the average moment of hypothesis  $H$  exclusion when it is not true (that is for  $\theta \notin D$ ) is:

$$E_{\theta} v(\alpha, D) \approx \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, D)}.$$

The right side of this approximate coincides with the right side of inequality (9.75), so moment (9.77) is approximately (for  $\alpha \ll 1$ ) optimal.

With construction of the moment of exclusion  $v(\alpha, D)$  constructed by rule (9.77), we can further to solve problems of test for composite hypotheses.

### 15.3.2 Sequential Criterion for Composite Hypotheses Test

Suppose we need to test two composite hypotheses of the type (9.70) for some function (reliability index)  $R(\theta)$  of vector parameter  $\theta = (\theta_1, \dots, \theta_m)$ . These hypotheses can be written in the form

$$H_0: \theta \in D_0, \quad H_1: \theta \in D_1 \quad (9.80)$$

where  $D_0$  and  $D_1$  are areas in the parameter space which is determined as  $D_0 = \{\theta: R(\theta) \leq R_0\}$  and  $D_1 = \{\theta: R(\theta) \geq R_1\}$ . The intermediate area  $I = \{\theta: R_0 < R(\theta) < R_1\}$  is called the "area of indifference".

Let us construct moments of exclusion for hypotheses  $H_0$  and  $H_1$ , respectively, on the basis (9.77)

$$\tau_0 = \tau(\tau, D_0), \quad \tau_1 = \tau(\tau, D_1) \quad (9.81)$$

where  $\tau$  and  $\tau$  are given risks of type I and II. Sequential criterion for tests of these hypotheses is determined as follows. Let us continue the test until such moment (step) when one of hypotheses will be excluded, that is, until moment

$$\tau = \min(\tau_0, \tau_1). \quad (9.82)$$

At this moment test is stopped and the remaining hypotheses is accepted. In other words, the rule of decision making at the stop moment (9.82) has the form

$$\begin{aligned} \text{if } \tau = \tau_1 < \tau_0 \text{ then hypothesis } H_0 \text{ is accepted,} \\ \text{if } \tau = \tau_1 > \tau_0 \text{ then hypothesis } H_1 \text{ is accepted,} \end{aligned} \quad (9.83)$$

(If  $\tau_1 = \tau_0$ , that is, both hypotheses are excluded simultaneously, either of these hypotheses may be accepted.) In accordance with (9.77) the sequential criterion (9.82), (9.83), determined as shown above, can be formulated in the following form. The area of test continuation is given by the following inequalities

$$\begin{aligned} U_n - V_{0n} &< \ln\left(\frac{1}{\alpha}\right), \\ U_n - V_{1n} &< \ln\left(\frac{1}{\beta}\right), \end{aligned} \quad (9.84)$$

where

$$U_n = \sum_{1 \leq r \leq n} \ln f(x_r, \hat{\theta}_{r-1}) \quad (9.85)$$

$$V_{in} = \max_{\theta \in D_i} \sum_{1 \leq r \leq n} \ln f(x_r, \theta), \quad i = 0, 1 \quad (9.86)$$

The test continues until the first violation of at least one of the inequalities in (9.84). If the first one is violated, then hypothesis  $H_1$  is accepted. If the second one is violated, then hypothesis  $H_0$  is accepted.

It is easy to find that the sequential criterion constructed in such a way for test hypotheses (9.80) has risks of type I and II not larger than  $\tau$  and  $\tau$ , respectively. Indeed, let  $\tau \leq \tau_0$ , that is, hypothesis  $H_0$  is true. Then by construction of the criterion, the probability of erroneous decision (acceptance of hypothesis  $H_1$ ) satisfies inequality

$$P_{\theta} \{ \text{to accept } H_1 \} \leq P_{\theta} (v_0 < \infty) \leq \alpha$$

Analogously, let  $\theta \in D_1$ , that is, hypothesis  $H_1$  is true. Then the probability of erroneous acceptance of hypothesis  $H_0$  satisfies inequality

$$P_{\theta} \{ \text{to accept } H_0 \} \leq P_{\theta} (v_1 < \infty) \leq \beta$$

(For more details see p.3 of Appendix to the chapter.) Besides, this sequential criterion for  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$  is asymptotically optimal by average test volume  $N(\theta) = E_{\theta} n$  for all  $\theta$ . The average test volume,  $N(\theta)$ , for small values of  $\alpha$  and  $\beta$  can be estimated by the following approximate formulae (see *ibid.*, Theorem 9.2):

$$N(\theta) \approx \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, D_1)} \quad \text{for } \theta \in D_0 \quad (9.87)$$

$$N(\theta) \approx \frac{\ln\left(\frac{1}{\beta}\right)}{\rho(\theta, D_0)} \quad \text{for } \theta \in D_1 \quad (9.88)$$

$$N(\theta) \approx \frac{\ln\left(\frac{1}{\alpha}\right)}{\Lambda(\theta)} \quad \text{for } \theta \in I \quad (9.89)$$

where

$$\Lambda(\theta) = \max[\rho(\theta, D_0), \rho(\theta, D_1)]$$

## 15.4 Sequential Confidence Limits

### 15.4.1 Construction of Sequential Confidence Limits

Let  $R(\theta) = R(\theta_1, \dots, \theta_m)$  be some function (reliability index) depending on  $m$ -dimensional vector parameter  $\theta = (\theta_1, \dots, \theta_m)$  whose true value is unknown. Assume that during the test, results are obtained (9.8) at each step  $n$ . On the basis of test results  $x_1, \dots, x_n, \dots$ , we construct the confidence limit (for instance, lower):  $\underline{R}_n = \underline{R}_n(x_1, \dots, x_n)$  for index  $R(\theta)$ . Random sequence

$$\underline{R}_n = \underline{R}_n(x_1, \dots, x_n), \quad n=1, 2, \dots \quad (9.90)$$

is called *sequential  $\alpha$ -LCL* for index  $R(\theta)$  if it satisfies the following conditions:

(a) Sequence  $\underline{R}_n$  monotone non-decreasing in  $n$ :

$$\underline{R}_1 \leq \underline{R}_2 \leq \dots \leq \underline{R}_n \leq \underline{R}_{n+1} \leq \dots$$

for any test results  $(x_1, \dots, x_n, \dots)$ .

(b) For any Markovian stop moment  $\square$  such that  $P_\square(\square < \square) = 1$ , inequality

$$P_\theta \{ \underline{R}_\square \leq R(\theta) \} \geq \gamma \quad (9.91)$$

for all  $\square$  where  $\underline{R}_\square = \underline{R}_\square(x_1, \dots, x_\square)$  is the LCL (9.90) calculated at the stop moment  $\square$ . Notice that due to the monotonicity of sequence  $\underline{R}_n$  condition (b) can be substitute by the weaker one

(c) For any fixed step  $n$  inequality

$$P_\theta \{ \underline{R}_n \leq R(\theta) \} \geq \gamma, n = 1, 2, \dots \quad (9.92)$$

holds for all  $\square$ .

Indeed, from monotonicity of sequence  $\underline{R}_n$  follows that there exists (for any test results  $x_1, \dots, x_n, \dots$ ) limit for  $n \rightarrow \square$

$$\underline{R} = \lim \underline{R}_n$$

and due to (9.92) this limit satisfies inequality

$$P_\theta \{ \underline{R} \leq R(\theta) \} = \lim P_\theta \{ \underline{R}_n \leq R(\theta) \} \geq \gamma,$$

and, taking into account monotone decreasing of  $\underline{R}_n$ , it follows that for arbitrary (not necessarily Markovian) stopping moment  $v$  inequalities  $\underline{R}_v \leq \underline{R}$  and

$$P_\theta \{ \underline{R}_v \leq R(\theta) \} \geq P_\theta \{ \underline{R} \leq R(\theta) \} \geq \gamma,$$

hold. The latter inequality is true for any moment  $\square$  such that  $P_\square(\square < \square) < 1$  if  $\underline{R}_\square = \lim \underline{R}_n = \underline{R}$ . Thus, if sequence (9.90) is monotone non-decreasing (in all  $x_1, \dots, x_n, \dots$ ), then for demonstration, that (b) is true, it is enough to prove condition (c).

Let us now construct the sequential confidence limit for  $R(\square)$  based on the exclusion moment  $\square(\square, D)$  for composite hypotheses given in (9.77). For this purpose, consider composite hypothesis

$$H_C: R(\square) \leq C \quad (9.93)$$



where  $C$  is a constant. The exclusion moment for this hypothesis is denoted as

$$\alpha_c = \alpha(\alpha, D_C) \quad (9.94)$$

where  $D_C$  is the area in space of parameters  $\alpha$  of the form  $D_C = \{\alpha: R(\alpha) \leq C\}$ . The LCL  $\underline{R}_n = \underline{R}_{\alpha}(\alpha)(x_1, \dots, x_n)$  for  $R(\alpha)$  at step  $n$  is defined as

$$\underline{R}_n = \max \{C: \alpha_c \leq n\}, \quad n=1, 2, \dots \quad (9.95)$$

that is as maximum value  $C$  for which hypothesis of type (9.93) is found untrue at step  $n$ .

It is easy to find that r.v.'s  $\underline{R}_n$ ,  $n=1, 2, \dots$  defined in (9.95) form  $\alpha$ -LCL for  $R(\alpha)$  where  $\alpha \alpha = 1 - \alpha$ . Indeed, let  $\alpha'$  be an arbitrary point in the space of parameters. Denote  $b = R(\alpha')$ . For each fixed  $n$  due to (9.95) the following relation is valid:

$$\{\alpha_b > n\} \cap \{\underline{R}_n \leq b\} = \emptyset \quad (9.96)$$

Taking into account that  $\alpha'$  belongs to area  $D_b = \{\alpha: R(\alpha) \leq b\}$  and due to (9.78) the following inequality holds:

$$P_{\alpha'}(\alpha_b > n) = 1 - P_{\alpha'}(\alpha_b \leq n) \geq 1 - P_{\alpha'}(\alpha_b < \alpha) = 1 - P_{\alpha'}\{\alpha(\alpha, D_b) < \alpha\} \geq 1 - \alpha.$$

from here, due to (9.96) follows that

$$P_{\alpha'}\{\underline{R}_n \leq R(\alpha')\} = P_{\alpha'}(\underline{R}_n \leq b) \geq \frac{P_{\alpha'}(\alpha_b > n)}{P_{\alpha'}(\alpha_b > n)} \geq 1 - \alpha$$

for each fixed  $n=1, 2, \dots$  (that is, condition C holds). Monotone non-decreasing of sequence  $\underline{R}_n$  (condition A) follows directly from its definition (9.95). Thus r.v.'s (9.95) form sequential  $(1 - \alpha)$ -LCL for  $R(\alpha)$ .

Analogously define the UCL. The sequence of r.v.'s

$$\bar{R}_n = \bar{R}_n(x_1, \dots, x_n), \quad n = 1, 2, \dots$$

is called the sequential  $\alpha$ -UCL for  $R(\alpha)$  if the following condition holds:

A') Sequence of  $\bar{R}_n$  is monotone non-increasing:

$$\bar{R}_1 \geq \bar{R}_2 \geq \dots \geq \bar{R}_n \geq \bar{R}_{n+1} \geq \dots$$

for any test results  $x_1, \dots, x_n, \dots$

B') For any (Markovian) stop test  $\alpha$  such that  $P_{\alpha}(\alpha < \alpha) = 1$  inequality

$$P_{\theta} \{ \bar{R}_v \geq R(\theta) \} \geq \gamma$$

holds for all  $\square$  where

$$\bar{R}_v = \bar{R}_v(x_1, \dots, x_v).$$

Analogously, we can show that if the sequence  $\bar{R}_n, n=1,2, \dots$ , satisfies to the first condition of monotonicity A' than for showing that the second condition B weaker condition maybe proved: C') For arbitrary fixed  $n=1, 2, \dots$  the inequality

$$P_{\theta} \{ \bar{R}_n \geq R(\theta) \} \geq \gamma$$

holds for all  $\square$ .

In an analogous way we define two-sided sequence  $\square$ -confidence interval  $(\underline{R}_n, \bar{R}_n)$  for  $R(\square)$ . It is easy to show that if r.v.'s  $\underline{R}_n$  and  $\bar{R}_n$  form, respectively,  $(1-\square)$ -LCL and  $\square \square \square \beta \square$ -UCL for  $R(\square)$  then  $(\underline{R}_n, \bar{R}_n)$  form sequential  $(1-\square-\square)$ -confidence interval for  $R(\square)$ . For constructing sequential  $(1-\square)$ -UCL for  $R(\square)$ , consider again composite hypotheses of the type

$$H'_C: R(\square) \geq C \quad (9.97)$$

The exclusion moment for hypothesis (9.97) is denoted by

$$\square'_C = \square(\square, D'_C) \quad (9.98)$$

where  $D'_C = \{ \square: R(\square) \geq C \}$ . At test step  $n$  let us set

$$\bar{R}_n = \min \{ C: v_C' \leq n \}, \quad n=1,2,\dots \quad (9.99)$$

The random sequence defined in such a way forms sequential  $\square \square \square \beta \square$ -UCL for  $R(\square)$ . The proof of this is similar to the one given above for (9.95).

On the basis of definition of the exclusion moment  $\square(\square, D)$  in (9.77) the sequential  $\square$ -LCL (9.95) and  $\square$ -UCL (9.99) can be written in more detail as follows

$$\underline{R}_n = \max_{1 \leq k \leq n} R'_k$$

$$\bar{R}_n = \min_{1 \leq k \leq n} \bar{R}'_k \quad (9.100)$$

where values  $\underline{R}'_n$  and  $\overline{R}'_n$  are determined at test step  $n$  from the following equations:

$$V_{0n}(C) = U_n - \ln\left(\frac{1}{\alpha}\right) \quad (9.101)$$

$$V_{1n}(C) = U_n - \ln\left(\frac{1}{\beta}\right)$$

where  $U_n$  is determined above in (9.85),

$$V_{0n}(C) = \max_{R(\theta) \leq C} \sum_{1 \leq r \leq n} \ln f(x_r, \theta), \quad (9.102)$$

$$V_{1n}(C) = \max_{R(\theta) \geq C} \sum_{1 \leq r \leq n} \ln f(x_r, \theta).$$

Sequential confidence limits, constructed in (9.100)-(9.102), are asymptotically optimal for  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$  in sense of the average time to reach the specified accuracy (see p.3 Appendix to the chapter).

#### 15.4.2 Test Reliability Hypotheses with the Help of Sequential Confidence Limits

Suppose we need to test, on the basis of sequential tests (9.8), two composite hypotheses of the following type

$$H_0: R(\square) \leq R_0, \quad H_1: R(\square) \geq R_1,$$

where  $R_0 < R_1$  are given critical levels of index  $R$ . Let  $\alpha$  and  $\beta$  be given risks of type I and II. One can see directly from (9.81)-(9.86) (9.94), (9.95), (9.98)-(9.102) that sequential criterion, constructed above in Section 9.3, with risks  $\alpha$  and  $\beta$  for hypotheses  $H_0$  and  $H_1$  can be written via sequential confidence limits as follows. The area of test continuation at step  $n$  is given by two inequalities

$$\underline{R}_n < R_0, \quad R_1 < \overline{R}_n \quad (9.103)$$

where  $\underline{R}_n$  is sequential  $(1-\alpha)$ -LCL,  $\overline{R}_n$  is sequential  $\beta$ -UCL for  $R(\square)$  constructed above in (9.100). The test continues, that is, we go to the next step  $n+1$  if at step  $n$  both inequalities in (9.103) hold. The test is stopped at such step where the first time at least one of inequalities in (9.103) is violated. If the first inequality is violated, the hypothesis  $H_1$  is accepted. If the second inequality is violated the hypothesis  $H_0$  is accepted.

In other words, this decision rule based on the sequential confidence limits means the following. Let, for instance,  $R=R(\square)$  be an index of the type of failure probability or failure rate. Then the critical levels  $R_0 < R_1$  are, respectively, equivalent to the levels of "acceptance"  $R_0$  and "rejection"  $R_1$ . The test continues if the confidence interval  $(\underline{R}_n, \bar{R}_n)$  at step  $n$  contains both critical levels  $R_0$  and  $R_1$ . If the  $(1-\alpha)$ -LCL  $\underline{R}_n$  crosses the level of "acceptance"  $R_0$  then the test is stopped and hypothesis  $H_1$  is assumed true ("rejection"). If the  $\alpha$ -UCL  $\bar{R}_n$  crosses the level of "rejection"  $R_1$  then the test is stopped and hypothesis  $H_0$  is assumed true ("acceptance").

Remark 9.5.1. This sequential decision rule preserves its meaning in the case where the intermediate area of "indifference"  $I$  is absent, that is, if critical levels are coincide,  $R_0=R_1$ . In this case risks of types I and II  $\alpha$  and  $\beta$  are preserved but the average test volume increases if the value of index  $R(\square) \rightarrow R_0$ , that is,  $N(\square) \rightarrow \infty$  if  $R(\square) \rightarrow R_0$  as it can be seen from (9.87)-(9.89). (For details, see p.3 of Appendix to the chapter.)

### 15.4.3 Scheme of Dependent Tests

Consider a case where the test process

$$x_0, x_1, \dots, x_n, \dots$$

represent a homogeneous Markov chain "starting" from the initial state  $x_0$ . Let  $f(x|y, \square)$  is the transitive density of distribution, in other words, the density of r.v.  $x_{n+1}$  observed at step  $n+1$  for given value of parameter  $\square=(\square_1, \dots, \square_m)$  and under condition that at step  $n$  the value of r.v.  $x_n=y$  was observed.

Consider again composite hypothesis:

$$H: \square \in D \tag{9.104}$$

where  $D$  is some area in the space of parameters  $\square$ . The moment of "exclusion" of hypothesis (9.104) we determine analogously to (9.77) in the scheme with independent tests, namely, in the following way

$$v(\alpha, D) = \min \{n: \prod_{1 \leq r \leq n} f(x_r | x_{r-1}, \hat{\theta}_{r-1}) \geq \frac{1}{\alpha} \max_{\theta \in D} \prod_{1 \leq r \leq n} f(x_r | x_{r-1}, \theta_{r-1})\} \tag{9.105}$$

where  $\hat{\theta}_n = \hat{\theta}(x_1, \dots, x_n)$  is again the point estimate of parameter  $\square$  based on the test results  $x_1, \dots, x_n$  at first  $n$  steps. It is possible to show (see p.4 of Appendix to the chapter) that this moment, as in the scheme of independent tests above, satisfies inequality

$$P_{\square} \{v(\square, D) < \infty\} \leq \alpha \text{ for all } \square \in D \tag{9.106}$$

After this, sequential confidence limits for  $R=R(\square)$  are constructed on the basis of the moment of exclusion of composite hypotheses (9.105) by previous formulae (9.94) and (9.95) for the lower limit, and by formulae (9.98) and (9.99) for the upper limit. The lower and upper limits at test step  $n$  are determined analogously to (9.100)-(9.102) by formulas:

$$\frac{R_n}{\bar{R}_n} = \frac{\max_{1 \leq k \leq n} R'_k}{\max_{1 \leq k \leq n} \bar{R}'_k} \quad (9.107)$$

where values  $\underline{R}'_n$  and  $\bar{R}'_n$  are determined at test step  $n$ , respectively, for the following levels of  $C$

$$\begin{aligned} V_{0n}(C) &= U_n - \ln\left(\frac{1}{\alpha}\right), \\ V_{1n}(C) &= U_n - \ln\left(\frac{1}{\beta}\right), \end{aligned} \quad (9.108)$$

where

$$U_n = \sum_{1 \leq r \leq n} \ln f(x_r | x_{r-1}, \hat{\theta}_{r-1}) \quad (9.109)$$

$$V_{0n}(C) = \max_{R(\theta) \leq C} \sum_{1 \leq r \leq n} \ln f(x_r | x_{r-1}, \theta) \quad (9.110)$$

$$V_{1n}(C) = \max_{R(\theta) \geq C} \sum_{1 \leq r \leq n} \ln f(x_r | x_{r-1}, \theta)$$

The random sequences  $\underline{R}_n$  and  $\bar{R}_n$ ,  $n=1,2, \dots$ , constructed in such a way, form sequential  $(1-\alpha)$ -LCL and sequential  $(1-\beta)$ -UCL for  $R(\square)$ . (The proof of this fact is completely coincides with that given in Section 9.4 for the scheme of independent tests.)

### ***15.5 Sequential Confidence Limits and Test of Hypotheses for Systems on the Base of Unit Tests***

Consider now some application of the results obtained above. Begin with a simplest "binomial model" of system unit tests (see above Example 9.1.7).

### 15.5.1 Binomial Model.

Let a system consist of  $m$  units of different types. At the current  $n$ th test step (cycle)  $N_i$  units of type  $i$  are simultaneously tested. Each unit has reliability parameter (failure probability)  $q_i$ ,  $1 \leq i \leq m$ . Failures of different units are independent. As the results of the test, we observe a sequence of i.i.d. random vectors

$$x_1, x_2, \dots, x_n, \dots \quad (9.111)$$

where random vector  $x_n$  represents a set of all numbers of failures for units of different types that were observed at step  $n$ :

$$x_n = (d_{1n}, d_{2n}, \dots, d_{mn}), \quad n=1, 2, \dots$$

where  $d_{in}$  is the number of failures of  $N_i$  tested units of type  $i$  at step (cycle)  $n$ . Thus, r.v.  $d_{in}$  has binomial distribution with parameters  $(N_i, q_i)$  where value of parameter  $q_i$  is unknown. This model, obviously, is a multi-dimension analogue of the classical scheme of sequential independent Bernoulli trials. Bernoulli scheme is a particular case of this general scheme for  $N_i=1$  and  $m=1$ .

We need, on the basis of test results (9.111), to construct confidence limits or to test hypotheses about some system reliability index

$$R = R(\mathbf{q}) = R(q_1, \dots, q_m)$$

where  $\mathbf{q} = (q_1, \dots, q_m)$  is vector of unknown parameters of unit reliability.

Denote  $M_{in} = d_{i1} + \dots + d_{in}$ , the total number of unit  $i$  failed during  $n$  test steps and

$$\hat{q}_i(n) = \frac{M_{in} + r_i}{n + k_i} \quad (9.112)$$

is the point estimate of parameter  $q_i$  at step  $n$ . Values  $k_i$  and  $r_i$  take into account some prior information. For instance, if there is information about previous tests, then  $k_i$  might be the number of tested units and  $r_i$  the number of observed failures,  $1 \leq i \leq m$ . Denote  $\hat{p}_i(n) = 1 - \hat{q}_i(n)$  the point estimate for parameter  $p_i = 1 - q_i$  at test step  $n$ .

Applying general formulae (9.100)-(9.102), we obtain that sequential (1- $\alpha$ )-LCL  $\underline{R}_n$  and (1- $\alpha$ )-UCL  $\overline{R}_n$  for parameter  $R = R(\mathbf{q})$  can be found at the  $n$ th test step as

$$\underline{R}_n = \max_{1 \leq l \leq n} R'_l \quad (9.113)$$

$$\overline{R}_n = \min_{1 \leq l \leq n} \overline{R}'_l$$

where values  $\underline{R}'_n$  and  $\overline{R}'_n$  are found at step  $n$  from solution of the following equations in respect to  $C$ :

$$V_{0n}(C) = U_n - \ln\left(\frac{1}{\alpha}\right) \quad (9.114)$$

$$V_{1n}(C) = U_n - \ln\left(\frac{1}{\beta}\right)$$

where

$$U_n = \sum_{1 \leq i \leq m} \sum_{1 \leq r \leq n} \{d_{ir} \cdot \ln \hat{q}_i \cdot (r-1) + (N_i - d_{ir}) \cdot \ln \hat{p}_i \cdot (r-1)\},$$

$$V_{0n}(C) = \max_{R(\mathbf{q}) \leq C} H_n(\mathbf{q}),$$

$$V_{1n}(C) = \max_{R(\mathbf{q}) \geq C} H_n(\mathbf{q}),$$

$$H_n(\mathbf{q}) = \sum_{1 \leq i \leq m} \{M_{in} \ln q_i + (nN_i - M_{in}) \ln(1 - q_i)\}.$$

Suppose we need to test two composite hypotheses on the basis of test results (9.111)

$$H_0: R(\mathbf{q}) \leq R_0, \quad H_1: R(\mathbf{q}) \geq R_1.$$

Sequential criterion with given risks of types I and II ( $\square$  and  $\square$ ) for hypotheses  $H_0$  and  $H_1$  is constructed in accordance with (9.84)-(9.86) as follows. The area of test continuation on step  $n$  is given by inequalities

$$U_n - V_{0n}(R_0) < \ln\left(\frac{1}{\alpha}\right) \quad (9.115)$$

$$U_n - V_{1n}(R_1) < \ln\left(\frac{1}{\beta}\right)$$

The test continues, if both inequalities (9.15) hold. The test is stopped if at least one of these inequalities is violated. The violation of the first one leads to acceptance of hypothesis  $H_1$ , and the second one to acceptance of hypothesis  $H_0$ .

In accordance with (9.113), this sequential criterion can be also written in a more compact form via sequential confidence limits (9.113). Namely, the area of test continuation at step  $n$  is given by inequalities

$$\underline{R}_n < R_0, \quad R_1 < \bar{R}_n \quad (9.116)$$

that are equivalent to previous inequalities (9.115). The decision rule is formulated in this case similarly. The test continues, if both inequalities (9.16) hold and stops if at least one of these inequalities is violated. The violation of the first one leads to acceptance of hypothesis  $H_1$ , and the second one to acceptance of hypothesis  $H_0$ .

This sequential criterion is asymptotically optimal for  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$  in sense of the average test volume (see p.3 of Appendix to the chapter, Theorem 9.2). Let us write formulae for the average test volume, assuming in the sake of simplicity  $\beta = \alpha$ . Let  $\tau$  be the moment of the first violation of the inequalities above and  $N(\mathbf{q}) = E_q \tau$  is the average test volume for given vector of parameters  $\mathbf{q}$ . In accordance with (9.87)-(9.89) the average test volume,  $N(\mathbf{q})$ , can be estimated for  $\alpha = \beta \rightarrow 0$  with the help of the following formulae

$$N(\mathbf{q}) \approx \frac{\ln\left(\frac{1}{\alpha}\right)}{\Lambda_1(\mathbf{q})} \quad \text{for } R(\mathbf{q}) \leq R_0,$$

$$N(\mathbf{q}) \approx \frac{\ln\left(\frac{1}{\alpha}\right)}{\Lambda(\mathbf{q})} \quad \text{for } R_0 < R(\mathbf{q}) < R_1,$$

$$N(\mathbf{q}) \approx \frac{\ln\left(\frac{1}{\alpha}\right)}{\Lambda_0(\mathbf{q})} \quad \text{for } R(\mathbf{q}) \geq R_1,$$

where

$$\Lambda(\mathbf{q}) = \max[\Lambda_0(\mathbf{q}), \Lambda_1(\mathbf{q})],$$

$$\Lambda_0(\mathbf{q}) = \max_{R(\mathbf{q}') \leq R_0} \rho(\mathbf{q}, \mathbf{q}'),$$

$$\Lambda_1(\mathbf{q}) = \max_{R(\mathbf{q}') \geq R_1} \rho(\mathbf{q}, \mathbf{q}'),$$



$$\rho(\mathbf{q}, \mathbf{q}') = \sum_{1 \leq i \leq m} N_i \left[ q_i \ln \left( \frac{q_i}{q'_i} \right) + (1 - q_i) \ln \left( \frac{1 - q_i}{1 - q'_i} \right) \right],$$

$\mathbf{q} = (q_1, \dots, q_m), \mathbf{q}' = (q'_1, \dots, q'_m)$ .

In practice, for often used  $0.01 \leq \alpha \leq 0.2$ , these approximate formulae give satisfactory estimates for average test volume. A more precise the value of the average test volume,  $N(\mathbf{q})$ , can be estimated with the help of Monte Carlo method (see below Examples 9.5.1 and 9.5.2).

### Gain in the average test volume with unit test information

Consider a series-parallel system (see Example 9.1.7 above) which is characterized by

$$R = R(\mathbf{q}) = 1 - \prod_{1 \leq i \leq m} (1 - q_i^{N_i}) \quad (9.117)$$

where  $N_i$  is the number of parallel units in the  $i$ th redundant group,  $1 \leq i \leq m$ . The test of hypotheses  $H_0$  and  $H_1$  for (9.117) can be performed on the basis of test results for the system, that is, on the basis of sequence

$$\square_1, \square_2, \dots, \square_n, \dots \quad (9.118)$$

where  $\square_n$  is system failure indicator at step  $n$ :  $\square_n = 1$  if at least one redundant group has failed at the step  $n$ , that is at least for one  $i$ ,  $1 \leq i \leq m$ ,  $d_{in} = N_i$ . Otherwise,  $\square_n = 0$ .

In this case the problem is reduced to the standard test of hypothesis of type

$$H_0: R \leq R_0, \quad H_1: R \geq R_1$$

for binomial parameter (failure probability)  $R$  on the basis of results of Bernoulli trials (9.118). This problem can be solved on the basis of sequential Wald criterion (see Section 9.2 and Example 9.2.1 above). The average test volume in this case is estimated with the help of known Wald's formulae (9.63).

The serious deficiency of such a simple method, based directly on system failures (9.118), is that all information about unit's failures is not taken into account. It is clear on an intuitive level that this fact should lead to increase in the average test volume especially for the highly reliable systems. For instance, in this case we consider as equivalent cases where at the test step  $n$  we observe  $d_{in} = 0$  or  $d_{in} = N_i - 1$  though these two cases are quite different.

Below we suggest numerical examples with comparison of average test volume (for the same values of  $R_0, R_1, \alpha$  and  $\beta$ ) for the sequential Wald's criterion based on system's failures

(9.118) and for the sequential method (9.115)-(9.116) based on complete information (9.111) taking into account complete data of unit's failures.

From these examples one can see that sequential criterion (9.115)-(9.116) gives a substantial gain in average test volume.

Example 9.5.1 (Series-parallel system)

Consider a series-parallel system with reliability index of form (9.117). The system consists of  $m=2$  parallel subsystems connected in series. The number of units within the subsystems are  $N_1=N_2=2$ . The critical levels are  $R_0=0.05$  and  $R_1=0.2$ , and risks of type I and II are  $\alpha=\beta=0.1$ . Using formula (9.63), we obtain that for  $R=R_0$  the average test volume for Wald's criterion based on

$$\begin{aligned} \text{the system's failures is equal to } N_w(R_0) &\approx \frac{\omega(\alpha, \beta)}{\rho(R_0, R_1)} = \frac{(1-\alpha) \ln\left(\frac{1-\alpha}{\beta}\right) + \alpha \ln\left(\frac{\alpha}{1-\beta}\right)}{R_0 \ln\left(\frac{R_0}{R_1}\right) + (1-R_0) \ln\left(\frac{1-R_0}{1-R_1}\right)} = \\ &= \frac{0.9 \cdot \ln 9 - 0.1 \cdot \ln 9}{0.95 \cdot \ln\left(\frac{0.95}{0.80}\right) - 0.05 \cdot \ln 4} = 18.9. \quad \blacksquare \end{aligned}$$

For the sequential criterion (9.115)-(9.116) based on the complete information, the average test volume,  $N(\mathbf{q})$ , is equal to 8.3 for the value of reliability index  $R(\mathbf{q})=R(q_1, q_2)=R_0$  the average test volume found with the help of Monte Carlo simulation with computer. (The average test volume was estimated for a symmetrical point  $q_1=q_2$ ,  $R(q_1, q_2)=R_0$ , that is for equally reliable units for both subsystems.) Thus, in this case, the average test volume can be decreased more than in two times due to use sequential criterion (9.115)-(9.116) with taking into account complete information.

Example 9.5.2 (Parallel system)

Consider a parallel system consisting of  $m=2$  units of different types. The reliability index (in this case the failure probability) has the form

$$R=R(\mathbf{q})=q_1q_2 \quad (9.119)$$

where  $q_i$  is the failure probability of the  $i$ th unit,  $\mathbf{q}=(q_1, q_2)$  is vector of unit parameters. Let, as in the previous example, at each step a single system be tested. It means that the number of tested units of the first and second types are equal  $N_1=N_2=1$ . Vector of test results,  $x_n$ , at step  $n$  has the form:

$$x_n=(d_{1n}, d_{2n}), \quad n=1, 2, \dots \quad (9.120)$$

where  $d_{in}=\{0 \text{ or } 1\}$  is the failure indicator for a unit of the  $i$ th type at step  $n$ . The system failure indicator at step  $n$  equals  $\square=1$  if  $d_{1n}=d_{2n}=1$ , that is both units have been failed, and  $\square=0$ , otherwise. Suppose we need to test two composite hypotheses for reliability index (9.119):

$$H_0: R \leq R_0, \quad H_1: R \geq R_1$$

where critical levels  $R_0=0.01$  and  $R_1=0.05$ . Given risks of type I and II are  $\alpha=\beta=0.05$ .

Test of hypotheses  $H_0$  and  $H_1$  can be done either by Wald's criterion applied to the sequence of system failure indicators

$$\square_1, \square_2, \dots, \square_n, \dots \quad (9.121)$$

or on the basis of criterion (9.115)-(9.116), applied to sequence (9.120). Using (9.63), we obtain the average test volume for Wald's criterion for  $R=R_0$  is equal

$$N_w \approx \frac{\omega(\alpha, \beta)}{\rho(R_0, R_1)} = \frac{0.95 \cdot \ln 19 - 0.05 \cdot \ln 19}{0.99 \cdot \ln\left(\frac{0.99}{0.95}\right) - 0.01 \cdot \ln\left(\frac{0.05}{0.01}\right)} = 110.$$

For sequential criterion (9.115)-(9.116) the average test volume for  $R(\mathbf{q})=q_1q_2=R_0$  (that is at symmetrical point  $q_1=q_2=\sqrt{R_0}$ ) gives, with the help of Monte Carlo simulation, the value of 44. Again we have a substantial gain in the average test volume due to the use of complete information about unit failures and applying sequential criterion (9.115)-(9.116). ■

Example 9.5.3 (Sequential UCL for the failure probability of a parallel system)

Consider a parallel system consisting of  $m$  different units. Failure probability of such system as the form

$$R = R(\mathbf{q}) = \prod_{1 \leq i \leq m} q_i \quad (9.122)$$

Construct  $(1-\alpha)$ -UCL for reliability index of type (9.122) using formulae (9.113)-(9.114). In this case function  $V_{1n}(C)$  has the form

$$V_{1n}(C) = \max H_n(\mathbf{q}) \quad (9.123)$$

where maximum is taken under the following restrictions for vector of parameters  $\mathbf{q}=(q_1, \dots, q_m)$ :

$$\prod_{1 \leq i \leq m} q_i \geq C \quad (9.124)$$

$$0 \leq q_i \leq 1, \quad 1 \leq i \leq m.$$

It is easy to see that maximum in (9.123) is attained in the inner point of the area specified by restriction (9.124), since  $H(\mathbf{q}) \rightarrow 0$  for  $q_i \rightarrow 1$ . Necessary condition for maximum (the system of Lagrange equations) in this case has the form

$$\frac{\partial H_n(\mathbf{q})}{\partial q_i} = \frac{M_{in}}{q_i} - \frac{n - M_{in}}{1 - q_i} = \frac{b}{q_i}, \quad 1 \leq i \leq m,$$

$$\prod_{1 \leq i \leq m} q_i = C$$

where  $b > 0$  is a Lagrange multiplier. It gives, after simple transformations, that  $(1 - \beta)$ -UCL for the probability of system failure at step  $n$  is determined by the formula

$$\bar{R}_n = \min_{1 \leq l \leq n} \bar{R}'_l \quad (9.125)$$

where

$$\bar{R}'_n = \prod_{1 \leq l \leq m} \left( \frac{M_{in} + b}{n + b} \right), \quad (9.126)$$

and value  $b > 0$  is found from equation

$$\sum_{1 \leq l \leq m} \left\{ M_{in} \ln \left( \frac{M_{in} + b}{n + b} \right) + (n - M_{in}) \ln \left( n - \frac{M_{in}}{n + b} \right) \right\} = U_n - \ln \left( \frac{1}{\beta} \right). \quad \blacksquare$$

Example 9.5.4. (Test of parallel system up to the first failure)

In conditions of the previous Example consider a case where system tests are continues until such random step  $n$  when the first failure of at least one unit has occurred, that is up to the step

$$\tau = \min \{n: d_{in} > 0 \text{ at least for one unit } i, 1 \leq i \leq m\}.$$

Take the point estimate (9.112) for parameter  $q_i$  (the probability of failure of unit  $i$ ) the standard estimate of maximum likelihood

$$\hat{q}_i(n) = \frac{M_{in}}{n}, \quad 1 \leq i \leq m.$$

In this case, by the definition of the test stop moment  $\square$  for all numbers of failures of units equals  $d_{in}=0$  and, consequently,  $\hat{q}_i(n) = 0$ ,  $\hat{p}_i(n) = 1$ . From (9.125)-(9.127) we have

$$\bar{R}'_n = \left( \frac{b}{n+b} \right)^m, \quad 1 \leq n \leq \sigma - 1$$

where  $b > 0$  can be found from equation

$$mn \cdot \ln \left( \frac{n}{n+b} \right) = -\ln \left( \frac{1}{\beta} \right)$$

It follows that sequential  $(1-\square)$ -UCL for the probability of system failure at test step  $n$  is determined by formula

$$\bar{R}_n = \left( 1 - \beta^{\frac{1}{mn}} \right)^m, \quad 1 \leq n \leq \sigma - 1. \quad (9.128)$$

UCL (9.128) decreases fast with increasing  $n$ . At the test step  $\square$  this limit has the form

$$\bar{R}_\sigma = \left( 1 - \beta^{\frac{1}{m(\sigma-1)}} \right)^m. \quad \blacksquare$$

### 15.5.2 General Parametrical Model

Consider a system consisting of  $m$  units of different types. Each unit has d.f.  $F_i(x, \square_i)$  and distribution density  $f_i(x, \square_i)$  where  $\square_i$  is unknown parameter (in general case, a vector). There are  $N_i$  units of type  $i$ ,  $1 \leq i \leq m$ . All unit failures are independent. A failed unit is instantaneously replaced by a new one (see Example 9.1.6 above). During the test we observe  $N_1 + \dots + N_m$  independent renewal processes among which there are  $N_i$  processes of type  $i$ .

Let  $R = R(\square) = R(\square_1, \dots, \square_m)$  be some system reliability index depending on vector of unit parameter. Using results of Section 9.4, it is easy to construct sequential confidence limits for  $R = R(\square)$  for more general model with continuous time. The following formulae for the sequential confidence limits can be obtained by using the results of Section 9.4.3 by dividing the time axis into intervals  $h$  and by setting then  $h \rightarrow 0$  (see Pavlov, 1983b and 1988 for details).

Let us introduce the following notation for test results:

$N_{\bar{t}}$  = the number of units of type  $i$  tested at moment  $t$  (in this model  $N_i$  is constant for any  $t$ )

$t_{ij}$  = the moment of the  $j$ th failure of unit of type  $i$ ,

$s_{ij}$  = the total test time of the  $i$ th unit at moment  $t_{ij}$ ,

$D_i(t)$  = the number of failures of units of type  $i$  on time interval  $(0, t]$ , that is, the number of failure moments  $t_{ij}$  such that  $t_{ij} \leq t$ ,

$S_{ie}(t)$  = the test time of the  $e$ th unit of type  $i$  at a current moment  $t$ ,  $1 \leq e \leq N_i$ .

We also introduce the following notation for the standard unit characteristics:

$$\lambda_i(x, \theta_i) = \frac{f_i(x, \theta_i)}{1 - F_i(x, \theta_i)}$$

is the failure rate of the  $i$ th type unit for a given value of parameter  $\theta_i$  (further we will assume that this function is continuous in  $x$  for each  $1 \leq i \leq m$ ,

$$\Lambda_i(x, \theta_i) = \int_0^x \lambda(u, \theta_i) du = -\ln[1 - F_i(x, \theta_i)]$$

is the "resource function" of the  $i$ th unit (in other terminology, "leading function").

Sequential  $(1-\alpha)$ -LCL and  $(1-\alpha)$ -UCL for  $R=R(\theta)$  are found for time moment  $t$  by formulae

$$\begin{aligned} \underline{R}_t &= \max_{u \leq t} \underline{R}'_u, \\ \bar{R}_t &= \max_{u \leq t} \bar{R}'_u, \end{aligned} \quad (9.129)$$

where values of  $\underline{R}'_t$  and  $\bar{R}'_t$  are found from the following equations in respect to  $C$ :

$$V_{0t}(c) = U_t - \ln\left(\frac{1}{\alpha}\right) \quad (9.130)$$

$$V_{1t}(c) = U_t - \ln\left(\frac{1}{\beta}\right)$$

where

$$V_{0t}(c) = \max_{R(\theta) \leq C} H_t(\theta), \quad (9.131)$$

$$V_{1t}(c) = \max_{R(\theta) \geq C} H_t(\theta),$$

$$H_t(\theta) = \sum_{1 \leq i \leq m} \left\{ \sum_{1 \leq j \leq D_i(t)} \ln \lambda_i(s_{ij}, \theta_i) - \sum_{1 \leq j \leq D_i(t)} \Lambda_i(s_{ij}, \theta_i) - \sum_{1 \leq e \leq N_i} \Lambda_i[S_{ie}(t), \theta_i] \right\} \quad (9.132)$$

$$U_t = \sum_{1 \leq i \leq m} \left\{ \sum_{1 \leq j \leq D_i(t)} \ln \lambda_i(s_{ij}, \hat{\theta}_i) - \int_0^t \sum_{1 \leq e \leq N_i} \lambda_i[S_{ie}(u), \hat{\theta}_i(u)] du \right\} \quad (9.133)$$

where  $\hat{\theta}_i(t)$  is the point estimate of parameter  $\theta_i$  obtained by test results on time interval  $(0, t]$ ,  $\hat{\theta}_{ij} = \hat{\theta}_i(t_{ij} - 0)$  is the left limit of function  $\hat{\theta}_i(t)$  at point  $t_{ij}$ .

#### Example 9.5.5 ("Exponential model")

Consider a particular case where units have exponential d.f. of TTF:  $F_i(x, \theta_i) = 1 - \exp(-\theta_i x)$ ,  $i=1, \dots, m$ . In this case reliability parameter of unit of type  $i$  is  $\theta_i = \lambda_i$ , vector of unit reliability parameters  $\theta = \lambda = (\lambda_1, \dots, \lambda_m)$ , and reliability index of the system,  $R = R(\theta) = R(\lambda_1, \dots, \lambda_m)$ , is some function of parameter vector  $\theta$ . Denote

$$\hat{\lambda}_i(t) = \frac{D_i(t) + r_i}{N_i t + T_i}$$

the point estimate of parameter  $\theta_i$  obtained on the basis of test results up to moment  $t$  where values of  $T_i$  and  $r_i$  allow to take into account information about parameter  $\theta_i$  if it exists (for instance, test duration  $T_i$  and number of observed failures  $r_i$ ). On the basis of previous formulas (9.129)-(9.133) we obtain that in this case  $R = R(\theta)$  sequential  $(1 - \alpha)$ -LCL and  $(1 - \alpha)$ -UCL are determined at moment  $t$  by formulae (9.129)-(9.130) where

$$V_{0t}(c) = \max_{R(\lambda) \leq c} H_t(\lambda),$$

$$V_{1t}(c) = \max_{R(\lambda) \geq c} H_t(\lambda),$$

$$H_t(\lambda) = \sum_{1 \leq i \leq m} \{ D_i(t) \ln \lambda_i - N_i \lambda_i t \},$$

$$U_t = \sum_{1 \leq i \leq m} \left\{ \sum_{1 \leq j \leq D_i(t)} \ln \hat{\lambda}_{ij} - N_i \int_0^t \hat{\lambda}_i(u) du \right\}$$

where  $\hat{\lambda}_{ij} = \hat{\lambda}_i(t_{ij} - 0)$  is the point estimate of the left limit of estimate  $\hat{\lambda}_i(t)$  at point  $t_{ij}$ .

### 15.5.3 Markov Model of Tests with Censorship and Unit Renewal

We have considered particular cases of tests: failed units are always instantaneously replaced and no unit tests are stopped before the total test completion. However these results can be extended onto more general models.

In Section 3.6 we introduced a general Markov model [MMR] of tests of identical units with TTF d.f.  $F(x, \square)$ . This model allows us to consider test censorship as well as unit renewal. Let us now introduce more general model for which we use notation  $[MMR]_m$  which differs from model [MMR] by following: there are units of  $m$  different types. In other words,  $[MMR]_m$  represents a multi-dimensional analogue of [MMR].

At moment  $t=0$  we begin to test  $N_i=N_i(0)$  identical units of type  $i$ . TTF of each of them is non-negative r.v.  $\square_i$  with d.f.  $F_i(x, \square_i)$  where  $\square_i$  is an unknown vector parameter,  $1 \leq i \leq m$ . The model is given by sequences

$$(\square_{i1}, n_{i1}), \dots (\square_{ik}, n_{ik}), \dots$$

$$(\tilde{\sigma}_{i1}, \tilde{n}_{i1}), \dots (\tilde{\sigma}_{il}, \tilde{n}_{il}), \dots$$

where  $\square_{i1} < \square_{i2} < \dots < \square_{ik} < \dots$  are Markov ("independent on future") moments of termination of test of units of  $i$ th type,  $\square_{i1} < \square_{i2} < \dots < \square_{il} < \dots$  are Markov moments of placing on test new units of  $i$ th type;  $n_{ik}$  is the number of units of the  $i$ th type whose test is terminated at moment  $\square_{ik}$ ,  $\tilde{n}_{il}$  is the number of new units of the  $i$ th type whose test begins at moment  $\square_{il}$ . R.v.  $n_{ik}=0, 1, 2, \dots$  might depend on prehistory of test process before moment  $\square_{ik}$  but does not depend on the future developing of the process for  $t > \square_{ik}$ . Analogous r.v.  $\tilde{n}_{il}=0, 1, 2, \dots$  might depend on behavior of test process before moment  $\square_{il}$  but does not depend on the future developing of the process for  $t > \square_{il}$ . (More detailed and accurate formal definition are given in Section 3.6 above and in pp.1, 2, and 7 of Appendix to Chapter 3.)

Denote

$$0 < t_{i1} < t_{i2} < \dots < t_{ij} < \dots$$

sequential failure moments of units of the  $i$ th type, where  $t_{ij}$  is the moment of failure of unit  $j$  of type  $i$ .

Introduce the following notation:

$D_i(t)$  is the number of  $i$ th unit failures on interval  $(0, t]$ , that is, the number of failure moments  $t_{ij}$  such that  $t_{ij} \leq t$ ,

$N_i(t)$  is the number of units of the  $i$ th type which began to be tested and whose test has not been terminated before  $t$ :

$$N_i(t) = N_i(0) + B_i(t) - D_i(t) - L_i(t)$$

where



$B_i(t) = \sum_{l: \sigma_{il} \leq t} \tilde{n}_{il}$  is the number of units of the  $i$ th type which began to be tested on interval  $(0, t]$ ,

$L_i(t) = \sum_{k: \tau_{ik} \leq t} n_{ik}$  is the number of units of the  $i$ th type whose test has been terminated (before failure) on interval  $(0, t]$ ,

$u_{ir}$  is test time of unit  $r$  of type  $i$  which test has been terminated on interval  $(0, t]$  before failure,  $1 \leq r \leq L_i(t)$ ,

$s_{ij}$  is test time of unit of type  $i$  which has failed at moment  $t_{ij}$ ,

$S_{ie}(t)$  is the total test time of unit  $e$  of type  $i$  which is on the test at moment  $t$ ,  $1 \leq e \leq N_i(t)$ .

Let  $R=R(\square)=R(\square_1, \dots, \square_m)$  be a function (reliability index) from unit parameter vector  $\square=(\square_1, \dots, \square_m)$ . Sequential  $(1-\square)$ -LCL and seq.  $(1-\square)$ -UCL for  $R=R(\square)$  are calculated for this model by formulae analogous to those in previous Section 9.5.2 (see also Pavlov, 1985 and 1988):

$$\underline{R}_t = \max_{u \leq t} \underline{R}'_u, \tag{9.134}$$

$$\overline{R}_t = \max_{u \leq t} \overline{R}'_u$$

where values  $\underline{R}'_t$  and  $\overline{R}'_t$  are determined during the test process at each current time moment  $t$  from the corresponding equations in respect to  $C$ :

$$V_{0t} = U_t - \ln\left(\frac{1}{\alpha}\right) \tag{9.135}$$

$$V_{1t} = U_t - \ln\left(\frac{1}{\beta}\right)$$

where

$$V_{0t} = \max_{R(\theta) \leq C} H_t(\theta), \tag{9.136}$$

$$V_{1t} = \max_{R(\theta) \geq C} H_t(\theta),$$

where, in turn,

$$H_t(\theta) = \sum_{1 \leq i \leq m} \left\{ \sum_{1 \leq j \leq D_i(t)} \ln \lambda_i(s_{ij}, \theta_i) - \sum_{1 \leq j \leq D_i(t)} \Lambda_i(s_{ij}, \theta_i) - \sum_{1 \leq e \leq L_i(t)} \Lambda_i[u_{ir}, \theta_i] - \sum_{1 \leq e \leq N_i(u)} \Lambda_i[S_{ie}(t), \theta_i] \right\} \quad (9.137)$$

$$U_t = \sum_{1 \leq i \leq m} \left\{ \sum_{1 \leq j \leq D_i(t)} \ln \lambda_i(s_{ij}, \hat{\theta}_{ij}) - \int_0^t \sum_{1 \leq e \leq N_i(u)} \lambda_i[S_{ie}(u), \hat{\theta}_i(u)] du \right\} \quad (9.138)$$

where  $\hat{\theta}_i(t)$  is, as before, the point estimate of parameter  $\square_i$  obtained by test results on time interval  $(0, t]$ ,  $\hat{\theta}_{ij} = \hat{\theta}_i(t_{ij} - 0)$  is the left limit of function  $\hat{\theta}_i(t)$  at point  $t_{ij}$ .

Formulas (9.134)-(9.138) are similar to (9.129)-(9.133) of previous Section and are their generalization.

Consider now some particular cases.

Example 9.5.6. (Markov model [MMR] for identical units)

Consider a particular case of model [MMR]<sub>m</sub> for  $m=1$ , that is where all tested units are identical with the same TTF d.f.  $F(x, \square)$  where  $\square$  is some vector parameter. Remember that in Chapter 3 we considered non-parametrical case related to model [MMR]. Formulae (9.134)-(9.138) allow to obtain corresponding results for parametrical case. Let  $R=R(\square)$  be some reliability index, for instance, the unit PFFO,

$$R(\square) = 1 - F(t_0, \square) \quad (9.139)$$

or MTBF

$$R(\theta) = \int_0^{\infty} [1 - F(x, \theta)] dx \quad (9.140)$$

Using formulae (9.134)-(9.138) with  $m=1$ , we find that sequential  $(1-\square)$ -LCL and  $(I-\square)$ -UCL for  $R=R(\square)$  are calculated for each current moment of time  $t$  as follows:

$$\underline{R}_t = \max_{u \leq t} R'_u \quad (9.141)$$

$$\bar{R}_t = \min_{u \leq t} \bar{R}'_u$$

where values of  $\underline{R}'_t$  and  $\bar{R}'_t$  for moment  $t$  are defined from the following equations in respect to  $C$ :

$$V_{0t} = U_t - \ln\left(\frac{1}{\alpha}\right) \quad (9.142)$$

$$V_{1t} = U_t - \ln\left(\frac{1}{\beta}\right)$$

where

$$V_{0t}(c) = \max_{R(\theta) \leq c} H_t(\theta), \quad (9.143)$$

$$V_{1t}(c) = \max_{R(\theta) \geq c} H_t(\theta),$$

where, in turn,

$$U_t = \sum_{1 \leq j \leq D(t)} \ln \lambda(s_j, \hat{\theta}_j) - \int_0^t \sum_{1 \leq e \leq N(u)} \lambda[S_e(u), \hat{\theta}(u)] du \quad (9.145)$$

$$H_t(\hat{\theta}) = \sum_{1 \leq j \leq D(t)} \ln \lambda(s_j, \hat{\theta}) - \sum_{1 \leq e \leq N(t)} \Lambda(s_j, \hat{\theta}) -$$

$$- \sum_{1 \leq r \leq L(t)} \Lambda[u_r, \hat{\theta}] - \sum_{1 \leq e \leq N(t)} \Lambda[S_e(t), \hat{\theta}] \quad (9.144)$$

where  $\lambda(x, \theta)$  is unit failure rate,  $\Lambda(x, \theta) = -\ln [1 - F(x, \theta)]$  is unit resource function,  $t_j$  is the moment of the  $j$ th failure during the test,  $s_j$  is the test time of unit failed at moment  $t_j$ ,  $D(t)$  is the number of failures on interval  $(0, t]$ ,  $N(t)$  is the number of tested units at current moment  $t$ ,  $L(t)$  is the number of unit whose test have been terminated (without failure) on interval  $(0, t]$ ,  $u_r$  is time which unit  $r$  was tested on interval  $(0, t]$  until stopping the test (without failure),  $1 \leq r \leq L(t)$ ,  $S_e(t)$  is test time for unit  $e$  which is under test at current moment  $t$ ,  $1 \leq e \leq N(t)$ ,  $\hat{\theta}(t)$  is point estimate of parameter  $\square$  by test results on interval  $(0, t]$ ,  $\hat{\theta}_j = \hat{\theta}(t_j - 0)$  is left limit of estimate  $\hat{\theta}(t)$  at moment  $t_j$ . ■

Example 9.5.7. (Weibull-Gnedenko distribution)

In conditions of previous example, let us consider a case where d.f. of unit TTF has Weibull-Gnedenko distribution, that is,

$$F(x, \square) = F(x, \square, \square) = 1 - \exp(-\square x^\square).$$

In this case  $\square$  is two-dimensional parameter:  $\square = (\square, \square)$ . Any chosen reliability index is a function  $R = R(\square, \square)$  of this two-dimensional parameter. For instance, reliability index of type (9.139), that is, the unit PFFO during time  $t_0$ , in this case has the form

$$R(\square, \square) = \exp(-\square t_0^\square).$$

Another standard reliability index of type (9.140), that is, the unit MTTF, is written in this case as

$$R(\lambda, \alpha) = \int_0^\infty e^{-\lambda x^\alpha} dx = \frac{\Gamma\left(1 + \frac{1}{\alpha}\right)}{\lambda^{\frac{1}{\alpha}}}$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

is the Gamma function.

For this d.f. the failure rate and "resource function" have the following form

$$h(x, \square) = \square \square x^{\square-1} \quad \text{and} \quad H(x, \square) = \square x^\square.$$

By substituting these functions into (9.144)-(9.145), we find that confidence limits  $\underline{R}_t$  and  $\bar{R}_t$  for reliability index  $R = R(\square, \square)$  for Weibull-Gnedenko distribution are determined for current moment  $t$  by (9.141)-(9.142) where

$$V_{0t}(c) = \max_{R(\lambda, \alpha) \leq c} H_1(\lambda, \alpha),$$

$$V_{1t} = \max_{R(\lambda, \alpha) \geq c} H_t(\lambda, \alpha),$$

where, in turn,

$$H_t(\lambda, \alpha) = D(t) \cdot \ln(\lambda \alpha) + (\alpha - 1) \sum_{1 \leq j \leq D(t)} \ln s_j - \lambda \sum_{1 \leq j \leq D(t)} s_j^\alpha - \lambda \sum_{1 \leq r \leq L(t)} u_r^\alpha - \lambda \sum_{1 \leq e \leq N(t)} S_e^\alpha(t),$$

$$U_t = \sum_{1 \leq j \leq D(t)} \ln(\hat{\lambda}_j \hat{\alpha}_j) + \sum_{1 \leq j \leq D(t)} (\hat{\alpha}_j - 1) \cdot \ln s_j - \int_0^t \sum_{1 \leq e \leq N(u)} \hat{\lambda}(u) \hat{\alpha}(u) [S_e(u)]^{\hat{\alpha}(u)-1} du$$

where  $\hat{\lambda}(t)$  and  $\hat{\alpha}(t)$  are point estimates of parameters  $\lambda$  and  $\alpha$  obtained by test results on time period  $(0, t]$ ,  $\hat{\lambda}_j = \hat{\lambda}(t_j - 0)$  and  $\hat{\alpha}_j = \hat{\alpha}(t_j - 0)$  are left limits of these estimates at moment  $t_j$ . ■

## **16.**

**16.1** sssssssssss

**16.2** sssssssssssssss

**16.3** sssssss

**16.4** sssssss

**16.5** sssssss

16.5.1 sssssss

16.5.2 sssss

16.5.3 sssss

**Example 9.5.8** (Renewal process)

In conditions of Example 9.5.6 let us consider a particular case of the renewal process with d.f.  $F(x, \square)$ .

In this case replacement is instant, that is,  $L(t)=0$  always. The number of units  $N(t)$  under testing at any current moment  $t$  is also constant,  $N(t)=1$ . Sequential LCL and UCL for reliability index  $R=R(\square)$  at moment  $t$  are determined by (9.141)-(9.145). In this case (9.144)-(9.145) can be simplified:

$$H_t(\theta) = \sum_{1 \leq j \leq D(t)} \ln \lambda(s_j, \theta) - \sum_{1 \leq j \leq D(t)} \Lambda(s_j, \theta) - \Lambda[S(t), \theta],$$

$$U_t = \sum_{1 \leq j \leq D(t)} \ln \lambda(s_j, \hat{\theta}_j) - \int_0^t \lambda[S(u), \hat{\theta}(u)] du,$$

where  $S(t)$  is the test time of the unit under test at current moment  $t$ , that is,  $S(t)=t-t_j$  where  $t_j$  is the moment of the last failure in time period  $(0, t]$ .

On the basis of sequential confidence limits, we can construct corresponding criteria for test of hypotheses of types  $H_0: R \leq R_0$  and  $H_1: R \geq R_1$  in respect to reliability index  $R=R(\square)$ . ■

**16.5.4** Sequential confidence limits for availability coefficient of renewable unit

Consider a unit with d.f. of TTF equal to  $F_1(x, \square_1)$  and d.f. of renewal time equal to  $F_2(x, \square_2)$  where  $\square_1$  and  $\square_2$  are unknown parameters, in general case, vectors. Consider a simple test scheme with test results in the form of alternating independent random intervals

$$(s_{1n}, s_{2n}), \quad n=1, 2, \dots \tag{9.146}$$

where  $s_{1n}$  is TTF of the unit, and  $s_{2n}$  is its random renewal time at the  $n$ th step of test. All r.v.'s are independent. Thus (9.146) represent a alternating renewal process. All r.v.'s  $s_{1n}$  have d.f.  $F_1(x, \square_1)$ , and all  $s_{2n}$  have d.f.  $F_2(x, \square_2)$ . The test results of such a test are sequential moment of failures and renewals of the unit:

$$t_{1n} = \sum_{1 \leq j \leq n-1} (s_{1j} + s_{2j}) + s_{1n},$$

$$t_{2n} = \sum_{1 \leq j \leq n} (s_{1j} + s_{2j})$$

where  $t_{1n}$  is the moment of the  $n$ th failure, and  $t_{2n}$  is the moment of the  $n$ th renewal,  $n=1, 2, \dots$ . Graphical illustration of this process is given in Figure 9.11 where  $I(t)$  is an indicator function such that  $I(t)=0$  if a unit is in renewal state and  $I(t)=1$  if a unit is in operational state at moment  $t$ .

Figure 9.11

(Test process of renewal unit:  $I(t)$  is indicator function)

A standard reliability index for a renewable unit is its availability coefficient, that is, the stationary probability of operational state of the unit

$$K = \lim_{t \rightarrow \infty} P\{I(t) = 1\}.$$

This reliability index can be found by well known formula

$$K = K(\theta_1, \theta_2) = \frac{T_1(\theta_1)}{T_1(\theta_1) + T_2(\theta_2)} \quad (9.147)$$

where

$$T_1(\theta_1) = \int_0^{\infty} [1 - F_1(x, \theta_1)] dx$$

is the unit's TTF, and

$$T_2(\theta_2) = \int_0^{\infty} [1 - F_2(x, \theta_2)] dx$$

is the unit's mean renewal time.

Applying general formulae (9.134)-(9.138), we obtain that sequential  $(1-\alpha)$ -LCL,  $\underline{K}_t$ , and  $(1-\alpha)$ -UCL,  $\overline{K}_t$ , for reliability index (9.147) can be calculated for a current moment  $t$  in the following manner:

$$\underline{K}_t = \max_{u \leq t} \underline{K}'_u \quad (9.148)$$

$$\overline{K}_t = \min_{u \leq t} \overline{K}'_u$$

where values  $\underline{K}'_t$  and  $\overline{K}'_t$  are found from the following equations

$$V_{0t}(C) = U_t - \ln\left(\frac{1}{\alpha}\right) \quad (9.149)$$

$$V_{1t}(C) = U_t - \ln\left(\frac{1}{\beta}\right)$$

where

$$V_{0t}(C) = \max_{K(\theta_1, \theta_2) \leq C} H_t(\theta_1, \theta_2) \quad (9.150)$$

$$V_{1t}(C) = \max_{K(\theta_1, \theta_2) \geq C} H_1(\theta_1, \theta_2)$$

where, in turn,

$$H_t(\theta_1, \theta_2) = \sum_{1 \leq j \leq D_1(t)} \ln \lambda_1(s_{1j}, \theta_1) - \sum_{1 \leq j \leq D_1(t)} \Lambda_1(s_{1j}, \theta_1) - \int_0^t \lambda_1[S_1(u), \hat{\theta}_1(u)][1 - I(u)] du - \int_0^t \lambda_2[s_2(u), \theta_2(u)] I(u) du \quad (9.152)$$

$$- [1 - I(t)] \Lambda_1[S_1(t), \theta_1] + \sum_{1 \leq j \leq D_2(t)} \ln \lambda_2(s_{2j}, \theta_2) -$$

$$- \sum_{1 \leq j \leq D_2(t)} \Lambda_2(s_{2j}, \theta_2) - I(t) \Lambda_2[s_2(t), \theta_2], \quad (9.151)$$

$$U_t = \sum_{1 \leq j \leq D_1(t)} \ln \lambda_1(s_{1j}, \hat{\theta}_{1j}) + \sum_{1 \leq j \leq D_2(t)} \ln \lambda_2(s_{2j}, \hat{\theta}_{2j}) -$$

where  $\square_i(x, \square_i) = -\ln [1 - F_i(x, \square_i)]$ ,  $i=1$  or  $2$ , are "leading functions" for failure and renewal,

$$\lambda_i(x, \theta_i) = \frac{d}{dx} \Lambda_i(x, \theta_i)$$



are corresponding failure and renewal rates,  $D_i(t)$  are corresponding numbers of failures (renewals) on interval  $(0, t]$ ,  $S_i(t)$  are total times when a unit was in operational (down) state if it is in operational (down) state at moment  $t$ ,  $\hat{\theta}_i(t)$  is point estimate of parameter  $\square_i$  on the base of test results on interval  $(0, t]$ ,  $\hat{\theta}_{1j} = \hat{\theta}_1(t_{1j} - 0)$  is left limit of estimate  $\hat{\theta}_1(t)$  and  $\hat{\theta}_{2j} = \hat{\theta}_2(t_{2j} - 0)$  is left limit of estimate  $\hat{\theta}_2(t)$  at the moment of the  $j$ th failure (renewal), that is,  $t_{1j}$  and  $t_{2j}$ , respectively.

Suppose we need to test two composite hypotheses

$$H_0: K \leq K_0 \text{ and } H_1: K \geq K_1$$

with respect to availability coefficient  $K = K(\square_1, \square_2)$ . Then the sequential criterion for test of hypotheses with risks of type I and II can be constructed on the basis of sequential confidence limits (9.148) in the same manner as it was done in Section 9.4.2. Namely, the area of test continuation at moment  $t$  is given by inequalities

$$\underline{K}_t < K_0, \quad K_t > \overline{K}_t \quad (9.153)$$

The test continues if both inequalities hold and are stopped at the moment when at least one of them is violated. If the first inequality is violated, hypothesis  $H_1$  is accepted, if the second is violated, then hypothesis  $H_0$  is accepted.

Notice that in this model with continuous time, moments of renewals,  $t_{2n}$ , are moments of regeneration for random process  $I(t)$ , and pairs of r.v.'s  $(s_{1n}, s_{2n})$ , defined in (9.146), form i.i.d. "regeneration cycles". Thus,  $I(t)$  is alternating renewal process formed with two-dimensional r.v.'s  $(s_{1n}, s_{2n})$ . Therefore properties of asymptotic optimality (for  $\square \rightarrow 0$  and  $\square \rightarrow \infty$ ) of sequential confidence limits (9.148) and sequential criterion of test of hypotheses (9.153) follow from corresponding results for the scheme of independent and identical tests (see p.3 of Appendix and Pavlov, 1988, 1990). In general case where one can not distinguish such regeneration moments, corresponding properties of sequential confidence limits above (9.134)-(9.138) need further investigations.

### Exponential model

Let us take a particular case of the test scheme considered above, where d.f. of unit's TTF is exponential:  $F_1(x, \square_1) = 1 - \exp(-\square_1 x)$  with unknown parameter  $\square_1$  (the failure rate), and d.f. of unit's renewal time is also exponential:  $F_2(x, \square_2) = 1 - \exp(-\square_2 x)$  with unknown parameter  $\square_2$  (the renewal rate). In this case  $\square_1 = \lambda_1$  and  $\square_2 = \lambda_2$ . Availability coefficient (9.147) has the form

$$K = K(\lambda_1, \lambda_2) = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Denote the total time of unit operation on interval  $(0, t]$  by  $W_1(t)$ , the total time of unit operation on this interval by  $W_2(t)$ . Obviously,  $W_1(t) + W_2(t) = t$ . These values are determined as follows:

$$W_1(t) = \sum_{1 \leq j \leq D_1(t)} s_{1j} + s_1(t)[1 - I(t)],$$

$$W_2(t) = \sum_{1 \leq j \leq D_2(t)} s_{2j} + s_2(t)I(t).$$

Let us also introduce

$$\hat{\lambda}_1(t) = \frac{D_1(t) + D_1^0}{W_1(t) + W_1^0},$$

$$\hat{\lambda}_2(t) = \frac{D_2(t) + D_2^0}{W_2(t) + W_2^0}$$

which are point estimate of parameters  $\square_1$  and  $\square_2$  based on the test results obtained on interval  $(0, t]$ . Values  $W_1^0$  and  $D_1^0$  allow us to take prior information into account (if available) about parameter  $\square_1$ . For instance,  $W_1^0$  might be the total test time and  $D_1^0$  the number of failures during some previous tests.  $W_2^0$  and  $D_2^0$  might play the same role in respect to parameter  $\square_2$ .

Applying formulae (9.148)-(9.252), we obtain that in this case  $(1-\square)$ -LCL,  $\underline{K}_t$ , and  $(1-\square)$ -UCL,  $\overline{K}_t$ , for the availability coefficient for moment  $t$  are calculated by formulae (9.148)-(9.149) where

$$V_{0t}(C) = \max_{K(\lambda_1, \lambda_2) \leq C} H_t(\lambda_1, \lambda_2) \quad (9.154)$$

$$V_{1t}(C) = \max_{K(\lambda_1, \lambda_2) \geq C} H_t(\lambda_1, \lambda_2) \quad (9.155)$$

where in turn

$$H_t(\lambda_1, \lambda_2) = D_1(t) \cdot \ln \lambda_1 + D_2(t) \cdot \ln \lambda_2 - \lambda_1 W_1(t) - \lambda_2 W_2(t),$$

$$U_t = \sum_{1 \leq j \leq D_1(t)} \ln \hat{\lambda}_{1j} + \sum_{1 \leq j \leq D_2(t)} \ln \hat{\lambda}_{2j} -$$

$$- \int_0^t \hat{\lambda}_1(u)[1 - I(u)]du - \int_0^t \hat{\lambda}_2(u)I(u)du,$$

where  $\hat{\lambda}_{1j} = \hat{\lambda}_1(t_{1j} - 0)$  is left limit of estimate  $\hat{\lambda}_1(t)$  and  $\hat{\lambda}_{2j} = \hat{\lambda}_2(t_{2j} - 0)$  is left limit of estimate  $\hat{\lambda}_2(t)$  at the moment of the  $j$ th failure (renewal), that is,  $t_{1j}$  and  $t_{2j}$ , respectively.

Maximum in (9.154)-(9.155) can be found in this case analytically. Let us find the first one. The restriction  $K(\square_1, \square_2) \leq C$ , for which maximum in (9.154) is being found, is equivalent to one of the following inequalities

$$\frac{\lambda_1}{\lambda_2} \geq \frac{1-C}{C}$$

or

$$\ln\left(\frac{\lambda_1}{\lambda_2}\right) \geq C'$$

where

$$C' = \ln \frac{1-C}{C}.$$

Taking into account the concavity of function  $H_t(\square_1, \square_2)$  in  $(\square_1, \square_2)$  and compiling the Lagrange function for this problem

$$L = H_t(\lambda_1, \lambda_2) + b \ln \frac{\lambda_1}{\lambda_2}$$

we obtain that maximum in (9.154) is attained at point which satisfies the following system of Lagrange equations:

$$\frac{D_1(t)}{\lambda_1} - W_1(t) = -\frac{b}{\lambda_1},$$

$$\frac{D_2(t)}{\lambda_2} - W_2(t) = \frac{b}{\lambda_2}$$

where  $b$  is Lagrange multiplier. After simple transformations, it gives the  $(1-\square)$ -LCL for availability coefficient at moment  $t$  in the form

$$\underline{K}_t = \max_{u \leq t} \underline{K}'_u$$

where value  $\underline{K}'_t$  is found at moment  $t$  by formula

$$\underline{K}'_t = \frac{W_1(D_2 - b)}{W_1(D_2 - b) + W_2(D_1 + b)}$$

where value  $b > 0$  is determined from equation

$$D_1 \ln\left(\frac{D_1 + b}{W_1}\right) + D_2 \ln\left(\frac{D_2 - b}{W_2}\right) - D_1 - D_2 = U_t - \ln\left(\frac{1}{\alpha}\right).$$

In these formulae and below we use short notation  $W_1 = W_1(t)$ ,  $W_2 = W_2(t)$ ,  $D_1 = D_1(t)$ ,  $D_2 = D_2(t)$ .

In an analogous way, by calculating maximum in (9.155) we obtain that sequential (1- $\alpha$ )-UCL for the availability coefficient at moment  $t$

$$\overline{K}_t = \min_{u \leq t} \overline{K}'_u$$

where value  $\overline{K}'_t$  is found at moment  $t$  by formula

$$\overline{K}'_t = \frac{W_1(D_2 + b)}{W_1(D_2 + b) + W_2(D_1 - b)}$$

where value  $b > 0$  is determined from equation

$$D_1 \ln\left(\frac{D_1 - b}{W_1}\right) + D_2 \ln\left(\frac{D_2 + b}{W_2}\right) - D_1 - D_2 = U_t - \ln\left(\frac{1}{\beta}\right).$$

In conclusion we would like to emphasize that Wald's plans of testing become an important part of modern test planning.

## 16.6 Appendices

### 16.6.1 Computation of accurate values of characteristics of sequential plans of test.

Assume that we observe a sequence of i.i.d. scalar r.v.'s

$$x_1, x_2, \dots, x_n, \dots$$

Denote the sum of these r.v.'s observed at first  $n$  steps by  $S_n = x_1 + \dots + x_n$ . Let us consider some sequential criterion (control plan) for hypotheses of type (9.49) for which the stop time has the form

$$\tau = \min \{n: S_n \in G_n\}$$

where  $G_n$  is the area of test stop at step  $n$ . Let  $W_n = \overline{G_n}$  be the area of test continuation at step  $n$  where  $\overline{G_n}$  is the area complement to  $G_n$ . Thus at step  $n$  the test continues if

$$S_n \in W_n \quad (9.156)$$

The test is stopped at first violation of condition (9.156), that is, at such step  $\tau = n$  when r.v.  $S_n$  first time reaches stopping area  $G_n$ . The decision rule at time stop  $\tau = n$  has the form

if  $S_n \in G_{n0}$  then hypothesis  $H_0$  is accepted  
 if  $S_n \in G_{n1}$  then hypothesis  $H_1$  is accepted

where  $G_{n0}$  and  $G_{n1}$  are non-intersected areas such that

$$G_n = G_{n0} + G_{n1}, \quad n = 1, 2, \dots$$

In other words, the area of stopping  $G_n$  is divided into two subareas:

$G_{n0}$  is the area of hypothesis  $H_0$  acceptance  
 $G_{n1}$  is the area of hypothesis  $H_1$  acceptance.

Main sequential control plans (Wald's, truncated Wald's and others) have such a form.

Now let us consider calculation of accurate characteristics such as risks of type I and II ( $\alpha$  and  $\beta$ ), operative characteristic  $L(\alpha)$  and average test volume  $N(\alpha)$ . At first consider a discrete case where r.v.  $x_n$  takes a finite set of values  $0, 1, \dots, J$ . Let

$$\alpha(j, \alpha) = P_{\alpha} \{x_n = j\}$$

be the probability that r.v.  $x_n = j$  for parameter value equal to  $\alpha$  where  $0 \leq j \leq J$ . Assume that the area of test continuation is restricted, that is, such finite  $n_m$  exists that  $W_n = \emptyset$  for  $n \geq n_m$  and, besides, area  $W_n$  is restricted for each  $n$ ,  $1 \leq n \leq n_m - 1$ . Value  $n_m$  represents the maximum possible value of stop time  $\tau$ .

Let us fix the value of parameter  $\theta$  and introduce a sequence of numbers

$$b_n(i, \theta), \quad 0 \leq i \leq n(l+1), \quad n=1, 2, \dots \quad (9.157)$$

which are recurrently calculated by

$$b_{n+1}(i, \theta) = \sum_{j \in W_n} b_n(j, \theta) \pi(i - j, \theta), \quad n = 1, 2, \dots \quad (9.158)$$

where at the first step we set  $b_1(j, \theta) = \pi(j, \theta)$ ,  $0 \leq j \leq l$ . Directly from construction of sequence (9.157) it follows that  $b_n(i, \theta)$ , determined as

$$b_n(i, \theta) = P_\theta(v > n - 1, S_n = i)$$

is the probability that the test process will not be terminated before step  $n$ , and at step  $n$  r.v.  $S_n = i$ . In other words,

$$b_n(i, \theta) = P_\theta(S_1 \in W_1, S_2 \in W_2, \dots, S_{n-1} \in W_{n-1}, S_n = i).$$

Notice that sum in (9.158) is taken over finite number of subscripts  $j$  because  $W_n$  has finite elements. Notice also that at step  $n+1$ , value of  $b_{n+1}(i, \theta)$  differs from zero only for  $i \in W_n$  and for  $i$  such that  $i \leq j+1$  where  $i \in W_n$ . So, the number of subscripts  $i$  for sum (9.158) calculation at step  $n+1$  is equal to  $k_n+1$  where  $k_n$  is the number of integer points in area of test continuation,  $W_n$ , at step  $n$ . Consequently, the volume of calculation at step  $n+1$  for this recurrent procedure (9.158) is proportional to value  $(k_n+1)(l+1)$ .

Let us introduce the following events

$$\begin{aligned} A_n &= \{\theta = n\}, \\ A_{n0} &= \{\theta = n \text{ and hypothesis } H_0 \text{ is accepted}\}, \\ A_{n1} &= \{\theta = n \text{ and hypothesis } H_1 \text{ is accepted}\} \end{aligned}$$

Their meaning is as follows:  $A_n$  is an event that the test is terminated at step  $n$ ,  $A_{n0}$  is an event that the test is terminated at step  $n$  and hypothesis  $H_0$  is accepted, and  $A_{n1}$  is an event that the test is terminated at step  $n$  and hypothesis  $H_1$  is accepted. The probabilities of these events for fixed value of parameter  $\theta$  are expressed via values  $b_n(i, \theta)$  as follows

$$P_{\theta}(A_n) = \sum_{i \in G_n} b_n(i, \theta),$$

$$P_{\theta}(A_{n0}) = \sum_{i \in G_{n0}} b_n(i, \theta),$$

$$P_{\theta}(A_{n1}) = \sum_{i \in G_{n1}} b_n(i, \theta).$$

It gives us the following expression for operative characteristic  $L(\square)$  for fixed value of parameter  $\theta$ :

$$L(\square) = P_{\square}(\text{to accept hypothesis } H_0) = \sum_{1 \leq n \leq n_m} P_{\theta}(A_{n0}) = \sum_{1 \leq n \leq n_m} \sum_{i \in G_{n0}} b_n(i, \theta) \quad (9.159)$$

Thus, it is easy to calculate operative characteristic  $L(\square)$  with the help of a computer, using (9.158) and (9.159). The precise values of risks of type I and II can be found via operative characteristic from formulas

$$\square = 1 - L(\square_0), \text{ and } \square = L(\square_1) \quad (9.160)$$

The average test volume,  $N(\square)$ , for given value of parameter  $\square$  can be also expressed via values  $b_n(i, \square)$  and found with the help of the same recurrent procedure (9.158) by formula

$$N(\theta) = E_{\theta} v = \sum_{1 \leq n \leq n_m} n P_{\theta}(A_n) = \sum_{1 \leq n \leq n_m} n \sum_{i \in G_n} b_n(i, \theta) \quad (9.161)$$

If the area of test continuation is not restricted (for instance, in Wald's plan without truncation), all characteristics, mentioned above, can be calculated on the basis of the same formulas (9.159)-(9.161) if  $n_m$  is sufficiently large.

Notice that the area of test continuation,  $W_n$ , the area of test termination,  $G_n$ , and areas  $G_{n0}$  and  $G_{n1}$  in most of cases have the form of intervals, or in other words, they are given as inequalities of the following type:

$$\begin{aligned} W_n: & g_n < S_n < h_n, \\ G_n: & g_n \geq S_n, < h_n \leq S_n, \\ G_{n0}: & g_n \geq S_n, \\ G_{n1}: & h_n \leq S_n. \end{aligned}$$

where  $g_n$  and  $h_n$  are boundaries of the area of test continuation at step  $n$ ,  $n=1, 2, \dots$ , and these values do not depend on the concrete test plan.

Example 9.A.1. (Binomial plan)

Consider a binomial test where r.v.  $x_n$  is a failure indicator at step  $n$ , parameter  $\theta = q$  is the failure probability, r.v.  $S_n = x_1 + \dots + x_n$  is a random number of failures during  $n$  steps. (See Example 9.2.1 above.) In this case  $l=1$ , r.v.  $x_n$  only takes values 0 or 1, and values  $\phi(j, \theta)$  have the form

$$\phi(j, \theta) = \phi(j, q) = \begin{cases} q & \text{if } j=1 \\ 1-q & \text{if } j=0. \end{cases}$$

In this case sum in (9.158) is taken over two subscripts and recurrent procedure (9.158) becomes extremely simple.

If r.v.  $x_n$  has continuous d.f.  $F(x, \theta)$ , then recurrent procedure (9.158) allows one also to calculate main characteristics  $L(\theta)$ ,  $N(\theta)$ ,  $\phi$ ,  $\psi$  with needed accuracy if we use a discrete approximation of continuous d.f.  $F(x, \theta)$ . Naturally, the average test volume increases with a more accurate approximation. For instance, if r.v.  $x_n$  is approximated by a discrete r.v. with  $l+1$  different states, then the computational burden on the basis of (9.158) roughly increases as  $l^2$ .

Now consider the case of continuous time for Poisson process with parameter  $\theta$  (see Example 9.2.3). In this case all above formulas can be applied if we use standard approximation. Divide the time axes onto equal small intervals  $h$ . In this case we denote the number of failures on interval  $[(n-1)h, nh]$  by  $x_n$ ,  $n=1, 2, \dots$ . In this case in formulas (9.158)-(9.161) let us take:  $l=2$ , parameter  $\theta = \theta$ , and define values  $\phi(j, \theta) = \phi(j, \theta)$ ,  $j=0, 1, 2$  as follows:

$$\phi(0, \theta) = e^{-\theta h}, \quad \phi(1, \theta) = \theta h e^{-\theta h}, \quad \phi(2, \theta) = 1 - e^{-\theta h}(1 + \theta h).$$

Obviously, we neglect the probability of occurrence of two or more failures on the interval  $h$ . The error of such approximation has the order  $o(h^2)$ . The main characteristics of different sequential test plans can be calculated with the help of recurrent procedure (9.158) and formulas (9.159)-(9.161). The accuracy of the result of calculation is defined by the size of value  $h$ . Corresponding average test volume and number of steps of the test increases proportionally to  $(1/h)$ . Of course, some more modifications for specific cases can allow to gain even more.

**Test plan allowing specified risks  $\alpha$  and  $\beta$**

Sequential Wald's test plan is characterized by two parameters  $a = \ln A$  and  $b = \ln(1/B)$  which determine the boundaries of the area of test continuation. If these parameters are given and fixed, approximate values of risks  $\alpha$  and  $\beta$  can be found by approximate Wald's formulas (9.33), and



precise ones with the help of computer with using the recurrent procedure described above. More complicated is the inverse problem: to find plan parameters  $a$  and  $b$  such that precise values of risks  $\alpha$  and  $\beta$  coincide with needed values. Notice that in general case, the proof of the fact, that a plan with precise risks  $\alpha$  and  $\beta$  for any  $\alpha + \beta < 1$  exists, is not trivial. Nevertheless, approximate solution is always available with the help of computer on the basis the following simple heuristic arguments. Let  $\alpha_0$  and  $\beta_0$  are given levels of risks. Let us consider value values

$$a_1 = \ln\left(\frac{1-\beta_0}{\alpha_0}\right)$$

and

$$b_1 = \ln\left(\frac{1-\alpha_0}{\beta_0}\right)$$

found by Wald's approximate formulas (9.33) as the first iteration.

Then with the help of computer, calculate precise values of risks

$\alpha_1$  and  $\beta_1$  for the plan with parameters  $a_1$  and  $b_1$ . Then we change parameters  $a$  and  $b$  in such a way that the precise values of risks become close to the given values of  $\alpha$  and  $\beta$ . Here we can use the property of the operative characteristic  $L(\alpha)$ : for any fixed  $\beta$  this function is monotone increasing in  $a$  for fixed  $b$ , and, on the contrary, this function is monotone decreasing in  $b$  for fixed  $a$ . Notice that the risk of type I,  $1-L(\alpha_0)$ , is sensitive to the variation of parameter  $a$  and substantially less sensitive to the variation of parameter  $b$ . The risk of type II,  $L(\beta_1)$ , is sensitive, on the contrary, to the variation of parameter  $b$  and substantially less sensitive to the variation of parameter  $a$ . It means that if we increase parameter  $a$  for fixed  $b$ , then  $1-L(\alpha_0)$  decreases and, simultaneously,  $L(\beta_1)$  increases though with slower rate. On the contrary, if we increase parameter  $b$  for fixed  $a$ , then  $L(\beta_1)$  decreases and  $1-L(\alpha_0)$  increases slowly. Knowing about these monotone dependencies, we can alternately change parameters  $a$  and  $b$  and calculate corresponding precise values of risks on a computer, and iteratively adjust plan parameters  $a$  and  $b$  in such a way that these risks coincide at last with given values  $\alpha$  and  $\beta$ .

Analogously, the same dialogue procedure with a computer helps in cases of other test plans (for instance, truncated Wald's plan).

## 16.6.2 Wald's Equivalency

Assume that we observe a sequence of i.i.d. r.v.'s

$$z_1, z_2, \dots, z_n, \dots$$

with the mathematical expectation  $E|z_n| < \infty$ . Let  $\mathcal{F}_n$  be some Markov moment (in respect to a system of  $\sigma$ -algebra  $F_n = \sigma(z_1, \dots, z_n)$ ,  $n = 1, 2, \dots$ , related to the sequence  $z_n$ ) such that

$$P(\tau < \infty) = 1.$$

Consider the mathematical expectation of the following sum

$$E \sum_{1 \leq n \leq \nu} z_n = E(z_1 + \dots + z_\nu).$$

This value can be presented in the form

$$E \sum_{1 \leq n \leq \nu} z_n = E \sum_{1 \leq n < \infty} z_n I(\nu \geq n)$$

where  $I(\nu \geq n)$  is indicator of event  $\{\nu \geq n\}$ . Event  $\{\nu \geq n\} \in \mathcal{F}_{n-1}$ , in other words, occurrence of this event depends only on values of  $z_1, \dots, z_{n-1}$  observed at first  $n-1$  steps. It follows that r.v.'s  $z_n$  and  $I(\nu \geq n)$  are independent. So,

$$E \sum_{1 \leq n < \infty} z_n = \sum_{1 \leq n < \infty} E \{z_n I(\nu \geq n)\} = \sum_{1 \leq n < \infty} E z_n \cdot E I(\nu \geq n).$$

Since mathematical expectation  $E z_n = E z$  does not depend on  $n$ , it follows that

$$E \sum_{1 \leq n \leq \nu} z_n = E z \sum_{1 \leq n < \infty} E I(\nu \geq n) = E z \sum_{1 \leq n < \infty} P(\nu \geq n) = E z E \nu$$

It gives us the Wald's equivalency.

### 16.6.3 Sequential optimal test criterion for composite hypotheses

Let us observe sequence of i.i.d. r.v.'s

$$x_1, x_2, \dots, x_n, \dots \quad (9.162)$$

R.v.  $x_n$  takes its values from a measurable space  $(X, \mathcal{B})$ , where  $X$  is complete separable metrics space, and  $\mathcal{B}$  is its Borel's  $\sigma$ -algebra. Distribution of  $x_n$  is given by density  $f(x, \theta)$  in respect to some  $\sigma$ -finite measure  $\mu$  on  $(X, \mathcal{B})$ . Here  $\theta = (\theta_1, \dots, \theta_m)$  is vector parameter taking its values from some subset  $\Theta$  of  $m$ -dimensional Euclid space  $R_m$ . We assume that set  $\{x: f(x, \theta) > 0\}$  does not depend on  $\theta \in \Theta$ .

Let  $(\Omega, \mathcal{F}, P_\theta)$  be the probabilistic space on which the process (9.162) is defined,  $\mathcal{F}_n = \sigma(x_1, \dots, x_n)$ ,  $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ ,  $n=1, 2, \dots$ , be a system of  $\sigma$ -algebra related to process (9.162),  $P_\theta$  and  $E_\theta$  be

the probabilistic measure and the mathematical expectation for given value of parameter  $\theta \in \Theta$ . Denote standard Kullback-Leibler "information distance" between points  $\theta$  and  $\varphi$  from  $\Theta$  by

$$\rho(\theta, \varphi) = E_{\theta} \ln \left[ \frac{f(x_n, \theta)}{f(x_n, \varphi)} \right].$$

The value

$$\rho(\theta, D) = \inf_{\varphi \in D} \rho(\theta, \varphi)$$

is called the "distance" from point  $\theta$  to set  $D \subset \Theta$ .

Let set  $\Theta$  of all possible values of parameter  $\theta$  be divided into  $l+1$  non-intersected subsets:

$$\Theta = D_1 \cup D_2 \cup \dots \cup D_l \cup I$$

where  $I$  is the area of "indifference". Let there be a correspondence between each point  $\theta$  and index  $K(\theta)$  of "closest" set  $D_i$ , i.e.,  $K(\theta)$  is defined by condition

$$\rho(\theta, D_{K(\theta)}) = \min_{1 \leq i \leq l} \rho(\theta, D_i)$$

Introduce the value

$$\Lambda(\theta) = \min_{i \neq K(\theta)} \rho(\theta, D_i) \quad (9.163)$$

assuming that  $\Lambda(\theta) > 0$  for all points  $\theta \in D_1 \cup D_2 \cup \dots \cup D_l$ .

Sequential criterion for test of  $l$  composite hypotheses

$$H_i: \theta \in D_i, \quad 1 \leq i \leq l \quad (9.164)$$

by pair  $(\tau, d)$  where  $\tau$  is Markovian (in respect to the system of  $\sigma$ -algebra  $F_n, n=1, 2, \dots$ ) stop time, and  $d$  is a decision made at the stop time  $\tau$ , or in other words,  $d$  is  $F_{\tau}$ -measurable function taking  $l$  values. Hypothesis  $H_i$  is accepted if  $d=i, 1 \leq i \leq l$ .

Introduce  $\theta = (\theta_1, \dots, \theta_l)$ . Denote by  $K_{\theta}$  a class of all  $(\tau, d)$  such that

$$P_{\theta}(d=i) \leq \alpha_i \text{ for all } \theta \in D_i, \quad 1 \leq i \leq l \quad (9.165)$$

In other words,  $K_{\alpha}$  is a class of all criteria  $(\alpha, d)$  with the probabilities of errors not larger given values  $\alpha_1, \dots, \alpha_l$ . Further we construct criterion  $(\alpha^*, d^*) \in K_{\alpha}$  which is asymptotically optimal in class  $K_{\alpha}$  in sense of the average test volume for  $\alpha_i \geq 0, 1 \leq i \leq l$ .

Further discussion is based on the following idea. At first, for any hypothesis  $H_i$ , we construct a Markovian (in respect to system  $F_n$ ) moment of exclusion of this hypothesis  $\alpha_i$  such that

$$P_{\alpha}(\alpha_i < \alpha) \leq \alpha_i \text{ for all } \alpha \in D_i.$$

This inequality means that the probability of exclusion of hypothesis  $H_i$  (that is, to decide that it is untrue) at some test step does not exceed  $\alpha_i$  if the hypothesis is true.

Let  $\alpha(\alpha, D)$  be the moment of exclusion of a composite hypothesis of a general type  $H: \alpha \in D$ . (This moment is defined in (9.77).) The moment of exclusion of hypothesis  $H_i: \alpha \in D_i$  is defined as

$$\alpha_i = \alpha(\alpha_i, D_i), \quad 1 \leq i \leq l. \quad (9.166)$$

Sequential criterion  $(\alpha^*, d^*)$  for test of composite hypotheses (9.164) can be constructed on the basis of the moments of exclusion (9.166) as follows. Denote ordered in time moments of exclusion (9.166) by

$$\alpha_{(1)} \leq \alpha_{(2)} \leq \dots \leq \alpha_{(l-1)} \leq \alpha_{(l)}.$$

Now define the time stop,  $\alpha^*$ , and the decision made at this moment,  $d^*$ , as follows:

$$\alpha^* = \alpha_{(l-1)} \quad (9.167)$$

$$d^* = \arg \max_{1 \leq i \leq l} v_i$$

that is, the test continues until moment  $\alpha^*$  when all hypotheses except one are excluded and a single one, which is not excluded, is accepted. In particular case  $l=2$  sequential criterion of composite hypotheses (9.167) takes the form

$$\alpha^* = \min(\alpha_1, \alpha_2)$$

$$d^* = \arg \max_{1 \leq i \leq 2} v_i = \begin{cases} 1, & \text{if } v_2 < v_1 \\ 2, & \text{if } v_1 < v_2 \end{cases}$$

that is, the test continues until the moment of exclusion of one of hypotheses, and at the moment the remaining hypothesis is accepted. If  $\alpha_1 = \alpha_2$  at some moment, then any of these two hypotheses is accepted.

Using results (Pavlov, 1985, 1990), we can show that for general enough conditions, moment of exclusion  $v(\alpha, D)$  is asymptotically optimal for  $\alpha \rightarrow 0$ . It follows that sequential criterion  $(\alpha^*, d^*)$  is also optimal (see Theorems 9.1-9.3 below).

Lemma 9.1. Moment  $v(\alpha, D)$  satisfies inequality

$$P_\theta \{v(\alpha, D) < \infty\} \leq \alpha \text{ for all } \theta \in D \quad (9.168)$$

Proof. Let each  $\alpha$  correspond to moment

$$v_\alpha = \min \left( n: \xi_n(\theta) \geq \frac{1}{\alpha} \right)$$

where

$$\xi_n(\theta) = \frac{\prod_{1 \leq r \leq n} f(x_r, \hat{\theta}_{r-1})}{\prod_{1 \leq r \leq n} f(x_r, \theta)}.$$

For each fixed  $\alpha$  sequence  $\alpha_n(\alpha)$ ,  $n=1, 2, \dots$  represents a non-negative martingale with respect to system  $F_n$ ,  $n=1, 2, \dots$  with mathematical expectation  $E_\alpha \alpha_n(\alpha) = 1$ . Applying Doob-Kolmogorov inequality, for each  $n=1, 2, \dots$  we have

$$P_\alpha (v_\alpha \leq n) = P_\alpha \left( \max_{1 \leq r \leq n} \xi_r(\theta) \geq \frac{1}{\alpha} \right) \leq \alpha,$$

and then

$$P_\alpha (v_\alpha < \infty) \leq \alpha.$$

for all  $\alpha \in (0, 1]$ . Moreover,

$$v(\alpha, D) \leq \sup_{\theta \in \Theta} v_\alpha.$$

It follows that for each  $\alpha \in (0, 1]$  inequality

$$P_{\theta}\{v(\alpha, D) < \infty\} \leq P_{\theta}\{\sup_{\theta \in D} v_{\theta} < \infty\} \leq P_{\theta}\{v_{\theta} < \infty\} \leq \alpha$$

holds. This proves (9.168). ■

The following lemma gives the LCL for the mathematical expectation of the moment of exclusion of a composite hypothesis  $H: \theta \in D$  in case where it is untrue, that is,  $\theta \in \bar{D}$ .

**Lemma 9.2** Let  $D$  be some subset  $\Omega$  and  $\rho$  be a Markovian moment (in respect to system  $F_n$ ,  $n=1, 2, \dots$ ) such that inequality  $P_{\theta}(\rho < \alpha) \leq \alpha$  for any  $\theta \in D$  where  $0 < \alpha < 1$ . Then mathematical expectation of this moment satisfies inequality

$$E_{\theta} v \geq \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, D)} \quad (9.169)$$

for any  $\theta \in \bar{D}$  where  $\bar{D} = \Omega \setminus D$ .

**Proof.** Let  $\theta \in \bar{D}$ , that is, hypothesis  $H: \theta \in D$  is untrue. If  $P_{\theta}(\rho < \alpha) < 1$  then  $E_{\theta} \rho = \alpha$  and inequality (9.169) is trivial. Let  $P_{\theta}(\rho < \alpha) = 1$ . Let us choose some point  $\theta_0 \in D$ . Consider two simple hypotheses  $h_0$  and  $h_1$  which corresponds points  $\theta_0$  and  $\theta$ . Let us fix some moment  $n < \infty$  and consider the following sequential criterion for test of these simple criteria. Test continues until truncated Markovian moment

$$\rho_n = \min(\rho, n).$$

If  $\rho_n < n$ , then hypothesis  $h$  is accepted. If  $\rho_n = n$ , then hypothesis  $h_0$  is accepted. For this criterion probability of error of type I (that is, to accept hypothesis  $h$  when hypothesis  $h_0$  is true) satisfies the following relations:

$$P_{\theta_0}(v_n < n) = P_{\theta_0}(v < n) \leq P_{\theta_0}(v < \infty) \leq \alpha.$$

The probability of error of type II (that is, to accept hypothesis  $h_0$  when hypothesis  $h$  is true) is denoted by:

$$\beta_n = P_{\theta}(v_n = n) = P_{\theta}(v \geq n).$$

Since  $P_{\theta_0}(\theta < \theta_0) = 1$ , sequence  $\theta_n \rightarrow \theta_0$  for  $n \rightarrow \infty$ . Using the lower limit (9.48) of the average test volume for any criterion for two simple hypotheses  $h_0$  and  $h$ , we have

$$E_{\theta} v_n \geq \frac{\omega(\beta_n, \alpha)}{\rho(\theta, \theta_0)} \quad (9.170)$$

where

$$\omega(\beta, \alpha) = (1 - \beta) \ln\left(\frac{1 - \beta}{\alpha}\right) + \beta \cdot \ln\left(\frac{\beta}{1 - \alpha}\right).$$

For  $n \rightarrow \infty$  we have  $E_{\theta} \theta_n \rightarrow E_{\theta} \theta$  and  $\rho(\theta, \theta_n) \rightarrow \rho(\theta, \theta_0) = \ln(1/\alpha)$ . From here, we obtain, taking limit in (9.170) for  $n \rightarrow \infty$ , that

$$E_{\theta} v \geq \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, \theta_0)}.$$

Since  $\theta_0$  is an arbitrary point in  $D$ , (9.169) follows. That proves the lemma. ■

Notice that from a formal viewpoint, Lemma 9.2 and inequality (9.169) are true if  $\theta \in \theta_0 D$ . In this case inequality (9.169) is trivial, because  $\rho(\theta, D) = 0$  and  $E_{\theta} \theta = \theta_0$ .

Denote the estimate of maximum likelihood of parameter  $\theta$  on the basis of test results  $x_1, \dots, x_n$  at the first  $n$  steps by  $\hat{\theta}_n = \hat{\theta}_n(x_1, \dots, x_n)$ . Let us introduce the following two groups of conditions. The first one is:

- (1)  $\theta$  is compact
- (2)  $X$  is compact
- (3)  $f(x, \theta) > 0$  for all  $(x, \theta)$  and continuous in  $(x, \theta)$ , function  $\rho(\theta, \Phi) > 0$  for all  $\theta \neq \theta_0 \in \Phi$ .
- (4) For any estimate  $\hat{\theta}_n$  which is equivalent to maximum likelihood estimate  $\hat{\theta}_n$ , the following equality holds:

$$P\left\{\lim_{n \rightarrow \infty} |\hat{\theta}_n - \theta'_n| = 0\right\} = 1.$$

The second group of conditions does not use the condition of  $X$  compactness. This group is as follows:

- (1')  $f(x, \theta)$  belongs to exponential family of distribution densities of the type

$$f(x, \theta) = \exp \left[ \sum_{1 \leq i \leq m} \theta_i T_i(x) - b(\theta) \right]$$

and set

$$\Phi = \left\{ \theta: \int \exp \left[ \sum_{1 \leq i \leq m} \theta_i T_i(x) \right] d\mu(x) < \infty \right\}$$

is an open subset of  $m$ -dimensional Euclid space  $R_m$ .

- (2')  $\Omega$  is compact subset of  $\Phi$
- (3') coincides with Condition (4).

Conditions (1)-(4) or (1')-(3') listed above lead to the results presented below. These results state the asymptotic optimality of the moment of exclusion,  $\Omega(\Omega, D)$ , and corresponding sequential criteria and confidence limits (Theorems 9.1-9.4). For a detailed proof of these Theorems see in [Pavlov (1985, 1990)]. One can also find results for weaker conditions there. Following theorem with inequality (9.169) gives the asymptotical optimality for  $(\Omega \neq \emptyset)$  of the moment of exclusion  $\Omega(\Omega, D)$  constructed in (9.77) for a composite hypothesis  $H: \Omega \neq \emptyset$ .

Theorem 9.1 If  $\Omega \neq \emptyset$

$$E_{\theta} \nu(\alpha, D) \leq \frac{\ln \left( \frac{1}{\alpha} \right)}{\rho(\theta, D)} [1 + o(1)] \quad (9.171)$$

for any  $\Omega \in \bar{D}$ . ■

The following inequality gives the system lower limit for average test volume for any sequential criterion  $(\Omega, d) \in K_a$ .

Lemma 9.3. Let  $\mathbf{a} = (\Omega t_1, \dots, \Omega t_l)$  where  $t_1, \dots, t_l$  are arbitrary positive constants. Then for any  $(\Omega, d) \in K_a$  inequality

$$E_{\theta} \tau \geq \frac{\ln \left( \frac{1}{\alpha} \right)}{\Lambda(\theta)} [1 + o(1)] \quad (9.172)$$



is valid for all  $\alpha > 0$  if  $\alpha > 0$ , where  $\rho(\alpha)$  is a function defined in (9.163). ■

It is easy to check that sequential criterion  $(\phi^*, d^*)$  belongs to class  $K_a$ , in other words, it satisfies inequalities (9.165). Indeed, taking into account that  $\rho(\alpha, D)$  is monotone decreasing in  $\alpha$ , it follows from (9.171) that for any  $0 < \alpha < 1$

$$E_{\theta} \rho(\alpha, D) < \alpha$$

for arbitrary  $\alpha$  such that  $\rho(\alpha, D) > 0$ . It follows that for arbitrary  $0 < \alpha_i < 1$ ,  $1 \leq i \leq l$ , inequality

$$P_{\theta}(\phi^* < \alpha) = 1 \quad (9.173)$$

holds for any  $\alpha$  such that  $\rho(\alpha) > 0$ . Let  $\alpha_i = D_i$ , then, due to Theorem 9.1, inequality

$$P_{\theta}(\alpha_i < \alpha) = P_{\theta}\{\rho(\alpha_i, D_i) < \alpha\} \leq \alpha_i$$

follows. From here, taking into account (9.173), we obtain

$$P_{\theta}(d^* = i) \geq P_{\theta}(\phi^* < \alpha_i) \geq P_{\theta}(\phi^* < \alpha, \alpha_i = \alpha) = P_{\theta}(\alpha_i = \alpha) \geq 1 - \alpha_i$$

for all  $1 \leq i \leq l$ . Thus, sequential criterion  $(\phi^*, d^*) \in K_a$ , or in other words, it has probability of errors not larger than specified values  $\alpha_1, \dots, \alpha_l$ .

The following theorem accompanied by inequality (9.172) gives asymptotical optimality of sequential criterion  $(\phi^*, d^*)$  in sense of average test volume within class  $K_a$  for all criteria with probability of errors not larger than specified values  $\alpha_1, \dots, \alpha_l$ .

**Theorem 9.2.** Let  $\mathbf{a} = (\alpha_1, \dots, \alpha_l)$  where  $\alpha_1, \dots, \alpha_l$  are arbitrary positive constants. Then average test volume for sequential criterion  $(\phi^*, d^*)$  satisfies inequality

$$E_{\theta} \tau^* \leq \frac{\ln\left(\frac{1}{\alpha}\right)}{\Lambda(\theta)} [1 + o(1)] \quad (9.174)$$

for all  $\alpha > 0$  if  $\alpha > 0$ . ■

Notice that Lemma 9.3 and Theorem 9.2 are also valid if the area of indifference,  $I$ , is absent, for instance, for the case  $l=2$  of composite hypotheses of the form:

$$H_1: R(\theta) \leq R_1, \quad H_2: R(\theta) \geq R_2$$

where critical levels  $R_1$  and  $R_2$  coincide:  $R_1=R_2$ . In this case  $\phi(\theta)=0$  and average test volume  $E_{\theta}n$  for  $R(\theta)=R_1=R_2$ .

### Sequential confidence limits

Let  $\underline{R}_n=\underline{R}_n(x_1, \dots, x_n)$ ,  $n=1, 2, \dots$  be sequential  $\phi$ -LCL for  $R=R(\theta)$ . (Definition of sequential confidence limits was given in Section 9.4.) Assume that  $R(\theta)$  is continuous in  $\theta$ . Introduce a random moment (step)  $T_c$  where limit  $\underline{R}_n$  crosses a fixed level  $c$ :

$$T_c = \min \{n: \underline{R}_n \geq c\} \quad (9.175)$$

We assume that  $T_c < \infty$  if  $\underline{R}_n < c$  for any finite  $n=1, 2, \dots$ .

If level  $c < R(\theta)$ , then  $T_c$  characterizes the test step where  $\underline{R}_n$  reaches the specified accuracy for the first time, in other words, the specified deviation  $\phi = R(\theta) - c$  from the true value of  $R(\theta)$ . The mathematical expectation of this moment,  $E_{\theta}T_c$ , characterizes the speed of attaining the true  $R(\theta)$  by the confidence limit  $\underline{R}_n$  or  $c < R(\theta)$ . Smaller the value of  $E_{\theta}T_c$ , the more is the effective sequential  $\phi$ -LCL  $\underline{R}_n$ . The following lemma gives the LCL for  $E_{\theta}T_c$ .

**Lemma 9.3.** Let  $\underline{R}_n$ ,  $n=1, 2, \dots$  is sequential  $\phi$ -LCL for  $R(\theta)$ . Then mathematical expectation of moment (9.175) satisfies inequality

$$E_{\theta}T_c \geq \frac{\ln(1-\gamma)^{-1}}{\rho(\theta, D_c)} \quad (9.176)$$

for all  $\theta \in D_c$  where  $D_c = \{\theta: R(\theta) < c\}$ .

**Proof.** First consider the case where  $R(\theta) \leq c$ . In this case (9.176) is trivial since  $\phi(\theta, D_c) = 0$ , and directly from the definition of sequential LCL follows that  $E_{\theta}T_c = \infty$ . Let now  $R(\theta) > c$ . From the definition of the LCL follows the relation

$$\{T_c < \infty\} \subset \{\underline{R}_{T_c} \geq c\}.$$

It follows that at any point  $\theta \in D_c$  the following chain of inequalities is valid:

$$P_{\theta}(T_c < \infty) \leq P_{\theta}(\underline{R}_{T_c} \geq c) \leq P_{\theta}\{\underline{R}_{T_c} > R(\theta)\} = 1 - P_{\theta}\{\underline{R}_{T_c} \leq R(\theta)\} \leq 1 - \gamma.$$

The proof of Lemma 9.2 follows. ■

Let sequential  $(1-\phi)$ -LCL for  $R(\theta)$  be defined by (9.95). The following theorem accompanied by the previous lemma shows that this limit is asymptotically optimal (for  $\phi < 0$ ) in sense of the average time,  $E_{\theta}T_c$ , of attaining the specified accuracy. Introduce notation

$$T_c^* = \min \{n: \underline{R}_n \geq c\}$$

where  $\underline{R}_n$  is the confidence limit defined in (9.95).

**Theorem 9.3.** Inequality

$$E_\theta T_c^* \leq \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, D_c)} [1 + o(1)] \quad (9.177)$$

holds for all  $\alpha > 0$  if  $\alpha > 0$ .

**Proof.** For  $c \geq R(\alpha)$  inequality (9.177) is trivial, since  $\rho(\alpha, D_c) = 0$ . Let  $c < R(\alpha)$ . By construction of sequence  $\underline{R}_n$ ,  $n=1, 2, \dots$  in (9.95), the following relations are valid:

$$\{c \leq \underline{R}_n\} \subseteq \{\underline{R}_n \geq c\} \subseteq \{T_c^* \leq n\}.$$

It follows that  $T_c^* \leq \underline{R}_n$ . Since  $c < R(\alpha)$ ,  $\theta \in \bar{D}_c$ . From Theorem 9.1 follows

$$E_\theta T_c^* \leq E_\theta \underline{R}_n \leq \frac{\ln\left(\frac{1}{\alpha}\right)}{\rho(\theta, D_c)} [1 + o(1)]$$

that proves (9.177). ■

Analogous results for sequential UCLs are formulated and obtained in the same manner.

### 16.6.4 Scheme of dependent tests

Assume that we observe a random sequence

$$x_1, x_2, \dots, x_n, \dots \quad (9.178)$$

where in contrast with (9.162) r.v.'s at different steps can be dependent. Let  $x^{(n)} = (x_1, \dots, x_n)$  be the set of observations at first  $n$  steps and

$$p_n(x^{(n)}, \alpha) = p_n(x_1, \dots, x_{n-1}, x_n, \alpha, \alpha), \quad n=1, 2, \dots, \quad (9.179)$$

is the density of the distribution of  $x^{(n)}$  in respect to measure  $\alpha^n$  on  $(X^n, B^n)$ ,  $n=1, 2, \dots$ . System of finite-measurable densities (9.179) must satisfy standard conditions of "accordance":

$$\int p_n(x_1, \dots, x_{n-1}, x_n, \theta) d\mu(x_n) = p_{n-1}(x_1, \dots, x_{n-1}, \theta) \quad (9.180)$$

for all  $\theta \in \Theta$ ,  $n=1, 2, \dots$ . Assume that set  $\{x^{(n)}: p_n(x^{(n)}, \theta) > 0\}$ , on which the density is defined, coincides with  $X^n$  for all  $\theta \in \Theta$ ,  $n=1, 2, \dots$ . Let us introduce the following notation:

$$p_n(x_n | x_1, \dots, x_{n-1}, \theta) = \frac{p_n(x_1, \dots, x_{n-1}, x_n, \theta)}{p_{n-1}(x_1, \dots, x_{n-1}, \theta)}$$

is the conditional density of distribution of  $x_n$  under condition that test results  $x_1, \dots, x_{n-1}$  at first  $n-1$  steps are known, and

$$B_n = \prod_{1 \leq r \leq n} p_r(x_r | x_1, \dots, x_{r-1}, \hat{\theta}_{r-1})$$

where  $\hat{\theta}_n = \hat{\theta}_n(x_1, \dots, x_n)$  is point estimate of parameter  $\theta$  based on observations on first  $n-1$  steps.

Consider composite hypothesis  $H: \theta \in D$  in respect to parameter  $\theta$  where  $D$  is subset of parameters space  $\Theta$ . Introduce moment of exclusion of this hypothesis,  $v(\alpha, D)$ , which is analogue of moment (9.77) in the scheme of independent tests:

$$v(\alpha, D) = \min \left\{ n: B_n \geq \frac{1}{\alpha} \sup_{\theta \in D} p_n(x^{(n)}, \theta) \right\}$$

Let us show that this moment satisfies inequality

$$P_{\theta} \{ v(\alpha, D) < \infty \} \leq \alpha \text{ for any } \theta \in D. \quad (9.181)$$

Indeed, from the definition of this moment, it follows that

$$v(\alpha, D) \geq \sup_{\theta \in D} v_{\theta} \quad (9.182)$$

where

$$v_{\theta} = \left\{ \min n: B_n \geq \frac{1}{\alpha} p_n(x^{(n)}, \theta) \right\}$$

Consider random sequence

$$\xi_n(\theta) = \frac{B_n}{p_n(x^{(n)}, \theta)}, \quad n = 1, 2, \dots$$

For each fixed  $\alpha$ , sequence  $\alpha_n(\alpha)$ ,  $n = 1, 2, \dots$ , is a non-negative martingale (in respect to system  $F_n = \sigma(x_1, \dots, x_n)$ ,  $n = 1, 2, \dots$ ) with mathematical expectation  $E_{\theta} \alpha_n(\alpha) = 1$  for all  $n$ . Applying Doob-Kolmogorov inequality for non-negative martingales, we obtain

$$P_{\theta}(v_{\theta} \leq n) = P_{\theta} \left\{ \max_{1 \leq r \leq n} \xi_r(\theta) \geq \frac{1}{\alpha} \right\} \leq \alpha$$

for each  $n = 1, 2, \dots$  from where follows that

$$P_{\theta}(v_{\theta} < \infty) = \lim_{n \rightarrow \infty} P_{\theta}(v_{\theta} \leq n) \leq \alpha \quad (9.183)$$

From (9.182) and (9.183), it follows that for any  $\alpha \in D$  inequality

$$P_{\theta} \{v(\alpha, D) < \infty\} \leq P_{\theta} \left\{ \sup_{\theta \in D} v_{\theta} < \infty \right\} \leq P_{\theta} \{v_{\theta} < \infty\} \leq \alpha$$

holds, that proves (9.181). ■

In particular case, if sequence (9.178) is Markovian with "transitive density"  $f_n(x_n | x_{n-1}, \alpha)$ , then the formulas above take the form:

$$p_n(x^{(n)}, \theta) = p_n(x_1, \dots, x_n, \theta) = \prod_{1 \leq r \leq n} f_r(x_r | x_{r-1}, \theta),$$

$$p_n(x_n | x_1, \dots, x_{n-1}, \theta) = f_n(x_n | x_{n-1}, \theta),$$

$$B_n = \prod_{1 \leq r \leq n} f_r(x_r | x_{r-1}, \hat{\theta}_{r-1})$$

Moment  $\square(\square, D)$  in this case coincides with (9.105), and inequality (9.181) is equivalent to (9.106).

## 17. MONTE CARLO SIMULATION

### 17.1 NATURE AND PURPOSE

Monte Carlo simulation is used to imitate the behavior of complex systems whose operational processes are difficult or impossible to describe using analytical models. Computer simulations may also be used if an analytical model is available but numerical solution for this model requires more time than a simulation.

After building a model formally (in terms of formal description of interrelations between system's states and processes), we should develop appropriate software or adapt one that may be available. Sometimes the model itself needs to be modified and made compatible with available software. We could do this by introducing some reasonable assumptions, which simplify or modify initial model. Sometimes the formal model needs corrections due to the lack of appropriate input data.

After these essential steps, we are ready to perform actual simulations. Monte Carlo simulation (runs) is a statistical imitation of possible behaviors of the investigated system in the frame of accepted assumptions and constraints. Data obtained as the result of simulation are processed in the same way as real data obtained during the system operation or field tests.

A reliability simulation model is commonly a discrete model, describing sequences of discrete events and their interactions. In reliability analysis, these events are failures and repairs of system's units, switching from main units to redundant units, interaction with external events (traffic in telecommunication networks, lightning in electric power systems, floods and hurricanes for various terrestrial systems, etc.). Development of a formal model requires creation of an algorithm that transforms a set of initial data into a sequence of discrete output events, which are subject to further analysis. Statistical inferences are made about the behavior of the system after the runs of the model are completed.

A Monte Carlo simulation has the following important components:

- Strong algorithmic description of the behavior of the investigated system and the inter-relation of system's units;
- Software allowing one to perform the process of imitation of the behavior of an investigated system (probably, including some special analytical blocks);

- Input data characterizing time-probabilistic properties of the system's units
- Generators of different necessary random variables with needed properties
- Computer tools for obtaining statistical inferences about output data.

Usually, system imitating, generator of random variables and tools for processing output data are combined into a united software tool.

## 17.2 GENERAL DESCRIPTION OF A DISCRETE SIMULATION MODEL

A discrete model of the investigated system replicates the structure and units' interaction. Any model reflects only some essential sides of a real system, which are interesting in the undertaken research. The model cannot be complete. As physicists joke, "the best model of a cat is a cat, but the very best model of a cat is the same cat".

We will consider a discrete model of a system consisting of some units operating in continuous time. As a matter of fact, in a computer model time is also discrete, but its discreteness can be neglected.

Thus, the modeled process is represented by the sequence of discrete events that are caused by unit's transition from state to state. The moments of those transitions are determined by random values that are generated by computer.

We will denote the system's units as  $A_1, A_2, \dots$ . Each unit,  $A_j$ , is characterized with a set of attributes,  $a_j$ . Attributes include the state,  $s_j$ , which describes the dynamics of the system's transition in time, and some special auxiliary variables,  $\square_j$ . Thus,  $a_j = (s_j, \square_j)$ .

Specified events  $e_j$  may occur with unit  $A_j$ . For example, the unit might fail or repair/replacement might renew failed units. The state  $s_j$  defines the moment of the event, and its content is defined by attributes  $a_j$ . The event  $e_j$  may follow a change of attributes of units. A set of units determining the event depends on the nature of the event.

One of the attributes,  $a_j$ , is the real variable  $\square_j$ , which is the time until the occurrence of a new event for this unit if there are no intervening event in the system which could change the behavior of this particular unit. At the initial time  $t=0$ ,  $s_j = s_j^0$  and  $\square = \square_j^0$  for each unit  $A_j$ . The first event occurs after a period of time  $\square_1 = \min \square_j$  over all  $j$ . If  $\tau_{j_1} = \theta_1$ , the template event occurs in unit  $A_{j_1}$ . Event  $e_{j_1}$  occurring at time  $t_1 = \square_1$  changes attributes of one or more units. A new state and new residual time  $\square_j^1$  must be found for each affected unit. For each unaffected unit the residual time is given by  $\square_j^1 = \square_j^0 - \square_1$ . The procedure then continues, and we find a new inter-occurrence time  $\theta_2 = \min_j \tau_j^1 = \tau_{j_2}^1$ , that is the second event is  $e_{j_2}$  and it occurs at time  $t_1 +$

$\square_2$ . Again, the attributes and the residual times of one or more affected units are changed depending on the nature of the event  $e_{j_2}$ . For unaffected units, the new residual time is  $\square_j^2 = \square_j^1 -$

$\square_2$ . This procedure continues on.

As the result of this procedure, we have the so-called governing sequence  $(t_1, e_{j_1}), (t_2, e_{j_2}), \dots$ , with corresponding attributes which allows one to reconstruct the entire trajectory of the simulation process.

### 17.3 DETAILED EXAMPLE OF ALGORITHMIC DESCRIPTION

In this section we consider a simple example that illustrates some details of Monte Carlo simulation. Consider a system consisting of  $m$  main and  $n$  standby redundant units. All units are identical and independent. There are  $l$  servers for repair,  $1 \leq l \leq n+m$ . The FIFO (first in, first out) discipline is supposed. The time to failure of each unit is a r.v. with d.f.  $F(x)$  and the repair time is a r.v. with d.f.  $G(x)$ .

Thus the system has  $n+m$  units  $A_1, A_2, \dots, A_{n+m}$  with respective attributes  $a_i = (s_i, \square_i)$ ,  $i=1, 2, \dots, n+m$ . The unit's state is defined as follows:

$$s_i = \begin{cases} \dots\dots & \\ 3 & \text{if a unit is the second in the line for repair} \\ 2 & \text{if a unit is the first in the line for repair} \\ 1 & \text{if unit } i \text{ is in repair} \\ 0 & \text{if unit } i \text{ is operating} \\ -1 & \text{if unit } i \text{ is first in the line for replacement} \\ -2 & \text{if unit } i \text{ is second in the line for replacement} \\ \dots\dots & \end{cases}$$

(Here we assume that the line of spare units is arranged in accordance with order of completion of their repair.)

The residual time  $\square_i$  is defined only for states 0 or 1, and in the first case it is the residual time to failure and in the second case is the residual repair time. In other cases let us assume that  $\square_i = \square$ .

Event  $e_i$  can take only values 0 (failure) and 1 (repair). Let us assume that for unit  $A_i$  the attribute  $\square_i$  is defined by the number of failures that have occurred at the current moment of time.

Let the model be described by the vector  $[(s_1^k, \square_1^k, \square_1^k), (s_2^k, \square_2^k, \square_2^k), \dots, (s_{n+m}^k, \square_{n+m}^k, \square_{n+m}^k)]$  at the time  $t_k$ . The nearest occurrence time then is  $t_{k+1} = t_k + \theta$ , where  $\theta = \min_{1 \leq i \leq n+m} \tau_i^k$  is the shortest residual time for all four considered units at time  $t_{k+1}$ . Let  $\square = \square_r^k$  which means that event  $e_r$  occurs at time  $t_{k+1}$ .

Consider following two cases:

(a) For the first case, let  $e_r = 0$  and, consequently,  $s_r^k = 0$ . Then for all standby units  $A_{\square}$  (defined by the condition  $s_{\square} < 0$ ), we can write

$$(s_{\square}^{k+1}, \square_{\square}^{k+1}, \square_{\square}^{k+1}) = \begin{cases} (s_{\square}^k + 1, \square_{\square}^k, \square_{\square}^k) & \text{if } s_{\square}^k < -1, \\ (0, \square_{\square}^k, \square_{\square}^k) & \text{if } s_{\square}^k = -1. \end{cases}$$

Thus one of the standby units (if any) becomes operating and others advance in the queue. Here  $\{\square_k\}$  are times to failure of the unit  $A_{\square}$  (All of these r.v.'s are i.i.d. with a d.f.,  $F$ ).



Let  $\gamma^k$  be the number of repair servers busy with the repair at moment  $t_k$  (and hence at time  $t_{k+1}$  because no repair is finished until this moment). Thus  $\gamma^k$  is the number of  $A_j$  for which  $s_j^k = 1$ . If there is at least one empty repair server, that is,  $\gamma^k < 1$ , then

$$(s_r^{k+1}, \pi_r^{k+1}, \tau_r^{k+1}) = (1, \pi_r^k + 1, \eta_k)$$

where  $\{\eta_k\}$  are i.i.d. r.v.'s with a d.f.,  $G(t)$ , and  $\square_k$  is the repair time for unit  $A_r$ . When a repair is completed, the auxiliary variable  $\square_r$ , increases by one.

If  $\gamma^k = 1$

$$(s_r^{k+1}, \pi_r^{k+1}, \tau_r^{k+1}) = (\max_{1 \leq i \leq n+m} s_i^k + 1, \pi_r^k + 1, \infty).$$

The unit joins the repair queue in position with the number  $\max_{1 \leq i \leq n+m} s_i^k + 1$ . For units  $A_{\square}$  ( $\square \square \square \square r$ ) that are failed and a main unit (for which  $s_{\square}^k \geq 1$ ), we make the following changes:

$$(s_{\square}^{k+1}, \square_{\square}^{k+1}, \square_{\square}^{k+1}) = (s_{\square}^k, \square_{\square}^k, \square_{\square}^k - \square).$$

Here we take into account that for units  $A_{\square}$  in the queue for which  $s_{\square}^k > 1$ ,  $\square_{\square}^k$  is equal to  $\square$ .

(b) For the second case, let  $e_r = 1$ . It means that the repair of unit  $A_r$  has just been completed, i.e.,  $s_r^k = 1$ . In this case the repair server who just completed the repair takes the first unit in the queue (if any), and other units advance one position. The units under repair stay at their repair places, but all residual repair times decrease by  $\square$ . Thus

$$(s_{\square}^{k+1}, \square_{\square}^{k+1}, \square_{\square}^{k+1}) = \begin{cases} (s_{\square}^k - 1, \square_{\square}^k, \square) & \text{if } s_{\square}^k > 2, \\ (1, \square_{\square}^k, \square_k) & \text{if } s_{\square}^k = 2, \\ (s_{\square}^k, \square_{\square}^k, \square_{\square}^k - \square) & \text{if } s_{\square}^k = 1. \end{cases}$$

Let  $\psi^k$  be the number of operable units at time  $t_k$ , that is, the number of units  $A_j$  for which  $z_j^k \leq 0$ . If  $\psi^k < m$ , then the repaired unit is directed to occupy the position of a main unit:

$$(s_r^{k+1}, \pi_r^{k+1}, \tau_r^{k+1}) = (0, \pi_r^{k+1}, \square_k).$$

If  $\psi^k = m$ , unit  $A_i$  becomes the last unit in the queue of spare units:

$$(s_r^{k+1}, \pi_r^{k+1}, \tau_r^{k+1}) = \max_{1 \leq i \leq m+n} (s_i^k - 1, \pi_r^k, \infty).$$

All main and standby units stay in their places. Residual time of all main units decrease by  $\square$ :

$$(s_{\square}^{k+1}, \square_{\square}^{k+1}, \square_{\square}^{k+1}) = (s_{\square}^k, \square_{\square}^k, \square_{\square}^k - \square) \text{ for } s_{\square}^k \leq 0.$$

These relationships entirely describe the chosen model.

Notice that this description of the model does not give us an opportunity to analyze the behavior of repair servers because they were considered only as a group of objects without their individual attributes. This simplification of the model was done for a concise explanation of the essential features.

In Figure 10.1 we represented a sample of trajectories of the system's units. The system under consideration consists of one main and two standby units with a single repair server.

FIGURE 10.1

Let us comment on changing of system's states represented in this figure.

Thus, the system has three units  $A_1$ ,  $A_2$  and  $A_3$ . The trajectories of changing of states for each unit are represented as three staircase functions.

Consider an initial state of units. The first unit  $A_1$  is on working position (state  $s_1=0$ ), its time to failure is generated by the counter of random values. This time equals to  $\square_1^1=t_1$ . The interval on which a unit is working is denoted by a double line on the upper trajectory. During period  $(t_0, t_1)$ , the second unit is the first in the line for replacement (state  $s_2=-1$ ) and the third one is on the second place in this line ( $s_2=-2$ ). Intervals where an unit is waiting for installation into the working position ( $s_i=-1$  or  $s_i=-2$ ) are denoted by thin lines.

A random value of TTF of unit  $A_1$  is generated. Moment  $t_1$  is defined as the moment of failure of unit  $A_1$ . At this moment unit  $A_1$  is directed to a repair shop, and its state becomes  $s_1=1$ . The duration of the repair is generated equal to  $\square_1^1$ . We denote the interval of repair by a bold line. Unit  $A_2$  becomes a working one (state  $s_2=0$ ) and its generated TTF equals  $\square_2^1$ . At the same moment of time  $A_3$  change its state to  $s_3=-1$ .

Moment  $t_2$  is defined as  $\min(\square_1', \square_2')$ . (Here we use sign "prime" to denote a residual value.) In our case, it happens that  $\square_1^1 > \square_2^1$ . At moment  $t_2$ , unit  $A_2$  has failed and moves to the queue for the repair shop (the failed unit  $A_1$  is still repaired), that is its state becomes  $s_2=2$ . The interval of waiting in a queue is denoted by thin line. Unit  $A_3$  becomes working (state  $s_3=0$ ). The residue time of repair of unit  $A_1$  equals  $\square_1^1 - (t_2 - t_1) = t_3 - t_2$ .

Moment  $t_3$  is defined as  $\min(\square_1', \square_3')$ . (Here we use sign "prime" to denote a residual value.) In our case, it happens that  $\square_1^1 < \square_3^1$ . At the moment  $t_3$  unit  $A_1$  has been repaired and takes the state  $s_1=-1$  because unit  $A_3$  keeps working. Unit  $A_2$  which was waiting for repair enters the repairshop (change state  $s_2=2$  for  $s_2=1$ ). And so on.

We will not continue the description of the trajectories any further. Note only that at the moment  $t_{10}$  unit  $A_3$  has failed and at the same time unit  $A_1$  is under repair and unit  $A_2$  is failed and waiting for repair. The failure of  $A_3$  means that the system as a whole has failed (here we use conditional notation: a cross on the level  $s_3$ ).

Thus realization of Monte Carlo simulation is represented in a form close to that obtained in a real testing: one records system failures, times to and between failures, duration of repair. All this data can be used in an ordinary way to obtain different reliability indexes.

Of course the description of the above model could be different. It might contain more or less details. For instance, we could only be interested in numbers of repaired and standby units (without description of their individual behavior). It makes sense if all units are identical and independent. If we have several repair servers, it might be interesting to obtain information about their loading. The reader can find detailed description of some related models in "Handbook of Reliability Engineering", pp.459-461.

#### 17.4 RANDOM NUMBERS

Monte Carlo simulation uses random values with required distributions. Many modern software libraries have random number generator which produces uniformly distributed random numbers. As a matter of fact, a computer generates the so-called pseudo-random numbers. The procedure of such a generation is a recurrent computation with withdrawing some intermediate numbers, for instance, digits on the positions from  $k$  to  $k+n$  in  $N$ -digit number,  $N \gg k+n$ . It is clear that recurrent procedure generates cycles when initial state repeats during a calculation. The main problem is to make such a cycle long enough to cover a Monte Carlo experiment with "independent" pseudo-random numbers. At the same time it is very important to avoid regular dependence on neighboring numbers.

The idea of generating of pseudo-random numbers with the help of recurrent calculations belongs, most probably, to John von Neuman who used it more than 50 years ago. Now this type of a procedure of random number generating is standard.

However, Monte Carlo models need not only uniformly distributed random numbers. The problem of generation of random numbers with arbitrary given distribution can be done using the so-called probability integral transform. Let us explain this considering a simple example. Suppose that we have a huge sample,  $S_N$ , from distribution  $F(x)$ :  $X_1, X_2, \dots, X_N$ . Order these realizations of an r.v. and draw a histogram (see Figure 10.2).

Figure 10.2

It is clear that if  $N \rightarrow \infty$  the ladder-like function  $\hat{F}(x)$  will converge to  $F(x)$  (this follows from Glivenko's Theorem, see Volume 1). Now let us take a sample of size  $n$ ,  $S_n$  from the original sample  $S_N$ , assuming that  $N \gg n$ . To take this sample one should use a uniformly distributed random numbers in the following way.

Let  $N$  be a  $k$ -digit number (for the sake of simplicity) with a value from 0 to  $10^k-1$ . Consider a set of  $n$   $k$ -digit uniformly distributed random numbers. To form sample  $S_n$ , let us pick up values  $X_{j_1}, X_{j_2}, \dots, X_{j_n}$  whose subscripts coincide with random numbers chosen above (see Figure 10.1).

In other words, we take a uniformly distributed ordinate of the function  $\hat{F}(x)$  (which represents an empirical distribution) and obtain in response a realization of an r.v. having this distribution.

After these obvious explanations, we can formulate the following rule. If there is a given arbitrary d.f.  $F(x)$  and a set of uniformly distributed r.v.'s  $X_1, X_2, \dots, X_n$ , in  $[0, 1]$  then one can obtain  $n$  r.v.'s  $Y_1, Y_2, \dots, Y_n$  with the distribution  $F(x)$  by solving the equation

$$F(Y_j) = X_j,$$

or in an equivalent form

$$Y_j = F^{-1}(X_j).$$

Let us now describe this problem in a more strict way. Suppose that an r.v.  $X$  has a d.f.  $F$  and let  $Y = F(X)$ . Let us show that the distribution of  $Y$  is uniform on the interval  $(0,1)$ . By the definition of the distribution function  $0 < F(x) < 1$  for  $-\infty < x < \infty$ . Let  $x_0$  be a number such that  $F(x_0) = y$  where  $0 < y < 1$ . Since by the definition  $F$  is strictly increasing, there exists a unique number  $x_0$  such that  $F(x_0) = y$ . However, if  $F(x) = y$  over an entire interval of values of  $x$ , then we can choose  $x_0$  arbitrarily from its interval. If  $G$  denotes the d.f. of  $Y$ , then

$$G(y) = \Pr(Y \leq y) = \Pr(X \leq x_0) = F(x_0) = y.$$

It follows that  $G(y) = y$  for  $0 < y < 1$ . Since this function corresponds to the uniform distribution on the interval  $(0,1)$ ,  $Y$  has this uniform distribution.

Now suppose that  $X$  is an r.v. with d.f.  $F$  and that  $G$  is some other d.f. It is required to construct a random variable  $Z = r(X)$  for which the d.f. will be  $G$ . For any value of  $y$ ,  $0 < y < 1$ , let  $z = G^{-1}(y)$  be any number such that  $G(z) = y$ . We can now define the random variable  $Z$  in the following way:

$$Z = G^{-1}[F(X)].$$

To verify that the d.f. of  $Z$  is actually  $G$ , we note that for any number  $z$  such that  $0 < G(z) < 1$ ,

$$\Pr(Z \leq z) = \Pr\{G^{-1}[F(X)] \leq z\} = \Pr[F(X) \leq G(z)].$$

It follows from the probability integral transformation that  $F(X)$  has a uniform distribution and, consequently, that

$$\Pr[F(X) \leq G(z)] = G(z).$$

Hence,  $\Pr(Z \leq z) = G(z)$ , that is  $G$  is the distribution of  $Z$ . Thus, the theorem is proven.

Generation of Bernoulli's r.v.'s which are frequently used in reliability modeling, in principle, is similar. However, it has a simpler explanation. Let us construct a sample of a Bernoulli's sequence with the probability of success (denoted as "0") equal to  $p$  and the probability of failures (denoted as "1") equal to  $q = 1 - p$ . Suppose that a random number generator produces uniformly distributed values  $\square_j$ ,  $0 \leq \square_j \leq C$ . Introduce a threshold  $A$  such that  $A = qC$ . Bernoulli's random variable,  $B_j$ , is generated by the rule

$$\begin{cases} 0 & \text{if } \square_j \leq A. \end{cases}$$

$$B_j = \begin{cases} 1 & \text{if } \square_j > A. \end{cases}$$

Obviously, probability transform of such kind is not the only way of generating of r.v.'s with a given distribution on the basis of uniform distributed r.v.'s. For instance, in many software implementation, a normally distributed r.v. is obtained as a sum of a very restricted number of uniformly distributed r.v.'s. This method is based on the Central Limit Theorem (see Volume 1). If the number of uniformly distributed r.v.'s in the sum equals  $k$ , the generated quasi-normal distribution will have the mean equal to  $k/2$  and the variance equal to  $k/12$ .

### 17.5 SAMPLE OF COMPUTER SIMULATION

Monte Carlo simulation is especially important if we intend to analyze a non-stationary process. Analytical results for this case are available only for simple Markov models. Markov model implies that all distributions within the model are assumed exponential.

Consider a simple model: a single renewable unit with the exponential distribution of time between failures and constant repair time. For an illustration let us choose MTBF and mean repair time equal. Consider the mechanism of Monte Carlo simulation on an example with two realizations.

Each realization of the unit operation process might be presented as alternating process where 1, as above, corresponds to the failure state and 0 corresponds to the operational state.

FIGURE 10.3

In Figure 10.3 we represented two separate realizations for two units. There is also a superposition of these two processes in a form of a staircase-like function:

- “2” corresponds to moments of time where both units have been failed
- “1” correspond to the moments where one unit is up and another is down
- “0” means that both units are operating.

It is clear that increasing the number of superimposed realization makes the resulting function more "continuous", i.e., discrete increments become smaller and smaller.

FIGURE 10.4

Figure 10.5

In Figure 10.4 we represent results of computer simulation for different number of realizations: 165 (see *a*), 1,293 (see *b*) and 11,013 (see *c*). (The duration of simulation period is 10 MTBF.) In Figure 10.5 the same curves are represented in a smoothed form: each realization is presented in the form of the running average with time window about 0.25 of the unit of time.

This illustrative example is simple by its nature. However it demonstrates dependence of results accuracy on the number of realizations.

## 17.6 MODELLING NETWORK RELIABILITY

Advantage of Monte Carlo simulation for continuous processes of a complex nature is almost obvious: sometimes we have no way to solve a huge system of integro-differential equations even with the use of powerful iterative procedures.

Another case requiring the use of the Monte Carlo method is analysis of networks of complex structure. We will not consider a queuing network, i.e., a network whose nodes can generate an input traffic of messages and process transit traffic messages (as servers of a queuing system). In a general sense, such a model will be very close to that described in Section 10.3. For the sake of simplicity, we will consider a two-pole network with links and nodes subjecting to failures. Let us focus on the probability of connectivity of the network.

In general, the procedure of simulation of a network in this case consists, of the following steps:

- Description of the two-pole network (a matrix of connectivity: what nodes are connected by direct links)
- Definition of criteria of connectivity of the two distinguished poles
- Procedure of generating of random states of the network
- Procedure of checking the connectivity of the network for each realization of state
- Collection of output data
- Statistical inferences.

If the system's units are highly reliable, the direct "event-by-event" simulation takes too much computer time because enormous number of realizations may be required before obtaining necessary number of events of interest for getting the required level of confidence. (Planning of test volume is outside the topics of this book.)

Consider a two-pole network. Such a network is supposed to operate successfully if its input and output nodes are connected by some chain of links. The system is assumed to have  $n$  nodes and  $k$  links, each is characterized by the probability of successful operation  $p_i$ . The system can be in  $2^{n+k}$  different state. Of course, enumerating all states, evaluation of their probability and checking for each state if the network connectivity has taken place or not is enormous calculating problem. For example, even for a moderate size network of 20 nodes and 50 links the total number of states is about  $10^{21}$ . In this case Monte Carlo method is obviously more effective than a direct calculation. Usually several thousands realizations are enough for statistical estimate. (In this particular case, each realization needs the generation of 70 Bernoulli's r.v.'s and checking every time for network connectivity.)

For illustrative purpose, we consider a simple two-pole network, namely, the so-called "bridge structure". All links are assumed independent and identical in their reliability parameters. Let us assume that the probabilities of the link's successful operation,  $p_i=0.5$  for all  $i$ . (We take this condition only for simplicity of checking the result obtained.) Assume that the structure nodes are absolutely reliable. In Table 10.1 there are three digit random numbers,  $\square_i$ , uniformly distributed.

TABLE 10.1

Table 10.2 contains Bernoulli's r.v.,  $B_i$ , for units of the bridge structure. These r.v.'s are formed in accordance to criterion

$$B_i = \begin{cases} 1 & \text{if } \square_i \leq 1000p_i, \\ 0 & \text{if } \square_i > 1000p_i, \end{cases}$$

that is  $B_i=1$  corresponds to operational state and  $B_i=0$  to failure of the  $i$ th link.

TABLE 10.2

Example 10.1 Each row of Table 10.2 contains five realizations of Bernoulli's r.v.'s corresponding to links of the considered bridge structure. These realizations of link states are used for obtaining the system's states realizations,  $B_{\text{sys}}$  (see column 6 of the table). These realizations are obtained in accordance with the rule

$$B_{\text{sys}} = (B_1 \wedge B_4) \vee (B_2 \wedge B_5) \vee (B_1 \wedge B_3 \wedge B_5) \vee (B_2 \wedge B_3 \wedge B_4) \quad (10.1)$$

i.e., if at least one path between input and output exists, the system is supposed to be operational.

From the statistical experiment performed, we obtain the result:  $\hat{P}_{\text{sys}} = \frac{9}{20} = 0.45$ . We intentionally choose  $p_i=0.5$  for all  $i$  because in this case the probability of the bridge structure connectivity can be easily calculated. Indeed,

$$\begin{aligned} P_{\text{sys}} &= \binom{5}{0} p^5 + \binom{5}{1} p^4 q + \left[ \binom{5}{2} - 2 \right] p^3 q^2 + 2 p^2 q^3 \\ &= (1 + 5 + 8 + 2) \cdot 0.5^5 = \frac{16}{32} = 0.5. \end{aligned}$$

The reader can find more details about network structures and, in particular, about the bridge structure, in Volume 1. ■

Of course, where the units are equally or almost equally reliable, the procedure can be improved. Particularly, analytical and Monte Carlo methods can be mixed to reduce the total time of simulation. Such a combination of methods is especially useful in cases of analysis of highly reliable systems.

### 17.6.1 Simulating with Fixed Failed States

Let a network include many links, which are highly reliable and almost identical (in probabilistic sense). Besides, assume that the network is redundant, that is, the system failure cannot occur if the number of link's failures has not exceeded, say,  $M$ . In this case many realizations will have no useful information. For instance, if the number of failed links  $k$  in some realization less than  $M$ , there is no real information in this particular test: it is clear that  $k < M$  failures never leads to the system failure.

Moreover, it may occur that "informative states" appear very seldom. In this case we can use the following procedure: first to compute analytically the probabilities of different system states with fixed numbers of failed units and then to estimate conditional probabilities of system failure under condition that the number of links is fixed by simulation. The general procedure follows:

1. Analytically calculate probabilities,  $P_{(k)}$ , of the states, which have exactly  $k$  failed units. Under condition of almost identical units this probability is

$$P_{(k)} = \binom{n}{k} q^k p^{n-k}.$$

2. Use Monte Carlo simulation to estimate each conditional probability,  $\hat{\Phi}_{(k)}$ , that the system in state  $H_{(k)}$  is operational.
3. Compute the total probability of system successful operation:

$$P_{syst} = \sum_{0 \leq k \leq M} P_{(k)} + \sum_{M+1 \leq k \leq n} \hat{\Phi}_{(k)} P_{(k)}.$$

**Example 10.2.** Consider the same bridge system and use the same Table 10.1 of random numbers. As we know this system can not fail if the number of failed units is less than two. It means that  $\hat{\Phi}_{(0)} = \hat{\Phi}_{(1)} = 1$ . At the same time the system is failed if it's four or five units are failed, that is  $\hat{\Phi}_{(4)} = \hat{\Phi}_{(5)} = 0.663$ . Thus we need to estimate the conditional probabilities of the system failure for the cases where two or three units are failed.

Let us use the first 10 rows of Table 10.1 for simulation of two failures in the bridge structure and the next 10 rows for simulation of three failures. In the first case let us choose two smallest numbers within each row and consider them as failed units. As the result we construct Table 10.3 with exactly two failed links and using the same formula (10.1) to determine the sixth column (system's state)

TABLE 10.3



Analyzing Table 10.3 we conclude that  $\hat{\Phi}_{(2)} = 0.7$ .

Table 10.4 contains analogous results of simulation with exactly three failed links. We choose three largest numbers within each row of the second part of Table 10.1 (rows from 11 to 20) and consider them as failed units.

TABLE 10.4

Analyzing Table 10.4 we conclude that  $\hat{\Phi}_{(3)} = 0.4$ . Thus the resulting mixed estimate, which was obtained using both analytical calculations and experimental estimation is found as follows:

$$\tilde{P} = (0.5)^5 + 5 \cdot (0.5)^5 + (0.7) \cdot 10 \cdot (0.5)^5 + (0.4) \cdot 10 (0.5)^5 = 17 \cdot (0.5)^5 \approx 0.53.$$

### 17.6.2 Modeling Link Failures Until System Failure

In this case links of the network are excluded one by one. The failure criterion is again the loss of network connectivity. When the system fails, the number of links,  $k_s$ , which were excluded in the  $s$ th realization is stored in the computer memory. After a sufficient number of experiments,  $N$ , we use the following estimate for the conditional probability of the loss of connectivity

$$\Phi_{(k)} = \frac{1}{N} \sum_{1 \leq s \leq N} \delta(k_s)$$

where we use an indicator function

$$\delta(k_s) = \begin{cases} 1 & \text{if } k_s \leq k, \\ 0 & \text{otherwise} \end{cases}$$

Example 10.3. Let us illustrate the method on a bridge structure. We will use the same table of uniformly distributed numbers. Consider each row of the table and let us exclude links in order corresponding to the increasing of the numbers. The numbers in Table 10.5 show the order of link's failures.

TABLE 10.5

From Table 10.5 we can see that in 10 cases of the total 20, structure can stand three failure and fails only after the fourth failure has occurred. This gives us  $\hat{\Phi}_{(3)} = \frac{10}{20}$ . Only in 3 cases the system failure has occurred after the failure of the second link. It means that  $\hat{\Phi}_{(2)} = \frac{17}{20}$ .

Note: All examples were purely illustrative. Besides, the reader should keep in mind that the use of the same table of random numbers makes all results dependent of each other.

### 17.6.3 Accelerated Simulation

Direct Monte Carlo simulation is convenient due to simplicity of the model. But, as we mentioned above, if the system's units are highly reliable or/and the system's structure is highly redundant, direct simulation may be ineffective, since many realizations will be non-informative. The solution may be obtained by simulation of similar structures but with less reliable units and transformation of the results afterwards.

#### Direct transform

Consider a network state where there are  $z$  failed links belonging to some set  $u$ . For instance, it might be some network cut disjoining two specified nodes (say, input and output). The frequency of the system failures due to this cut tends to its probability

$$P_u = \prod_{i \in u} p_i \prod_{i \in u} (1 - p_i).$$

If  $n$  denotes the total number of the network links then

$$P_u = \prod_{1 \leq i \leq n} p_i \prod_{j \in u} \frac{1 - p_j}{p_j}.$$

Let us now introduce into the network units with lesser reliability,  $p'_i$ , for which the condition

$$\frac{1-p_{i'}}{p_{i'}} = \gamma \frac{1-p_i}{p_i}.$$

holds. This condition can be rewritten in form

$$p_{i'} = p_i [p_i + \gamma(1-p_i)]^{-1}.$$

After substitution of new values for all of the network links, the probability of the occurrence of the network failure due to failure of cut  $u$  will change and be equal to

$$P_{u'} = \prod_{1 \leq i \leq n} p_i [p_i + \gamma(1-p_i)]^{-1} \prod_{j \in u} \gamma \frac{1-p_i}{p_j}.$$

Let us now introduce the coefficient of modeling acceleration,  $\square$ , which characterizes how the frequency of occurrence of a cut increases:

$$\delta = \frac{P_{u'}}{P_u} = \gamma^z \prod_{1 \leq i \leq n} [p_i + \gamma(1-p_i)]^{-1} \quad (10.2)$$

where  $z$  is the power of set  $u$ , that is, the number of links belonging to this cut. Notice that the second multiplier in (10.2) is a constant for specified network:

$$K = \prod_{1 \leq i \leq n} [p_i + \gamma(1-p_i)]^{-1}$$

Thus, for cuts of the same power the relative frequency of occurrence is preserved. At the same time, the relative frequency of cuts of power  $z + \square$  increases in  $\square_\square$  times. Thus, during Monte Carlo simulation we are able to collect satisfactory statistics for cuts of larger power which occurs in regular simulation with a negligible probability.

### Inverse transform

Inverse calculation of the probability of occurrence of failure of cut  $u$  can be performed by the following formula

$$P_u = \frac{1}{\delta} P_{u'} = \frac{1}{\gamma^z K} P_{u'}. \quad (10.3)$$

We emphasize that the power of a corresponding cut plays an essential role in the inverse transform. From (10.3), we see that the frequency of failure of cut  $u$  with power  $z$  in a transformed model is in  $\square$  times higher than the same value in an initial model. If during the simulation process, the transformed system has had  $m$  failures, then for the initial system the number of failures can be calculated by formula

$$m' = \frac{1}{K} \sum_{1 \leq i \leq m} \frac{1}{\gamma^{z_i}}$$

where  $z_i$  is the number of units which cause the  $i$ th failure of the system. After a current failure of transformed system, the probability  $P_m$  should be calculated as follows

where  $M$  is the total number of realization taken into account.

Naturally, parameter  $\square$  must be chosen in a way that allows one to accelerate the most critical events. Usually, such an event corresponds to a failure of a set of the system units whose power equals to the power of the minimal cut, that is, the minimal set of the system's units which causes the system failure.

## Explanations:

Figure 10.4. Results of Monte Carlo simulations for different sizes of samples: (a) small sample, (b) average sample, (c) large sample.

This figure is compiled of Figures 19.2(a), 19.3(a) and 19.4(a) from "Handbook of Reliability Engineering", pp. 469-471 (John Wiley & Sons, 1994)

Figure 10.5. Smoothed results of Monte Carlo simulations for different sizes of samples: (a) small sample, (b) average sample, (c) large sample.

This figure is compiled of Figures 19.2(b), 19.3(b) and 19.4(b) from  
“Handbook of Reliability Engineering”, pp. 469-471 (John Wiley & Sons,  
1994)