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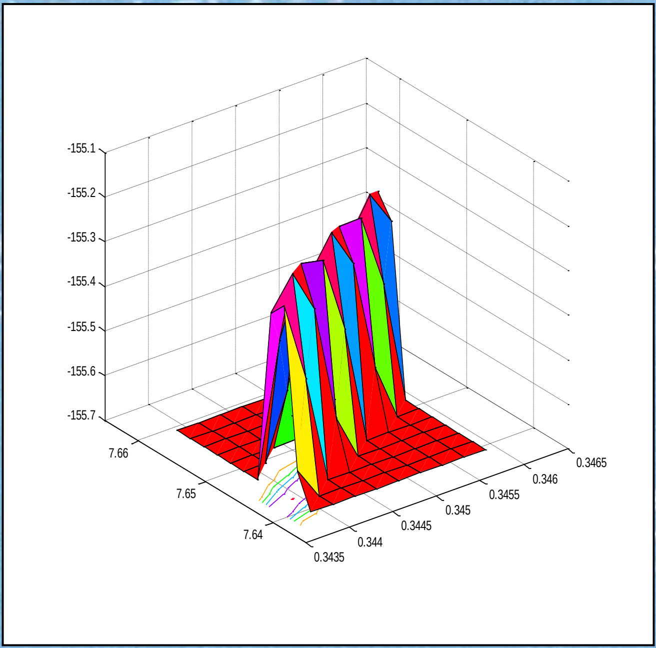
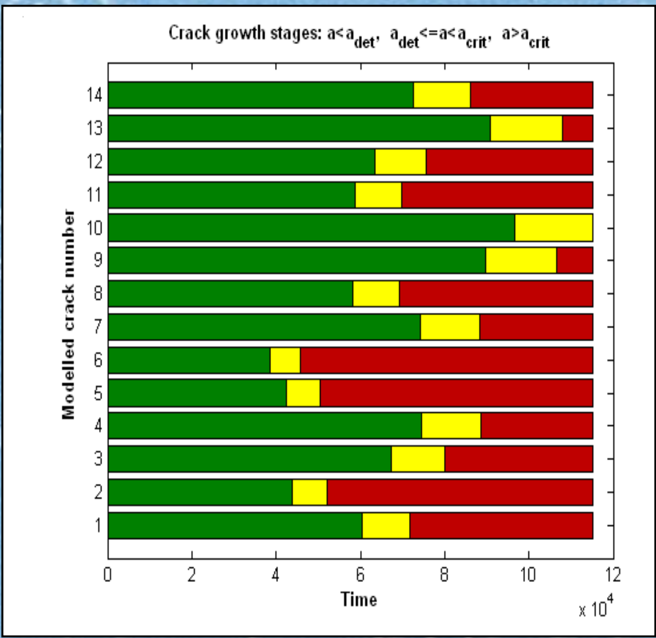
Yu. Paramonov, A. Kuznetsov, M. Kleinhofs

RELIABILITY of fatigue-prone airframes and composite materials





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Destination states

Source states	E_1	E_2	E_3	…	E_{n-1}	E_n	E_{n+1} (SL)	E_{n+2} (CD)	E_{n+3} (FF)
E_1	0	u_1	0	…	0	0	0	v_1	q_1
E_2	0	0	u_2	…	0	0	0	v_2	q_2
…	…	…	Q	…	…	…	R	…	…
E_{n-2}	0	0	0	…	u_{n-2}	0	0	v_{n-2}	q_{n-2}
E_{n-1}	0	0	0	…	0	u_{n-1}	0	v_{n-1}	q_{n-1}
E_n	0	0	0	…	0	0	u_n	v_n	q_n
E_{n+1} (SL)	0	0	0	…	0	0	1	0	0
E_{n+2} (CD)	0	0	0	…	0	0	0	1	0
E_{n+3} (FF)	0	0	0	…	0	0	0	0	1

Q, R, 0, I

$$u_i = \frac{\Phi\left(\dfrac{\ln(C_d/t_i)-\theta_0}{\theta_1}\right)}{\Phi\left(\dfrac{\ln(C_d/t_{i-1})-\theta_0}{\theta_1}\right)}$$

$$q_i = \frac{\Phi\left(\dfrac{\ln(C_d/t_{i-1})-\theta_0}{\theta_1}\right)-\Phi\left(\dfrac{\ln(C_c/t_i)-\theta_0}{\theta_1}\right)}{1-\Phi\left(\dfrac{\ln(C_d/t_{i-1})-\theta_0}{\theta_1}\right)}$$

$$v_i + u_i + q_i = 1$$

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Yu. Paramonov, A. Kuznetsov, M. Kleinhofs. Reliabilty of fatigue-prone airframes and composite materials. Riga, Aviation institute of Riga Technical University, 2011 – 122 pages.

There are two parts in this book. Reliability of airframe, aircraft specified life nomination and inspection program development are considered in first part. Reliability of series, parallel and series-of-parallel systems with defected items and its applications to statistical analysis of static strength and fatigue life of composites are considered in the second part. Modern methods of probability theory, Markov chains and mathematical statistics are used in the book for solution of the problem under consideration.

The book is meant for students of Aviation Institute of Riga Technical University.

Grāmatā ir divas daļas. Pirmajā daļā tiek izskatīts lidmašīnas noteikta resursa aprēķins un apskašu programmas plānošana. Kompozītā materiāla statiskās stiprības analīze un tās saistība ar materiāla nogurumu ir aplūkotas otrajā daļā. Lai risinātu nosauktās problēmas, ir izmantotas varbūtības teorija, Markova ķēdes un matemātiskās statistikas modernās metodes.

Paredzēta Rīgas Tehniskās universitātes Aviācijas institūta studentiem.

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Contents

Preface

This book is intended to help students of Aviation Institute of Riga Technical University to study the problem of reliability of an airframe. It describes modern mathematical solutions of this problem and its historical background. The book consists of two parts.

Airframe designers have always been seriously concerned with an issue of preventing of an airframe from a fatigue failure. Several approaches to solution of this problem are offered: safe-life, fail-safe and damage-tolerance. Mathematical aspects of these approaches are considered in the first part of the book.

There used to be a delusion that only metals suffer from fatigue and that this problem would disappear for composite airframe. In November 2001 the composite tail fin of an Airbus A300 had broken away just before the plane crashed soon after take-off from the New York airport. Now the problem of fatigue of composite materials is being substantially and seriously studied. Of course, investigation of static strength and its relation to fatigue life of composite material is very impotant. These problems are discussed in the second part of the book.

This book is not „An Introduction to Reliability”. The introductory courses in both the theory of probability and mathematical statistics are prerequisites for this book. Formal treatment of the mathematical models considered in the book requires the use of some basic facts from Markov chain theory.

It is worth to mention that some new specific definitions are introduced in this book: p-set function, Byes-Fidicial approuch, MinMaxDM distribution family. New solutions of the problems with wide field of application are offered: the most uniformly poweful test for testing statistical hypotheses (Weibull distribution against lognormal distribution); maximization of conditional expectation of estimate of p-quantile (as maximization of specified life under condition of limitation of aircraft failure probability by a very small value); specific coordinates for fitting test data (estimates of mean value of ordered statistics vs sample ordered statistics. But in general this book represents an application of well-known methods of the theory of probability and mathematical statistics to a specific problem related to reliability analysis of both airframe and composite material.

Part 1. Nomination of Specified Life and Inspection Program for Fatigue-Prone Airframe

1. Nomination of Specified Life

1.1. Introduction

There are two categories of Structural Significant Items (SSI) of airframe, the failure of which is considered as failure of an airframe in service. For the first one the service reliability is ensured by discarding an SSI from service when its service life exceeds Specified Maximum Permitted Life (SMPL), that is the time specified by an appropriate authority after which a particular airframe item must be removed from service. We call this category of SSI a *safe-life-SSI* (SLSSI). For the second category of SSI the fatigue damage is detected during planned inspections, then the failure danger is eliminated. We call this category of SSI an *inspection-dependent-SSI* (IDSSI). Here based on p-set function definition we consider the solution of two corresponding problems (for fatigue-prone airframe): the choice of SMPL for SLSSI and inspection program (IP) choice for IDSSI.

We suppose that SLSSI is characterized by a random variable, $\boldsymbol{T}_c$, where $\boldsymbol{T}_c$ is critical lifetime (time to failure). IDSSI is characterized by a random vector (r.v.), ($\boldsymbol{T}_d$, $\boldsymbol{T}_c$), where $\boldsymbol{T}_c$ is the time to failure, $\boldsymbol{T}_d$ is service time, when some damage (fatigue crack) can be detected with probability equal to unit. So if within an interval ($\boldsymbol{T}_d$, $\boldsymbol{T}_c$) some inspection is made, the failure of the IDSSI will be eliminated. We suppose that a service life of IDSSI is also limited by Limited Service Life (LSL), the service life of SSI at which it is no longer physically feasible to repair the item to an acceptable standard. Hereafter we use for short the same abbreviation (SL) and the same notation (t_{SL}) for both cases (SMPL and LSL), because in both cases we should do the same: discard the SSI from service. We suppose that the c.d.f. type for r.v. ($\boldsymbol{T}_d$, $\boldsymbol{T}_c$) is known, while unknown parameter should be calculated using results of full-scale fatigue test.

In section 1 we consider the problem of SL nomination. It should be mentioned also that SL can be chosen as (1) some number from $[0,\infty)$ and as (2) some number from the set of two numbers $\{0,\ t^*_{SL}\}$. This corresponds to (1) nomination of SL, t_{SL}, and (2) rejection or acceptance of predetermined (required) SL, t^*_{SL}.. Here we consider the problem of fatigue failure probability limitation and economical approach, when a fatigue failure leads to some economical losses.

In section 2 we consider the inspection program development.

1.2. Definition of p-set function and p-bound for random vector and variables

To make common approach to solution of the reliability problem for SLSSI and IDSSI possible we need to introduce the *p*-set function definition. It is a special statistical decision function, which, in fact, is generalization of *p*-bound for random variable, the definition of which was introduced by the author in 1976 [1,2]. As the development of *p*-bound the *p*-set function definition was offered in 1999 [3] in the following way.

Definition 1. Let Z and X be random vectors (r.v.) of m and n dimensions and suppose that the class is known $\{ P_\theta \ ,\ \theta \in \Theta \}$ to which the probability distribution of the random vector *W=(Z,X)* is assumed to belong. Of the parameter θ, which labels the distribution, it is presumably known only that it lies in a certain set Θ, the parameter space. Let $S_Z(x) = \bigcup_{i=1}^{r} S_{Z,i}(x)$ denote some set of disjoint sets of *z* values as function of *x*. If

$$\sup_\theta \sum_{i=1}^{r} P(Z \in S_{Z,i}(X)) = p$$

then statistical decision function $S_Z(x)$ is p-set function for r.v. Z on the base of a vector $x=(x_1,x_2,...,x_n)$.

Remark. Later on the value x, observation of the vector X, would be interpreted as a result of some test (for example, the result of full-scale fatigue test of airframe) or parameter estimates after processing of x); Z would be interpreted as some random variable (for example, $\log(\boldsymbol{T}_c)$ or some vector (for example, $(\log(\boldsymbol{T}_d),\log(\boldsymbol{T}_c))$).

For the most important case, when m=1 and Z is a random scalar, there are several useful definitions of special types of p-set functions $S_Z(x)$ which we denote for this special case by $\tau(x)$.

Definition 2. P-set function $\tau(x)$ is called a p-bound for r.v. Z if

$$\sup_{\theta\in\Theta} P_\theta\{Z<\tau(X)\}=p. \tag{1.1}$$

Definition 3. P-set function $\tau(x)$ is called a parameter-free (p.f.) p-set function for r.v. Z if

$$P_\theta\{Z<\tau(X)\} = p \text{ for all parameters } \theta\in\Theta. \tag{1.2}$$

Definition 4. A p-bound for r.v. Z is right-hand binary (r.h.b. p-bound), if for each possible observation x of r.v. X, function $\tau(x)$ assigns only one of two decisions:

$$\tau(x) = -\infty \text{ if } x\in S;\ \tau(x)=\tau^*, \text{ if } x\in S^*, \tag{1.3}$$

where τ^* is some number, S^* and S are two complementary regions of the sample space.

In framework of definition of p-set function we can consider a very wide spectrum of problems related to prediction interval, tolerance region, testing statistical hypotheses.

It is easy to see, that in definition 2 the set $S_Z(x)$ is, in fact, some interval $(-\infty, \tau(x))$ and if Z can be interpreted as "future observation" with the same c.d.f as c.d.f. of all the components of the vector $X=(X_1,X_2,...,X_n)$ then $\tau(x)$ is some right-hand bound of prediction interval (or β-expected bound [4] for $\beta=p$). But, for example, it may be that $X=(X_1,X_2,...,X_n)$ is some vector of (stationary) ages of some renewal parallel system of n items, Z is durability of one item of the same type. Cumulative distribution function of r.v. Z and X may have the same unknown parameter but corresponding c.d.f. are different. It seems that the term "p-bound" in this case is more appropriate than the term "prediction interval". We can say also that p.f. p-bound $\tau(x)$ is a p-quantile estimate and, as function of p, it is an estimate of inverse cumulative distribution function $F_Z^{-1}(p)$, but very specific estimate: expectation value $E(F_Z(\tau(X)))=p$.

If $Z=Y_{(k)}$ is order statistic of independent observations taken on Y, say $Y_1, Y_2, \ldots, Y_m$, and $F_Y(y, \theta)$ strictly increasing c.d.f. of Y has the same unknown parameter θ, $k=[\beta m]$, where $0<\beta<1$, $[x]$-is a maximum integer less than or equal to x, and if $m\to\infty$ then approximately

$$P\{Y_{(k)}<\tau(X)\} = P\{F_Y^{-1}(\beta)<\tau(X)\} = P\{F_Y(\tau(X))>\beta\} = p$$

and $(-\infty, \tau(x)]$ *is* β - content tolerance region at confidence level p.

The binary p-set function has, evidently, some close connection with testing statistical hypotheses: S^* and S are two complementary regions of the sample space just the same as S_0 and S_1 in the problem of hypotheses testing (see [5]).

1.3. P-bound for distribution with location and scale parameters

It is easy to get $\tau(x)$ for distribution with location and scale parameters. As the main application of the problem under consideration we will consider a problem of SL nomination for some fatigue-prone airframe structure. We suppose to have observations of fatigue lives of some identical units as a result of full-scale fatigue tests. Usually for fatigue life data processing both lognormal and Weibull distributions are used. If we use logarithm scale (if we use $X=\ln(T)$ instead of *T*), both these distributions will become distributions with location and scale parameters. So we can say, that r.v. *X* has the following structure:

$$X=\theta_0+\theta_1 \overset{0}{X}, \qquad (1.4)$$

where θ_0, θ_1 are unknown parameters, r.v. $\overset{0}{X}$ has either standard normal c.d.f. $F_{\overset{0}{X}}(x)=\Phi(x)$, where $\Phi(x)$ is c.d.f. of standard normal distribution or standard smallest extreme value (s.e.v) c.d.f., $F_{\overset{0}{X}}(x)=1-\exp(-\exp(x))$, for lognormal or Weibull distributions of *T* correspondingly. In this case for the specified life nomination problem the following theorem can be used (we give it without proof).

Theorem 1. Let

$$F_{X_i}(x,\theta)=F_{\overset{o}{X}}\left(\frac{x-\theta_0}{\theta_1}\right),\ \mathrm{i=1,...,n}, \qquad F_Z(x,\theta)=F_{\overset{o}{Z}}\left(\frac{x-\theta_0}{\theta_1}\right),$$

where $F_{\overset{0}{X}}(\cdot)$, $F_{\overset{0}{Z}}(\cdot)$ are known c.d.f. of $\overset{0}{X}$, $\overset{0}{Z}$, θ_0,θ_1 - are unknown location and scale parameters. And let random variables, estimates of θ_0,θ_1, as function of $X=(X_1, X_2,..., X_n)$ be *correct* estimates. This means that they have the following structures

$$\hat{\theta}_0=\theta_0+\theta_1\overset{o}{\theta}_0,\quad \hat{\theta}_1=\theta_1\overset{o}{\theta}_1, \qquad (1.5)$$

where $\overset{o}{\theta}_0, \overset{o}{\theta}_1$ are random variables, in accordance with the estimates of θ_0,θ_1 using a sample of the same size *n* but when $\theta_0=0$, $\theta_1=1$. And let τ^* be some constant.

Then p.f. and r.h.b. p-bounds are described accordingly by the formulae

$$\tau_1(x)=\hat{\tau}_1,\quad \tau_2(x)=\begin{cases}-\infty, & \hat{\tau}_2\le\tau^*,\\ \tau^*, & \hat{\tau}_2>\tau^*,\end{cases} \qquad (1.6)$$

where $\hat{\tau}_i=\hat{\theta}_0+t_i\hat{\theta}_1,\ i=1,2,$

t_1 is p-quantile of r.v. $V_Z=(\overset{0}{Z}-\overset{0}{\theta}_0)/\overset{0}{\theta}_1$,

t_2 is the root of equation: $\xi(t)=p$,

$$\xi(\mathrm{t})=\sup_{\mathrm{C}} F_{\overset{O}{Z}}(c)(1-F_{\overset{O}{\tau}(t)}(c))=\sup_{\mathrm{C}} F_{\overset{O}{Z}}(c)F_{V_C}(t),$$

$$\overset{o}{\tau}(t)=\overset{o}{\theta}_0+\overset{o}{\theta}_1 t,\qquad \mathrm{V_C}=(c-\overset{o}{\theta}_0)/\overset{o}{\theta}_1.$$

2. If one of the parameters θ_1 or θ_0 is known, then, as usually, we can transform the initial data ($x'_i = x_i/\theta_1$ *or* $x'_i = x_i - \theta_0$, $i \in 1,...,n$) in such a way that in previous formulae for τ we can put $\hat{\theta}_1 = 1$ *or* $\hat{\theta}_0 = 0$, and then

2.1 If it is known that the scale parameter $\theta_1 = 1$ then V_Z, V_C should be replaced by

$$U_Z = \overset{o}{Z} - \overset{o}{\theta}_0, \quad U_C = c - \overset{o}{\theta}_0;$$

function $\xi(t)$ should be replaced by the function $\xi_1(t) = \max_C F_{\overset{0}{Z}}(c) F_{U_C}(t)$,

but for $\hat{\theta}_0 = X_{(1)} = \min(X_1, X_2, ..., X_n)$ by function $\xi_1^1(t) = \max_C F_{\overset{0}{Z}}(c)(1 - F_{\overset{0}{X_1}}(c-t))^n$.

2.2. If it is known that the location parameter $\theta_0 = 0$ then V_Z, V_C should be replaced by

$$W_Z = \overset{o}{Z} / \overset{o}{\theta}_1, \quad W_C = c / \overset{o}{\theta}_1,$$

function $\xi(t)$ by the function $\xi_0(t) = \max_C F_{\overset{o}{Z}}(c) F_{W_C}(t)$, but if additionally $\hat{\theta}_1 =$ $X_{(1)} = \min(X_1, X_2, ..., X_n)$ by function $\xi_0^1(t) = \max_C F_{\overset{o}{Z}}(c)(1 - F_{\overset{0}{X}}(c/t))^n$.

Let us remind that for the purpose of approximate calculation of c.d.f. for V_C, U_C, W_C the Monte Carlo method or normal approximation of distributions of estimations $\overset{o}{\theta}_0, \overset{o}{\theta}_1$ can be used.

1.4. Application of the specified life nomination to the problem

There are two types of problems:

(1) nomination of specified life, t_{SL},

(2) acceptance or rejection of predetermined (required) specified life, t_{SL}^*, for the *m* aircraft in operation.

1.4.1. Nomination of specified life

Let $X = (X_1, X_2, ..., X_n)$, where X_i, $i = 1,...,n$, are fatigue lives of aircraft in (full-scale) laboratory test, $Z = \min(Y_1, Y_2, ..., Y_m)$,where Y_j, $j = 1,...,m$, are fatigue lives of aircraft in operation, $F_{X_i}(t) = F_{Y_j}(t)$, $i = 1,...,n$, $j = 1,...,m$; *p*- allowed probability of failure in operation of at least one aircraft.

It is important to distinguish two cases: the requirement of safety can be defined either by inequality

$(a) \quad P(Y_{(1)} < T_{SL}^1) \le \varepsilon$

or by inequality

(b) $P(Y_{(k)} < T_{SL}^2) \le \varepsilon, \qquad 1 << k << m,$

Where T_{SL}^1, T_{SL}^2 are some functions of $X = (X_1, ..., X_n)$

It is easy to see that the solution of the problem number (1) is defined by the p. f. p-bound $\tau_1(x)$ for $Z=Y_{(1)}$ (for the case (a)) or for $Z=Y_{(k)}$ (for the case (b)) and $p=\varepsilon$. The solution of the problem number 2 is defined by the r.h.b. p-bound $\tau_2(x)$ and for the same Z and p.

If $Z=Y_{(1)}$ then

$$F_{\overset{o}{Z}}(z) = 1 - (1 - F_{\overset{o}{Y}}(z))^m .$$

If $Z=Y_{(k)}$, $m \to \infty$, $k/m = q$, q - is not too close to 0 or 1, then approximately

$$F_{\overset{o}{Z}}(z) = \Phi\left(\frac{z - a_{km}}{\sigma_{km}} \right),.$$

where a_{km}, σ_{km} are expectation value and standard deviation of order statistics $\overset{o}{Y}_{(k)}$, $1 << k << m$. If in addition we can assume normal approximation of $\hat{\theta}_0, \hat{\theta}_1$ (if sample is large enough) then approximately

$$F_{V_Z}(t) = \Phi\left(\frac{t - a_{km}}{\sigma} \right),.$$

where $\sigma^2 = \sigma_{km}^2 + \sigma_0^2 + 2rt\sigma_0\sigma_1 + t^2\sigma_1^2$, σ_0, σ_1, r are standard deviations of $\overset{o}{\theta}_0, \overset{o}{\theta}_1$ and their correlation coefficient. As it has already been mentioned, for fatigue life distribution description the two models— log-normal and Weibull distributions are in wide use. If we put the Y_i in the logarithm of fatigue life, $i = 1,...,m$, then we shall have in both cases the distribution with the location and scale parameters. In both cases the approximate distribution of random variables $\overset{o}{\tau}$ or V_Z, V_C, U_Z, W_C can be obtained using the asymptotically normal distribution of the maximum likelihood estimation of parameters, [2]. For the Weibull distribution

$$\varsigma^1(t) = \sup_c F_{\overset{0}{Z}}(c)(1 - F_{\overset{\circ}{X}}(c - t))^n = \max \varsigma^1(c,t) ;$$

where $\varsigma^1(c,t) = (1-r)r^a$,

$r = \exp(-m\exp(c))$, $a = n\exp(-t)/m$.

Let us denote by ρ the value of r corresponding to the maximum of $\varsigma^1(c,t)$.

From equation $\dfrac{d(\varsigma^1(c,t))}{d(r)} = 0$ we have $\rho = a/(a+1) = n/(m\exp(t)+n)$.

And finally

$$\varsigma^1(t) = \max_c (1 - \exp(-m\exp(c)))\exp(-n\exp(c-t))$$

$$= \quad (1-\rho)\rho^{\frac{\rho}{1-\rho}} = \frac{1}{a+1}\left(\frac{a}{a+1}\right)^a ,$$

where $\rho = n/(m\exp(t)+n)$, $a = n\exp(-t)/m$

It is useful to note that the same formula defines the solution for the class of distributions with monotone increasing hazard rate function [2].

1.4.2. Numerical examples of test time nomination

Let us consider numerical example of the problem of SL nomination. Assume, for example, that simultaneously fatigue tests of 6 airframes of the same type of aircraft have been made but only up to 4-th fatigue failure. So we know only 4 first minimal fatigue lives: ($t_{(1)}$, ..., $t_{(4)}$)=(59971; 72600; 77630; 80863) and correspondingly for $x_{(i)} = \ln(t_{(i)})$, i=1,...4, we know x =($x_{(1)}$, ..., $x_{(4)}$) = (11.002; 11.193; 11.260, 11.3005).There are m = 100 aircraft in operation and there is a requirement, that the probability of at least one fatigue failure before t^*_{SL} = 50000 cycles should not exceed $p \approx 0.05$. Then τ^* = log(50000) = 10.82. In accordance with the r.h.b. p-bound definition we can be sure of the required reliability if $\hat{\theta}_0 + t_2\hat{\theta}_1 > \tau^*$

Let us consider at first the log-normal distribution of the fatigue life. Then using the Lifereg procedure of SAS system we can easily get ML estimations $\hat{\theta}_0 = 11.26, \ \hat{\theta}_1 = 0.145$. And then using Monte Carlo method to get V_C c.d.f. (5000 samples) we have, that t_2= -7.055 is the root of the equation

$$\xi(t) = \max(1-(1-\Phi(c))^m)F_{V_C}(t) = p = 0.05$$

Where $V_C = (c - \overset{0}{\theta}_0)/\overset{0}{\theta}_1$. Accordingly $\hat{\tau}_2 = \hat{\theta}_0 + t_2\hat{\theta}_1 =$ 11.26 -7.055*0.145=10.237.

This value is less than required τ^* = 10.82. So the required reliability is not provided. Now let us consider the case when θ_1 = 0.346 is known and a new fatigue test after some structure retrofit has to be made. And we have to know time limit of fatigue test without failure, which will be enough to be sure of the required reliability. In this case t_2^1 =-2.04 is the root of the equation

$$\zeta^1(\mathrm{t}) = p = 0.05 \ ,$$

where $\varsigma^1(t) = \max_c(1-(1-\Phi(c))^m)(1-\Phi(c-t))^n$, m=100; n=6.

So for required time limit of fatigue test (in logarithm scale), we have

$$x_{(1)} = \tau^* - t_2^1\theta_1 = 10.82 - (-2.0425) \times 0.346 = 11.52648$$

or in natural scale

$$t^* = \exp(11.52648) = 101365 \quad .$$

For the case of Weibull distribution, using the same Lifereg procedure of SAS system, we can easily get ML estimates

$$\hat{\theta}_0 = 11.3, \quad \hat{\theta}_1 = 0.093.$$

And then it can be found that $t_2 = -14.303$ is the root of the equation

$$\xi(\mathrm{t}) = \sup_{\mathrm{c}}(1-\exp(-m \ \exp(c)))F_{V_C}(t) = 0.05.$$

Accordingly

$$\hat{\tau}_2 = \hat{\theta}_0 + t_2\hat{\theta}_1 = 11.3 - 14.303 \times 0.093 = 9.97.$$

And again this value is less than required τ^* = 10.82. So we do not provide the required reliability. If then, again, we know standard deviation of logarithm of fatigue life σ{logT}=0.346 and correspondingly θ_1=0.346($\sqrt{6}/\pi$)=0.270 then we can find that t_2^1=-4.74 is the root of the equation

$$\zeta^1(t) = (1-\rho)\rho^{\frac{\rho}{1-\rho}} = 0.05,$$

where $\rho = n/(me^t + n) = 6/(100\,e^t + 6),$

and required time limit of fatigue test without failure of 6 aircraft (in order to provide required reliability of 100 aircraft in operation) is equal to

$\exp(\tau^* - t^1{}_2\theta_1) = \exp(10.82+4.74.0.27) = 179836.$

It is worth to mention, that in case of Weibull distribution of fatigue life the needed test time is more than in the case of log-normal distribution.

1.4.3. Optimality criterion for p-bound used for airframe specified life nomination

In application to the problem number 1 (SL nomination) we should get the maximum of expectation value of $\tau(X)$ provided that reliability requirements are met, it is if $\tau(X)$ is a *p*-bound for *Z*. To study the optimality of $\tau(x)$ we can use the Jensen's inequality [6].This inequality says that the function of complete sufficient statistics, which is unbiased estimation of its own mathematical expectation, provides the minimal risk if the correspondent loss-function is convex. Consider the simplest case, when θ_1 is the known parameter. Let $\theta_t = \theta_0 + t\theta_1$ be some quantile. Random variable $\hat{\theta}_t = \tau(x) = \hat{\theta}_0 + t\theta_1$ is unbiased estimate of its own expectation (which in general case does not equal to θ_t). In problem under consideration the function $F_Z(\tau)$ can be considered as the loss-function. Then the expectation $E_X\{F_Z(\hat{\theta}_t)\} = P(Z < \tau(X))$ is the risk function. For normal and smallest extreme value (sev) distributions of Y_j $j = 1,...,m$, $F_Z(\tau)$ is convex (and the increasing one) if its value is small enough and we have minimum of $E_X\{F_Z(\hat{\theta}_t)\} = P(Z < \tau(X)) = p$ at the fixed expectation value of $\hat{\theta}_t = \tau(X)$, if $\tau(x)$ is a function of sufficient statistic. And, on the contrary, if $\tau(x)$ is a function of sufficient statistic and $P(Z < \tau(X)) = p$, then we have maximum of expectation value of $\tau(X)$ if p is small enough and probability $P(\tau(X) < c)$ is high enough for such *c*, that $F_Z(z)$ is convex if $z < c$. For example, for normal distribution $\Phi(z)$ is convex if $z<0$.

For the case when parameter θ_1 is also unknown we can make similar statement. The generalization of the Jensen's inequality for the case of multivariate sufficient statistic can be found in [6].

For the problem number 2 (acceptance or rejection of predetermined (required) Specified Life, t^*_{SL}), if instead of $\tau(x)$ we consider $\exp(\tau(x))$ (in this way we can avoid difficulty of calculation of expectation $E\{\tau_2(X)\}$, when $P\{\tau_2(X) = -\infty)\}>0$, we also need to increase expectation $E\{\exp(\tau_2(X))\}$ and for this purpose we need to use sufficient statistic. In fact, this means, that we have to increase the probability to make decision $\tau(x) = \tau^*$. Of course, it is very near to the requirement to increase the power of corresponding test, if we consider this problem as the test of statistical hypothesis.

For the case when sufficient statistic coincides with the sample itself (for example, for the Weibull distribution) usually for calculation of prediction interval the Monte Carlo (MC) method is used [11]. Here we show that for the problem of p.f. p-bound, $\tau(x)$, calculation an analytic solution can be found using Bayes-fiducial (BF) approach.

1.4.4. Bayes-fiducial approach

This approach was offered in 1976 [1]. Its development was offered in [2,3,7,8]). It was shown that using this approach we can get Pitmen's estimates of location and scale parameters [4] and most powerful invariant test for testing statistical hypotheses $H_0 : F(x) = F_0((x-\theta_0)/\theta_1)$; $H_1 : F(x) = F_1((x-\theta_0)/\theta_1)$). It can be used also for unbiased estimation [9]. BF estimate, $\tau_X(x)$, of some function of parameter $\tau_\theta(\theta)$ is a function, which minimizes BF risk

$$\rho_{BF}(\tau_\theta, \tau_X) = \int L(\tau_\theta(\theta), \tau_X(x)) dF_{\tilde{\theta}}(\theta) \ ,$$

where $L(\tau_\theta(\theta), \tau_X(x))$ is loss function, $F_{\tilde{\theta}}(\theta)$ is fiducial distribution on parameter space [2,10].

There are two advantages of BF approach:

1. As in case of using a maximum likelihood (ML) estimates BF solution is always a function of sufficient statistic, but in contrast to ML the BF solution takes into account the loss function.

2. In contrast to the usual Bayes solution we do not need to have a priori distribution of unknown parameters.

Using BF approach for p.f. p.-bound calculation

Let the problem be to estimate p-quantile $\tau_p(\theta)$ for c.d.f. $F_Z((x-\theta_0)/\theta_1)$ and loss function $L(\tau_p(\theta), \tau_X(x)) =$ $(F_Z((\tau_p - \theta_0)/\theta_1) - F_Z((\tau_X(x) - \theta_0)/\theta_1))^2 = (p - F_Z((\tau_X(x) - \theta_0)/\theta_1))^2$

when we have sample $x = (x_1, x_2, ..., x_n)$ from population with c.d.f $F_X((x-\theta_0)/\theta_1)$.

Let us denote by $\tau_X(x, p)$ the solution of equation , corresponding to the considered loss function

$$E_{\tilde{\theta}}\{F_Z((\tau_X(x,p) - \tilde{\theta}_0)/\tilde{\theta}_1\} = p, \tag{1.7}$$

where $\tilde{\theta} = (\tilde{\theta}_0, \tilde{\theta}_1)$, r.v. $\tilde{\theta}_0$, $\tilde{\theta}_1$ have fiducial distribution. Here $E_X(f(X))$ is an expected value of $f(X)$ in accordance with c.d.f. of *X*.

We can simplify solution of Eq.1.7. Instead of vector $x = (x_1, ..., x_n)$ without loss of information we can consider vector $\varpi = (\hat{\theta}_0, \hat{\theta}_1, w_1, ..., w_{n-2})$, where $\hat{\theta}_0, \hat{\theta}_1$ are correct parameter estimates (see (1.5)), $w_i = (x_i - \hat{\theta}_0)/\hat{\theta}_1$, $i = 1, ..., n$-2 , are components of maximal invariant. Then conditional fiducial distribution (at the fixed invariant $(w_1, ..., w_{n-2})$) of random variables $\tilde{\theta}_0, \tilde{\theta}_1$ is defined by equation [1,2]

$$f_{\tilde{\theta}_0, \tilde{\theta}_1 | w_1, ..., w_n}(s_0, s_1) = h \frac{\hat{\theta}_1^{\,n-1}}{s_1^{n+1}} \prod_{i=1}^{n} f\left(\frac{\hat{\theta}_0 + \hat{\theta}_1 w_i - s_0}{s_1}\right) ds_0 ds_1 \ ,$$

where *h* is just normalization factor. (Note: w_{n-1}, w_n are functions of $w_i = (x_i - \hat{\theta}_0)/\hat{\theta}_1$, $i = 1, ..., n-2$, components of maximal invariant ϖ).

If in (1.7) we use new notations:

$U_0 = (\hat{\theta}_0 - s_0)/\hat{\theta}_1$, $U_1 = \hat{\theta}_1 / s_1$, $\overset{0}{\tau}(x,p) = (\tau(x,p) - \hat{\theta}_0)/\hat{\theta}_1$,

then instead of (1.7) we get equation

$$E_{W_1,\ldots,W_n} E_{U_0 U_1 | w_1,\ldots,w_n} \left(F((\overset{0}{\tau}(x,p) - U_0)/U_1) \right) = p \,. \tag{1.8}$$

where random variables U_0, U_1 have conditional p.d.f, which do not depend on unknown parameters

$$f_{U_0,U_1|w_1,\ldots,w_n}(u_0,u_1) = h_w u_0^{n-2} \prod\nolimits_{i=1}^{n} f(u_0 + w_i u_1)\,, \tag{1.9}$$

where h_w is just a normalization factor which depends only on invariant vector $w = (w_1,\ldots,w_{n-2})$.

If $\overset{0}{\tau}(x,p)$ is the solution of the equation

$$E_{U_0 U_1 | w_1,\ldots,w_n} \left(F((\overset{0}{\tau}(x,p) - U_0)/U_1) \right) = p \tag{1.10}$$

then

$$\tau_X(x,p) = \hat{\theta}_0 + \overset{0}{\tau}(x,p)\hat{\theta}_1 \tag{1.11}$$

is the solution of Eq. (1.8) and Eq.(1.7) also because equation (1.10) takes place for every vector $w = (w_1,\ldots,w_{n-2})$, c.d.f. of which does not depend on $\theta = (\theta_0, \theta_1)$. So if (1.10) is true then (1.7) is true also.

It is very important that $\overset{0}{\tau}(x,p)$ in (1.10) does not depend on parameter $\theta = (\theta_0, \theta_1)$ and for solution of this equation we can set $\theta_0 = 0$, $\theta_1 = 1$. If $\hat{\theta}_0, \hat{\theta}_1$ have correct structures defined by (1.5) then the probability $P(Z < \tau(X,p))$ does not depend on $\theta = (\theta_0, \theta_1)$ and we can find p_1 for which

$P(Z < \tau(X, p_1)) = p$.

So $\tau_X(x, p_1)$ is p-bound for random variable Z.

As it is easy to see (see p.84 in [2]) that the p.d.f (1.9) is conditional p.d.f of $\hat{\theta}_0, \hat{\theta}_1$ at the fixed $w = (w_1,\ldots,w_{n-2})$ for the case when $\theta_0 = 0$, $\theta_1 = 1$. This means that the values p_1 and p coincide.

It is very important also that a result does not depend on the choice of the type of correct statistics $\hat{\theta}_0, \hat{\theta}_1$ (see (1.13), (1.14))), because vector $x = (x_1,\ldots,x_n)$ and vector $\varpi = (\hat{\theta}_0, \hat{\theta}_1, w_1,\ldots,w_{n-2})$ have one-to-one mapping at any choice of correct statistics.

Example. P-bound for lognormal distribution.

Let $t = (t_1, t_2, t_3)$ =(45 952, 54 143, 65 440) be the sample from this distribution. Then r.v. $X = \log(T)$ has a normal distribution $N(\theta_0, \theta_1^2)$ and $x = (x_1, x_2, x_3)$ = (10.735 10.899 11.089) is the sample from this distribution. The problem is to calculate the p.f. p-bound for independent r.v. $Z = \min(Y_1,\ldots,Y_m)$, where r.v. Y_i, $i = 1,\ldots,m$, has the normal distribution

$N(\theta_0, \theta_1^2)$ also. We consider here only the case, when m=1, because for this case there is a general analytical solution (see, for example, p. 172 in [2])

$$\tau(x) = \hat{\theta}_0 + \hat{\theta}_1 t_{n-1,p}(1+1/n)^{1/2}, \tag{1.12}$$

where

$\hat{\theta}_0 = \bar{x}$, $\hat{\theta}_1 = (\sum (x_i - \bar{x})^2 /(n-1))^{1/2}$

are estimates of expected value and standard deviation, $t_{k,q}$ is q-quantile from Student's distribution with *k* degrees of freedom. So we can make comparison of this solution with the solution which we get, using new approach.

For considered data, using equation (1.12) and Student's distribution table for *p*=0.01 we calculate t_{St}=exp($\tau(x)$) = 13 162, which is the value of p-bound for r.v. *T* on the base of observations (t_1, t_2, t_3) .

Now let us consider a new approach. For normal distribution the conditional p.d.f. has the following form

$$f_{U_0,U_1|w_1,\dots,w_n}(u_0, u_1) = h_w u_0^{n-2} \prod_{i=1}^{n} \varphi(u_0 + w_i u_1),$$

where $\varphi(x) = \exp(-x^2/2)/(2\pi)^{1/2}$. After transformation the equation (1.10) has the following form

$$1 - a(\overset{0}{\tau}, \bar{z}, D_z)/\Gamma((n-1)/2) = p,$$

where

$$a(\overset{0}{\tau}, \bar{z}, D_z) = \int_0^{\infty} u^{(n-3)/2} \exp(-u)\Phi\left((2u/D_z(n+1))^{1/2}(\bar{z} - \overset{0}{\tau}) \right) du, \ \bar{z} = \sum_1^n z_i / n,$$

$D_z = \sum_{i=1}^{n} (z_i - \bar{z})^2 / n$, $\Gamma(\cdot)$ is gamma function, $\Phi(\cdot)$ is c.d.f. of standard normal distribution.

Let us consider two types of statistics $\hat{\theta}_0, \hat{\theta}_1$, which for considered data have the following values:

a) $\hat{\theta}_{0a} = \bar{x}$ =10.908, $\hat{\theta}_{1a} = (\sum (x_i - \bar{x})^2 / (n-1))^{1/2}$ =0.177 , (1.13)

b) $\hat{\theta}_{0b} = x_{1,n}$ =10.735, $\hat{\theta}_{1b} = x_{n,n} - x_{1,n}$ =0.354, (1.14)

where $x_{i,n}$ is its order statistic of vector $x = (x_1, \dots, x_n)$.

In case a) we have $\overset{0}{\tau}_a$ = -7.889, in case b) we have $\overset{0}{\tau}_b$ = - 3.560.

Corresponding values of p-bound for r.v. *T* on the base of observations (t_1, t_2, t_3) are:

t_a = exp($\tau_a(x)$) = 13 523, t_b = exp($\tau_b(x)$) = 13 050.

It seems that the difference between t_a, t_b and t_{St}=13 162 is produced only by the problem to get required calculation accuracy.

Example. P-bound for Weibull distribution.

Let us have the same sample $t=(t_1,t_2,t_3)$=(45 952, 54 143, 65 440) or $x=(x_1,x_2,x_3)$ = (10.735 10.899 11.089) but r.v. T has a Weibull distribution and, correspondingly $X=\log(T)$ has distribution of the smallest extreme value with c.d.f. $F_X(x)=1-\exp(-\exp((x-\theta_0)/\theta_1)$. In this case the equation (1.10) has the following form

$$1-a(\overset{0}{\tau},\bar{z},D_z)/b(\bar{z},D_z)=p\ ,$$

where

$$a(\overset{0}{\tau},\bar{z},D_z)=\int_0^{\infty}u^{(n-2)}\Big(\ \exp(-u\sum_{i=1}^{n}z_i)/(\sum_{i=1}^{n}\exp(uz_i)+m\exp(u\overset{0}{\tau}))^n\ \Big)du\ ,$$

$$b(\bar{z},D_z)=\int_0^{\infty}u^{(n-2)}\Big(\ \exp(-u\sum_{i=1}^{n}z_i)/(\sum_{i=1}^{n}\exp(uz_i))^n\ \Big)du\ ,$$

$$\bar{z}=\sum_{1}^{n}z_i/n\ ,\ D_z=\sum_{i=1}^{n}(z_i-\bar{z})^2/n\ .$$

For m=1, p=0.01, using statistics (1.13) we get $\overset{0}{\tau}$ = -11.929, using statistics (1.14) we get $\overset{0}{\tau}$ = -5.424. Corresponding values of p-bound for r.v. T on the base of observations (t_1,t_2,t_3) are:

t_a = exp($\tau_a(x)$) = 6 616, t_b = exp($\tau_b(x)$) = 6 752.

For m=500, p=0.2 using statistics (1.13) we get $\overset{0}{\tau}$ = -12.889, using statistics (1.14) we have $\overset{0}{\tau}$ = - 5.970. Corresponding values of p-bound for r.v. T based on the observations (t_1,t_2,t_3) are: t_a = exp($\tau_a(x)$) = 5 584, t_b = exp($\tau_b(x)$) = 5 568.

Again, it seems that the difference between t_a and t_b is produced only by the problem to get required calculation accuracy.

Considered data in fact was studied in several papers. Mee and Kushary (1994) for m=500, p=0.2 have got t = 5225 [11] . They say that Lowless (1973) obtained t =5623. In both cases for necessary calculation the Monte Carlo method was used.

1.4.5. P-bound as function of order statistics

Let us denote expectation value $E\{\tau(X)\}$ by ν. The straight way to get approximate solution of the problem to get maximum of ν provided that reliability requirements are met, can be found if $\tau(x)$ is a linear function of order statistics: $\tau(x)=a\ x_{(1,\dots,n)}$, where a = $(a_1,\dots,a_n)$ is row vector, $x_{(1,\dots,n)}$ = $(x_{(1)},x_{(2)},\dots,x_{(n)})^T$ is column vector of order statistics (here the transpose (of a vector or of a matrix) is denoted by a capital superscript T). If $a\xi=1$, where $\xi=(1,\dots,1)^T$ is column vector of units then $\tau(x)$ is p.f. p-bound for r.v. Z for some p because in this case $\tau(x)$ has the following structure:

$$\tau(x)=\theta_0+\theta_1\overset{0}{\tau}\ ,\ \overset{0}{\tau}=\tau(\overset{0}{X})=a\ \overset{0}{X}_{(1,\dots,n)}\ ,$$

where $\overset{0}{X}_{(1,\dots,n)}$ has the same type of c.d.f. as $X_{(1,\dots,n)}$ but θ_0 =0, θ_1 =1.

Let $\overset{0}{\nu}$ be expectation value of $\overset{0}{\tau}$. The probability $P(Z < \tau(X))$ is the function of $\overset{0}{\nu}$. Let us define the function

$$\psi(\overset{0}{\nu}) = P(Z < \tau(X)) = P(U < 0),$$

where $U = \overset{0}{Z} - \overset{0}{\tau}$. And let us denote the root of equation by $\overset{0}{\nu}_p$

$$\psi(\overset{0}{\nu}) = p\,.$$

Temporarily we assume that U has normal distribution. Then

$$\overset{0}{\nu}_p = \mu_{\overset{0}{Z}} + (\sigma^2_{\overset{0}{Z}} + \sigma^2_{\overset{0}{\tau}})^{1/2}\Phi^{-1}(p),$$

where μ_X, σ_X are expectation value and standard deviation of variable *X*, $\Phi(\cdot)$ is c.d.f. of standard normal distribution function.

We see that for *p*<0.5 the value of $\overset{0}{\nu}_p$ increases if standard deviation of $\overset{0}{\tau}$ decreases. In application to the problem of specified life nomination the value of *p* is much smaller than 0.5. Thus for *p*<0.5 and

for fixed both expectation value and standard deviation of $\overset{0}{Z}$ we can increase the value of $\overset{0}{\nu}_p = \overset{0}{\nu}_p(a)$ if we choose a in such a way that the standard deviation of $\overset{0}{\tau}$ decreases. Using the theorem 1.f.1(II) in [10] we can get vector $a\ (\overset{0}{\nu})$, which provides the minimum of variance of $\overset{0}{\tau} = \tau(\overset{0}{X}) = a\ \overset{0}{X}_{(1,\ldots,n)}$ for fixed $\overset{0}{\nu}$ and under condition that $a\xi = 1$. These two conditions (fixed $\overset{0}{\nu}$, condition for parameter free p-bound) can be written as the equality

$$B^T a = W,$$

where *B*=($\overset{0}{\mu}, \xi$), $\overset{0}{\mu} = (\overset{0}{\mu}_{(1)},\ldots,\overset{0}{\mu}_{(n)})^T$ is column vector of expectation of vector of standard order statistic, $\overset{0}{X}_{(1,\ldots,n)}$, column vector *W*=$(\overset{0}{\nu},1)^T$. The variance of $a\,\overset{0}{X}_{(1,\ldots,n)}$ is equal to $V = a^T\,\overset{0}{D}\,a$, where $\overset{0}{D}$ is covariance matrix of order statistics $\overset{0}{X}_{(1)},\ldots,\overset{0}{X}_{(n)}$. In accordance with the mentioned theorem minimum of *V* takes place for

$$a(\overset{0}{\nu}) = ((\overset{0}{D})^{-1} B S^{-1} W)^T,$$

where matrix *S* = $B^T\ (\overset{0}{D})^{-1}\ B$,

Now (in general case, using Monte Carlo method) we can find $\overset{0}{\nu} = \overset{0}{\nu}(p)$ for which

$$P(\overset{0}{Z} < a(\overset{0}{\nu})\,\overset{0}{X}_{(1,\ldots,n)}) = p\,.$$

Corresponding $\tau(x) = a\ (\overset{0}{\nu}(p))\ x_{(1,\ldots,n)}$ is p.f. p-bound. It has maximum of $E\{\tau(X)\}$ if *U* has normal distribution. Some conditions under which the sum of order statistics has approximately normal distribution are given in [12]. For general case we have got only an approximate solution corresponding to minimum of variance of p.f. p-bound $\overset{0}{\tau} = \tau(\overset{0}{X}) = a$

$\overset{0}{X}_{(1,\ldots,n)}$ at the fixed $\overset{0}{\nu}$. True value of $P(Z < a(\overset{0}{\nu})\overset{0}{X}_{(1,\ldots,n)}) = p$ can be estimated using, for example, Monte Carlo method.

P-bound as function of order statistics. Numerical examples.

Suppose that simultaneous fatigue tests of 6 airframes of the same type of aircraft have been made but the test was finished at the 4-th fatigue failure. So we know only 4 first minimal fatigue lives: $t_{(1)},...,t_{(4)}$ = 59971; 72600; 77630; 80863 and correspondingly in (natural) logarithm scale $x_{(1)},...,x_{(4)}$ =11.002; 11.193; 11.260, 11.3005. There are m=100 aircraft in operation but the probability of at least one fatigue failure up to specified life should not exceed p=0.05.

If it is supposed that the lognormal distribution of fatigue life takes place then we have normal distribution of r.v. $X = \ln(T)$.

Expectation values and covariance matrix for the first four order statistics of 6 observations from standard normal distribution can be found in special tables or may be calculated using Monte Carlo (MC) method. Then we can calculate the function $\psi(\overset{0}{\nu})$ and for p=0.05 we find that $\overset{0}{\nu}_p$=-7.0, vector a=[3.8883; 1.5865; 0.3789; -4.8537] and, finally, τ = 9.9539. In natural scale t_{SL}=21 035.

1.4.6. Influence of cumulative distribution function type on specified life nomination

As it was mentioned already, usually lognormal or Weibull cumulative distribution functions are used for fatigue life sample, $t_1,...,t_n$, processing. If we use the logarithm scale and define $x_i = \log(t_i)$ i=1,2,…,n, then in both cases we have c.d.f. with location, θ_0, and scale, θ_1, parameters: $F_0((x-\theta_0)/\theta_1)$ or $F_1((x-\theta_0)/\theta_1)$, where $F_0(x) = 1-\exp(-\exp(x))$, $F_1(x) = \Phi(x)$. So later we consider the sev and normal distributions. There are the following connections

$$\mu = \theta_0 + \theta_1 \overset{0}{\mu},\ \sigma = \theta_1 \overset{0}{\sigma}$$

of parameters θ_0 and θ_1 with expectation values $\mu = E(\overset{0}{X})$ and standard deviation $\sigma = \sigma(\overset{0}{X})$, where random variable $\overset{0}{X}$ has c.d.f. $F_0(.)$ or $F_1(.)$, $\overset{0}{\mu}$, $\overset{0}{\sigma}$ are expectation value and standard deviation of random variable $\overset{0}{X}$, when θ_0=0 and θ_1=1.

For normal distribution $\overset{0}{\mu}$=0, $\overset{0}{\sigma}$=1. For sev distribution $\overset{0}{\mu} = -\gamma$, $\overset{0}{\sigma}=\pi/\sqrt{6}$, where γ=0.5772156649 is Eulerian constant. We suppose that SL is defined by the allowable failure probability ε and should be equal to p-quantile (p=ε) of corresponding c.d.f.. In both cases SL is defined by equation

$\tau = \theta_0 + t\theta_1 = \mu + h\,\sigma$,

where t=$F^{-1}(p)$, $F^{-1}(.)$ is function, which is inverse to function $F(.)$, $h = (F^{-1}(p) - \overset{0}{\mu})/\overset{0}{\sigma}$.

For the purpose of comparison of two values of SL at the same μ and σ it is enough to make comparison of two values of h. The results of calculations of

$h_0 = (\log(-\log(1-p)) + \gamma)\sqrt{6}/\pi$ for s.e.v and $h_1 = \Phi^{-1}(p)$ $h_1=\Phi^{-1}(p)$ for normal distributions are given in table 1, where delta = h_1-h_0.

Table 1.4.1. Calculation and comparison of h_0 and h_1

p	0.001	0.01	0.1	0.2
h_1	-3.090	-2.326	-1.281	-0.841
h_0	-4.935	-3.136	-1.304	-0.719
delta	1.845	0.8103	0.0230	-0.122

We see, that if p is small (p<0.1) then delta is positive and SL for normal distribution is greater than for s.e.v. distribution. So if required reliability is high enough and we don't know which of two considered c.d.f. is true then we should use s.e.v. distribution in order to provide the required reliability in any case. But this means that we will have some significant loss if really normal distribution is true. Of course, the desire appears to make statistical hypothesis testing of c.d.f. type before calculation of SL.

The use of statistical hypothesis testing for c.d.f. type choice

There is uniformly most powerful invariant test for the case, when we need to make choice of one of two c.d.f. with unknown location and scale parameters [2,5]. For the case, when hypothesis H_0 is s.e.v. distribution and alternative H_1 is normal distribution the statistic of this criterion is

$\aleph_{SN} = f_N / f_S$;

where $f_N = (\pi)^{(n-1)/2}\Gamma((n-1)/2)/2n^{n/2}$; $f_S = \Gamma(n)\int_0^\infty t^{n-2}dt / (\sum_{i=1}^{n}\exp(t(z_i)))^n$

where $z_i = (x_i - \bar{x})/s$, $i = 1,...,n$, $\bar{x} = \sum_{i=1}^{n} x_i / n$, $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 / n$.

Calculation of c.d.f. for the statistics of the corresponding test is difficult enough. More simple test, which has nearly the same power, is offered in [2]. The critical region in this case is described by inequality

$$\rho(x) = \sum_{i=1}^{n}\exp(\pi(x_i - \bar{x})/s\sqrt{6})/n > C,$$

So we will calculate p-bound or SL (in the logarithm scale) by the use of normal distribution function

$$x_{pL} = \hat{\theta}_{0N} + \hat{\theta}_{1N}\Phi^{-1}(p),$$

if $\rho(x)>C$. And we will calculate SL (in the logarithm scale) by the use of sev distribution function

$$x_{pW} = \hat{\theta}_{0W} + \hat{\theta}_{1W}\ln(-\ln(1-p))$$

if $\rho(x) \le C$, where $\hat{\theta}_{0L}, \hat{\theta}_{1L}, \hat{\theta}_{0W}, \hat{\theta}_{1W}$ are estimates of corresponding parameters. As it was shown, if p is small enough, then $x_{pL} > x_{pW}$. So the requirement for reliability, R>(1-ε), will be met in any case, if it is met for the s.e.v distribution. Corresponding requirement can be written in the following way:

$$E\{(1-\exp(-\exp((X_{pW}-\theta_{0W})/\theta_{1W}),\rho\le C\}+$$
$$E\{(1-\exp(-\exp((X_{pL}-\theta_{0W})/\theta_{1W}),\rho>C\}\le\varepsilon. \quad (1.15)$$

Here $E\{Z,A\}$ means $E\{Z\mid A\}P\{A\}$ - conditional expectation value of random variable Z, if event A takes place, multiplied by probability of event A. Results of calculation by the use of Monte Carlo method are given in the table 4.2 for ε=0.01, sample size n=20, Monte Carlo sample number, N_{MC}, is equal to 5000. In the table the following values, corresponding to this procedure, are given: level of significance α; value of *C*; power of test, β; corresponding value of *p* and expectation value of $\hat{X}_{pL}$ (for $\theta_{0L}=0,\ \theta_{1L}=1$). If α increases (and simultaneously β), we more often use normal distribution and we have larger value of SL, but at the same time we should use much smaller value of p in order to provide reliability in case of s.e.v distribution. We see, that for n=20 and ε=0.01 optimal α^* is approximately equal to 0.1.

Table 1.4.2. Calculation of expected value $E(\hat{X}_{pL})$

α	C	β	p	$E(\hat{X}_{pL})$
0,00	-	0	0.0050	-3.6230
0,01	2,4179	0,192	0,0140	-3,4214
0,05	2,1826	0,428	0,0101	-3,2507
0,10	2,0867	0,573	0,0067	-3,2294 *
0,15	2,0318	0,659	0,0045	-3,2665
0,20	1,9893	0,728	0,0031	-3,2958
0,25	1,9565	0,785	0,0021	-3,3338
0,30	1,9272	0,833	0,0015	-3,3488
0,40	1,8788	0,898	0,0008	-3,4017
0,50	1,8363	0,936	0,0005	-3,4406
0,60	1,7920	0,968	0,0003	-3,4917
0,70	1,7535	0,982	0,0002	-3,5556
0,80	1,7062	0,993	0,0002	-3,5157

When α=0 then we always use s.e.v distribution and for n=20 $E(\hat{X}_{pL})$=-3.6230. It is worth to mention, that if distribution parameters are known but distribution function type is unknown, then to provide reliability in any case without test we should choose SL calculated for sev distribution and then h_0= -3.136 (see table 1). Value of $E(\hat{X}_{pL})$ is much less than h_0, because in this case we make estimation of parameters by the use of random sample and $\hat{X}_{pL}$ is random variable. The final result is defined by specific features of s.e.v. distribution function.

For optimal α*=0.1 we have $E(\hat{X}_{pL})$=-3,2294. For calculations, connected with fatigue life, very often the value of standard deviation σ= σ(log(*T*))=0.346 is used. Then in time scale increasing of SL is equal to

$k=\exp(\mu+\sigma\ E(\hat{X}_{pL}(\alpha^*)))/\exp(\mu+\sigma E(\hat{X}_{pL}(0)))=1.482$

The value *p*=0.01 was chosen to make calculation results clearer, but really we need much less value of allowable failure probability and, for example, if p=0.001, then for big enough n, when test power is quite near to 1, k is equal to 1.89. So in the considered case using the offered procedure we will make SL nearly two times bigger.

1.4.7. Economic approach

This section is devoted to the economic approach to the problem of SL nomination. The problem will be studied both for the case when distribution parameters are known and for the case when they are not known. Numerical example will be given for the case of lognormal distribution of fatigue life with known scale parameter.

Choice of specified life when fatigue life distribution function is known

If we measure the income of aircraft successful service during time t as equal to t, but in case of failure we suppose to have a loss, which is equal to b, then of one aircraft service income, r.v. U is defined by formula

$$U = \begin{cases} t_{SL}, & if\ T > t_{SL}, \\ T - b, & if\ T \le t_{SL}, \end{cases} \tag{1.16}$$

where T is r. v., fatigue life of SSI, t_{SL} is some SL.

Let $F_T(t,\theta)$, $\theta=(\theta_0,\theta_1)$, be c.d.f. of r.v. T. Then u, expectation value of r.v. U, as function of t_{SL}, θ, b is defined by formula

$$u(t_{SL},\theta, b) = \int_0^{t_{SL}} (t-b)\, d F_T(t,\theta) + t_{SL}(1- F_T(t_{SL},\theta)) \tag{1.17}$$

For the case of normal distribution of $X = \log(T)$

$$u(t_{SL};\theta,b)=exp(\theta_0+\theta_1^2/2)\Phi(z_{SL}-\theta_1)+ t_{SL} - (t_{SL}+b)\Phi(z_{SL}), \tag{1.18}$$

where $z_{SL} = (\log(t_{SL}) - \theta_0)/\theta_1$.

An example of this function, as function of t_{SL}, and function q*50 000,where $q=F_T(t_{SL},\theta)$ (the factor 50 000 is used for clear visual purpose), for fixed b=346 000, θ_1=0.346 and θ_0=log(20909)= 9.94793 is given in Fig.1.1.

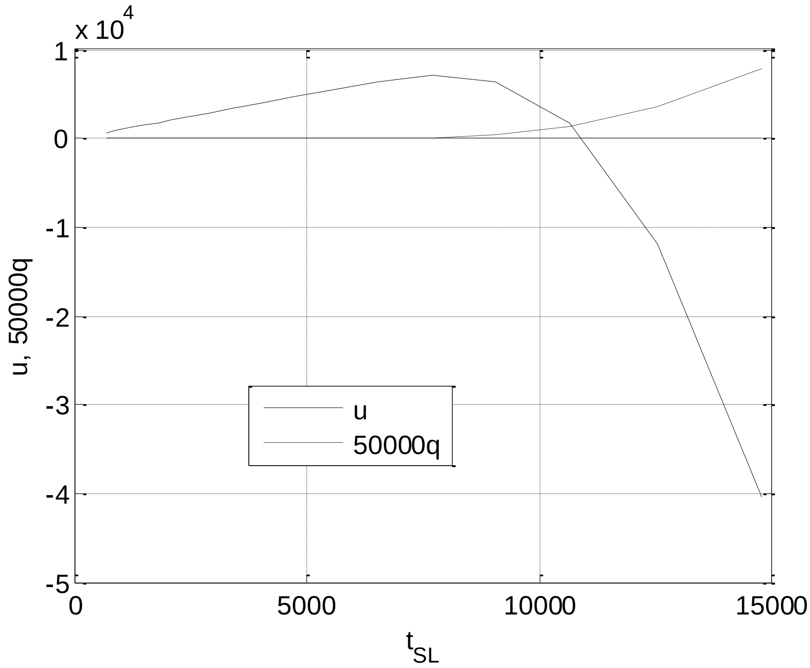

Fig.1.1. Expectation value of r.v. U, function $u(t_{SL};\theta,b)$, and $50000q$, where $q=F_T(t_{SL},\theta)$ (the factor 50 000 is used for clear visual purpose), for b=346 000, θ_0= 9.95, θ_1=0.346.

Using Fig.1.1 we see that maximum value of u corresponds to t_{SL} which is approximately equal to 8000. More precise result is t_{SL}=7936. It is interesting to note that the value of t_{SL}=7936 corresponds (at considered θ_0= 9.94793, θ_1=0.346) to the probability failure (0.0026). This can be interpreted in following way. Failure of 2.6 from 1000 aircraft can be considered as loss of 346000 hours of operation time or loss of 346000/7936=43.6 aircraft of the same type (each aircraft of this type has t_{SL}=7936).

In general case maximum of $u(t_{SL}, \theta, b)$ is reached at t^*_{SL}, which is the root of the equation

$$bf_T(t)/(1-F_T(t,\theta))=1. \quad (1.19)$$

For normal distribution of $X=\log(T)$ equation (1.19) can be written in the following form

$$b\lambda(z)/t\theta_1=1 \quad (1.20)$$

where $\lambda(z)=\phi(z)/(1-\Phi(z))$, $\phi(z)=\exp(-x^2/2)/\sqrt{2\pi}$.

Using equations

$t_{SL}=b\lambda(z_{SL})/\theta_1$, $\theta_0=x_{SL}-z_{SL}\theta_1$,

where $x_{SL}=\log(t_{SL})$, we can get optimal value of t^*_{SL} and θ_0 as functions of $z_{SL}=(x_{SL}-\theta_0)/\theta_1 t^*_{SL}$ (at the fixed θ_1) and then t^*_{SL} as function of θ_0 (at the fixed θ_1 and b):

$$t^*_{SL}=S^*(\theta_0;\theta_1,b). \quad (1.21)$$

An example of calculations of t^*_{SL} as function of θ_0 for the case of normal distribution of X for $b=346000$, θ_1=0.346 is given in Fig.1.2 (here t^*_{SL} is denoted by t_SL, θ_0 is denoted by Th0).

If θ_1 is known, θ_0 is not known, but there is sample $x = (x_1,...,x_n)$ then using maximum likelihood method we can get an estimate of $\theta_o = \bar{x} = \sum_{i=1}^{n} x_i / n$ and then calculate

$t^*_{SL} = S^*(\hat{\theta}_0;\theta_1,b)$. This time it is random variable.

It appears that much better solution can be obtained using Bayes-Fiducial approach.

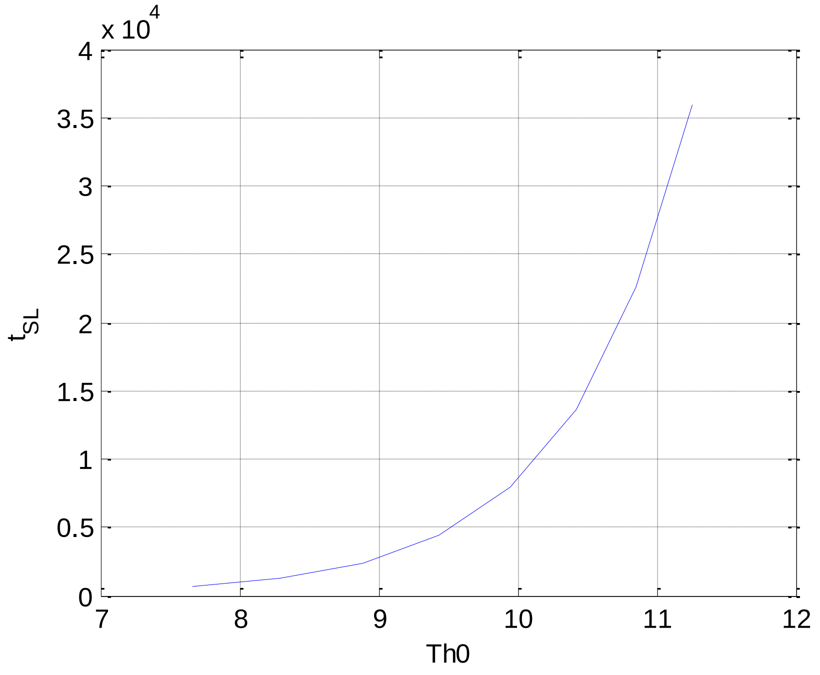

Fig.1.2. Example of calculations of t^*_{SL} as function of θ_0 for the case of normal distribution of $X = \log(T)$ for $b = 346000$, θ_1=0.346.

Choice of specified life using Bayes-fiducial approach

In accordance with Bayes approach the parameter θ is considered as r.v.. For the case of fatigue life of airframe it can be interpreted in the following way. Design stress analysis of an airframe should be fulfilled in accordance with some requirements (FAR,…). These requirements in fact define only some mean value of $X = \log(T)$ and corresponding θ_0. Of course, in every case there are some "occasional mistakes" and we have some specific (random) value of θ_0 for every designed aircraft type. Also, there is a scatter of r.v. X (specific random fatigue life of some specific SSI) at this random θ_0. It seems that parameter θ_1 is function of technology level, and if one is not changed, then the parameter θ_1 is not changed neither.

Suppose that θ_1 is known constant, but θ_0 is random variable, which is denoted by $\tilde{\theta}_0$. Denote by $\pi(\theta_0)$ a priori distribution density of $\tilde{\theta}_0$, then c.d.f. of r.v. X will be

$$\tilde{F}_X(x) = \int_{-\infty}^{\infty} F_X((x-\theta_0)/\theta_1)\pi(\theta_0)d\theta_0.$$

It is well known that if θ_1 is constant, and r.v. $\tilde{\theta}_0$ has normal distribution with both mean τ_0 and standard deviation τ_1 known, then distribution of X is also normal with mean τ_0 and standard deviation $((\tau_1)^2+(\theta_1)^2)^{1/2}$. In this case SL, t^*_{SL}, again will be defined by equation (1.21), but θ_1 should be replaced by $((\tau_1)^2+(\theta_1)^2)^{1/2}$.

Table 1.4.3. Comparison of ML- and FB- approaches

B - approach to SL nomination.					ML - approach to SL nomination.			
10 Mean U for different sample size (k)					10 Mean U for different sample size (k)			
I \ k	1	2	4	100	1	2	4	100
1	1556.515	4212.891	4790.097	6918.316	-6120.11	3040.754	3800.639	6937.074
2	1391.894	3888.095	5944.583	6901.444	-9356.71	-463.272	3373.792	6933.383
3	4891.845	5274.869	5569.536	6917.047	-10404.4	701.5571	4494.191	6926.571
4	3291.739	4350.727	5428.912	6906.56	-2222.72	2953.583	5308.547	6939.215
5	4233.825	4347.703	5516.218	6891.818	-13569.7	-1189.63	5032.563	6953.464
6	-582.891	5257.01	6102.211	6892.028	-10781.4	1494.476	4363.197	6936.326
7	3261.272	589.4247	5807.507	6902.364	-18960	353.4555	5033.566	6925.607
8	2074.037	5379.541	5867.967	6901.553	-5359.58	544.8548	4588.427	6904.558
9	2520.228	2609.616	5262.105	6900.52	-3362.24	-292.094	4135.01	6951.803
10	463.1878	5310.673	5419.943	6912.179	-6102.06	949.7383	4090.15	6945.736
………	……….	………..	……….	……….	……….	……….	……….	……….
Average for I=1...100	2310.165	4122.055	5570.908	6904.383	-8623.89	809.3422	4422.008	6935.374

In fact we do not know a priori distribution of $\tilde{\theta}_0$. For this case a BF approach is offered. Instead of posterior distribution we offer to use fiducial distribution [10] of $\tilde{\theta}_0$. In the considered case fiducial distribution of $\tilde{\theta}_0$ is also normal with mean $\bar{x}$ and standard deviation $\theta_1\ (1+1/n)^{1/2}$. Then for the purpose of calculation t_{SL} we again can use the same equation (1.21), but θ_0, θ_1 should be replaced by $\hat{\theta}_0=\bar{x}$ and $\theta_1(1+1/n)^{1/2}$ correspondingly. So using sample $x=(x_1,...,x_n)$, the result of full-scale fatigue test, in case of ML approach the nominated SL is equal to $S^*(\bar{x},\theta_1,b)$, but for BF approach $t^*_{SL}=S^*(\bar{x},\theta_1(1+1/n)^{1/2},b)$.

Comparison of Byes-fiducial approach with direct use of ML estimates

The results of 10 calculations of t^*_{SL} and then average of 100 calculations (for every k=1, 2, 4, 100 observations $x_1,x_2,...,x_k$) by the use of Monte Carlo method for both ML and BF approaches are shown in table 1.4. We see that for ML-approach, when ML estimate of θ_0 and corresponding t_{SL} are used, the average of $U= u(t^*_{SL}(X);\theta, b)$ is a negative value (u decreases very rapidly (see Fig.1.1) if t_{SL} *is* more than optimal). Only at k=4 it has stabile positive value. When k=100 and standard deviation $\theta_1/n^{1/2}$ is very small we have nearly the same expected value of U as at known θ_0.

But if we use the BF-approach, then already at k=2 we have stabile positive value of u (nearly at the same value as at k=4 when ML-approach is used). As it should be expected, at k=100 of course the value of $U= u(t^*_{SL}(X);\theta, b)$ is nearly the same as at the known θ_0. So for small k the advantage of FB-approach over the use of ML estimates of θ_0 is obvious.

1.5. References

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2. Inspection program development

2.1. Introduction

Inspection program development should be made based on processing of lifetime test result. Usually a confidence interval is used for lifetime distribution parameter estimation and then for the reliability estimation. It is always very difficult to find compromise choice of required reliability and confidence probability. But if we process some approval test data making some redesign of the tested system in case some requirements are not met, then, as it will be shown later, it is possible to use minimax approach which provides required reliability independently of unknown parameters of lifetime distribution without the use of confidence probability. Here for this purpose the p-set function definition is used to solve the problem of development and control of inspection program. This time we consider the case when some Structural Significant Item (SSI), the failure of which is considered a failure of the whole system under consideration, is characterized by a random vector (r.v.) (T_d, T_c), where T_c is critical lifetime (up to failure), T_d is service time, when some damage (fatigue crack) can be detected. So we have some time interval, provided that if during this interval some inspection is made the failure of the SSI will be eliminated. We suppose also that a required operational life of the system is limited by a so-called Specified Life (SL), t_{SL}, when a system is discarded from service.

2.2. P-set function and inspection program

Recall p-set function definition.

Let Z and X be random vectors of m and n dimensions and suppose that a class {Pθ, θ ∈ Ω} to which the probability distribution of the random vector W=(Z,X) is assumed to belong is known. Regarding the parameter θ, which labels the distribution, it is assumably known only that it lies in a certain set Ω, the parameter space. If

$$S_Z(x) = \bigcup_i S_{Z,i}(x)$$

is such set of disjoint sets of z values as function of x that

$$\sup_{\theta} \sum_i P(Z \in S_{Z,i}(X)) \le p \qquad (2.1)$$

then statistical decision function Sz(x) is p-set function for r.v. Z based on a sample x=(x1,...,xn).

Here vector $Z = (T_d, T_c)$ is interpreted as random vector-characteristic of some SSI in service, but instead of x, observation of the vector X, the result of approval test, the estimate $\hat{\theta} = \hat{\theta}(x)$ of parameter θ will be used; the set of sets, $S_Z(x) = \bigcup_i S_{Z,i}(x)$, is a sequence of intervals at the end of which inspections have to be done; this sequence defines the inspection program under consideration. Technological parameter of the program (value of detectable fatigue crack size, a_d), is fixed.

2.3. Minimax approach to inspection program development

Now the problem is to find (in general case) a vector function $t(\hat{\theta})$, where $t=(t_1,t_2,...,t_n)$, t_i is time moment of inspection, *i=1,2,…,n, n* is inspection number, $t_{n+1}=t_{SL}$, in such a way, that failure probability of SSI under consideration

$$p_f(\theta,t)=\Sigma^r{}_{i=1}P(T_{i-1}\le T_d<T_c<T_i)\,, \tag{2.2}$$

does not exceed some small value ε:

$$\sup_{\theta} p_f(\theta,t)\le\varepsilon\,, \tag{2.3}$$

where $T_1,...,T_n$ are random moments of inspections: r.v. $T=(T_1,...,T_n)=\ t(\hat{\theta})$; $T_0=0$; $T_{n+1}=t_{SL}$. This means that vector function $t(\hat{\theta})$ in fact defines some p-set function for vector $(T_d,T_c\ )$ at *p=ε*. The choice of a structure of the sequence $t=(t_1,t_2,...,t_n)$ is a special problem. For the known parameter a choice of optimal sequence is discussed, for example, in [1]. We do not study this problem here. Our problem is the choice of a sequence t when parameter θ is not known but its estimate is known. The offered minimax approach can be used for any structure of sequence t. In real practice, usually the following sequence is used: $t_i=t_1+d(i-1)$, $d=(t_{SL}-t_1)/n$, $i=1,2,...,n$. In this case we should choose only t_1 and n. For simplicity purpose we put $t_1=d$ (in general case t_1 can be chosen, for example, as parameter-free p-bound for $T_{c,}$ or as the value corresponding to the minimum of expectation value of n at a fixed required reliability, etc). Now the probability of failure will be function of θ and n and we denote it by $p_f(\theta,n)$. We suppose that $p_f(\theta,n)$ is such that $\lim_{n\to\infty} p_f(\theta,n)=0$ for all θ, and for every small value ε there is minimal inspection number $n(\theta,\varepsilon)$ such that $p_f(\theta,n)\le\varepsilon$ for all $n\ge\ n(\theta,\varepsilon)$. But true value of θ is not known. So $\hat{n}=n(\hat{\theta},\varepsilon)$ and $\hat{p}_f\ =p_f(\theta,\hat{n})$ are random variables. It is supposed that we begin the commercial production and operation **only** if some specific requirements are met. For example, there are the following requirements: 1) $\hat{n}\le n_R$, 2) $\hat{t}_c>t_R$, …, where n_R,t_R are some constants, defined in specific documents, $\hat{t}_c$ is estimate of expected value of T_c. If these requirements are met let us denote in general case this event as $\hat{\theta}\in\Theta_0$, where $\Theta_0,\Theta_0\subset\ \Theta$, is some part of parameter space. We suppose, that if $\hat{\theta}\notin\Theta_0$ (estimate of required inspection number for some fixed ε exceeds some threshold, n_R, or estimate of expected value of T_c, $\hat{t}_c$, is too small in comparison with some bound t_R,…), then we make redesign of the SSI in such a way, that probability of failure after this redesign will be equal to zero.

Let us define
$$\hat{p}_{f0}=\begin{cases} p_f(\theta,\hat{n})\ \ if\ \ \ \hat{\theta}\in\Theta_0, \\ 0\ \ if\ \ \ \hat{\theta}\notin\Theta_0. \end{cases} \tag{2.4}$$

For this type of strategy (it is demonstrated in Fig. 2.1) the mean probability of fatigue failure $w(\theta,\varepsilon)=E_\theta(\hat{p}_{f0})$ is a function of θ and ε. If for limited t_{SL} it has a maximum, depending on ε then the choice of maximal value of $\varepsilon=\varepsilon^*$ for which $w^*=$

$\max_{\theta} w(\theta,\varepsilon^{*}) \le 1-R$ and the strategy, which defines the inspection number $n = n(\hat{\theta},\varepsilon^{*})$ is such a strategy (decision function) for which required reliability R is provided.

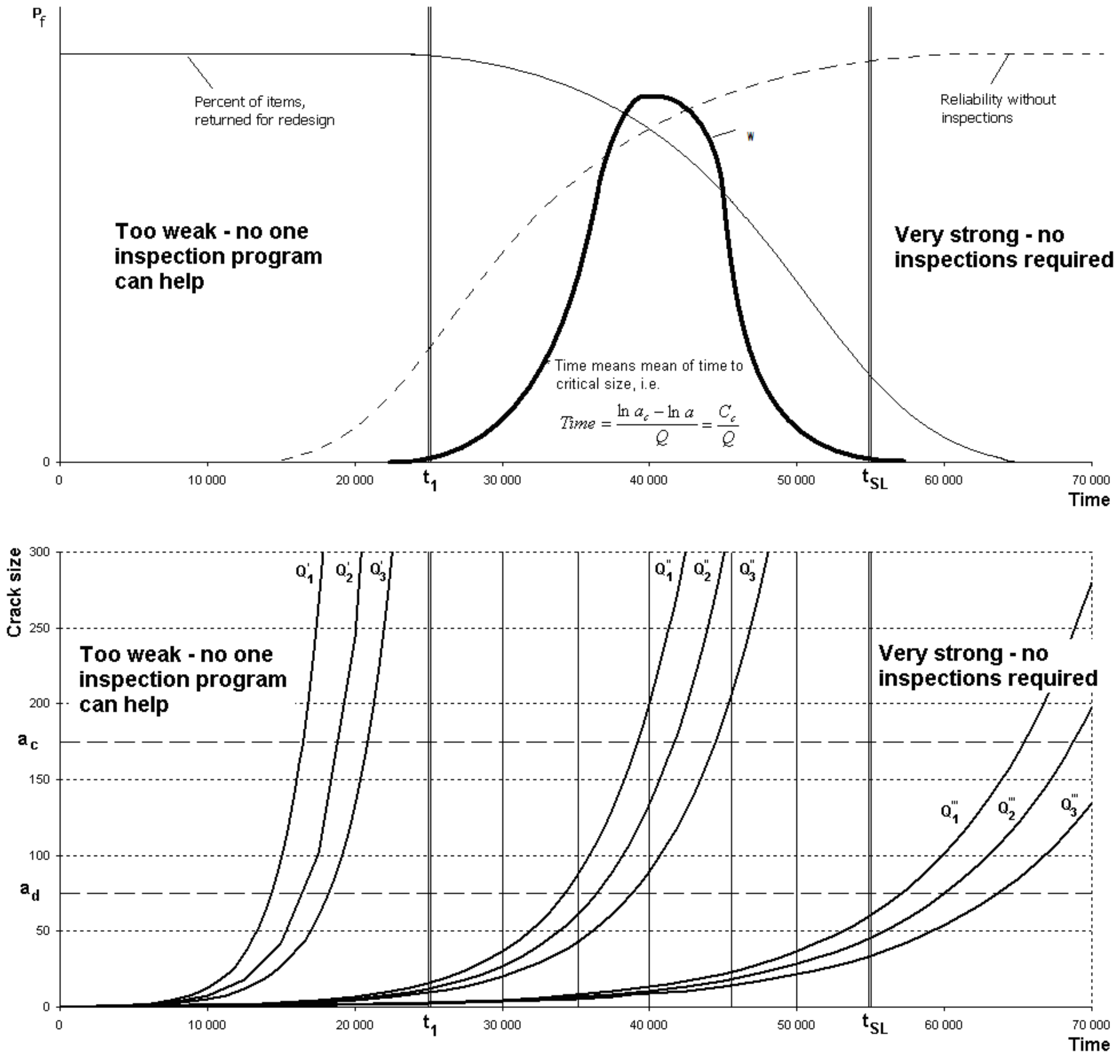

Fig. 2.1. The value of w= $w(\theta,\varepsilon) = E_{\theta}(\hat{p}_{f0})$ as function of $E(T_C)$ for three design versions (three different expected values of Q, $E(Q)$, and corresponding random fatigue crack growth example sets.

2.4. Exponential approximation of fatigue crack growth function

The following approximations are used usually [2,3] for the function of fatigue crack propagation:

$$a(t) = a(0)/\left(1-\mu(a(0))^{\mu}Qt\right)^{1/\mu} = \alpha\exp(-(\log(1-\mu\omega Qt))/\mu) \text{ ,if } \mu \ne 0 \text{ ,} \tag{2.5}$$

or

$$a(t) = \alpha\exp(Qt) \text{, if } \mu = 0 \text{,} \tag{2.6}$$

where $a(t)$ is fatigue crack size at time t (the number of flights); $\alpha = a(0)$, μ, Q are fatigue crack growth model (FCGM) parameters, $\omega = \alpha^{\mu}$. Parameter α is a so called Equivalent Initial Flow Size (EIFS) [2,3]; μ depends on material characteristics, technology and structure, but Q depends also on the loading mode. Observations of 10 fatigue cracks discovered during full-scale fatigue tests of some aircraft are shown in Fig.2.2.

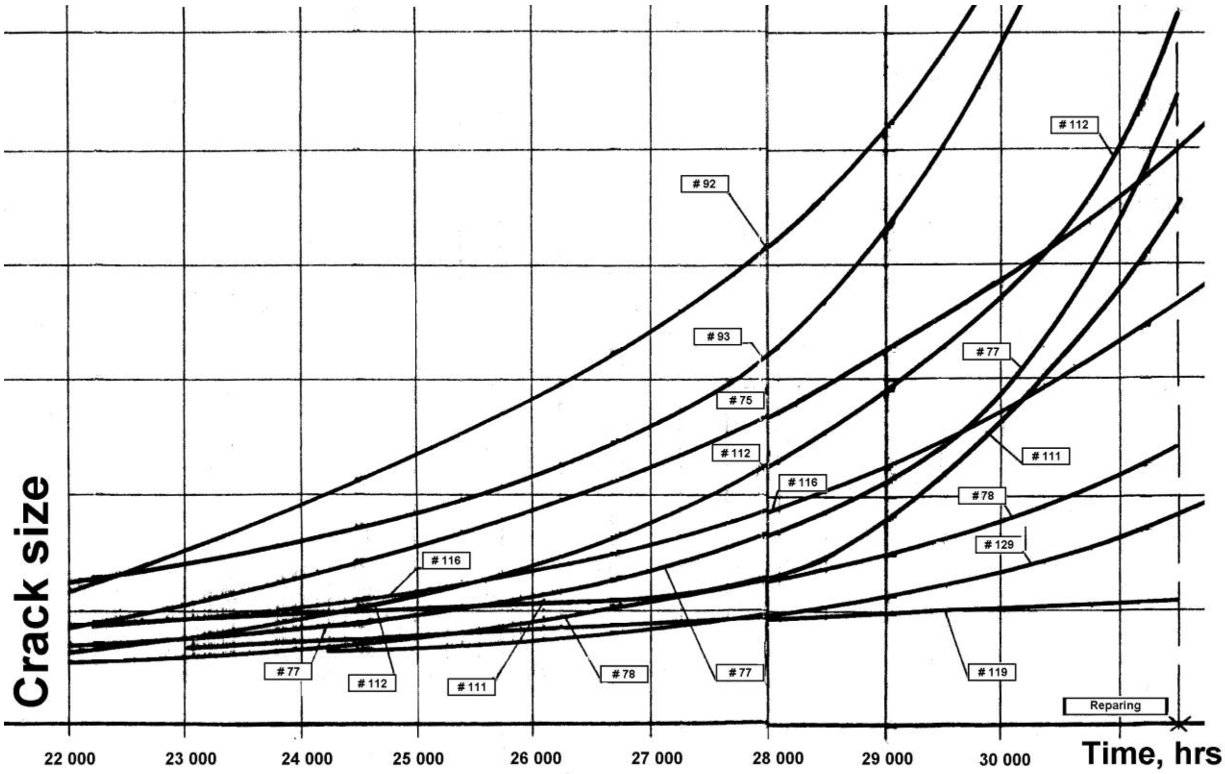

Fig. 2.2. Observations of 10 fatigue cracks discovered during full-scale fatigue tests of some aircraft.

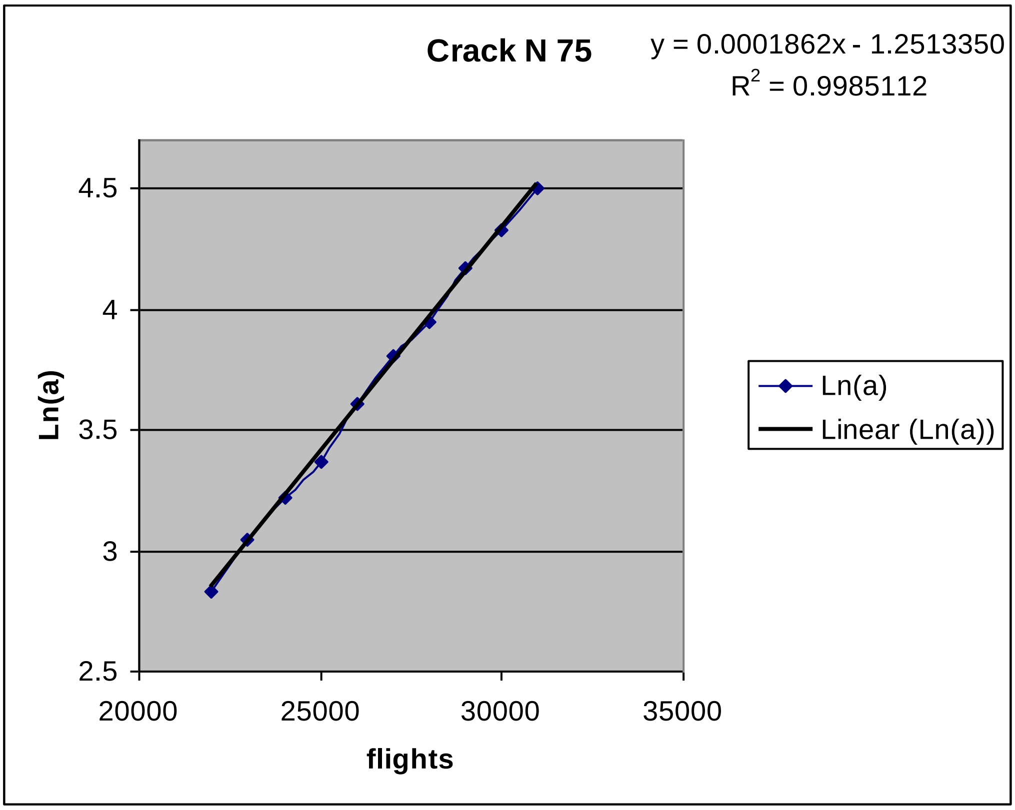

Fig.2.3. Example of fatigue crack growth model (FCGM) parameters estimates

Usually we suppose that all structural significant items (SSI) have the same level of stress. This assumption allows us to consider these fatigue cracks as observations of the same random process and corresponding vector-parameter (of fatigue crack) estimates, $(\hat{\alpha}, \hat{Q})$, as observations of the random vector with the same c.d.f.. We should know this c.d.f. for modeling of result of fatigue cracks inspection. It is very difficult for three dimension vector (α, μ, Q). So we consider only the case of exponential fatigue crack (equation (2.6)).

For this case, where $\mu = 0$, we have

$$\log(a(t)) = \log(\alpha) + Qt \text{ or } y = \gamma_0 + \gamma_1 x, \quad (2.7a)$$

where $y = \log(a(t))$, $\gamma_0 = \log(\alpha)$, $\gamma_1 = Q$, $x = t$.

Estimations of parameters γ_0, γ_1 can be easily obtained by the use of regression analysis.

The results of the processing of 8 fatigue crack growth observations are given in table 2.1 [4,5]. In fact the growth of 10 fatigue cracks was examined but two of them were considered as too specific and they were not taken into account.

In [2,3] there are also the results of fatigue test of specimens with fixed initial flow size d. So in this case we should estimate only Q. We have equations:

$$\log(a(t)/\alpha) = Qt, \text{ or } y = \gamma_1 x, \quad (2.7b)$$

where $y = \log(a(t)/\alpha)$, $\gamma_1 = Q$, $x = t$.

Estimate of Q again can be easily obtained by the use of regression analysis. For example, the results of test of 7475-T7351 aluminium specimens under F-16 fighter 400 hour block spectrum with maximum stress of 34ksi have been analysed statistically and presented in

[2,3]. Two types of specimens were used: WPF – no loads transfer, XWPF – 15% of loads transfer.

It is worth to notice that the mean and standard deviation of log(Q) estimates by the values -8.58733, 0.1557, which are given in table 2.1, are very close to the estimates given in table 2.2 for WPF specimens.

Table 2.1 Airframe fatigue crack parameter estimates

Crack #	log(α)	Q	log(Q)
	γ_0	γ_1	
75	-1.2513	0.000186	-8.58976388
77	-1.8768	0.000195	-8.542511
78	-1.2445	0.000161	-8.73410619
92	-1.697	0.00022	-8.42188301
93	-1.5102	0.000207	-8.48279176
112	-2.5329	0.000228	-8.38616493
116	-0.6479	0.000154	-8.77855796
129	-1.4226	0.000157	-8.75926475
Mean	-1.5229	0.0001885	-8.58688044
StdDev.	0.548084	2.89877E-05	0.155128668
Corr.coeff. of log(alfa) and log(Q) =			-0.794078505

Table 2.2. Fatigue crack parameter estimates for specimens WPF and XWPF

Specimens (sample size)	Mean Q (10^{-3})	Coeff. of variation	log(Q)	
			$\mu_{\log(Q)}$	$\sigma_{\log(Q)}$
WPF (33)	0.273	16.86 %	- 8.219	0.1552
XWPF (37)	0.3725	21.57 %	- 7.919	0.2197

Despite of all the simplicity, the equation (2.6) gives us rather comprehensible result in the range of observation $[T_d, T_c]$, where T_d is a time when the crack becomes detectable and T_c is the time when the crack reaches its critical size. We have

$$T_d = (\log a_d - \log \alpha)/Q = C_d / Q, \tag{2.8}$$

$$T_c = (\log a_c - \log \alpha)/Q = C_c / Q, \tag{2.9}$$

where a_d is a crack size, when the probability to discover it is equal to unit, a_c is a crack size, which corresponds to the minimum residual strength of an aircraft component allowed by special design regulation.

2.5. Failure probability calculation using Monte Carlo method

So let us define *failure* as the situation, when we were unable to discover crack of the size $a_{\det} \le a < a_{crit}$, or, in other words, if there were no inspections performed in [T_d; T_c] time interval and $T_c < t_{SL}$. As an example, you can see three such missed cracks in Figure 2.4 – no inspections were performed between time moments T_d and T_c for those cracks.

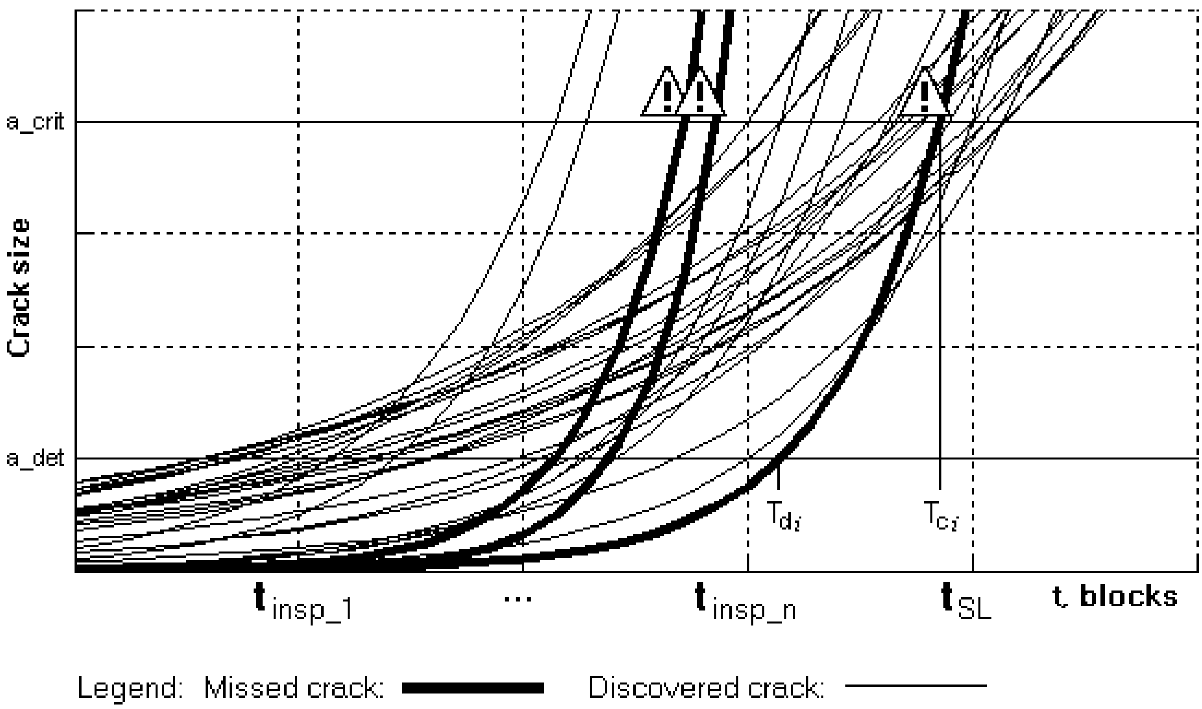

Figure 2.4. Demonstration of missed cracks.

Our approach is visually presented in 2.5. The beginning of the horizontal bar represents time moment T_d while the end of the bar – time moment T_c. Black bars represent missed cracks and are marked with the exclamation marks. Using equations (2.8) and (2.9) we can make modeling random values T_d and T_c.

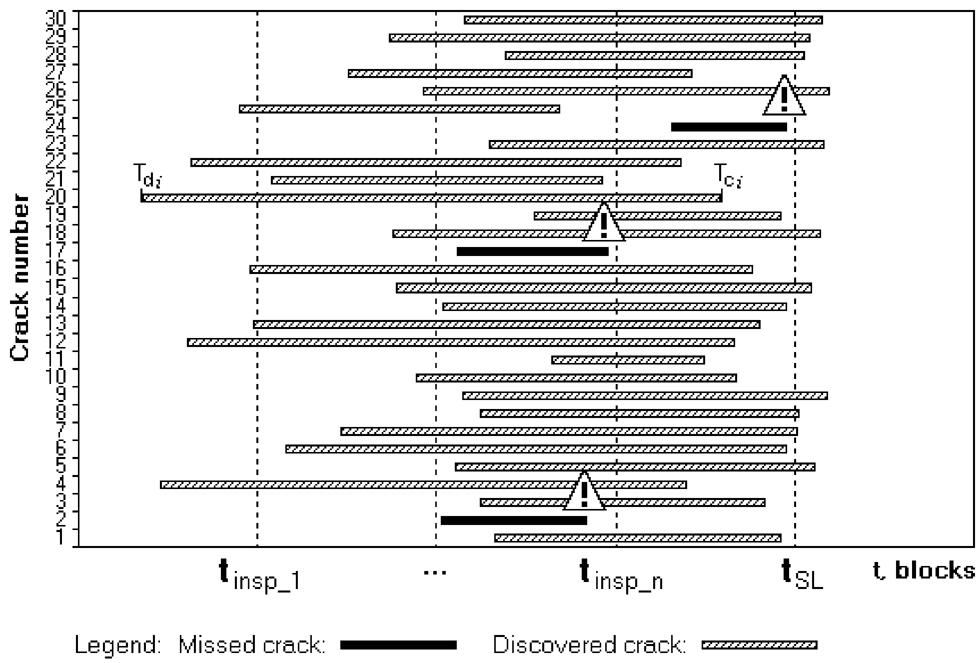

Figure 2.5. Defining the failure probability.

If the number of cracks observed is sufficiently big we could accept the part of missed cracks among all the cracks in the series as the estimate of probability of failure for a particular inspection program:

$$\hat{P}_f \underset{N_{total}\to\infty}{\to} P_f = \lim_{N_{total}\to\infty}\left(\frac{N_{missed}}{N_{total}}\right)$$

where N_{missed} is the number of missed cracks and N_{total} is the total numbers of cracks, i.e. $N_{\text{total}} = N_{\text{missed}} + N_{\text{discovered}}$.

Using Monte Carlo method we can take into account a so called "human factor". In general case it can be assumed that if inspection takes place within an interval $[T_d, T_c]$ then fatigue crack will be detected with probability *w*. If there will be r inspections during this interval then the probability of failure (the fatigue crack will not be discovered) is equal to $(1-w)^r$. The value of r depends on the interval $[T_d, T_c]$ and on inspection interval *d*. We suppose that every i-th aircraft, $i = 1, 2, ..., N$, $N = N_{\text{total}}$, can be characterized by interval $[T_d, T_c]_i$. The corresponding value of fatigue failure probability for fixed inspection number n for specific aircraft is

$$p_{fi}(n) = (1-w)^{r_i}, \quad i = 1, 2, ..., N.$$

Then we can calculate the mean value of failure probability as, for example, a function of inspection number, n, (now it is supposed that all inspection intervals are equal)

$$\hat{p}_f(n) = \sum_{i}^{N} p_{fi}(n) / N$$

2.6. Equations for failure probability calculation

Let us denote $X = \log Q$ and $Y = \log C_c = \log\left(\log\left(a_c/\alpha\right)\right)$. From the analysis of the fatigue test data it can be assumed, that the logarithm of time required the crack to grow to its critical size (logarithm of durability) is distributed normally. It comes from the additive property of the normal distribution that $\log T_c = \log C_c - \log Q$ could be normally distributed either if both $\log C_c$ and $\log Q$ ($C_c = \log a_c - \log\alpha$) are normally distributed (i.e. $X = \log Q \sim N(\mu_X, \sigma_X^2)$, $Y = \log C_c \sim N(\mu_Y, \sigma_Y^2)$) (with some coefficient of correlation r), or if one of them is normally distributed while another one is constant. In Figure 2.6 these two cases are called one- and two-parametric models.

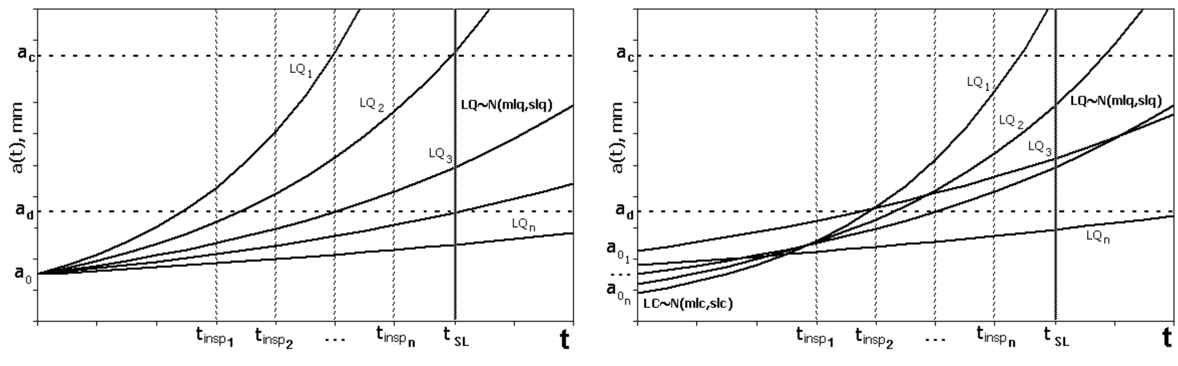

Figure 2.6. One- and two-parametric crack growth modelling (LQ=ln(Q)).

The particular inspection program is defined by sequence of inspection time moments, $t=(t_1,t_2,...,t_n)$, where t_i is time moment of ith inspection, *i=1,2,…,n, n* is inspection number, $t_{n+1}=t_{SL}$. For calculation of the failure probability for this inspection program we have to sum up all failure probabilities in all intervals:

$$p_f=\sum_{i=1}^{n+1} q_i, \tag{2.10 }$$

where

$$q_i=P(t_{i-1}<T_d\le T_c\le t_i)=P\left(t_{i-1}<\frac{C_d}{Q}\le\frac{C_c}{Q}\le t_i\right).$$

If, in a simplest case, we assume that C_c and C_d are some constants, $C_c\ge C_d$, then

$$q_i=\begin{cases}0,\ \text{ if } t_i\le t_{i-1}\dfrac{C_c}{C_d},\\ q_i^+,\ \text{ if } t_i>t_{i-1}\dfrac{C_c}{C_d},\end{cases} \tag{2.11}$$

where for normal distribution of random variable $X=\log Q$

$$q_i^+=P\left(\frac{C_c}{t_i}\le Q<\frac{C_d}{t_{i-1}}\right)=\Phi\left(\frac{\ln\left(C_d/t_{i-1}\right)-\theta_0}{\theta_1}\right)-\Phi\left(\frac{\ln\left(C_c/t_i\right)-\theta_0}{\theta_1}\right). \tag{2.12}$$

For two parametric model, assuming normal distribution of $X=\log Q$ and $Y=\log C_c$, we should take into account that

$$C_c-C_d=\log(a_c)-\log(a_d)=\log(a_c/a_d)=\delta,$$

where δ is some constant. Then

$$q_i^+=P\left(\log C_c-\log t_i\le\log Q<\log(C_c-\delta)-\log t_{i-1}\right)=\int_{\ln\delta}^{+\infty}\left(g_i^+(y)\right)d\Phi\left(\frac{y-\mu_Y}{\sigma_Y}\right),$$

where $g_i^+(y)=\max\left(0,\Phi\left(\dfrac{\left(\log\left(e^y-\delta\right)-\log t_{i-1}\right)-\mu_{X/y}}{\sigma_{X/y}}\right)-\Phi\left(\dfrac{\left(y-\log t_i\right)-\mu_{X/y}}{\sigma_{X/y}}\right)\right)$,

$$\mu_{X/y}=\mu_X+r\frac{\sigma_X}{\sigma_Y}\left(y-\mu_Y\right),\ \sigma_{X/y}=\sigma_X\sqrt{1-r^2}.$$

Here we suppose, that parameters σ_X,σ_Y and r depend on technology, which does not change (for a new aircraft), and these parameters can be estimated using information of previous designs. We suppose that they are fixed and they are known values. Then unknown parameter, θ, has only two components: $\theta=(\mu_X,\ \mu_Y)$.

The example of the linear regression analysis estimates of Q and α is shown in Fig. 2.3. (in logarithm scale : $\log(a(t))=Qt+\log(\alpha)$). The observation of only one fatigue crack was used for this estimation. Using these estimates and known a_c we can get estimates of μ_X and μ_Y: $\hat{\mu}_X$ is just equal to $\log(Q)$, $\hat{\mu}_Y$ is equal to $\log(C_c)$, where $C_c=\log(a_c)-\log(\alpha)$. In the following calculation the vector (σ_X , σ_Y , r) was considered as some constant. In fact, the corresponding values were taken from table 2.1.

2.7. Aircraft failure probability calculation using Markov chain theory

2.7.1. Failure probability calculation for fixed inspection program

To be able to use the theory of Markov chains the inspection program is presented as a process of several states: first $n+1$ states represent aircraft service in the appropriate interval between two consequent inspections, while three additional states represent aircraft withdrawal from the service due to

the successful end of service when the specified life period is over,

due to fatigue failure and

due to discovery of a crack:

E_i – aircraft service time t is in the $i-th$ inspection interval, $t \in (t_{i-1}, t_i]$, $i = 1, 2, \ldots, (n+1)$;

E_{n+2} – aircraft has successfully reached t_{SL} without cracks (SL-state);

E_{n+3} – fatigue failure, i.e. a crack has been missed (FF-state);

E_{n+4} – crack is detected during the inspection (CD-state).

Let the probability of crack detection during the inspection number i be denoted as v_i; probability of failure in service time interval $t \in (t_{i-1}, t_i]$, as q_i; and probability of successful service continuation as u_i. Since these three cases form a complete set $u_i + v_i + q_i = 1$. In our model we also assume that an aircraft is discarded from service at t_{SL} even if there are no cracks discovered by the time moment of t_{SL}. Inspection at the end of (n+1)-th interval (in state E_{n+1}) does not change the reliability but we do it in order to know the state of aircraft (whether there is a fatigue crack or there is no fatigue crack). The transition probability matrix of this process can be composed as it is presented in Fig.2.7.

	E_1	E_2	E_3	…	E_{n-1}	E_n	E_{n+1}	E_{n+2} (SL)	E_{n+3} (FF)	E_{n+4} (CD)
E_1	0	u_1	0	…	0	0	0	0	q_1	v_1
E_2	0	0	u_2	…	0	0	0	0	q_2	v_2
E_3	0	0	0	…	0	0	0	0	q_3	v_3
…	…	…	…	…	…	…	…	…	…	…
E_{n-1}	0	0	0	…	0	u_{n-1}	0	0	q_{n-1}	v_{n-1}
E_n	0	0	0	…	0	0	u_n	0	q_n	v_n
E_{n+1}	0	0	0	…	0	0	0	u_{n+1}	q_{n+1}	v_{n+1}
E_{n+2} (SL)	0	0	0	…	0	0	0	1	0	0
E_{n+3} (FF)	0	0	0	…	0	0	0	0	1	0

E_{n+4} (CD)	0	0	0	…	0	0	0	0	0	1

Fig.2.7. The transition probability matrix.

The state E_i itself can be reached only if until time t_{i-1} there was not failure and there was not fatigue crack detection during inspection at time t_{i-1}. But in fact it is enough to have just the latter event. The probability of the event equal to

$$P(T_d > t_{i-1})$$

The transition to the state E_{i+1} from E_i is possible only if there was not fatigue crack detection at time t_i.

As the event is included in previous event

$$\{T_d > t_i\} \subset \{T_d > t_{i-1}\}$$

then the conditional probability

$$u_i = P(T_d > t_i \mid T_d > t_{i-1}) = P(Q < C_d / t_i) / P(Q < C_d / t_{i-1}) \tag{2.13}$$

Under the considered condition the fatigue failure in the interval (t_{i-1},t_i) can occur if only simultaneously two events occur: a fatigue crack is not discovered at (i-1)-th inspection and fatigue life T_c is less than t_i.

The product of the events is included in the event $\{T_d > t_{i-1}\}$. So

$$q_i = P(t_{i-1} < T_d < T_c < t_i \mid Td > t_{i-1}) =$$
$$= \begin{cases} 0, \quad if \quad t_{i-1}C_c / C_d > t_i, \\ \dfrac{P(C_c / t_i < Q < C_d / t_{i-1})}{P(Q < C_d / t_{i-1})}, \quad if \quad t_{i-1}C_c / C_d \le t_i, \end{cases} \tag{2.14}$$

$$v_i = 1 - u_i - q_i$$

Here we suppose that C_c and C_d are constants (one parameter model) but random variable $\ln(Q)$ has normal distribution $N(\theta_0, \theta_1^2)$. Then conditional probabilities u_i, q_i are defined by equations

$$u_i = a_i / a_{i-1}, \; q_i = \max(0, (a_{i-1} - b_i) / a_{i-1}), \tag{2.15}$$

where $a_i = \Phi(\ln(C_d / t_i) - \theta_0) / \theta_1)$, $b_i = \Phi(\ln(C_c / t_i) - \theta_0) / \theta_1)$, $\Phi(.)$ is distribution function of standard normal variable. It is clear that

$$v_i = 1 - u_i - q_i . \tag{2.16}$$

It is necessary to mention, that if we consider a park of N aircraft of the same type and if we are interested to know the probabilities of the failure of at least one aircraft or crack discovery in at least one aircraft of the park then instead of q_i and u_i we should use

$$q_{i,N} = 1 - (1 - q_i)^N \text{ and } u_{i,N} = (u_i)^N . \tag{2.17}$$

Let us denote the corresponding matrix by symbol P_N .

The structure of considered matrices can be described in the following way:

Q	R
0	I

Fig. 2.8. Sub-matrices of transition probabilities matrix.

where I is a matrix of identity corresponding to absorbing states, 0 is a matrix of zeros. Then matrix of probabilities of absorbing in different absorbing states for different initial transient states is defined by formula

$$B = \left(I - Q\right)^{-1} R . \tag{2.18}$$

	E_{n+2} (SL)	E_{n+3} (FF)	E_{n+4} (CD)
E_1			
E_2			
E_3			
…			
E_{n-1}			
E_n			
E_{n+1}			

Fig. 2.9. The structure of the matrix B

The structure of the matrix B is shown in Fig. 2.9. The first row of the matrix B defines the probabilities of absorption in states SL, FF, CD, particularly item $B(1,2)$ defines the failure probability for new aircraft which begins operation within the first interval. For the park of aircraft, $B_N(1,2)$ is a probability of at least one aircraft failure in fleet of N aircraft. The following rows of the matrix B and B_N define the same probabilities for different initial states: for aircraft which begins operation in different time intervals. So, for example, the failure probability of at least one aircraft in fleet of N aircraft is equal to

$$p_f = aB_N b , \tag{2.19}$$

where vector row $a = (1,0,...,0)$ means that all aircraft begin operation within the first interval (state E_1), vector column $b = (0,1,0)'$.

2.7.2. Failure probability calculation for specific inspection program control

Markov Chains theory is especially attractive to model various scenarios of switching to the alternative inspection programs when the certain event takes place. For example, assume in the fleet at the beginning there were N aircraft. When the first crack is discovered in the fleet we can make repair of corresponding aircraft and change the frequency of inspections of the remaining (N-1) aircraft using for our decision not only initial but and also additional information on the discovered crack. Here we consider a specific decision: after the fatigue crack was discovered we double the remaining inspection number but in the following service time we do not change the inspection program for any aircraft.

For example, at first we have only one inspection and decide to decrease interval between inspections twice if a fatigue crack is discovered during an inspection, and continue service of the remaining (*N*-1) aircraft up to a specified life. We suppose that the aircraft with discovered fatigue crack will be repaired and then it will reach specified life without any fatigue crack. For this simple example the remaining interval is devided into two parts and one additional inspection will be required. Ordinary one-inspection strategy state transition diagram is shown in Fig. 2.10.

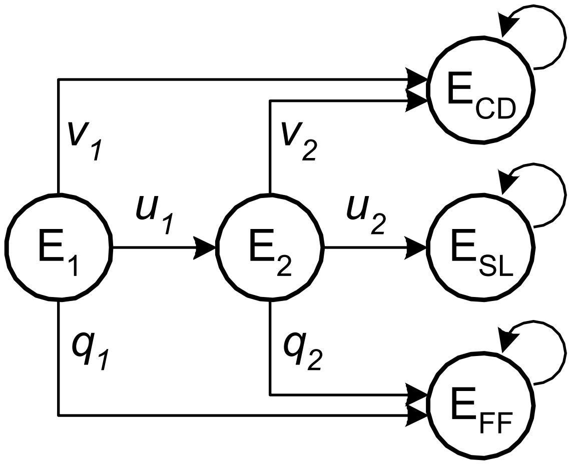

Figure 2.10. Ordinary one-inspection strategy state transition diagram

The decision to double inspection number (or, which is the same, decrease the inspection interval twice) graphically looks as it is shown in Fig.2.11

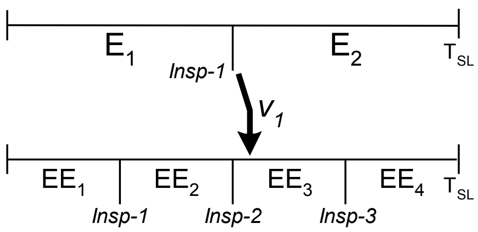

Fig. 2.11. Switching to doubled inspection number program (initial one-inspection model)

As you can see, this decision is equivalent to continuation of service in accordance with inspection plan based on 3 inspections. Thus, the resulting graph will look as it's shown in Fig. 2.12.

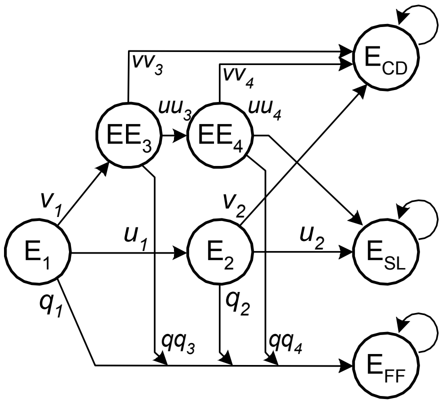

Fig. 2.12. Switching to the double inspection frequency state transition diagram.

For the purpose of fatigue failure probability calculation the latter can be reduced into the state transition diagram shown on Figure 2.13.

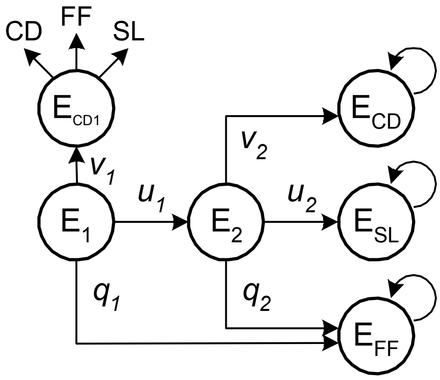

Figure 2.13. Switching to the double inspection frequency reduced state transition diagram (initially one-inspection model).

Similar transformations for initial number of inspections equal to 2 are shown in Fig. 2.14, …, 2.17 below.

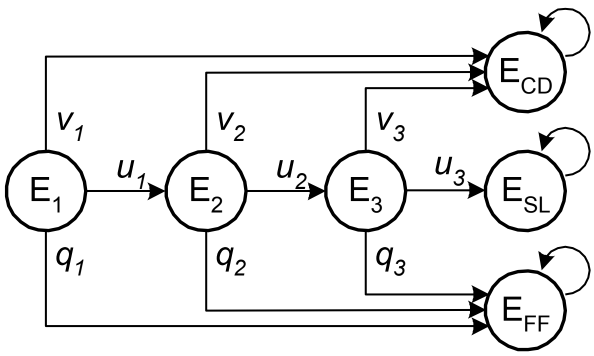

Fig. 2.14. Ordinary two-inspections strategy state transition diagram.

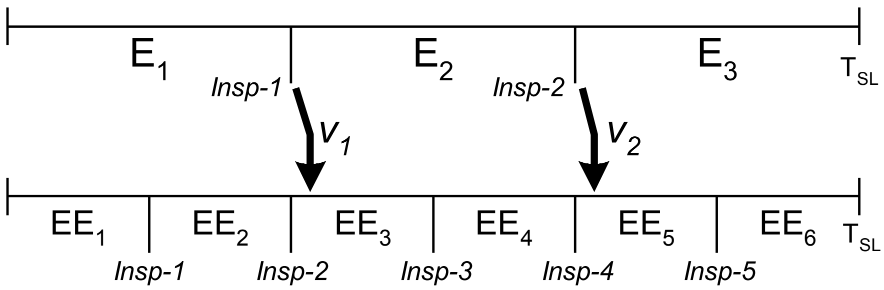

Fig. 2.15. Switching to double inspection frequency for initial two-inspection program.

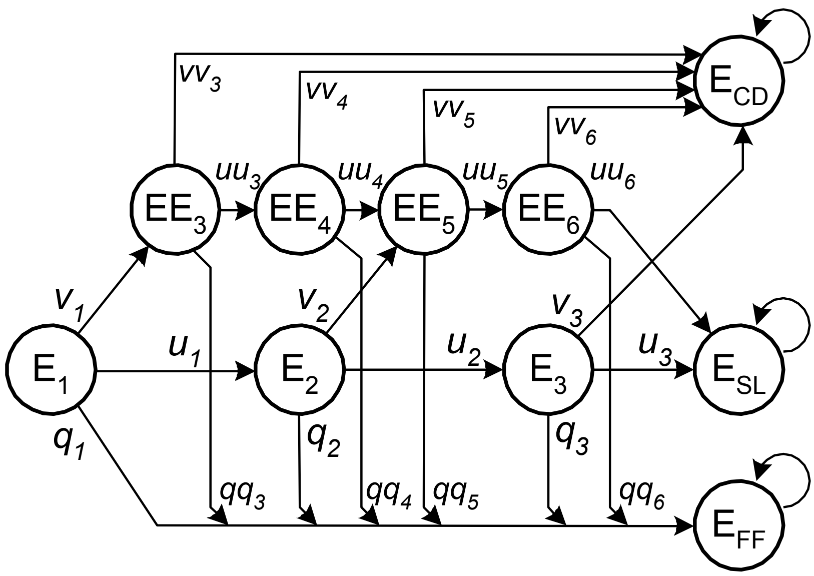

Fig. 2.16. Switching to the double inspection frequency state transition diagram for initial two-inspection program.

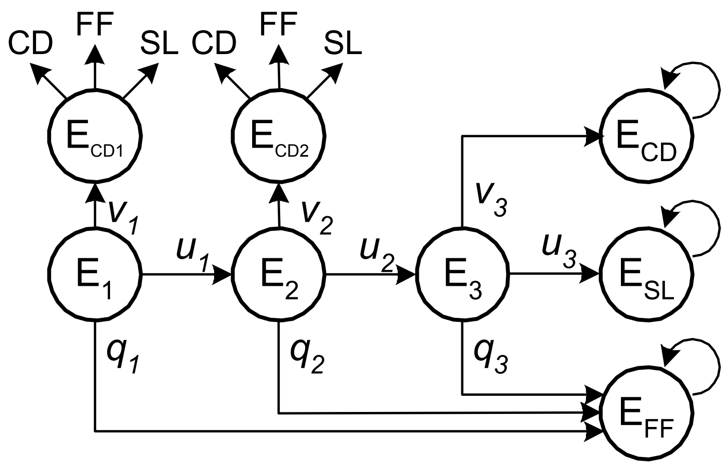

Fig. 2.17. Switching to the double inspection frequency reduced state transition diagram for initial two-inspection program

For failure probability calculation it is convenient to introduce new "quasi-absorbing states" *CD*1 and *CD*2 (see Fig.2.17), corresponding to states EE_3 and EE_5 (see Fig. 2.16) from the initial matrix. The states CD1, CD2 are quasi-absorbing states corresponding to "absorption" of "initial process" at the inspection 1 and inspection 2. These are the points of beginning of “new” process with different inspection program. Corresponding transition probability matrix is shown in Fig. 2.19.

	E_1	E_2	E_3	E_4 (SL)	E_5 (FF)	E_6 (CD)	E_7 (CD 1)	E_8 (CD 2)
E_1	0	$u_{1,N}$	0	0	$q_{1,N}$	0	$v_{1,N}$	0
E_2	0	0	$u_{2,N}$	0	$q_{2,N}$	0	0	$v_{2,N}$
E_3	0	0	0	$u_{3,N}$	$q_{3,N}$	$v_{3,N}$	0	0
E_4 (SL)	0	0	0	1	0	0	0	0
E_5 (FF)	0	0	0	0	1	0	0	0
E_6 (CD)	0	0	0	0	0	1	0	0
E_7 (CD1)	0	0	0	0	0	0	1	0
E_8 (CD2)	0	0	0	0	0	0	0	1

Fig.2.18. The matrix IP^0 corresponding to the Fig.2.17 and N aircraft in service.

As in the previous case the matrix of probabilities of absorbing in different absorbing states for different initial transient states is defined by formula (2.18).

It is obvious that the random inspection program $IP(\cdot)$ has in fact three possible realizations:

$IP^0: \quad \{t_1, t_2, t_{SL}\};$

$IP^1: \quad \{t_1, t_2, \frac{1}{2}\cdot(t_{SL}+t_2), t_{SL}\};$

$IP^2: \quad \{t_1, \frac{1}{2}\cdot(t_2+t_1), t_2, \frac{1}{2}\cdot(t_{SL}+t_2), t_{SL}\}.$

The probability of each scenario to be realized depends on the probability to discover a crack during the inspections of the basic scenario. Probability of IP^1 is equal to probability of absorption in state CD1, $p(CD1)$. Probability of IP^2 is equal to probability of absorption in state CD2, $p(CD2)$. Probability of IP^0, $p(IP^0)$, is equal to 1- $p(CD1)$ - $p(CD2)$. For every scenario, using already described approach we can calculate the probability of failure and then to calculate total probability of failure of at least one aircraft in a fleet:

$$p_f = B_N IP^0(1,2) +$$
$$+B_N IP^0(1,4) \bullet (1-(1-B_1 IP^1(2,2))^{N-1}) \,,$$
$$+B_N IP^0(1,5) \bullet (1-(1-B_1 IP^2(3,2))^{N-1})$$

where $B_N IP^r(i,j)$ is (i, j) –th element of matrix B for IP^r inspection program, $r = 1,2,3$, for fleet with N aircraft. State transition diagram for inspection program IP^0 is shown in Fig. 2.10. The structure of the matrix $B_N IP^0$ is shown in Fig. 2.19.

	BE_1 (SL)	BE_2 (FF)	BE_3 (CD)	BE_4 (CD 1)	BE_5 (CD 2)
BE_1 (SL)					
BE_2 (FF)					
BE_3 (CD)					
BE_4 (CD1)					
BE_5 (CD2)					

Fig. 2.19. Structure of the matrix $B_N IP^0$.

In general case there are n inspections in the initial inspection program and there are $(n+1)$ transient and $(n+3)$ absorbing states (initial absorbing states (SL, FF and CD) and n additional "quasi-absorbing" states in corresponding matrix of transition probabilities, see Fig. 2.20.

	E_1	E_2	$\cdots$	E_{n+1}	E_{SL}	E_{FF}	E_{CD}	E_{CD1}	$\cdots$	E_{CDn}
E_1	0	u_{1N}	$\cdots$	0	0	q_{1N}	v_{1N}	0	$\cdots$	0
E_2	0	0	$\cdots$	0	0	q_{2N}	0	v_{2N}	$\cdots$	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
E_{n+1}	0	0	$\cdots$	0	$u_{(n+1)N}$	$q_{(n+1)N}$	0	0	$\cdots$	$v_{(n+1)N}$
E_{SL}	0	0	$\cdots$	0	1	0	0	0	$\cdots$	0
E_{FF}	0	0	$\cdots$	0	0	1	0	0	$\cdots$	0
E_{CD}	0	0	$\cdots$	0	0	0	1	0	$\cdots$	0
E_{CD1}	0	0	$\cdots$	0	0	0	0	1	$\cdots$	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
E_{CDn}	0	0	$\cdots$	0	0	0	0	0	$\cdots$	1

Fig. 2.20. Modified transition probability matrix

The optimal changes of initial inspection program can be based on the analysis of new information realised after discovering the fatigue crack.

Thus, there are n possibilities to switch to the new inspection program, generating a set of $(n+1)$ realizations (or scenarios) of the random inspection program, $\{IP^0, IP^1, IP^2, ..., IP^n\}$. Let $b = \{b_0, b_1, b_2, \dots b_n\}$ be a vector of corresponding probabilities: v_i is a probability to discover a crack during i^{th} inspection in accordance with initial program, $i = 1, \dots, n$, b_0 is probability of realization of initial inspection program IP^0 (it is a probability of non-discovery of any crack at any first n inspections or probability to be absorbed in states SL, FF or CD in accordance with the initial inspection program)

$$b_i = B_N IP^0(1, 3+i),\ i = 1, \dots, n,\ b_0 = 1 - \sum_{i=1}^{n} b_i .$$

The total failure probability of the random inspection program can be presented as a sum of at least one failure probability of all scenarios multiplied by the probabilities of these scenarios to realize:

$$p_f = p_{f0} + \sum_{i=1}^{n} \left(b_i \cdot (1 - (1 - p_{fi})^{N-1}) \right),$$

where p_{f0} is a failure probability of at least one aircraft in accordance with initial inspection program; p_{fi}, $i = 1, \dots, n$, is a probability of failure of aircraft with the inspection program chosen after crack discovery at i-th inspection (in accordance with initial inspection program). The new inspection program is implemented for each of (N-1) aircraft.

2.8. Numerical examples

Here we limit ourselves by calculation of fatigue failure probability using only equations 2.10,…,2.12 for two cases:

- Random variable $\log(Q)$ has normal distribution $N(\theta_{0X},\theta_{1X}^2)$ but values C_c and C_d are constants (in fact we get the estimate $\hat{\theta}_{0X}$ of the parameter $\theta_{0X} = E(\log(Q))$ and the values C_c and C_d using equation (2.8), (2.9), the results are shown in Fig. 2.3 and specific values of a_d and a_c). There are N=100 aircraft in service.
- Random variable $\log(C_c)$ has also normal distribution $N(\theta_{0Y},\theta_{1Y}^2)$. Vector ($\theta_{1X},\theta_{1Y},r$) is supposed to be known. There are N=1 aircraft in service.

It is useful to note that similar calculations, but using theory of Markov chains, should be made in case of development of "dynamic" inspection program, which was discussed in the end of the section 2.7 (but here we do not consider the example of this type).

2.8.1. Random variable $\log(Q)$ has normal distribution $N(\theta_{0X},\theta_{1X}^2)$ but values C_c and C_d are constants. There are N=100 aircraft in service.

Example of the calculations of

- the function $w(\theta,\varepsilon)$,
- the probability of redesign,
- the reliability without inspection

as function of $(\theta_0 - \hat{\theta}_0)/\theta_1$ and corresponding initial data are shown in Fig.2.21. Let us remind that $\hat{\theta}_0$ can be considered as an estimate of speed of fatigue crack growth (in log-scale).

In considered example we have event $\hat{\theta} \notin \Theta_0$ if

- for ε_1=0.0001 (in this case probability of at least one failure in park $\varepsilon = 1-(1-\varepsilon_1)^N$ is approximately equal to $\varepsilon_1 N$=0.01) the required number of inspections $\hat{n} = n(\hat{\theta},\varepsilon\ )$ is more than 3 or
- value of $\hat{t}_c = C_c / Q$, estimate of mean T_c, lesser than t_{SL}=40000 (flights).

Because of the limitation of specific features of *plot*-command of MATLAB in Fig.2.21 the following notations are used: EsTc instead of estimate of $E(T_c)$, the "th0" instead of θ_0, Esth0 instead of $\hat{\theta}_0$, the th1 and theta1 instead of θ_1); N_{MK} trials is the number of Monte Carlo trials; P_r is redesign probability, R_{Winsp} is reliability without inspections. The maximum of the function $w(\theta,\varepsilon) = E_\theta(\hat{p}_{f0})$ for $\varepsilon = \varepsilon_1 N$=0.01 is equal to 0.0014. (It is worth to mention, that the maximum of this function exists because we make redesign of a"weak" structural significant item (when $\theta_0 =\ E(\log(Q))$, "the rate" of fatigue crack growth (in logarithm scale), is too high) and, on the other hand, we do not need any inspection if structural significant item is too strong (when θ_0 is too small)). So if required reliability (of one aircraft) is equal to 0.9986 then for the considered example for $\varepsilon = N\varepsilon_1$=0.01 we should choose the inspection number $\hat{n} = n(\hat{\theta},\varepsilon\ )$=2.

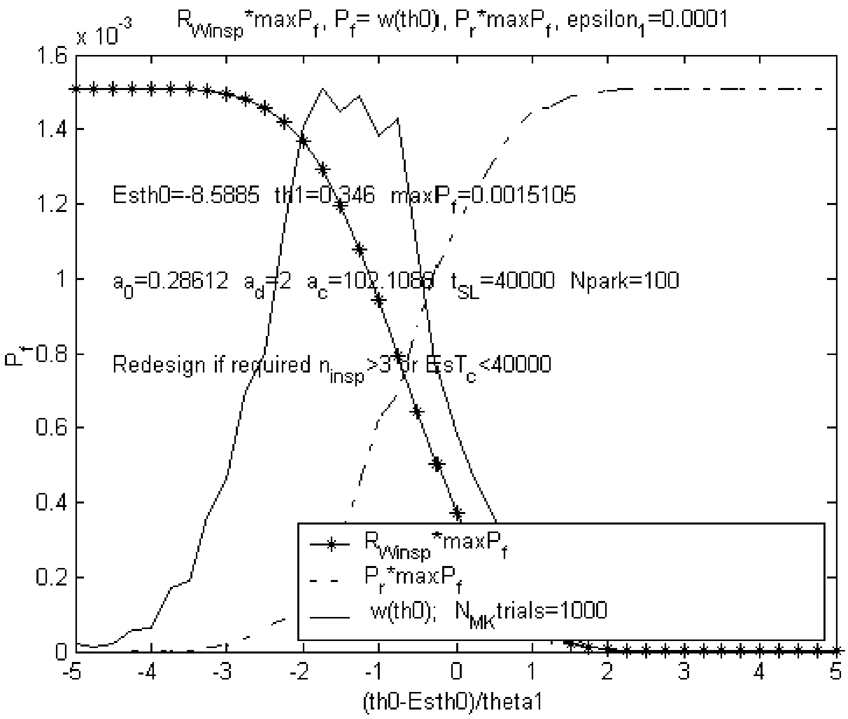

Fig. 2.21. Function $w(\theta,\varepsilon) = E_\theta(\hat{p}_{f0})$ for ε_1=0.0001 (notation are explained in text).

2.8.2. Random variable $\log(C_c)$ also has normal distribution $N(\theta_{0Y},\theta_{1Y}^2)$. Vector ($\theta_{1X},\theta_{1Y},r$) is supposed to be known. There are N=1 aircraft in service.

Around the point $(\hat{\theta}_{0X},\hat{\theta}_{0Y})$ we choose some area in plane $\{\theta_{0X},\theta_{0Y}\}$, where $\theta_{0X} = E(\log(Q))$, $\theta_{0Y} = E(\log(C_c))$ (it is supposed that Q and C_c are found during processing test data set). Using Monte Carlo method for modelling of other possible estimates or we make calculation for some set of θ=(θ_{0X},θ_{0Y}) in order to get the surface $w(\theta,\varepsilon) = E_\theta(\hat{p}_{f0})$. In the Fig. 2.22 and Fig.2.23 the results of modelling for ε=0.001 and 0.005 respectively are presented (in this examples we use inspection program with special choice of t_1 and evenly distributed time moments between t_1 and t_{SL}; the time moment of the first inspection is defined as $t_1 = t_{SL} - 5\cdot\theta_{1_X}$; the detectable and critical crack sizes are $a_d = 20mm$, $a_c = 237.84mm$). For these examples we assume that only one full-scale test was performed and we have data on just one single crack growth (crack #75 in Fig. 2.2): $\ln Q = -8.588527$, $\ln C_c = 1.905525$). Assume, that the probability of failure should not exceed 0.0326 and we will return all projects for redesign if required number of inspections exceeds n_R=5. If we perform modelling using various values of failure probability ε we will get a set of "surfaces" $w(\theta,\varepsilon) = E_\theta(\hat{p}_{f0})$.

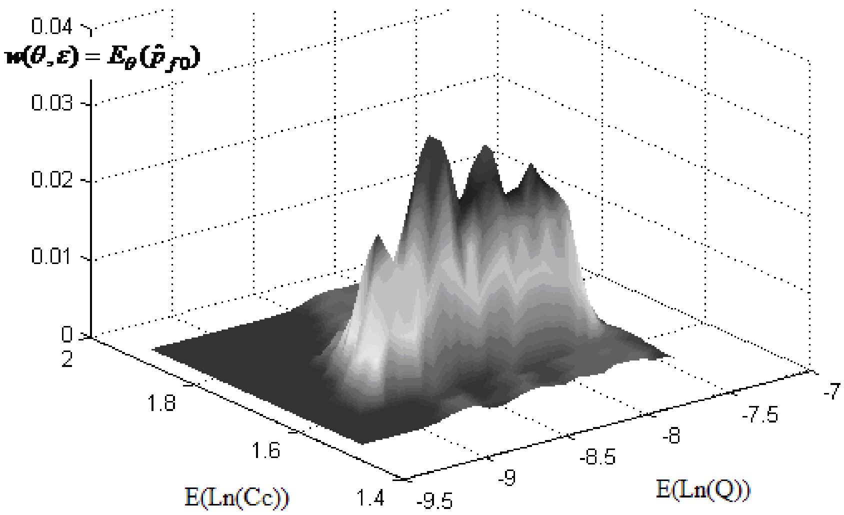

Fig. 2.22. Function $w(\theta,\varepsilon)=E_\theta(\hat{p}_{f0})$ for ε =0.001.

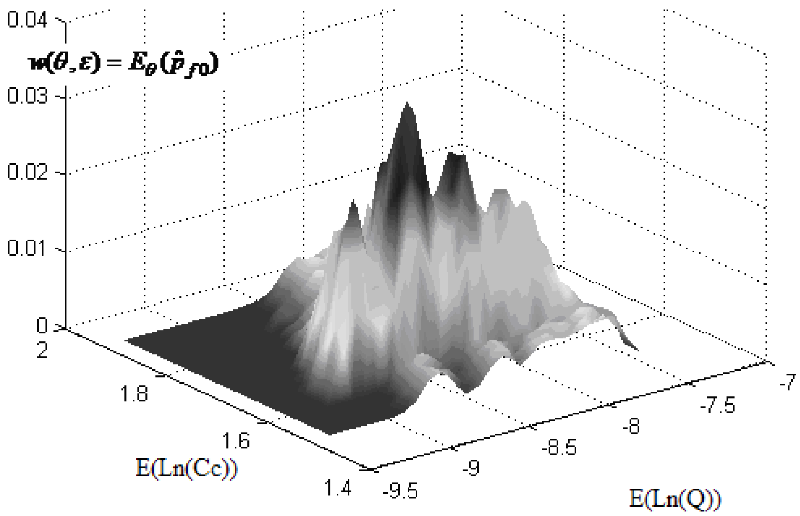

Fig. 2.23. Function $w(\theta,\varepsilon)=E_\theta(\hat{p}_{f0})$ for ε =0.005.

This time maximum values of the function $w(\theta,\varepsilon)$, $w^*(\varepsilon)=\max\limits_\theta w(\theta,\varepsilon)$, are equal to 0.030990 and 0.033874 for ε =0.001 and 0.005 correspondingly. Similar calculation gives $w^*(\varepsilon)=\max\limits_\theta w(\theta,\varepsilon)=0.032593$ for ε =0.003.

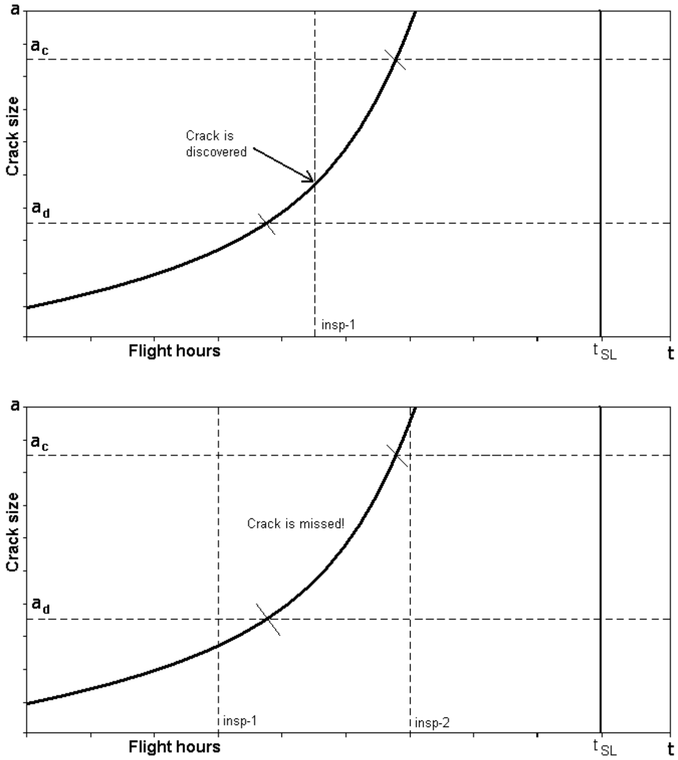

Fig. 2.24. Demonstration of non-monotonous nature of $p_f(\theta, n)$.

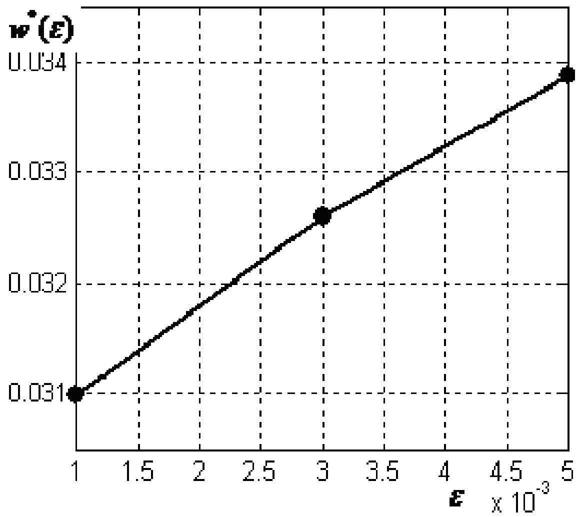

Fig. 2.25. Numerical example: The function $w^*(\varepsilon)$.

Complex form of the function $w(\theta,\varepsilon)$ is defined by the fact that $p_f(\theta, n)$ might be non-monotonous function of n. For relatively small n, $p_f(\theta, n)$ can grow with the increase of n. The reason of such "strange" effect comes from the fact of inspection time moments relocation with the change of n. Example in Fig. 2.24 demonstrates how a crack, discoverable with a single-inspection program, is missed if an inspection program with two inspections is applied. The function $w^*(\varepsilon)$ is shown in Fig.2.25. In our example we see, that to ensure the probability of failure not exceeding $w^* = 0.0326$ at the choice of n^*, the required number of inspections for our inspection program (using formula : n^*=min(n: $p_f(\theta, n) < \varepsilon$)) we have to use the value $\varepsilon = \varepsilon^*$=0.003 (it is worth to mention that w^* is ten times higher than ε^*!).

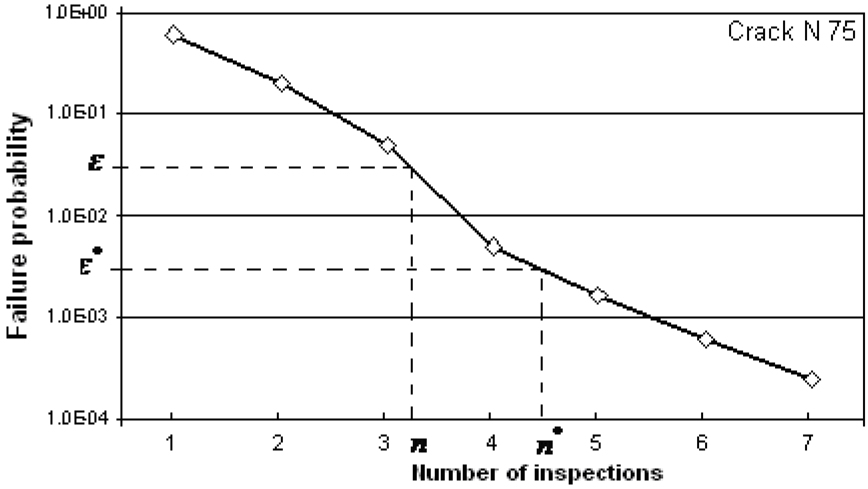

Fig. 2.26. Numerical example: determining required number of inspections.

The required number of inspections in our example is n^*=5 (the same data give the required number of inspections n=4 for ε=0.0326).

2.9. Reliability of airline

Here we consider the problem of the reliability of airlines and its connection with reliability of fatigue-prone aircraft. As it has already been mentioned, the reliability of a fatigue-prone aircraft (AC) and airline (AL) operations can be ensured by the implementation of a specific inspection program, which can be planned using full scale fatigue test data and the theory of Markov Chains (MC) and a Semi-Markov process (SMP). Remeber, that the process of the operation of AC is considered as absorbing MC with $(n+4)$ states. The states $E_1, E_2, ..., E_{n+1}$ correspond to AC operation in time intervals $[t_0, t_1), [t_1, t_2), ..., [t_n, t_{SL})$, where n is an inspection number, t_{SL} is specified life (SL), i. e. AC retirement time. States E_{n+2}, E_{n+3}, and E_{n+4} are absorbing states: AC is withdrawn from service when the SL is reached, fatigue failure (FF) or fatigue crack detection (CD) take place.

In the corresponding matrix for the operation process of AL the states E_{n+2}, E_{n+3} and E_{n+4} are not absorbing, but correspond to the return of the MC to state E_1 (AL operation returns to the first interval). In the matrix of transition probabilities of AC, P_{AC}, there are three units in the three last lines in the diagonal, but for corresponding lines in the matrix for AL, P_{AL}, the units are in the first column, corresponding to state E_1 (see Fig.2.27). Using P_{AC} we can get the probability of FF of AC and cumulative distribution function, mean and variance of AC life, and the same characteristics under the condition of absorption in a specific absorbing state. Using P_{AL} we can get the stationary probabilities of AL operation $\{\pi_1, ..., \pi_{n+1}, \pi_{n+2}, ..., \pi_{n+4}\}$ and the intensity of FF, λ_F, i.e. the number of FF in one time unit. It can be calculated also using the theory of Semi-Markov Process (SMP) with rewards [7,8]. Using this theory we can calculate also the gain of the process. The problem of inspection planning is the choice of the sequence $\{t_1, t_2, ..., t_n, t_{SL}\}$ (in the case of equal inspection intervals and fixed t_{SL}, choice of n) corresponding to the maximum gain, taking into account the limitations of the intensity of fatigue failure of AL or AC fatigue failure probability.

2.9.1. Formal setting of the problem

For the formal setting of the problem we should define the matrix of transition probabilities of MC and the matrix of rewards for SMP.

This matrix is very similar to the transition probability matrix of MC corresponding to the process of operation of one aircraft but now the process is not absorbed or stopped in states SL, FF or CD. If these states are reached, the process restarts from the initial state E_1. This means that an new aircraft is acquired. The modified transition probability matrix, P_{AL}, is shown in Fig.2.27. An example of the state transition diagram is shown in Fig.2.28.

	E_1	E_2	E_3	...	E_{n-1}	E_n	E_{n+1}	E_{n+2} (SL)	E_{n+3} (FF)	E_{n+4} (CD)
E_1	0	u_1	0	...	0	0	0	0	q_1	v_1
E_2	0	0	u_2	...	0	0	0	0	q_2	v_2
E_3	0	0	0	...	0	0	0	0	q_3	v_3
...	...	...	...	...	...	...	...	...	...	...
E_{n-1}	0	0	0	...	0	u_{n-1}	0	0	q_{n-1}	v_{n-1}
E_n	0	0	0	...	0	0	u_n	0	q_n	v_n
E_{n+1}	0	0	0	...	0	0	0	u_{n+1}	q_{n+1}	v_{n+1}
E_{n+2} (SL)	1	0	0	...	0	0	0	0	0	0
E_{n+3} (FF)	1	0	0	...	0	0	0	0	0	0
E_{n+4} (CD)	1	0	0	...	0	0	0	0	0	0

Fig. 2.27. The transition probability matrix for stationary AL operation process.

where probabilities v_i, q_i, u_i, $i = 1,...,n+1$, are the same as in section 4 (probability of detection of fatigue crack during inspection at time point t_i; probability of fatigue failure in interval (t_{i-1}, t_i); probability of absence of mentioned events).

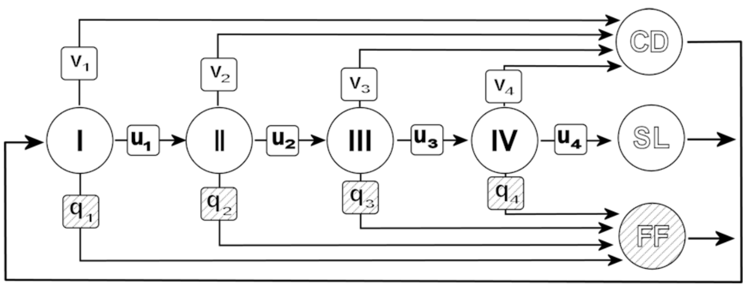

Fig. 2.28. State transition diagram for stationary AL operation process with three inspections.

Next we will consider economic analysis. The theory of Semi-Markov process with rewards is usually used to find the solution of similar problems [7,8]. A reward structure is described by the reward matrix R, the component of which, r_{ij}, describes the reward, connected with the transition from state E_i to state E_j; here *i*,*j*=1, 2,…, *n*+4. Let us define the reward, related to a successful transition from one operation interval to the following one by value $a(n)$; the reward related to transitions to state CD (or E_{n+4}) from any state $E_1, E_2, ..., E_{n+1}$ - by value b, to state FF (or E_{n+3}) – by value *c*; and from states SL, FF, CD to state E_1 (acquisition of new AC) – by value *d*.

Let us calculate the airline gain:

$$g(n) = \sum_{i=1}^{n+4} \pi_i g_i(n), \tag{2.20}$$

where $\pi = (\pi_1, ..., \pi_{n+4})$ is the vector of stationary probabilities, which is defined by the equation system

$$\pi P = \pi, \quad \sum_{i=1}^{n+4} \pi_i = 1; \tag{2.21}$$

$$g_i(n) = \begin{cases} a(n) \cdot u_i + b \cdot q_i + c \cdot v_i, & i = 1, ...n+1, \\ d, & i = n+2, ..., n+4 \end{cases}; \tag{2.22}$$

u_i, q_i, v_i, $i = 1, ..., n+1$, are probabilities of successful transitions from one to the following interval, to E_{n+3} and E_{n+4} states correspondingly;

$a(n) = a_0(n) + d_{insp} t_{SL}$, where $a_0(n) = a_1 t_{SL} / (n+1)$, - is the reward, related to successful transition from one operation interval to the following one, and $d_{insp} t_{SL}$ is the cost of one inspection (negative value) which is supposed to be proportional to t_{SL} if it is supposed that all intervals are equal, a_1 defines the reward of operation in one time unit (one hour or one flight); the dimension of $a(n)$ should coincide with dimensions of t_{SL};

$b = b_1 a_0(n)$ and $c = c_1 a_0(n)$ are the rewards related to transitions from any state E1,…,En+1 to state En+3 (FF takes place) and En+4 (CD takes place) which are supposed to be proportional to a_0; $d = d_1 t_{SL}$ is negative reward, the absolute value of which is the cost of new aircraft after events SL, FF or CD and transition to E_1 takes place (it is supposed to be proportional to t_{SL}).

If $a(n) = b = c = 1$, $d = 0$ and time transition to state E_1 are equal to zero, then $\pi_{tj} = \pi_j g_j(n) / g(n)$ defines the part of time which SMP spends in state E_j, $j = 1, ..., n+1$; $L_j = g(n) / \pi_j$ defines the mean return time for state E_j; specifically, L_1 is the mean time of renewal of AL operation in the first interval, L_{n+3} is the mean time between FF; so $\lambda_F = 1 / L_{n+3}$ is the intensity of fatigue failure. It should be remembered that the same value can be obtained using the theory of absorbing MC. This value is also equal to the ratio of aircraft failure probability to the mean life of new aircraft.

The problem is to maximize gain, $g(n)$, under limitations of probability of aircraft fatigue failure, p_f, or AL intensity of fatigue failure λ_F. In the following numerical example we consider the last version of the problem.

2.9.2. Choice of inspection number if parameter of fatigue crack growth trajectory is known. Numerical example

Usually in Aircraft Design Bureau the sequence of inspection time moments, $(t_1,...,t_n)$, is defined by equation $t_i = t_1 + (i-1)(t_{SL} - t_1)/n$, $i = 1,2,...,n+1$, with specific choice of t_1. Then we should choose only t_1 and n. To simplify the numerical example we suppose that all intervals of operation are equal. In Fig.2.29 and 2.30 we see numerical examples of calculations of gain, g, and of intensity of fatigue failure, λ_F, as functions of *n*.

The procedure of choice of *n* is the following. First we calculate $g(n)$ and choose the number of *n*, corresponding to the maximum of gain g: $n_g = \arg\max g(n)$. Then we calculate expected value of intensity of fatigue failure, $\lambda_F(n)$, which is a function of *n*, and choose n_λ in such a way that for any large inspection number $n \geq n_\lambda$ the function $\lambda_F(n)$ will be equal or less than λ_F^*, which is the value intensity of fatigue failure, allowed by specific aviation regulation

$n_\lambda = \min\left\{n : \lambda_F(n) \leq \lambda_F^*, \ \text{for all } n \geq n_\lambda\right\}$. And finally we should choose $n = \max(n_g, n_\lambda)$.

Consider the numerical example of calculation of gain, g, and intensity of fatigue failure, λ_F, for *n*=2. In this example we will use the same exponential fatigue crack growth model as in the previous section: $a(t) = \alpha \exp(Qt)$. But this time we consider the simplest case when $X = \ln Q$ has normal distribution, $N(\theta_0, \theta_1^2)$, with mean value θ_0 and standard deviation θ_1 and let the initial, $\alpha = a(0)$, detectable, a_d, and critical, a_C, crack sizes be known constants.

Let us have the following initial data: t_{SL} =40 000, θ_0 = -8.5885, θ_1=0,34600, a_0=0.2181; a_c = 102, a_d =20, $a_1 = 1$, $b_1 = 0.05$, $c_1 = -0.05$, $d_1 = -0.3$, $d_{insp} = -0.0001$.

Using equations (2.15-2.16) we have

$$P_{AL} = \begin{bmatrix} 0 & 0.94 & 0 & 0 & 0.00638 & 0.054 \\ 0 & 0 & 0.347 & 0 & 0.269 & 0.385 \\ 0 & 0 & 0 & 0.16 & 0.242 & 0.597 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall that in the corresponding matrix P_{AC} the first three lines are the same, but in the last three lines the units are placed diagonally.

Using P_{AL} and equations (2.21) we have stationary distribution for the case when time unit is one step in MC: π = (3.06e-001, 2.88e-001, 9.97e-002, 1.60e-002, 1.04e-001, 1.87e-001).

Let us define the matrix P_{uqv} with u_i, q_i, v_i in every line for $i = 1,...,n+1$. In the considered example

$$P_{uqv} = \begin{bmatrix} 0.93958 & 0.00638 & 0.054037 \\ 0.34655 & 0.269000 & 0.384450 \\ 0.16030 & 0.242400 & 0.597300 \end{bmatrix}$$

Then $g = \pi((P_{uqv}(a,b,c)')',d,d,d)'$. In considered case $g = 1\ 896$.

If the time dimension unit is the inspection interval then the vector of return times for every state

$L = (g_t / \pi_1,..., g_t / \pi_6)$ =(2.27, 2.41, 6.96, 4.34, 6.70, 3.72).

Here $g_t = \pi[1;1;1;0;0;0]$ is the mean airline gain if income is the time and the time unit is inspection interval. It is useful to note that $g_t = g$ if $a = b = c = 1$, $d = 0$. In considered case g_t = 0.694.

The return time of failure (to state FF), L_F, is equal to corresponding (n+3)-th component of vector $L : L_F = L(n+3)$ =6.7. The time length of one interval $t_1 = t_{SL} / (n+1)$ =13 333.

The intensity of falure in t-time unit (flight or flight hour) $\lambda_F = (1 / L_F) / t_1$= 0.0000112.

It is useful to note that the same values can be obtained using the theory of absorbing MC. For this case we calculate the matrix of absorption

$$B = \begin{bmatrix} 0.052196 & 0.33806 & 0.60975 \\ 0.055552 & 0.35300 & 0.59145 \\ 0.160300 & 0.24240 & 0.59730 \end{bmatrix}$$

Then we take from it the probability of absorption of new aircraft in state FF, p_f =0.338, calculate the mean time to absorption of new aircraft, $T_1 = \tau_1 t_1$ (where τ_1 =2.27 is the mean time to absorption if the time is measured by the number of inspection intervals; its value coincides with $L(1)$!) and calculate the ratio p_f / T_1 =0.0000112.

We see in Fig. 2.29 and 2.30 the results of similar calculations for n=1,…,9 after specific choice of designed failure intensity $\lambda_F = \lambda_{FD}$. Using this results we can make the choice of n (see section 2.9.3).

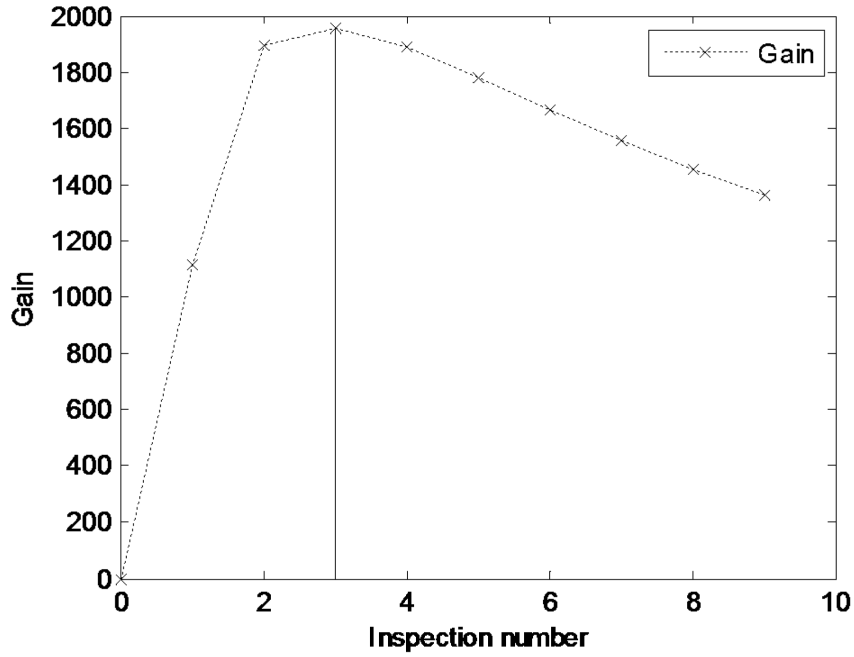

Fig. 2.29. Gain as function of inspection number.

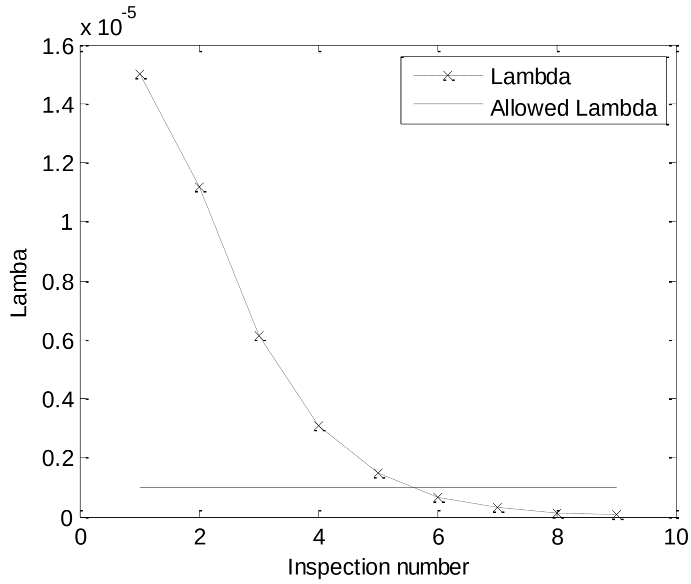

Fig. 2.30. Intensity of failure as function of inspection number.

2.9.3. Minimax choice of inspection number

For the considered strategy for the choice of inspection number the real intensity of failure is a function of θ and designed allowed value of intensity of fatigue failure, λ_{FD}. But in fact we do not know θ and we can only get some estimate of this parameter using test results, $\hat{\theta}$. So real intensity will be a random variable, $\lambda_F(\hat{\theta}, \lambda_{FD})$. We can limit the mean value of this function if again we take into account that really full-scale fatigue test is an approval test and redesign of airframe will be made (service of aircraft of tested design version of aircraft will not be allowed) if some requirement is not met (for example, if it appears, that estimate of mean time to failure, $\hat{t}_C = C_C / \exp(\hat{\theta})$, is too small or calculated required inspection number, based on using $\hat{\theta}$ instead of θ, is too great). If full-scale fatigue test is approved and there is no necessity to make airframe redesign let us write $\hat{\theta} \in \Theta_0$, where Θ_0 is some part of parameter space. Let us denote corresponding fatigue intensity function by

$$\lambda_F(\hat{\theta}, \lambda_{FD}, \Theta_0) = \begin{cases} \lambda_F(\hat{\theta}, \lambda_{FD}) & \text{if } \hat{\theta} \in \Theta_0, \\ 0 & \text{if } \hat{\theta} \notin \Theta_0 \end{cases}.$$

In Fig. 2.31 we see an example of calculation of expected value of this function, $E(\hat{\lambda}_F)$, where $\hat{\lambda}_F = \lambda_F(\hat{\theta}, \lambda_{FD}, \Theta_0)$, for different θ_0. In order to better understand connection of θ_0 and fatigue life in axis we see corresponding estimate of mean time to failure, $mean\_t_C = C_C / \exp(\theta_0)$ (where, recall, $C_C = \log(a_C / \alpha)$). The other initial data are the same as in previous example. Additionally, we define, that there is event $\hat{\theta} \in \Theta_0$ if $mean\_t_C = C_C / \exp(\theta_0) \geq k_R t_{SL}$. For Fig. 2.31 $k_R = 1$, λ_{FD} =0.000001.

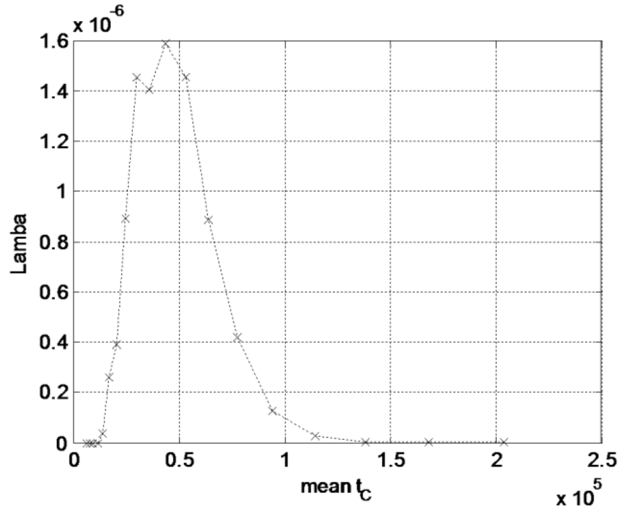

Fig. 2.31.Mean intensity of fatigue failure of airline as function of $mean\_t_C = C_C / \exp(\theta_0)$.

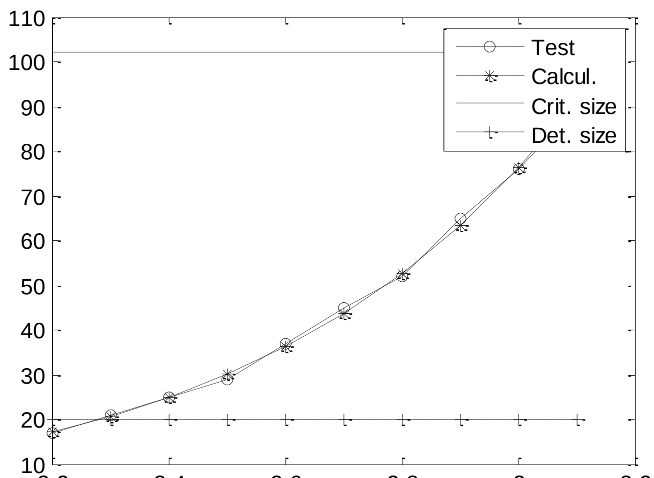

Fig. 2.32. Example of fatigue crack size as function of flight number (result of full scale fatigue test).

We see that there is maximum value of $E(\hat{\lambda}_F)$ because if design is too bad and test is not approved, then we redesign the airframe. But if it is too good then fatigue failure with a great probability does not take place before the specified life is reached. Maximum value of $E(\hat{\lambda}_F)$ is equal to 1.6*10-6, it is in 1.6 time more than designed λ_{FD} =0.000001. So we can use this value of λ_{FD} for calculation of inspection number only if allowed failure intensity $\lambda^* \geq 0.0000016$. Suppose that the required reliability is equal to 0.0000016.

In Fig. 2.32 we see an example of a fatigue crack, which was observed in full-scale fatigue test of some airframe. Using this crack data and regression analysis we can get estimates $\hat{\theta}$ =-8.5885. For these data we can make calculations of estimates of inspection intensity λ_F and gain g as a function of inspection number and, finally, can find required inspection number (see Fig. 2.29-2.30). In this example n_L = 6; $n_g = 3$; $n = \max(n_g, n_\lambda)$ = 6.

So, this section presents a method, based on MC and SMP with rewards theory, to solve the problem of inspection planning corresponding to maximum airline gain while airline fatigue failure intensity is limited. If the parameters of fatigue crack growth exponential models are unknown, but there are results of approval full-scale fatigue test of corresponding airframe, then the minimax approach (see [8-12]) should be used.

2.10. References

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Part 2. Statistical Analysis of Static Strength and Fatigue Life of Composite

3. Analysis of composite tensile strength

3.1. Introduction

There is significant dependence of static strength of a composite on the scatter of static strength of its components. It can be illustrated by the following example. Let us consider three parallel items with 10 N, 15 N and 30 N strength and identical stiffness. It may seem surprising that they will fail at the applied load of 30 N, as if the strength of every item is equal to 10 N. Why?

The reason is that under 30 N load, at first the weakest item will fail because its strength is equal to 10 N. At the uniform distribution of total loads, its load is equal to 10 N also. Now the load acting on each intact item is equal to 15 N. So the second item, the strength of which is equal to the same value of 15 N, fails. Now the load for the last strongest item is equal to 30 N. It fails also because its strength is just equal to this load. This process ("domino phenomenon") is shown in Fig.3.1. The same phenomenon takes place if element strengths are proportional to the terms of harmonic series: 1, ½, 1/3, …, 1/n, see Fig. 3.2.

So we see that the composite strength dependence on the strength scatter of its constituents can be very significant.

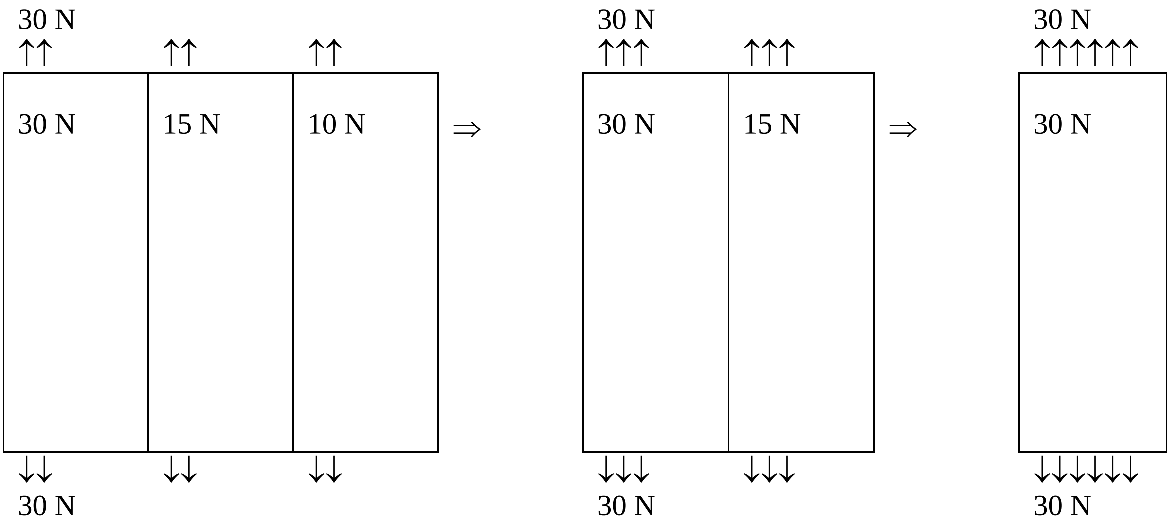


Fig. 3.1. This bundle of three parallel longitudinal items fails at the applied load of 30 N, as if the strength of every item is equal to 10 N.

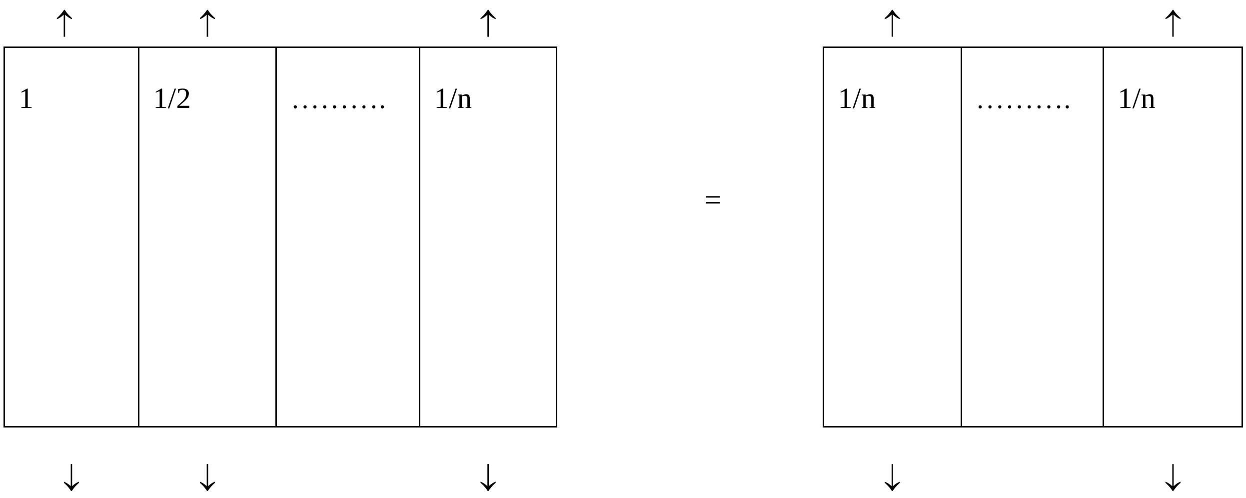


Fig. 3.2. This bundle of n parallel longitudinal items fails at the applied load of 1 N, as if the strength of every item is equal to 1/n.

A composite specimen for test of tensile strength can be considered as series system, every link of which is a parallel system, or more specifically, a bundle of n_C longitudinal items (fibers or bundles) immersed into composite matrix (CM). We make an assumption that the CM is a composition of the matrix itself and all the layers with stacking different from the longitudinal one. We make an additional assumption also that only longitudinal items (LIs) carry the longitudinal load but the matrix only redistributes the loads after the failure of some longitudinal items. We can suppose that the composite is divided into n_L parts ("links") of the same length, l_1. The total length of the composite specimen is equal to $L = n_L\, l_1$. It is supposed that the development of the process of fracture of a specimen takes place in one or in several of these parts. Let the process of monotonic tensile loading (i.e. the process of increase of the nominal stress in the specimen cross section (CS)) be described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$, and let $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, be the number of failures of LI in *i*-th link (CS) at the load x_t with n_C initial number of LI. Then the strength of *i*-th CS

$$X_i^* = \max(x_t : n_C - K_{Ci}(t) \ge 0), \tag{3.1}$$

but the ultimate strength of the specimen (which is the sequence of n_L CS) is

$$X = \min_{1 \le i \le n_L} X_i^* = \min_{1 \le i \le n_L} \max(x_t : n_C - K_{Ci}(t) \ge 0). \tag{3.2}$$

Different assumptions about the distribution of strength of one link and a priori distribution of the number of initial (technological) defects compose a family of the distributions of ultimate composite tensile strength.

3.1.1. Reliability of series systems with defects

Let us consider the case $n_C = 1$, when test specimen can be considered as series system some links of which are defected. The probability distributions of strength proposed by Weibull for brittle materials have found numerous applications [1]. The Weibull distribution complies with one of the three asymptotic forms of extreme value distributions. It is a stable distribution of the smallest extreme value in that the strength of a chain of identical links has the same type of distribution as the strength of a link; only the distribution parameters change. This feature makes the Weibull distribution a convenient and natural means of characterising the scale effect of strength, i.e. the decrease of the strength with increase of the specimen size. The two-parameter Weibull distribution is apparently the most widely used distribution function for fiber tensile strength; the fiber fracture probability dependence on LI length is given by cumulative distribution function

$$F(s) = 1 - \exp\left(-\frac{L}{l_1}\left(\frac{s}{\beta}\right)^{\alpha} \right) \tag{3.3}$$

where s is the applied tensile stress, L denotes LI length, l_1 is a normalizing parameter with length dimension, α and β stand for the shape and scale parameters respectively. There is a growing amount of experimental data suggesting that equation (3.3) might oversimplify the properties of fibers sampled randomly from a yarn – the shape parameter value determined from test results at a fixed gauge length is smaller than that obtained from average strength vs. fiber length data (see e.g. [2-6]). This phenomenon is analyzed in a great number of studies accompanied by extensive lists of references [7-14]. It has turned out that the better fitting of experimental data on probability paper is provided by the model originally suggested in [6,7]. According to this model, the c.d.f. of strength is described by formula

$$F(s) = 1 - \exp\left(-\left(\frac{L}{l_1}\right)^{\gamma}\left(\frac{s}{\beta}\right)^{\alpha} \right), \tag{3.4}$$

Where γ - is additional parameter. Distribution (3.4) {called the “power-law” Weibull model (PW) contrary to the “linear-law”(LW) model defined by equation (3.3)} and its modifications (for example, a volume instead of length is employed to account for the scatter of fiber diameter) are being widely applied to describing experimental data. But a good fitting of experimental data does not ensure a reliable prediction of changes in c.d.f. with changing length of a specimen.

3.1.2. Probability structure of series system with defected items

As it has already been mentioned, here we consider a special case of n_C =1 and two types of links (with and without defects). Let us denote the random number of “damaged” links by K_L, $0 \le K_L \le n_L$, with strength c.d.f. $F_Y(x)$ (we say that there are K_L of Y-type links), and let us denote by $F_Z(x)$ the strength c.d.f. of ($n_L - K_L$) links without defects (we say they are Z-type links). We suppose that the failure process of the considered system has two stages. In the first stage the process develops along the specimen and K_L links of Y-type appear. They can appear before loading in accordance with some a priori distribution, $\pi_L = (\pi_{L1}, \pi_{L2}, ..., \pi_{L(n_L+1)})$, where $\pi_{Lk} = P(K_L = k-1)$, or during the process of loading, when the stress in LI exceeds *a defect initiation stress* with c.d.f. $F_K(x)$. Then the second stage takes place: the process of accumulation of elementary damages in crosswise direction up to specimen failure. Two levels of differences between LI with and without defects and three groups (levels) of accuracy of description of the difference of strength inside these groups form the six types of corresponding probability structures (p.s.) which are shown in Table 3.1.

Table 3.1. Probability structures of specimen strength dependence on the strength of single links

A 1: $X = \min(Y_1, ..., Y_{K_L}, Z_1, ..., Z_{n_L - K_L})$;	B1: $X = \begin{cases} \min(Y_1, ..., Y_{K_L}), & K_L > 0, \\ \min(Z_1, ..., Z_{n_L}) & K_L = 0; \end{cases}$
A2: $X = \min(Y_1, ..., Y_{K_L}, Z)$;	B2: $X = \begin{cases} \min(Y_1, ..., Y_{K_L}), & K_L > 0, \\ Z, & K_L = 0; \end{cases}$
A3: $X = \min(Y, Z)$	B3: $X = \begin{cases} Y, \; K_L > 0, \\ Z, \; K_L = 0. \end{cases}$

In p.s. of type A it is assumed that the difference between the strength of links of Y and Z types is relatively small and the failure of the specimens can be caused by the failure of a link of either type. In p.s. of type B it is assumed that the difference between the strength of links of Y and Z types is very large and we must take into account the strength of the link of Z type only if there are no links of Y-type. In some way, the description of the group of type B is a limit of the description of the group of type A if the difference between the c.d.f. of strength of Y and Z type links increases. Really, all the description of p.s. different from A1 are some form of approximation of the description of the group A1. We suppose that the usefulness of considering this set of different p.s. is defined by the difference of materials, different requirement to accuracy of calculation and different size of the test sample.

Sometimes it is acceptable to assume that specimen failure can take place only in items with defects. This leads to the structures shown in Table 3.2. It is very essential that in this case we do not need to know the c.d.f. of LI without defects. But some limitation appears: for r.v. K_L we should use only such probability mass function (p.m.f.) that $P(K_L = 0) = 0$.

Table 3.2. Conditional probability structures of specimen strength dependence on the strength of single links

AC: $X = \min(Y_1, ..., Y_{K_L} \mid K_L > 0)$	BC: $X = (Y \mid K_L > 0)$

It is easy to connect the c.d.f. of the strength of the specimens and the c.d.f. of the strength of single LI. For example, for the "extreme" structures A1 and B3 we have the following equations (see details in Appendix 5.2.1)

$$F(x) = 1 - \sum_{k=0}^{n_L} p_k (1 - F_Y(x))^k (1 - F_Z(x))^{n_L - k}, \tag{3.5}$$

$$F(x) = (1 - p_0) F_Y(x) + p_0 F_Z(x), \tag{3.6}$$

where $\{p_k, k = 0,1,...,n_L\}$ is the probability distribution for the r.v. K_L. This probability distribution can be considered as a function of the applied nominal stress and if, for example, it is a binomial distribution, $b(k, p_L, n_L) = p_L^k (1-p_L)^{n_L-k} n_L!/k!(n_L-k)!$, then it can be assumed that $p_L = F_K(x)$, where $F_K(x)$ is c.d.f. of *defect initiation stress*. In this case we add the letter F to the notation of p.s. For example, processing of the test data in section 3.3 will be made using p.s. B3F. In this case

$$p_0 = (1 - F_K(x))^{n_L}. \tag{3.7}$$

The process of gradual (during loading) accumulation of defects and failure of a series system can be described by a Markov chain (MC). Then for notation of the p.s. we use additional letter M: MA1, MA2 and so on. If the process of monotonic tensile loading (i.e. the process of increase of the nominal stress (or mean load of one LI)) is described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$ then the number of links of Y-type and the strength of specimens are random functions of time, $K_L(t)$ and $X(t)$. For example, for the MA1 we have $X(t) = \min(Y_1, Y_2, ..., Y_{K_L(t)}, Z_1, Z_2, ..., Z_{n_L - K_L(t)})$. Let us consider a MC with $(n_L + 2)$ states. MC is in state i if there are $(i-1)$ of Y-type links, i=1,...,n_L+1. State i_{n_L+2} is an absorbing state corresponding to the fracture of specimen. The process of MC state change and the corresponding process $K_L(t)$ are described by the transition probabilities matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{13} & \cdots & p_{1(n_L+1)} & p_{1(n_L+2)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n_C+1)} & p_{2(n_L+2)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n_C+1)} & p_{3(n_L+2)} \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & p_{(n_L+1)(n_L+1)} & p_{(n_L+1)(n_L+2)} \\ 0 & 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix} \tag{3.8}$$

At the *t*-th step of MC the matrix *P* is a function of *t*, *t*=1,2,... A priori or initial distribution of MC states and r.v. K_L is represented by a row vector $\pi_L = (\pi_{L1}, \pi_{L2}, ..., \pi_{L,n+1,} \pi_{L,n+2})$, where $\pi_{L(n+2)}$=0.

Now the ultimate strength of specimen is defined by equation

$$X = x_{T^*}, \tag{3.9}$$

where

$$T^* = \max(t : X(t) > x_t). \tag{3.10}$$

The c.d.f. of ultimate strength, *X*, is defined by equation

$$F_X(x_t) = \pi_L (\prod_{j=1}^{t} P(j)) u, \tag{3.11}$$

where vector-column $u = (0,0,...,1)'$.

The examples of specifying of the matrix *P* for “extreme” p.s. MA1 and MB3, for n_C=1, are given in Appendix 5.2.2.

3.2. Specification of the models

Now we need to give the specification of the components of the considered structures: $\{p_k\}$, $F_Y(x)$, …First we consider the specification of the components common for all probability structures.

3.2.1. A source of defects. The distribution function of the number of defects

Defects can appear before loading (technological defect) and during loading. In this paper we consider the following hypothesis. For finite number of elements, *n*, it seems natural to assume the binomial p.m.f. of the number of defects, K_L, $b(k;p_L,n_L)=p_L^k(1-p_L)^{n_L-k}n_L!/k!(n_L-k)!$, where p_L is probability that defect appear in one link. Let us note that in this case for B3 $p_0=(1-p_L)^{n_L}$.

If *n* is large enough then as an approximation of binomial distribution the Poisson pmf is used usually:

$$p_k = p(k,\lambda) = \exp(-\lambda)\lambda^k / k!.$$

Parameters p_L and λ can either be taken as independent of load ("technological" defect), or $p_L = F_K(x)$, $\lambda = nF_K(x)$, where $F_K(x)$ is the c.d.f. of defect initiation stress during loading.

But there are some limitations. At large but limited *n* for p.s. A1-B3 conditional Poisson distribution should be used (under condition that $K_L \le n_L$) :

$$p_k = (\exp(-\lambda)\lambda^k / k!) / (\exp(-\lambda)\sum_{r=0}^{n_L}\lambda^r / r!) = (\lambda^k / k!) / (\sum_{r=0}^{n_L}\lambda^r / r!),\ k = 1,...,n_L. \tag{3.12a}$$

It is worth to note that (for B3)

$$p_0 = 1/(\sum_{r=0}^{n_L}\lambda^r / r!). \tag{3.12b}$$

Let us underline once again that in framework of p.s. AC and BC we suppose now that the failure of a specimen is defined only by failure of the damaged links. Now we call these links "damaged cross sections" (DCS). So using this term we describe the composite as a *series system of* DCS. We make an additional assumption that the distribution of the number of cross sections with defects is a conditional Poisson distribution under condition of zero probability that this number is zero. We denote this distribution by ZCPD. So for random number of DCS, K_L,

$$P(K_L = k) = p_k = \frac{e^{-\lambda}}{1-e^{-\lambda}}\frac{\lambda^k}{k!}, \tag{3.13}$$

$k = 1,2,...$

The c.d.f. of strength of this DCS we again denote by $F_Y(x)$. Then c.d.f. of specimen strength

$$F(x) = 1 - \sum_{k=1}^{\infty} p_k\left(1-F_Y(x)\right)^k = \frac{1-e^{-\lambda F_Y(x)}}{1-e^{-\lambda}}. \tag{3.14}$$

Let us note a specific feature of this distribution of r.v. K_L. It is easy to show that for this case for the expected value of K_L

$E(K_L) = \lambda / (1 - e^{-\lambda})$,

while the mean value of a random variable with the usual Poisson distribution is equal to λ. Recall that the natural assumption is: $\lambda = \lambda_1 * L$, where λ_1 is defect intensity, L is the length of specimen. So $E(K_L)$ is not proportional to the length of specimen contrary to the models, in which p.m.f. p_k is defined by usual binomial or Poisson distributions.

For the models based on using MC theory for a priory distribution of ("technological") defect number, K_L, again the binomial or Poisson p.m.f. can be used. Poisson p.m.f. $p(k, \lambda)$ should be "cutted" in point *n*; this means that $p_k = p(k, \lambda)$ for $k \le n$; $p_{n+1} = 1 - \sum_1^n p_k$ and $p_k = 0$ for $k \ge n + 2$; this procedure of "cutting" is necessary because the theory of final Markov chains is used. A priory distribution $\pi = (1,0,...,0)$ means that before beginning of loading the number of defects is equal to zero.

3.2.2. Cumulative distribution function of strength of a single link

Result of processing of test data set depends on c.d.f. of strength of a single link. In some cases [15] the lognormal in others [5] the Weibull distribution are used. Usually the answer to the question about the appropriate c.d.f. depends on a result of fitting data set in probability paper and visual analysis.

For example the conclusion on the good agreement of fiber strength at a fixed gauge length with the Weibull distribution was made in [5] based only on visual evaluation of standard probability plot ($p_i = (i - .3)/(n + .4)$ vs. $x_{(i)}$, where $x_{(i)}$ is order statistic, *i*=1,…,*n*). Confirmation of similar conclusions is made in Appendix 5.2.3. using OSPPTest (Test based on Probability Plot of Ordered Statistics, $x_{(i)}, i = 1,...,n$, versus expected values of standard order statistics, $E(\overset{o}{X}_{(i)}), i = 1,...,n$, corresponding to θ_0=0, θ_1=1) and ρ-approximation of most powerful invariant test [16-20].

In the following numerical example we suppose the Weibull distribution for a single link strength.

But if for a link strength, S, the Weibull distribution is appropriate, then for $Y = \log(S)$ the smallest extreme value (s.e.v.) distribution can be used with c.d.f.

$$F_Y(x) = 1 - \exp(-\exp((x - \theta_{0Y}) / \theta_{1Y})) . \quad (3.16)$$

The same type of distribution (but with specific parameters) can be used for defect initiation stress

$$F_K(x) = 1 - \exp(-\exp((x - \theta_{0K}) / \theta_{1K})) , \quad (3.17)$$

and for c.d.f. of strength of link without defects

$$F_Z(x) = 1 - \exp(-\exp((x - \theta_{0Z}) / \theta_{1Z})) . \quad (3.18)$$

In order to decrease the number of unknown parameters it can be accepted that $\theta_{1Y} = \theta_{1Z} = \theta_{1K}$. If the difference between strength of links with defects and without defects is very large then it can be assumed that

$$F_Z(x) = \begin{cases} 0, & x < C, \\ 1, & x \ge C, \end{cases} \quad (3.19)$$

where C is large enough (let us note, that (3.19) is a specific case of (3.18), if $\theta_{0Z} = C$, but $\theta_{1Z} \to 0$).

Then instead of (3.5) we get

$$F(x) = \begin{cases} 1 - \sum_{k=0}^{n} p_k (1 - F_Y(x))^k, & x < C, \\ 1, & x \geq C, \end{cases} \tag{3.20}$$

or

$$F(x) = \begin{cases} 1 - \exp(-\lambda F_Y(x)), & x < C, \\ 1, & x \geq C \end{cases}, \tag{3.21}$$

if Poisson distribution for K_L takes place. And instead of (3.6) we get

$$F(x) = \begin{cases} \{1 - (1 - F_K(x))^n\} F_Y(x), & x < C, \\ 1, & x \geq C. \end{cases} \tag{3.22}$$

3.2.3. Specification of sequence of loads (stresses) in framework of probability structures MA and MB

For processing of experimental data for this paper the sequence $\{x_1, x_2, ..., x_t, ...\}$, where for specific value x_1, the value $x_{t+1} - x_t = Const$ for all $t = 1,2,...$ can be used. The value of x_1 should be chosen in such a way that $F_Y(x) = 1 - \exp(-\exp(x_1))$ *is* small enough. The value of *Const* must also be small enough.

3.3. The processing of test data

Comparison of the models is made via processing the following tension test results

Data_1: glass fibers of lengths (L_1, L_2, L_3, L_4)=(10,20,40,80 *mm*) at the number of specimens (n_1, n_2, n_3, n_4)=(78,74,49,60) (see [5]).

Data_2: flax fibers of length (L_1, L_2, L_3)=(5, 10, 20 *mm*) at the number of specimens (n_1, n_2, n_3)=(90,70, 58) (see [9]).

Data_3: carbon fibers of length (L_1, L_2, L_3, L_4)=(1, 10, 20, 50 *mm*), at the number of specimens (n_1, n_2, n_3, n_4)=(57,64,70,66) (see [8]).

Data_4: epoxy-impregnated carbon fiber bundles of length (L_1, L_2, L_3, L_4)=(20, 50, 150, 300 *mm*), at the number of specimens (n_1, n_2, n_3, n_4)=(28,30,32,29) (see [8]).

Here we consider the fitting of the test data and estimation of c.d.f. parameters using test results at (L_1, L_2) and then predict the expected values of order statistics for the longest length, L_3 (for flax fibers) or L_4 (for all the other data set).

Note. The result of similar processing of these data was done in [16,17] but for „opposed prediction direction”: fitting of experimental data and estimation of c.d.f. parameters was made using test results at (L_2, L_3) (for flax fibers) or (L_3, L_4) (for all the others LI) but prediction of expected values of order statistics for the smallest length, L_1.

The maximum likelihood method (MLM) can be used for parameter estimation but it is excessively labor-consuming. The estimates of parameters θ_0 and θ_1 (at fixed other parameters) can be found easily using regression analysis of order statistics. Our purpose here is to investigate the possibility of using the considered models for prediction of fiber strength distribution changes when fiber length is varied and compare the models. So we have limited ourselves by the use of regression analysis.

Let r.v. X (logarithm of strength) have c.d.f. with a location and a scale parameter and let x_{ij} be *j*-th order statistic for $L = L_i$. Here $j = 1,2,...,n_i$, n_i is the number of specimens with $L = L_i$, $i = 1,2,...,k_L$, k_L is the number of different L_i. Let us denote by $E(X_{ij})$ the expected value of random order statistic X_{ij}, $E(\overset{0}{X}_{ij})$ is the same but for θ_0=0 and θ_1=1.

Then we have

$$E(X_{ij})=\theta_0+\theta_1 E(\overset{0}{X}_{ij}), \tag{3.23}$$

where $E(\overset{0}{X}_{ij})$ is a function of n_i, *j* and L_i (or λ_i for ZCPD).

For the LW model for processing data of test at different length L_i

$$E(X_{ij})=\theta_0 + \theta_1(-\log(L_i/l_1)+E(\overset{0}{X}_{ij})), \tag{3.24}$$

where $\theta_0=\log(\beta)$, $\theta_1 = 1/\alpha$.

For PW model we have equation with three unknown parameters $\theta_{00}=\log(\beta)$, θ_{01}=-γ/α and θ_1

$$E(X_{ij})=\theta_{00} + \theta_{01}\log(L_i/l_1)+\theta_1 E(\overset{0}{X}_{ij}) \tag{3.25}$$

This equation can be used for linear regression (LR) estimation of θ_0 and θ_1. Modern PC allows to use the Monte Carlo method for a very simple calculation of $E(\overset{0}{X}_{ij})$. If R is r.v. with a uniform distribution in interval (0,1) then r.v. $\overset{0}{X} = \log(-\log(1-R))$ has a standard s.e.v. distribution but if λ is known then r.v. $\overset{0}{X} = \overset{0}{F}{}^{-1}(-\log(1-R(1-e^{-\lambda}))/\lambda)$ has c.d.f.

$F(x) = (1-e^{-\lambda F_Y(x)})/(1-e^{-\lambda})$.

Here $\overset{0}{F}{}^{-1}(.)$ is the inverse function.

For numerical evaluation and comparison of different models, the following three additional statistics were calculated:

Statistic $OSPPt$ - for experimental data fitting. This statistic for the case of use the samples corresponding to L_1 and L_2 for parameter estimation is defined by equation

$$OSPPt_{1-2} = (\sum_{k=1}^{2}\sum_{j=1}^{n_k}(x_{kj}-\hat{x}_{kj})^2/\sum_{k=1}^{2}\sum_{j=1}^{n_k}(x_{kj}-\overline{x}_k)^2)^{1/2},$$

where x_{kj} is *j*-th order statistic of sample at $L = L_k$; $\hat{x}_{kj} = \hat{\theta}_0 + \hat{\theta}_1 E(\overset{o}{X}_{kj})$ is an estimate of $E(X_{kj})$; $\hat{\theta}_0$, $\hat{\theta}_1$ are estimates of θ_0, θ_1; $\bar{x}_k = \sum_{j=1}^{n_k} x_{kj} / n_k$, n_k is sample size at $L = L_k$.

As LR-estimates are used here, then in this case *OSPPt* -statistic coinsider with

$$\bar{R}_{LR} = (1 - R^2)^{1/2}, \tag{3.26}$$

where R^2 is the standard statistic of LR analysis (the coefficient of determination).

Statistic $OSPPt_k$, where $k = 3$ for flax fiber and $k = 4$ for other LIs, - for prediction of order statistic of test results at different length $L = L_k$,

$$OSPPt_k = (\sum_{j=1}^{n_k} (x_{kj} - \hat{x}_{kj})^2 / \sum_{j=1}^{n_k} (x_{kj} - \bar{x}_k)^2)^{1/2}, \tag{3.27}$$

Statistic Q_1 - for prediction of mean strength at different L

$$Q_1 = (\sum_{i=1}^{k_L} (\bar{x}_i - \hat{x}_i)^2 / \sum_{i=1}^{k_L} (\bar{x}_i - \bar{x})^2)^{1/2}, \tag{3.28}$$

where $\hat{x}_i = \sum_{j=1}^{n_i} x_{ij} / n_i$, $\bar{x} = \sum_{i=1}^{k_L} \bar{x}_i / k_L$.

There is a great deal of variations of the models in framework of considered family. First, we consider four basic models with $F_Y(x)$, $F_K(x)$ and $F_Z(x)$ defined by (3.16 – 3.18), with binomial (for “Markov’s ” model) or Poisson distribution of defect number with following p.s.:

1. A1with the c.d.f. specified by (3.5).
2. B3 with the c.d.f. specified by (3.6).
3. MA1 with the c.d.f. ispecified by (3.11); the matrix *P* is described in Appendix 5.2.2.1; $F_Z(x)$ is defined by (3.19) with $C = \infty$.
4. MB3 with the c.d.f. specified by (3.11); the matrix *P* is described in Appendix 5.2.2.2; $F_Z(x)$ is defined by (3.19) with $C = \infty$.

We study also the model with AC probability structure with the c.d.f. specified by (3.14).

For prediction stability it is very important to minimize the number of unknown parameters. For this purpose we set $l_1 = L_1$ for all models and materials.

At the same time the calculation for both LW and PW (if possible) models was done.

The results of processing the test data are shown in Table 3.3 in four blocks. In the first one, the results (quality statistics and parameters) corresponding to the best of four considered models are given. In the following blocks there are the results (quality statistics and parameters) for AC p.s. with ZCPD as a priori distribution with the c.d.f. specified by (3.14) and for PW, and LW models. In the two last blocks, parameter estimates, using only data at L_1, L_2 (_LRA_2) or whole data set (_LRA_ k_L) both linear regression and maximum likelihood (_ML) estimates are given (if there were these estimates in original papers [5,9,8]). We think (see [17]) that the difference between the estimates _LRA_ k_L - and maximum likelihood _ML- estimates is not significant taking into account that actually the likelihood function has no distinct maximum. So we are not too far from ML-estimates if we use LR-estimates of parameters.

Let us note again that for every data set with a specific L it was accepted that parameter l_1 is equal to L_1. As the estimate of nonlinear parameters $\lambda_1 = \lambda / L_1$ and p we take the value of

Table 3.3. Criteria statistics. Estimates of parameters

№	Criteria statistics. Estimates of parameters	Glass fibers. [5]	Flax fibers. [9]	Carbon fibers [8]	Bundles 1000 of epoxy-impreg. carbon fibers [8]
1	($\overline{R}_{LR}$, OSPPt, Q_1)_MMDM	0.156 **0.174 0.124** MB	**0.119 0.197 0.316**	0.182 0.284 0.230	*0.417 0.578 0.433*
	Structure θ_0, θ_1 p λ_1	MB 7.63 0.246 0.15	BF 6.76 0.479	MA 8.436 0.152 0.99	MA 7.97 0.044 0.025
2	($\overline{R}_{LR}$,OSPPt, Q_1)_ZCPD	0.1567 0.3800 0,4184	0.1394 0.6483 1.2933	**0.1614** 0.3246 0.2499	**0.2107** 0.7054 0.5855
	$\hat{\theta}_0, \hat{\theta}_1$ λ_1	7.8499 0.1661 0.142	7.0978 0.2893 0.27	8.5047 0.1498 1.655	7.9778 0.0448 0.0198
3	($\overline{R}_{LR}$,OSPPt, Q_1)_PW	**0.1525** 0.2155 0.1644	0.1534 0.3332 0.5121	0.1705 0.4026 0.2809	0.2109 1.1647 1.1228
	(β, α γ)_LRA_2	2 381 5.440 0.601	1 068 3.15 0.580	4 543 6.23 0,887	2 896 20.7 0.149
	(β, α γ)_LRA_ k_L	2 394 5.159 0, 608	1 076 3.09 0.623	4 582 6.15 1.022	2 965 16.6 0.732
	(β, α γ)_ML	3 030 5.43 0. 580	1 400 2.80 0.460	4 630 5.31 0.9	3 250 16.8 0.580
4	($\overline{R}_{LR}$,OSPPt, Q_1)_LW	0.1855 0.4760 0.6702	0.1890 0.7937 1.6501	0.1803 **0.2778 0.2268**	0.3680 **0.5386 0.3793**
	(β, α)_LRA_2	3 663 5.605	1 810 3.25	4 575 6.58	3 340 23.3
	(β, α)_LRA_ k_L	3 691 5.827	1 836 3.31	4 571 6.07	3 510 18.8
	(β, α)_ML	3010 8.99	1 836 3.31	-	-

Table 3.4. The best models, criteria statistics

№	Criteria	Glass fibers. [5]	Flax fibers. [9]	Carbon fibers [8]	Bundles 1000 of epoxy-impreg. carbon fibers [8]
1	$\overline{R}_{LR}$	**PW 0.1525**	**BF 0.119**	**ZCPD 0.1614**	**ZCPD 0.2107**
2	OSPPt	**MB 0.174**	**BF 0.197**	**LW 0.2778**	**LW 0.5386**
3	Q_1	**MB 0.124**	**BF 0.316**	**LW 0.2268**	**LW 0.3793**

parameters, which provide the best fitting (minimum of $\overline{R}_{LR}$). The best results (corresponding to minimum of specific criterion) are marked out in the table 3.3 by bold figures.

In table 3.4 we see the best models for every data set for every criterion.

Examples of detailed comparison of the models are shown in Fig. 3.3-3.4 The processing of single carbon fiber experimental data is shown in Fig. 3.3. Here we do not see a great difference between the models.

Processing of impregnated bundle data is shown in Fig. 3.4. Here it is worth to note that all three models provide nearly the same quality of fitting of data for L_1 and L_2 but there is a significant difference of prediction for L_4.

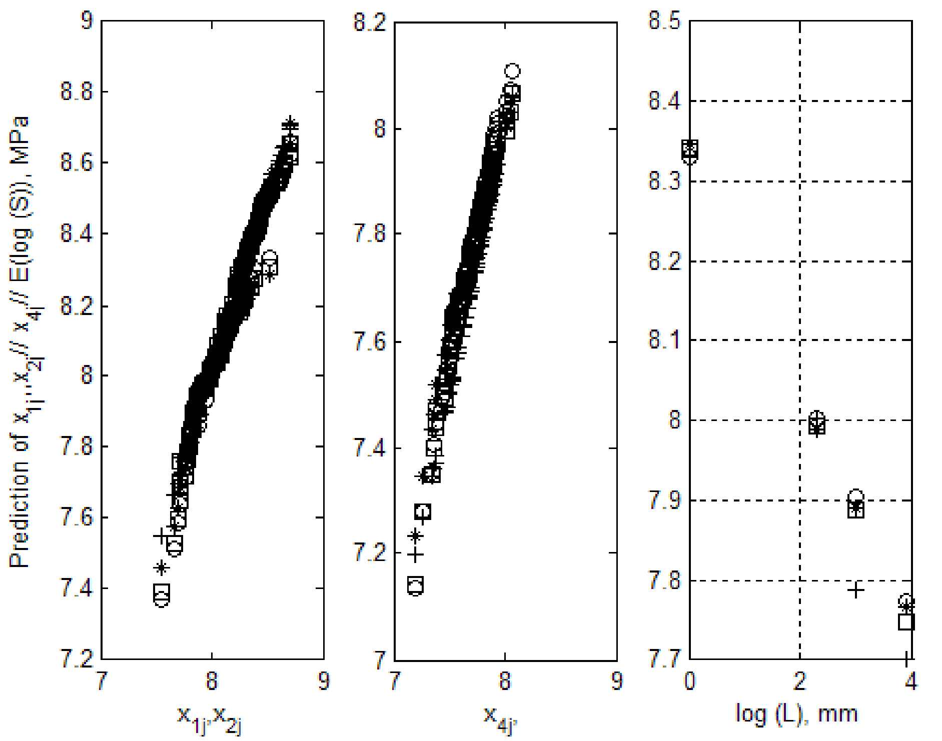


Figure 3.3. Single carbon fiber test data (+) ; predictions, $\hat{x}_{1j}$, $\hat{x}_{2j}$, $\hat{x}_{4j}$ and $\overline{x}_i$, using MA-model (*) ($\overline{R}_{LR}$ =0.1737, OSPPt-4=0.3863, Q_1 =0.2549), LW model (□) ($\overline{R}_{LR}$ =0.1803, OSPPt-4=0.278, Q_1 =0.2268) and PW model (o) ($\overline{R}_{LR}$ =0.1705, OSPPt-4=0.4026, Q_1 =0.2809). Initial data: samples with L=L_1 and L=L_2.

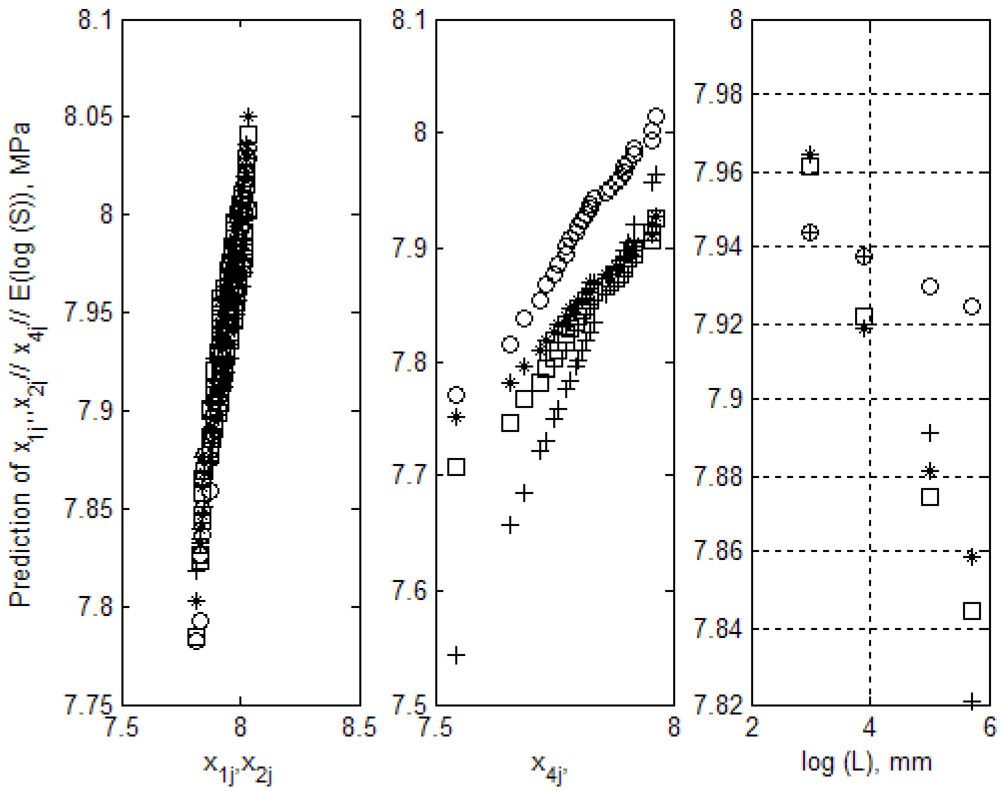


Fig. 3.4. Data of bundles of 1000 impregnated carbon fibers (+); predictions, $\hat{x}_{1j}$, $\hat{x}_{2j}$, $\hat{x}_{4j}$ and $\bar{x}_j$, using MA-model (*) ($\bar{R}_{LR}$ =0.4218, OSPPt-4=0.6981, Q_1 =0.4890), LW model (□) ($\bar{R}_{LR}$ =0.368, OSPPt-4=0.5386, Q_1 = 0.4890) and PW model (o) ($\bar{R}_{LR}$ =0.2109, OSPPt-4=1.1647, Q_1 = 1.1228). Initial data: samples with L=L_1 and L=L_2.

3.4. Reliability of series of parallel systems with defects. MinMaxDM distribution family

Here we consider the models of reliability of parallel systems and their connection with MinMaxDM distribution family (MMDF) and apply the models into processing of test data of fiber strength.

The most significant contribution to the solution of the considered problem was made by Peirce [21] and Daniels [22, 23]. A review of later papers devoted to this problem can be found in [10-12]. A deep analysis of the Daniels' results was given in [7]. It was shown that this model yields accurate results only in specific cases. Here a version of Daniels' model and model based on using the theory of Markov chains are considered. A solution of the same problem was considered in [14] using the theory of continuous Markov process. We suppose that using the theory of Markov chains allows us to make much broader analysis of the problem. The size effect of fibrous material was modeled in [24] but there was no comparison of the models with the test data. We consider similar modeling using another version of models and provide numerical estimate of the quality of prediction of the test data.

3.4.1. Randomized Daniels's model

Here we consider the models of failure of a parallel system with redistribution of load after failure of some LIs and a connection of the c.d.f. of the strength of the link and the c.d.f. of the strength of a single LI. Statistical description of the development of the process of fracture of one link (as a loose bundle of LIs (fibers or strands) or as a parallel system without initial defects with redistribution of load after failure of some LI) was studied by Daniels [22-23]. The corresponding model can be described in a following way. Let $(X_1,...,X_n)$ be random strength of intact LIs in some link and X_j be the *j*-th order statistics. If there is a uniform distribution of load between intact LI and the applied load increases monotonically, then the ultimate strength of this link

$$X^* = \max_{1\le j\le n} X_j (n-j+1)/n . \tag{3.29}$$

We consider the case when $n = n_C - K_C$. In this case mean strength of initial n_C LIs

$$X^* = \max_{1\le j\le n} X_j (n-j+1)/n_C . \tag{3.30}$$

Daniels studied the case K_C =0. In the general case for random number of (technological) failures, K_C, there is a priori distribution $\pi_C = (\pi_{C1}, \pi_{C2}, ..., \pi_{C(n_C+1)})$ (here $\pi_{Ck} = P(K_C = k-1)$, $\pi_{C(n_C+1)} = 0$). Then

$$F_{X^*}(x) = \pi_C \vec{F}(x) , \tag{3.31}$$

where vector column $\vec{F}(x) = (F_1(x),...,F_{n_{C+1}}(x))'$, $F_k(x)$, $k = 1,...,n_C$, is c.d.f. of X^* if $n = n_C + 1 - k$, $F_{n_C+1}(x)$ is identical with unity (there are no intact LI).

Now we consider the specification of distribution of strength of link based on a randomized Daniels' model with Weibull distribution of the strength of a single LI. If the number n in an equation (3.29) is sufficiently large then for r.v. X^* there is a convergence in probability to a constant μ defined by equation

$$\mu = \max_x x(1 - F_X(x)) ,$$

where $F_X(x)$ is the c.d.f. of strength of a LI. We consider the case of Weibull distribution of the single LI strength (without defects). Then using logarithmic scale (in order to use the advantage of s.e.v. distribution with the location and scale parameters) we can write the equation for μ in the following form

$$\mu = \max_x \exp(x)\exp(-\exp((x-\theta_0)/\theta_1)) .$$

We have the following solution of this equation

$$\mu = \theta_1^{\theta_1} \exp(\theta_0 - \theta_1) .$$

Daniels has shown that for a sufficiently large *n*, the r.v. X^* in equation (3.29) has approximately normal distribution. For the considered case, when K_C =0, the parameters of this distribution are μ and $\sigma = \mu(\exp(\theta_1 - 1)/n_C)^{1/2}$. But if there are K_C damaged LIs (i.e. there are only ($n_C - K_C$) intact LIs) then we should use $\mu_n = \mu(n_C - K_C)/n_C$ and $\sigma_n = \sigma(n_C - K_C)/n_C$ (the denominator is equal to n_C instead of ($n_C - K_C$) because the specimen strength is calculated taking into account the initial number of LI, n_C).

So if $n = n_C - K_C$, where random variable K_C has a truncated binomial a priori distribution (note that we should eliminate the case of $K_C = n_C$) with parameters (n_C, p_C), is large enough then c.d.f. of X^* is approximately defined by the equation

$$F_{X^*}(x) = \sum_{n=1}^{n_C} F_{X_n^*}(x) b(n_C - n, p_C, n_C) / (1 - b(n_C, p_C, n_C)), \quad (3.32)$$

where $F_{X_n^*}(x) = \Phi((x - \mu_n)/\sigma_n)$; $\Phi(.)$ is the c.d.f. of the standard normal distribution, $b(k, p, m) = p^k (1-p)^{m-k} m!/k!(m-k)!$.

Let us call this model as randomized Daniels'model and denote it by MinMaxDM_RDM or just RDM. Let us denote the usual nonrandomized Daniels' model by NRDM.

3.4.2. Description of reliability of a parallel system using Markov chain theory

Let us recall that the process of monotonic tensile loading (i.e. the process of increase of the nominal stress (or mean load of one LI)) is described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$, and let $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, be the number of random failures of LIs under the load x_t in *i*-th link with n_C - the initial number of LIs. There is a failure of *i*-th link if $K_{Ci} = n_C$. We again consider the process of accumulation of failures as an inhomogeneous finite Markov chain (MC) with finite state space. We say that MC is in state i if $(i-1)$ of LIs have failed, $i = 1, ..., n_C + 1$. State $n_C + 1$ is an absorbing state corresponding to the fracture of the link. The process of MC state change and the corresponding process $K_{Ci}(t)$ is described by transition probabilities matrix *P* and at the *t*-th step of MC the matrix *P* is a function of *t*, *t*=1,2,...

The c.d.f. of strength of link, X^*, is defined on the sequence $\{x_1, x_2, ..., x_t, ...\}$ by equation

$$F_{X^*}(x_t) = \pi_C (\prod_{j=1}^{t} P(j)) u, \quad (3.33)$$

where $P(j)$ is the transition probability matrix for *t*=*j*, column vector $u = (0, ..., 0, 1)'$.

Four main versions (hypotheses) of the structure of matrix *P*, denoted as P_a, P_{an_C}, P_b and P_c are considered in Appendix 5.2.34. Matrix P_a corresponds to an assumption that in one step of MC only one LI can fail and it is the nearest to the already failed LI (or it is extreme side one), P_{an_C} corresponds to a failure of the weakest item in the cross section considered, P_b corresponds to a binomial distribution of failure number at every step of MC, P_c corresponds to the case when we know the stress concentration function (see details in Appendix 5.2.34)

3.4.3. Modeling of reliability of parallel system using Monte Carlo method

Let $E_1, E_2, ..., E_n$ be the elastic moduli, $f_1, f_2, ..., f_n$ the cross-sectional areas of n LIs and $\bar{\varepsilon}_i = \varepsilon_i / \varepsilon$, where ε_i is the strain in an i-th LI at mean strain ε in the cross section and the distribution of r.v. $\bar{\varepsilon}_i$ does not depend on ε. Then we have a random stress-strain function

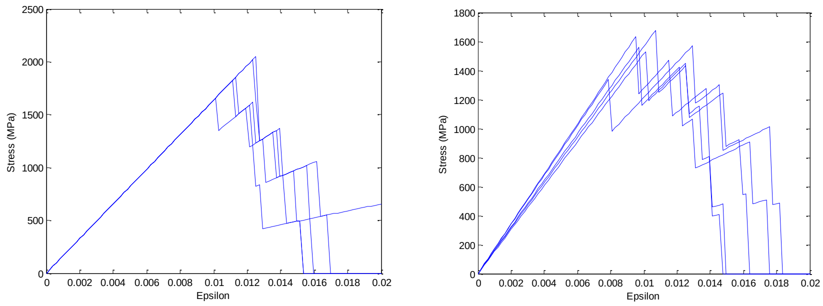
$$\sigma(\varepsilon) = \varepsilon \sum_{\varepsilon \bar{\varepsilon}_i E_i^p < X_i} E_i^p \bar{\varepsilon}_i f_i / \sum_{\varepsilon \bar{\varepsilon}_i E_i < X_i} f_i, \quad (3.34a)$$

where X_i is the strength of i-th LI, E_i^p is a r.v. which is equal to E_i with probability p_C and equal to zero with probability (1- p_C), $i = 1, ..., n_C$.

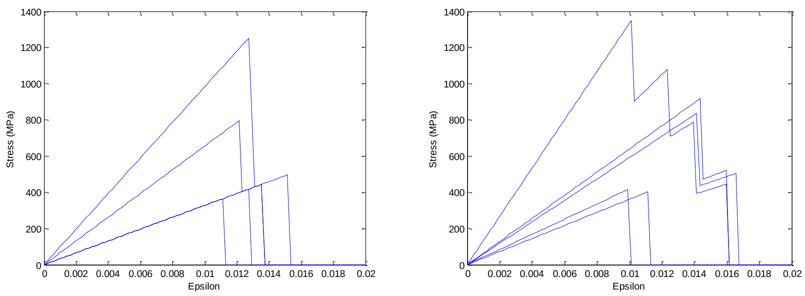
Random strength is defined by equation

$$\sigma = \max_{\varepsilon} \sigma(\varepsilon). \quad (3.34b)$$

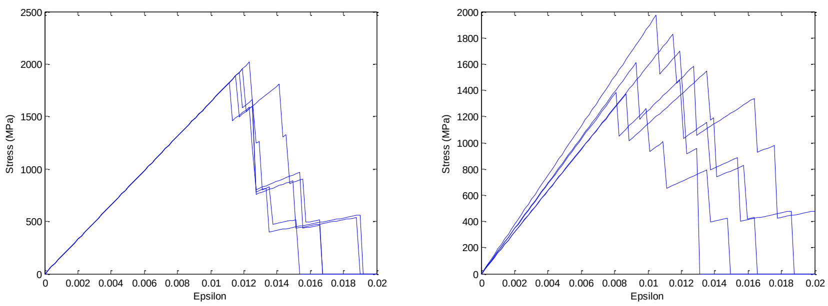
Using Monte Carlo method, these equations allow to obtain the distribution of CS strength easyly if we know the combined distribution of random variables E_i, $\bar{\varepsilon}_i$ and f_i, $i = 1,...,n_C$.



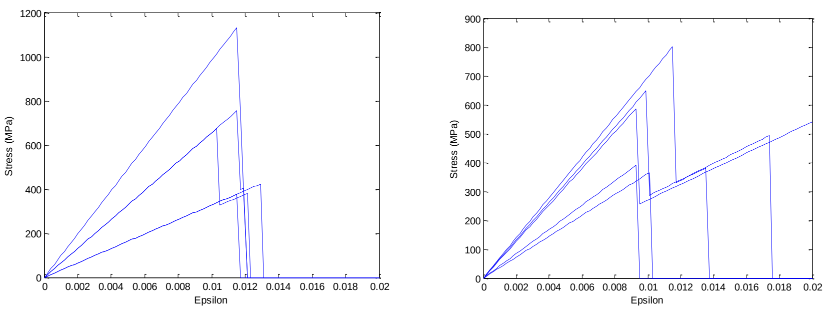
a) b)



c) d)



e) f)



g) h)

Fig. 3.5. Function $\sigma(\varepsilon)$ for different σ_{LE}, σ_{ε}, p_C :
a) (σ_{LE}, $\sigma_{L\varepsilon}$, p_C)=(0;0;0) ; b) (σ_{LE}, $\sigma_{L\varepsilon}$, p_C)=(0,2;0;0) ;
c) (σ_{LE}, $\sigma_{L\varepsilon}$, p_C)=(0;0;0,7) ; d) (σ_{LE}, $\sigma_{L\varepsilon}$, p_C)=(0,2;0;0,7) ;
e) (σ_{LE}, $\sigma_{L\varepsilon}$, p_C)=(0;0,1;0) ; f) (σ_{LE}, $\sigma_{L\varepsilon}$, p_C)=(0,2;0,1;0);

g) (σ_{LE} , $\sigma_{L\varepsilon}$, p_C)=(0;0,1;0,7) ; h) (σ_{LE} , $\sigma_{L\varepsilon}$, p_C)=(0,2;0,1;0,7).

In [25] there are examples of modeling of $\sigma(\varepsilon)$ for n_C =5 assuming normal distribution of $\log(X)$ and $\log(E)$ with correlation coefficient r, independent normal distribution of $\log(\bar{\varepsilon})$ with the expected value equal to zero and with $f_i = f = Const$, $i = 1,...,n_C$.

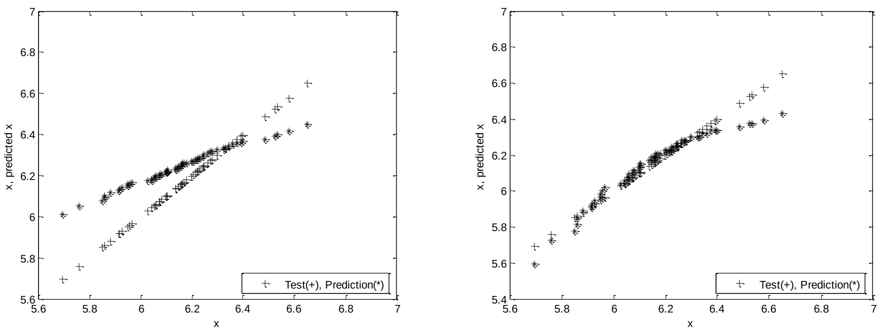
Figure 3.5 shows examples of the function $\sigma(\varepsilon)$ modeled on the assumption that $\log(X)$ and $\log(E)$ have a normal joint distribution with parameters $\mu_{LE}, \sigma_{LE}, \mu_{LX}, \sigma_{LX}, r$ and $\log(\bar{\varepsilon})$ has a normal distribution with a zero mathematical expectation and a nonzero standard deviation $\sigma_{L\varepsilon}$. To ensure the equilibrium of the cross section ($\sum_{i=1}^{n_C} \bar{\varepsilon}_i = n_C$) an additional condition was used:

$$\sum_{i=1}^{n_C} 1(\varepsilon\bar{\varepsilon}_i E_i < X_i) = \sum_{i=1}^{n_C} 1(\varepsilon E_i < X_i) .$$

In these examples, n_C =5 , $f_i = f$, $i = 1,...,n_C$, $\mu_{LE}, \mu_{LX}, \sigma_{LX}, r$ =(12.01; 7.69; 0.133; 0.255) and two values of each of the parameters $\sigma_{LE}, \sigma_{L\varepsilon}$ and p_C are considered. If all these three parameters are equal to zero then all realizations of the random function $\sigma(\varepsilon)$ in Fig. 3.5a coincide up to the moment of failure of the weakest LI. In Fig. 3.5b, for ($\sigma_{LE}, \sigma_{L\varepsilon}, p_C$)= (0.2, 0,0), they do not coincide because of the different elastic moduli. The following two figures (for 3.5c ($\sigma_{LE}, \sigma_{L\varepsilon}, p_C$)=(0;0;0.7) ; for 3.5d ($\sigma_{LE}, \sigma_{L\varepsilon}, p_C$)=(0,2;0;0,7)) differ from the two preceding ones by a random number of defect-free LIs and by the corresponding number of peaks on realizations of the function $\sigma(\varepsilon)$. The following four figures differ from the four previous ones in the scatter of strain: $\sigma_{L\varepsilon}$ = 0.1.

3.4.4. Numerical example

In [15], the results obtained in testing 64 bundles of carbon fibers separated from a monolayer and the same number of strips consisting of 10 similar 20-mm long bundles, are reported. The following estimates of distribution parameters of the left-hand extremum for the (natural) logarithm of strength, $\log(X)$, of separate bundles are obtained: $(\hat{\theta}_{0LX}, \hat{\theta}_{1LX})$ = (6.55, 0.132). For the corresponding estimates of the mean, $E(\log(X))$, and the standard deviation (s.d.), $\sigma(\log(X))$, we have $(\hat{\mu}_{LX}, \hat{\sigma}_{LX})$ = (6.4769, 0.1732). For the monolayer of 10 fiber bundles, $(\hat{\mu}_{LX}, \hat{\sigma}_{LX})$ = (6.156, 0.194). Figure 3.6 presents the results of strength tests on a monolayer and the results of prediction according to the nonrandomized Daniels model with data on



a) b)

Fig. 3.6. Comparison between the results of strength tests of a monolayer made of 10 fiber bundles (+) and the prediction (*) based on data on the strength of separate bundles for a nonrandomized (a) and randomized (b) Daniels models.

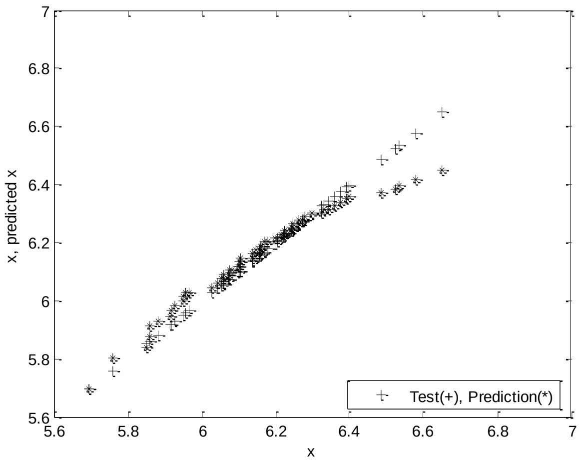
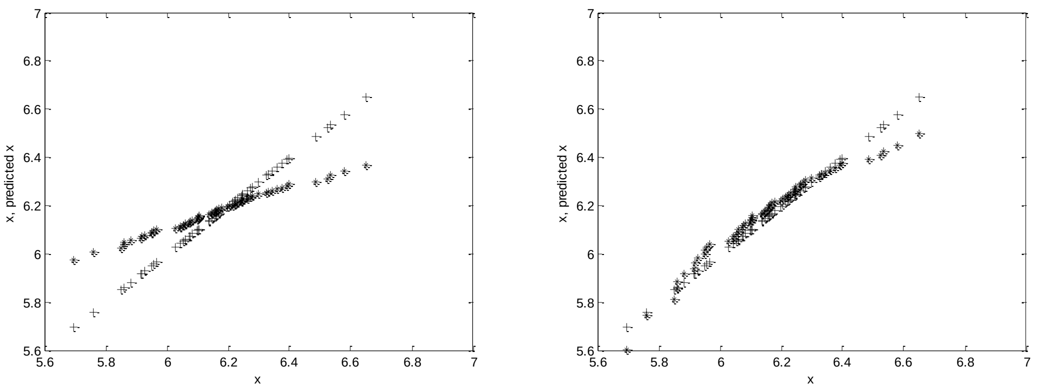


Fig. 3.7. Comparison between the data of strength tests on a monolayer of 10 fiber bundles (+) and the results of prediction (*) according to the model based on the theory of Markov chains with a matrix of type Pb.

The strength of separate fiber bundles. Here, x is the natural logarithm of test order statistics for the sample of results of strength tests on a monolayer, and $\hat{x}$ is the prediction (estimate) of the respective mathematical expectations made on the basis of the nonrandomized Daniels model on the assumption that the strength of one bundle has the Weibull distribution. The distribution of elastic modulus was neglected. The predicted estimates for the strip strength were $(\hat{\mu}_{LX}, \hat{\sigma}_{LX})$ = (6.245, 0.095), while the experimental results, as mentioned before, were (6.156, 0.194). In this case, the measure of prediction error according to the order statistics was OSPPt = 0.695 (the definition of OSPPt is given in Appendix 5.2.3). A comparison shows that the prediction error is relatively small for the average strength and rather significant for the Standard deviation (s.d.). The prediction can be improved (see Fig. 3.6.b) if the randomized Daniels model is used (see Eq. (3.32)): at p_C = 0.2, we have $(\hat{\mu}_{LX}, \hat{\sigma}_{LX})$ = (6.152, 0.173). Then, OSPPt = 0.3.

By using the model based on the theory of Markov chains with a matrix of the type Pb (Fig.3.7), we succeeded in decreasing the value of OSPPt to 0.27 and obtained that $(\hat{\mu}_{LX}, \hat{\sigma}_{LX})$ = (6.16, 0.157).

An average strength (but not s.d.!) can be also easily predicted by using the two-dimensional lognormal distribution of strength and elastic modulus with the following parameters of a separate fiber bundle: ($\mu_{LE}, \sigma_{LE}, \mu_{LX}, \sigma_{LX}, r$)= (11.9,0.332, 6.4769, 0.1732, 0.255). For estimates of the mean and s.d. of strip strength, we have $(\hat{\mu}_{LX}, \hat{\sigma}_{LX})$ = (6.17, 0.0844) in the logarithmic scale. As seen, the s.d. is underestimated. The corresponding value of OSPPt is rather great: 0.58 (Fig. 3.8.a). The prediction can be improved by selecting appropriate structural parameters. However, in such a case (as in all cases of using structural



a) b)

Fig. 3.8. Experimental values (+) and prediction (*) of the strength of a 10-bundle monolayer by using the NRDM (a) and RDM (b) with a two-dimensional normal distribution of log(X) and log(E) (the parameters are given in the text).

parameters for fitting experimental data), this is not a prediction, but a "conditional fitting," i.e., "fitting" with the help of structural parameters at fixed strength distribution parameters equal to those of individual fibers. An increase in the content of processing defects p_C markedly reduces the predicted average strength, but slowly raises the value of s.d. The effect of chage of $\sigma_{L\varepsilon}$ is similar. We should note that the quantity n_C can be considered not as an actual number of LEs in a cross section, but as an additional parameter — the critical size of a cluster, whose failure leads to the failure of the entire specimen. Its estimate $\hat{n}_C$ can be determined upon processing experimental results. A decrease in $\hat{n}_C$ markedly increases the average strength and s.d., but only within some limits. By simultaneously decreasing $\hat{n}_C$ to 4, increasing $\sigma_{L\varepsilon}$ to 0.1, and increasing $\hat{p}_C$ to 0.05, we managed to reduce OSPPt to 0.25 and to raise the estimate of $\hat{\sigma}_{LX}$ to 0.174 at its experimental equal to 0.194 (see Fig. 3.8.b).

Taking into account the results obtained in processing experimental data we se that the use of the Daniels model for describing the distribution of strength of a cross section (link) consisting of parallel LIs makes it possible to predict, with a sufficient accuracy, an average value of the strength of a monolayer consisting of 10 fiber bundles by employing data of the strength of one bundle of the same length. However, in this case, the s.d. of the strength is considerably underestimated, which can lead to a dangerous overestimation of the warranted strength (p-quantiles of distribution at low *p*). Recall that similar results were obtained in [7]. This also leads to a heightened value of the criterion for estimating the prediction error OSPPt of the order statistics of strength. Its magnitude can be reduced from 0.695 to 0.3 by using the randomized model with p_C = 0.2, with an independent distribution of the strength and elastic modulus. A further decrease in OSPPt down to 0.27 can be achieved by employing the model based on the Markov chains theory with a matrix of type P_b. When the joint normal distribution of the logarithms of strength and elastic modulus is utilized, the prediction becomes worse, and OSPPt reaches the value 0.6. This value can be lowered to 0.26 if we assume that a priori probability of a defect is 0.05 and additionally accept that the cross section fails just after the break of a critical cluster consisting of n_C = 4 LIs.

Based on the results of analysis of the foregoing examples of processing experimental data, we obviously can conclude that the models examined deserve a more extended testing.

3.5. Reliability of series of parallel systems with defects

3.5.1. MinMaxDM distribution family

For the case n_C =1, considered in section 3.1, the types of c.d.f. $F_Y(x)$ and $F_Z(x)$ should be chosen "a priori". But clearly, all the ideas considered in section 3.1 can be used also for the series system in which the links are parallel systems with n_C >1. C.d.f. $F_Y(x)$ and $F_Z(x)$ define now c.d.f. of strength of parallel systems of Y-type or Z-type correspondingly with n_C >1. In accordance with (3.2) the strength of specimen can be described by equation

$$X = \min_{1\le i\le n_L} X_i^* = \min_{1\le i\le n_L} \max_t (x_t : n_C - K_{Ci}(t) \ge 0),$$

where $K_{Ci}(t)$ is $K_C(t)$ for i-th link.

For building the c.d.f. of X in the numerical examples we suppose that the logarithm of strength of one LI (in one link) without defects has a s.e.v. distribution. Of course it is not the only possible assumption. Different assumptions about the distribution of strength of one link, a priori distribution of initial (technological) defects compose a family of the distributions of ultimate composite tensile strength. Taking into account (3.2) the corresponding family of distributions of X was denoted by abbreviation MinMaxD (in honour of Daniels). If for calculation of c.d.f. the MC theory is used then it is appropriate to use the abbreviation MinMaxM. The abbreviation MinMaxDM is appropriate for the unified family. It should be mentioned that, within the framework of assumption that the strength of defective LIs is equal to zero, it may be assumed that the c.d.f.s $F_Y(x)$ and $F_Z(x)$ differ only in the a priori distribution of the number of defective LIs in a cross section. The models of MinMaxDM family distribution contain two groups of parameters. The first group includes the strength and rigidity parameters of individual LIs (fibers, fiber bundles, etc.): for example, parameters θ_0 and θ_1 in Weibull c.d.f. of strength individual LI, $F(x) = 1 - \exp(-\exp((\log(x) - \theta_0)/\theta_1))$. The second group contains the structural parameters: $n_C, p_C, l_1, p_L, \sigma_{L\varepsilon}$, where $l_1 = L/n_L$, L is the specimen length, and n_L is the number of links. The structural parameters give us a numerical estimate of the quality of the technology used to produce the tested specimens.

3.5.2. Processing of test data using MinMaxD_RDM model

In [8] carbon fiber test data are presented (for every specimen) for $(L_1,...,L_4)$ =(1,10, 20, 50 mm) (Data_B1) and in [7] the mean values, μ_X, and standard deviations, σ_X, for dry bundle (of the same fibers) tests with $(L_1,...,L_4)$ =(5, 20, 100, 200 mm) are given (Data_B2). Just as in [7] we perform fitting and parameter estimation using Data_B1 for L=20 mm and attempt to predict the strength of bundles (Data_B2) for different lengths (note that the processing of Data_A we performed only for one L). Of course, we cannot consider all the versions of models in the framework of MinMaxDM distribution family. In Table 3.5 there are results of using randomized Daniels' model (MinMaxD_RDM) in comparison with similar processing of the same data using the model provided in [7].

Table 3.5. Summary of prediction of Data_B2 using parameter estimates obtained by processing Data_B1

L (mm)		5	20	100	200	Parameter estimates					
Number of observations		28	25	29	27	$\hat{\theta}_0$	$\hat{\theta}_1$	n_C	p_C	l_1	p_L
Mean (GPa)	Observed	1.92	1.68	1.58	1.38	7.883	0.182	1000	0	20	0
	NRDM[13]	2.19	1.71	1.28	1.14	7.883	0.182	1000	0	20	0
	MinMaxD_RDM	1.61	1.57	1.48	1.4	7.883	0.182	1000	0.2	20	0.1
Std (GPa)	Observed	0.07	0.1	0.13	0.11	7.883	0.182	1000	0	20	0
	NRDM[16]	0.031	0.024	0.018	0.016	7.883	0.182	1000	0	20	0
	MinMaxD_RDM	0.057	0.114	0.161	0.156	7.883	0.182	1000	0.2	20	0.1

In [7] a good agreement of NRDM prediction of mean strength of dry bundle with the same length, 20 mm, is observed, but this agreement does not extend to other values of L and there is a significant mismatch of standard deviation for all four lengths.

Using p.s. B3, when

$F(x) = (1 - p_0)F_Y(x) + p_0 F_Z(x)$,

where $p_0 = (1 - p_L)^{n_L}$,

and corresponding to RDM equation (3.32) with $F_Y(x) = F_{X^*}(.)$ with a specific p_C and $F_Z(x) = \Phi((x-\mu)/\sigma)$ corresponding to p_C =0 we obtained results shown in Table 3.5.

For the considered model we see much better estimations of σ_S and not too bad estimation of μ_S at least for L≥20 mm if we use strength parameters of single fibers and specific structural parameters mentioned above (see Table 3.5).

3.6. Conclusions

The analysis of Tables 3.3 and 3.5 shows that the considered models as part of MinMaxDM distribution family provide good fitting of the results of tensile strength tests and can explain some specific features of the strength of LI in the framework of a more complex structure. Good fitting is not surprising, of course, if the considered models have a large number of parameters which can be used for fitting of a test data. But some versions of this family models provide good fitting and at the two estimated parameters (see Table 3.3, flax fibers). Unlike other models, the parameters of the models in the framework of the MinMaxDM distribution family allow a natural interpretation and give us an additional numerical information about both the specific local parameters of fibers (strands) and the structure of composite: for example, parameter λ_1, parameters p_C and p_L provide us with estimates of defect intensity; model RDM allows to explane the increase of standard deviation in comparison with the theoretical value, corresponding to usual nonrandomized Daniels' model.

But as we see in Table 3.4 and 3.5, for prediction of size effect of specimen length the best results are obtained using models with minimum number of unknown parameters: models MB3, B3F and LW have only two unknown parameters.

It seems that MinMaxDM distribution family opens a broad field for study of the problem considered. We can study different versions of the strength distribution of a single fiber, different versions of a priori distribution of a defect number, different versions of matrix of transition probability for MC-models etc. The results obtained now should be regarded only as preliminary, but it seems that the MinMaxDM distribution family deserves to be studied much more thoroughly using more test data.

3.7. References

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4. Markov Model of Connection between the Distribution of Static Strength and Fatigue Life of Composite

In this section within the framework of a unified mathematical model based on the Markov chain (MC) theory, an attempt is made to describe the connection between the distribution of static strength and the fatigue curve, and the accumulation of fatigue damages in programming loading of composite. We consider different possible structures of transition probabilities matrix of MC. In the framework of the most complex structure it is assumed that the fatigue failure of a test specimen occurs after the destruction of its critical microvolume (MCV) consisting of two parts: elastic (brittle fibers) and plastic (matrix). In the second part, permanent strains accumulate as soon as the cyclic local stress exceeds some level. We take into account also the length of the fatigue test specimen. Here the theory of reliability of series system with partly damaged links is used. Numerical examples are presented.

4.1. Introduction

The distribution of a static strength, a fatigue curve and an accumulation of fatigue damages under a program loading are often described by poorly interconnected theories and hypotheses. The distribution of static strength is usually analyzed by the Weibull or lognormal distributions, while the fatigue curve is described by formal regression dependences. For example, in [1, p.139], the fatigue curve is presented by the equation of nonlinear regression

$$\lg N = 39{,}87 - 0{,}389S + 5{,}185^{-8} S^4,$$

where S is the stress amplitude and N is the corresponding average number of cycles.

In Table 4.1, taken from [2, p.299], the seven models for the fatigue curve quantile estimate are given.

Table 4.1. Models of fatigue curves [2]

Model	Quantile	Number of parameter
Little and Ekvall (1981)	$\log(\hat{y}_{ij}) = \hat{A} + \hat{B}x_j + z_{p_i}\hat{\sigma}$	3
Little and Ekvall (1981)	$\log(\hat{y}_{ij}) = \hat{A} + \hat{B}\log(x_j) + z_{p_i}\hat{\sigma}$	3
Spindel and Haibach (1981)	$\log(\hat{y}_{ij}) = \log(\hat{Y}_0) + \hat{A}\log\left(\frac{x_j}{\hat{X}_0}\right) + {} + B\left\{\log\left(\frac{x_j}{\hat{X}_0}\right) + \frac{1}{\hat{\alpha}}\log\left[1 + \left(\frac{x_j}{\hat{X}_0}\right)^{-2\hat{\alpha}}\right]\right\} + z_{p_i}\hat{\sigma}$	6
Bastenaire (1972)	$\log(\hat{y}_{ij}) = \hat{Y}_0 + \hat{A}\frac{\exp[-\hat{C}(x_j - \hat{X}_0)]}{x_j - \hat{X}_0} + z_{p_i}\hat{\sigma}$	5
Castillo at al. (1985)	$\log(\hat{y}_{ij}) = \log(\hat{Y}_0) + \frac{\hat{A}}{\log(x_j) - \log(\hat{X}_0)} + z_{p_i}\hat{\sigma}$	4

Castillo and Hadi (1995)	$\log(\hat{y}_{ij}) = \dfrac{p_i^{\hat{\alpha}} - \hat{D} - \hat{C}x_j}{\hat{A}x_j + \hat{B}}$	5
RFL model (1999)	$\log(\hat{y}_{ij}) = F_W^{-1}(p_i; \log(x_j), \hat{\theta})$	5

Here the following notations are used: x_j – is the j-th stress level, y_{ij} – is the estimate of p_i quantile at the jth stress level; W=log(N), $F_W(w;\ x,\ \theta)$ – c.d.f. of random variable W, corresponding to Random-Fatigue-Limit (RFL) model, offered in [2]:

$$\log(N)=\beta_0 + \beta_1 \log(S-\gamma) + \varepsilon,$$

where ε is an error term, V=log(γ) has normal or smallest extreme value probability distribution function (p.d.f.).

If we try to use these formulae for composite material we will see that parameters of these formulae have no connections with the parameters of static strength distribution of composite material component.

In other publications the equation, suggested by Weibull, is used frequently:

$$S - S_{-1} = C(N + B)^{-\alpha},$$

where S_{-1}, C, B, and α are some parameters. These parameters, as in the previous similar equations, are not connected with the parameters of distribution of static strength.

The summation of fatigue damages under a program loading, as a rule, is carried out by using the Palmgren-Miner hypothesis and its modifications. For example, in [3], containing a brief review of 50 papers, for describing the condition of failure in two-stage fatigue loading of woven carbon fibre reinforced laminates, the equation

$$(n_1 / N_1)^{\alpha_1} + (n_2 / N_2)^{\alpha_2} = 1$$

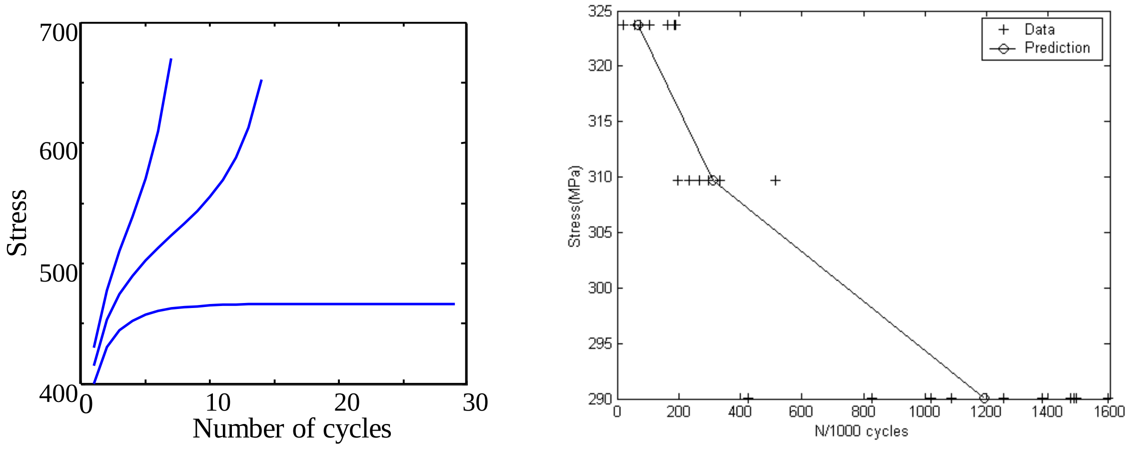
is suggested, where n_1 and n_2 are the numbers of loading cycles; N_1 and N_2 are average lives at the first and second loading levels; α_1 and α_2 are constants.

In the present study an attempt is made to unite the solution of the three above-mentioned problems within the framework of a unified mathematical model. This model is a development of the models described in [4-7], where the connection between the distribution of the static strength of fibers and the fatigue life of a unidirectional fibrous composite is considered neglecting the plastic properties of the matrix. The models presented in [5-7] give a rather plausible description of p-quantile fatigue curves (corresponding to the probability of failure p). However, upon processing the data of program tests, results that only slightly differ from those found with the help of the Palmgren-Miner linear hypothesis are obtained, although even in tests with a single change in the loading mode, the deviations from the Palmgren-Miner hypothesis are rather noticeable.

In [8], a relationship between the distribution functions of fiber strength and strength of an aggregate of parallel fibers at a uniform distribution of load between them was determined. "Developing" this model in time, we come to a sequence of local stresses $\{s_0, s_1, s_2, ...\}$ (in the cross section where the failure proceeds):

$$s_{i+1} = S/(1 - F(s_i)), \quad i = 0, 1, 2, ...,$$

where $s_0 = S$ is the initial rated stress in the undamaged specimen. An example of such sequences for different S is shown in Fig.4.1a.



a b

Fig.4.l. Growth of local stresses according to the quasi-stochastic model (a)
and the corresponding fatigue curve (b).

The behavior of the respective curves is very similar to that of the *S*-shaped curves describing changes in some physical parameters of a composite in fatigue tests (see, for example, Fig.2.30 in [l]).

Let us assume that

$$S/(1-F(s))-s=ds/dN$$

or

$$1/(S/(1-F(s))-s)=dN/ds$$

and expand the function ($S/(1-F(s))-s$) into the Taylor series at the point where its first derivative is equal to zero, restricting our calculations only to second-order terms. After integration, we arrive at the following approximation for the fatigue curve:

$$S=C_1-C_2\tan(N(S)/C_3-\pi/2).$$

An example of a rather satisfactory description of test data for CFRP specimens [11] is presented in Fig.4.1b. However, this result may be explained by the facts that only three stress levels are considered and equation derived contains just three constants. Two serious drawbacks of the model are obvious. First, Fig.4.1 shows that the number of cycles before failure grows rapidly and tends to infinity with fairly small changes in the amplitude of the cyclic stress, thus this model, giving a correct qualitative description of the process of accumulation of fatigue damages, can be used practically for approximating the fatigue curve only in a rather narrow range of changes in the stress amplitude. Second, the equation derived determines only an average strength. However, it is important to know also the distribution function of fatigue life. To overcome these drawbacks, it seems necessary to consider the fact that the failure of a specimen takes some time, even if its average static strength is lower than the maximum of the cyclic stress applied and that the number of components failing during one cycle is a random variable. The above-mentioned drawbacks, to some extent, may be overcome by using the theory of Markov chains (MC) for describing the process of accumulation of fatigue failure. Although this idea is not new (see [9]), the formulation of the relation between the parameters of distribution of static strength and fatigue life suggested in this study and described here is really new. Below numerical examples of approximation of the fatigue curve, prediction of the residual strength and the residual fatigue life found in tests with a single change in the loading mode are given. Finally possible directions of further investigations are outlined.

4.2. Unidirectional composite fatigue model based on the Markov chains theory

In this section following Daniels [8] we consider a model of unidirectional composite, which is called now the "classical model of bundle of *n* parallel fibers stretched between two clamps". Strands or some set of strands can be considered in general case instead of fibers. For all structural items of these types we use the terms "component" or "item". Destruction of the specimens under cycling fatigue loading is destruction of these components. It is a random process. It can be described as a Markov Chain (MC) with the following matrix of transition probabilities

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{13} & \cdots & p_{1(n_C+1)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n_C+1)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n_C+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p_{n_C(n_C+1)} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

corresponding to a MC with one (r+1)-th absorbing state and *r* nonrecurrent states.

The vector of cumulative distribution functions of number of steps to absorption, *T*, components of which correspond to different initial states of MC, $\{F_T(t)\} = (F_T^{(1)}(t), F_T^{(2)}(t), ..., F_T^{(r+1)}(t))'$, is defined in the following way.

$$\{F_T(t)\} = P^t b,\ t = 1,2,3,..., \tag{4.1}$$

where $b = (00...01)'$ is the column vector.

If the probability distribution on initial states of MC, $\pi = (\pi_1, \pi_2, ..., \pi_r, \pi_{(r+1)})$, is known, then for c.d.f. of *T* we have

$$F_T(t) = \pi P^t b = \pi\ (F_T^{(1)}(t), F_T^{(2)}(t), ..., F_T^{(r+1)}(t))'.$$

So if initial state is the first state ($r = 1$) then c.d.f. and probability mass function (p.m.f.) of *T* are defined by the formulae

$$F_T(t) = aP^t b,\ p_T(t) = F_T(t) - F_T(t-1)\ , \tag{4.2}$$

where $a = (100...0)$ is the row vector.

We consider the number of steps to absorption, *T*, as a life time or, more precisely, fatigue life of specimen measured in number of steps of MC, but we assume, that in general case one step in MC corresponds to k_M cycles in fatigue test and in processing of experimental data we will use this value as a scale factor. It is considered as some parameter of the model.

4.2.1. One-step Markov model

It is useful, of course, to consider the first of all the simplest model (*One-Step-Markov-Model* (OSMM) [9]), for which only transitions to the nearest 'senior' states are allowed:

$$P = \begin{bmatrix} q_1 & p_1 & 0 & & & \cdots & 0 \\ 0 & q_2 & p_2 & 0 & & \cdots & 0 \\ 0 & 0 & q_3 & p_3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & & \cdots & 0 & q_r & p_r \\ 0 & & & \cdots & 0 & 0 & 1 \end{bmatrix}, \; q_i = 1 - p_i \,.$$

Here r is the number of parallel longitudional items, i is equal to the number of failured longitudional items plus one, $i = 1,...,r+1$. The main characteristics of this type of MC are well known. Time to failure (time to absorption) $T=X_1+X_2+...+X_r$, where X_i (time the process spends in i-th state), are independent random variables with p.m.f. of geometric distribution

$$P(X_i = n) = (1-p_i)^{n-1} p_i \,, \; i = 1,2,...$$

Expectation value and variance are equal to

$$E(X_i) = 1/p_i \text{ and } V(X_i) = (1-p_i)/p_i^2 \,. \tag{4.3}$$

Probability generating function for random variable T (which can be used to obtain p.m.f. of T)

$$G_T(z) = \sum_{i=1}^{\infty} p_T(i) z^i = \prod_{i=1}^{r} \frac{zp_i}{1-z(1-p_i)} \,.$$

All these formulae are well known. A new step which we offer to do is the connection of probabilities p_i, i=1,...,r, with parameter of composite material component strength distribution and parameters of cycles of fatigue loading. Then we can get the fatigue curve equation.

Recall our assumption that in one step of Markov process (1 cycle or may be 1000 cycles) only one parallel structural item (for example strand) can fail. Then if there are (R-i) still intact parallel structural items (MC is in $(i+1)$-th state because in the first state the number of failed items is equal to 0) and the same c.d.f. $F(s)$ for every item, then the probability of failure of at least one additional item, p_{i+1}, is defined by the formula

$$p_{i+1} = 1 - (1 - F_i(s_i))^{R-i} \,,$$

where $F_i(.)$ is the conditional distribution function of the ultimate strength of yet intact items after i ("the weakest"!) items are failed (in the numerical example to decrease the time of calculation we make assumption that this function does not depend on i). R is an initial number of items, i is the number of items, which have already failed, s_i is the corresponding stress applied uniformly to all $(R$-$i)$ intact items.

We suppose also that

$$s_i = \frac{SR - S_f i}{R-i} = \frac{S(1 - S_f\, i/SR))}{1 - i/R} \,,$$

where S is an initial stress in every item (at the start of the test), S_f is a stress which already failed item is still able to carry (as at least at the beginning of damage accumulation the rupture of fibers can be found in different cross sections).

Let us consider the case when c.d.f. $F(s)$ has location and scale parameters:

$F(s)=F_0((g(s)-\theta_0)/\theta_1)$,

where g(.) is some known function, $F_0(.)$ is some known c.d.f. For example, below we will consider lognormal c.d.f. $F(s)$ and use normal c.d.f. instead of $F_0(.)$ and $\log(s)$ instead of $g(s)$. Now the considered model has 6 parameters: θ_0, θ_1, r, R, k_M, S_f. They have the following "physical" interpretation: θ_0, θ_1 are the parameters of c.d.f. of strength of composite item (strand or fiber); for example, if $g(s)=s$ (normal distribution of strength) then θ_0 is an expectation value and θ_1 is a standard deviation of item strength; R is the number of composite items in the critical volume, failure of which corresponds to the total failure of specimen; r is a critical number of failed elements inside this critical volume, corresponding to failure of this volume; the ratio r/R defines a part of the cross section of critical volume, the destruction of which is consideres to be a failure of specimen; the value r defines mainly the variance and coefficient of variation of fatigue life; k_M is the number of cycles corresponding to one step in MC; S_f is residual strength of failed item (it depends on the orientation and number of layers, the characteristics of matrix,...).

So from now on we will use $F_T(t;S,\eta)$ as the specific notation of c.d.f. of a random variable T instead of more general notation, $F_T(t)$.

4.2.2. Binomial Markov model

For the model, which we call as Binomial Markov Model (BMM) we consider a possibility of MC "jumps" from state i-th to any other j-th state, $j=i+1,i+2,...,r+1$. It is natural to use here the binomial distribution

$$p(i,j)=\binom{r+1-i}{k}p_{i,1}^{k}(1-p_{i,1})^{r+1-k},\quad k=j-i,$$

where $p_{i,1}=F_i(s_i)$, $F_i(.)$, s_i are the same as in the previous section.

The calculation of average and variance of time to failure can be calculated of course by the use of equation (4.2). But another approach can be used also. Again, let the structure of the matrix of transition probabilities be described in the way

$$P=\begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}, \tag{4.4}$$

Then, as it is known [10], the vector of average and variance of step number to absorption from different transient states is defined by formulae

$$\tau=N\cdot\xi,\ \tau_2=(2N-I)\tau-\tau_{sq} \tag{4.5}$$

where $N=(I-Q)^{-1}$, ξ is a column vector of units, $\tau_{sq}(i)=(\tau(i))^2, i\in I_T$, I_T is a set of transient states.

Note, that this version of the model has only 4 parameters (including k_M).

4.3. Estimation of the model parameters of OSMM

Formulae (4.2), (4.3) can be used in both directions: for calculation of mean and p-quantile fatigue curves, if parameters are known, or for nonlinear regression analysis for model parameters estimation, if fatigue life dataset is known. Mean and p-quantile fatigue curves are defined by formulae

$$E(T(S_j)) = \sum_{i=1}^{r} 1/\,p_i(S_j,\eta) \quad,\ t_p(S_j) = F_T^{-1}(p;S_j,\eta)\,. \tag{4.6}$$

where $E(T(S_j))$, $t_p(S_j)$ are mean value and p-quantile of fatigue life for stress S_j.

The parameters of the model can be estimated by the use of Maximum Likelihood Method (MLM), which is more preferable. For the profound investigation of this model nonlinear regression procedure of SAS system can be recommended. But in any case it is a very difficult task to find 6 unknown parameters. So we limit ourselves to only approximate solution of this problem. At first we consider OSMM. To decrease the number of estimated parameters we put: $k_M = 1$, $S_f = 0$.

An approximate estimate of parameter r can be found, if we assume, that approximately

$$p_i = 1-(1-F(s_i))^{R-i} \cong p \cong cF_0((g(S)-\theta_0)/\theta_1) \text{ for all i=1,2,…,r,} \tag{4.7}$$

where c is some constant.

Then for the expectation value, variance and coefficient of variation we have

$$E(T) \cong \frac{r}{p};\ V(T) \cong \frac{r}{p^2};\ C_V = \sqrt{V(T)}\,/\,E(T) \cong 1/\sqrt{r}\,.$$

And approximate estimate of parameter r is defined by formula

$$\hat{r} \cong]1/(\hat{C}_V)^2[\,+1,$$

where $\hat{C}_V$ is an estimate of coefficient of variation C_V;] x [is the nearest integer towards minus infinity.

Value $E(T) \cong \frac{r}{p}$ is very large (10^5-10^7!!!), while r is small enough, so the value of p is very small and $F(s)$ is very small too. All the above mentioned gives us the idea to make rather serious assumption defined by formula (4.7). Not too bad final result is the only justification of it! Let us denote $D_f = r/c$. Then we have an approximate formula

$$E(T(S)) = \frac{D_f}{F_0((g(S)-\theta_0)/\theta_1)}\,. \tag{4.8}$$

At the fixed D_f we can get the following linear regression model

$$y_i = F_0^{-1}(D_f / E(T(S_i))) = -\theta_0/\theta_1 + (1/\theta_1)g(S_i) = \beta_0 + \beta_1 x_i,\ x_i = g(S_i),\ i = 1,2,...,n\,.$$

Parameters β_0 and β_1 of this model can be estimated by the use of some statistical program of linear regression analysis at every fixed value of parameter D_f. And it is not too serious problem to find only one nonlinear parameter D_f. Then $\hat{\theta}_1 = 1/\hat{\beta}_1$, $\hat{\theta}_0 = -\hat{\beta}_0/\hat{\beta}_1$ are estimates for θ_1 and θ_0.

Estimate of parameter R can be obtained after estimation of ratio $\rho = r/R$. Recall that this ratio defines the part of the cross section area of critical volume, the destruction of which we consider as total failure of specimens. In the Daniels's model of static strength [8] this value corresponds to the value of $F(x^*)$, where $x*$ is such, that $x^*(1-F(x^*)) = \max\limits_x x(1-F(x))$. We can estimate $x*$, using estimates of θ_0 and θ_1. So we have

$\hat{\rho} = F(x^*)$, $\hat{R} =]1/((C_V)^2 \hat{\rho})[+1$.

Now we have approximate estimates of all four parameters θ_0 and θ_1, *r* and *R*. At the fixed estimates of *r* and *R* more precise estimates of θ_0 and θ_1 can be found by the use of MLM and formula (7.2). But for calculation of c.d.f. we should first calculate P^t. But this is very expensive. It is offered to use some approximation of $F_T(.)$. It appears, that lognormal approximation is good enough for our purpose [6]

$$F_T(t; S, \eta) \cong \Phi\left(\frac{\log(t) - \theta_{0LT}}{\theta_{1LT}}\right),$$

where θ_{0LT} , θ_{1LT} are such, that we have the same (as defined by formula (4.3)) expectation value and standard deviation

$$\theta_{0LT} = \log(E(T)) - (\log(C_V^2 + 1))/2, \ \theta_{1LT} = (\log(C_V^2 + 1))^{1/2}.$$

The time of fatigue test is usually limited and live observations are censored, so maximum likelihood function in logarithm scale is defined in the following way

$$l(\eta) = \ln(L(\eta)),$$

where $L(\eta) = \prod_{i=1}^{n} f_i^{A_i} (1 - F_i)^{1-A_i}$, f_i, F_i are probability density function and cumulative distribution function of a random variable T (for fixed η and S) ; A_i is equal to 1, if fatigue test is finished by the failure of specimens; A_i is equal to 0, if the time of test is limited (right censored observation).

For the BMM the same approach can be used, because in fact the probability $p(i, i+1)$ has the largest value in every i-th row of matrix *P*. So for approximate parameter estimation all the relevant previous formulae can be used.

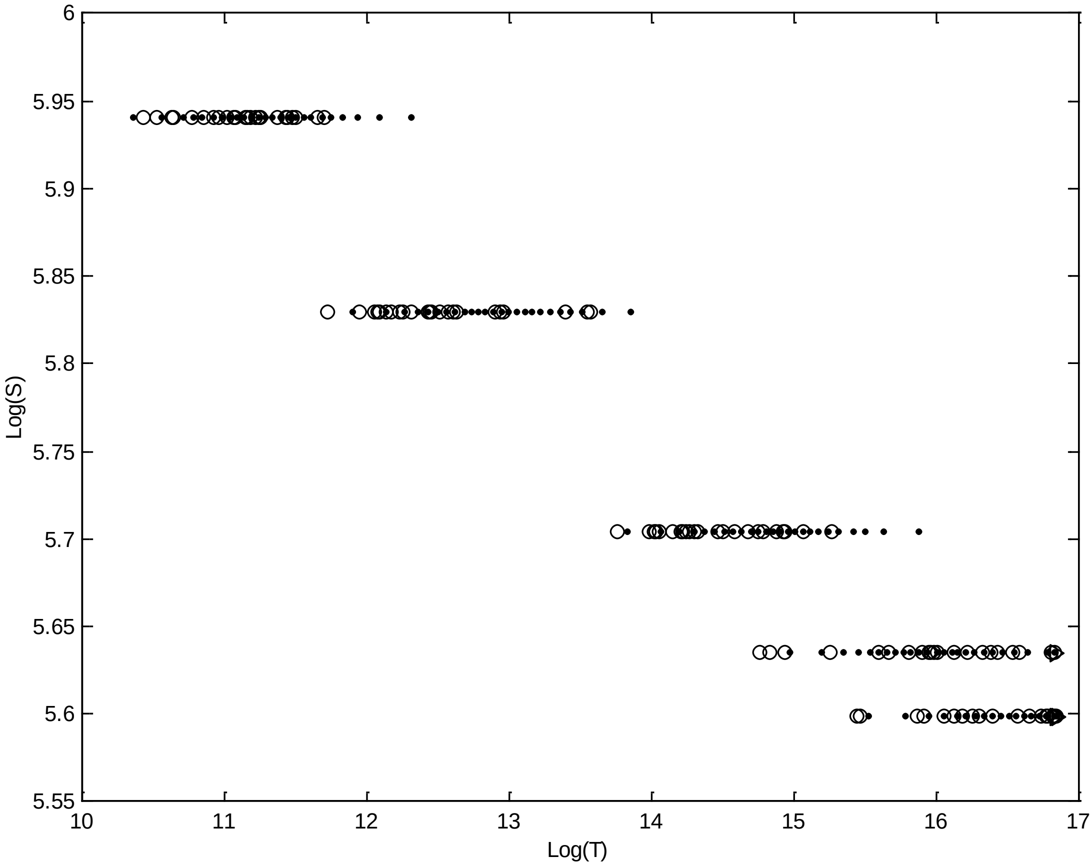


Fig.4. 2. Experimental data (•) and the results of calculations of the expectation values of relevant order statistics (o).

4.4. Processing of the experimental data of fatigue test of laminate panel

For numerical example we consider the problem to fit the experimental data of fatigue test of laminate panel. These data were kindly given to the authors by W.Q. Meeker, who studied them in paper [2] and gives the following description of these data: “the data come from 125 specimens in four-point out-of-plane bending tests of carbon eight-harness-satin/ epoxy laminate. Fiber fracture and final specimen fracture occurred simultaneously. Thus, fatigue life is defined to be the number of cycles until specimen fracture. The dataset includes 10 right censored observations (known as “runouts” in the fatigue literature)”.

Using these data in accordance with the approach considered in section 4.3 for OSMM we have got: $\hat{r}$ =3, $\hat{R}=15$, :$\hat{\theta}_0$ = 7.646, $\hat{\theta}_1$ =0.345, maximum of $\boldsymbol{l}(\eta)$ =-155.19. For the BMM: $\hat{r}$ =20, $\hat{\theta}_0$ = 7.46, $\hat{\theta}_1$ =0.326, maximum of $\boldsymbol{l}(\eta)$ =-152.09. It seems that BMM is more appropriate for this data and has some smaller number of parameters. The fatigue curves, more precisely, the experimental data (•) and the results of calculations of the expectation values of the relevant extreme order statistics (o) are shown in Fig. 4.2. It seems that it is not too bad. The considered models can be used for prediction of fatigue life under program loading, too. Preliminary calculations show a small difference from Miner’s law. Some experimental investigations do not contradict this law, but it seems that it takes place only for nearly unidirectional composite. But a significant deviation from this law can be expected if the role of composite matrix is more significant. Another type of model to take into account this phenomenon is offered in the next section. It is more appropriate for prediction of fatigue life under program loading.

4.5. Model of accumulation of fatigue damages based on the Markov chain theory taking into account matrix plasticity

4.5.1. Mathematical description of the model

Now we assume that the fatigue failure of a test specimen occurs after destruction of some its critical microvolume (CMV) consisting of perfectly elastic (brittle) longitudinal fibers (bundles) (the elastic part) and a matrix where plastic deformations accumulate during cyclic loadings (the plastic part) (Fig. 4.3 and Fig. 4.4).

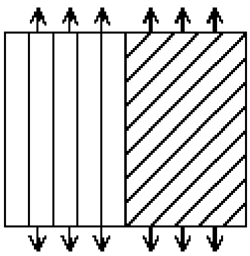


Fig.4.3. Model of the critical microvolume of a composite under a load

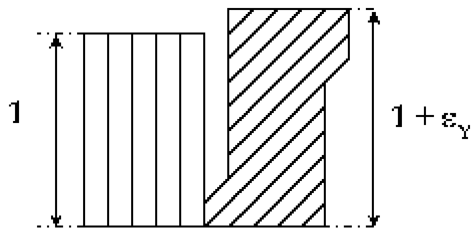


Fig. 4.4. Model of the critical microvolume of a composite after removal of the load.

And we assume that except for the longitudinal elements, a plastic part includes all other composite components, i.e., the matrix itself and all the layers with stackings different from the longitudinal one.

We assume that, if the number of intake elastic elements in the CMV (able to take up the longitudinal load) decreases by r_R during the cyclic loading, the elastic part of the specimen breaks down, which is followed by the failure of the specimen as a whole. The slanted hatching in Fig. 4.3 points symbolically to the possibility of accumulating an irreversible permanent strain, ε_Y (see Fig. 4.4). If permanent strain exceeds some critical level, ε_{YC}, the failure of the CMV and the specimen as a whole takes place. We emphasize that this graphic image, as applied to a composite, should be understood symbolically. It is more suitable for metals, where the accumulation of plastic strains is associated with some "act of yielding" (for metal — a shift over slip planes). We assume that an individual act of yielding in the mathematical description of the process, leads to a respective change in the state of MC, while in the physical description — to the appearance of a constant plastic strain ε_{Y1}. The failure of CMV takes place after the accumulation of a "critical" number of such acts, r_Y, i.e., after the accumulation of a critical permanent strain, the relation $\varepsilon_{YC} = \varepsilon_{Y1} r_Y$ is valid, where ε_{YC} and r_Y are model parameters.

Since the elastic and plastic parts are integrated in a unit, accumulation of permanent strain leads to the appearance of residual stresses: tension in the elastic and compression in the plastic part of the specimen.

Let us associate the process of a gradual failure of a specimen with a stationary Markov chain of two dimensions: the number of broken elastic elements and the number of acts of yield. We will consider the matrix of transition probability as a complex of blocks: $(r_Y + 1)$

blocks with (r_R +1) states within each of them. Then, the indices of input and output states, i and j, respectively, can be expressed in terms of the corresponding local indices i_Y, i_R, j_Y and j_R by the formulae

$$i = (r_R + 1)(i_Y - 1) + i_R\,; \quad j = (r_R + 1)(j_Y - 1) + j_R\,.$$

Table 4.1.Example of transition probabilities matrix

		j_Y	*1*			*2*			*3*		
		j_R	*1*	*2*	*3*	*1*	*2*	*3*	*1*	*2*	*3*
i_Y	i_R	i \ j	1	2	3	4	5	6	7	8	9
1	*1*	1	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	$p_{R2}p_{Y0}$	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$	$p_{R2}p_{Y1}$	$p_{R0}p_{Y2}$	$p_{R1}p_{Y2}$	$p_{R2}p_{Y2}$
	2	2	0	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	0	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$	0	$p_{R0}p_{Y2}$	$p_{R1}p_{Y2}$
	3	3	0	0	1	0	0	0	0	0	0
2	*1*	4	0	0	0	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	$p_{R2}p_{Y0}$	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$	$p_{R2}p_{Y1}$
	2	5	0	0	0	0	$p_{R0}p_{Y0}$	$p_{R1}p_{Y0}$	0	$p_{R0}p_{Y1}$	$p_{R1}p_{Y1}$
	3	6	0	0	0	0	0	1	0	0	0
3	*1*	7	0	0	0	0	0	0	1	0	0
	2	8	0	0	0	0	0	0	0	1	0
	3	9	0	0	0	0	0	0	0	0	1

Table 4.1 shows the example of (symbolic) filling of the matrix for the case where $r_Y = r_R = 2$. In this case, destruction of a specimen occurs if two longitudinal elements fail (event A), or two acts of yielding take place (event B), or events A and B take place simultaneously. To these events the absorbing states of the Markov chain correspond. In the example considered, there are $(r_Y + 1)(r_R + 1) = 9$ such states. The symbols $p_{R0}, p_{R1},...$ designate the probabilities of failure of the corresponding number of elastic (rigid) elements; $p_{Y0}, p_{Y1},...$ are the probabilities of the corresponding numbers of acts of yielding.

In the present study we assume that the number of elastic elements destroyed in one step has a binomial distribution. If we have n_R still intact elements, the probability of failure k_R of additional elements is defined by the equation:

$$p_R(i,j) = \binom{n_R}{k_R}\left(F_R\left(S_R\left(i_R, i_Y\right)\right)\right)^{k_R}\left(1 - F_R\left(S_R\left(i_R, i_Y\right)\right)\right)^{n_R - k_R},$$

where $n_R = r_R - i_R$, $k_R = j_R - i_R$, for $0 \le k_R \le n_R$, $1 \le n_R \le (r_R - 1)$, $F_R(.)$ is c.d.f. of strength of intact elastic element, $S_R\left(i_R, i_Y\right)$ is the stress in elastic part, when the process is in i – th state. The probability of the fact that at the same state of the process, an additional number of acts of yielding will be equal to k_Y, is described by a similar equation:

$$p_Y(i,j) = \binom{n_Y}{k_Y}\left(F_Y\left(S_Y\left(i_R, i_Y\right)\right)\right)^{k_Y}\left(1 - F_Y\left(S_Y\left(i_R, i_Y\right)\right)\right)^{n_Y - k_Y},$$

where $n_Y = r_Y - i_Y$, $k_Y = j_Y - i_Y$, at $0 \le k_Y \le n_Y$; $1 \le n_Y \le (r_Y - 1)$ and $F_Y(.)$ is c.d.f. of the yielding point, (j_Y-1) is the number of acts of yielding; $S_Y(i_R, i_Y)$ is the stress in the plastic part; the number of preceding acts of yielding is (i_Y-1), and the number of already destroyed elastic elements is ($i_{R,}$ -1).

The stress itself is the function of applied "brutto" stress and the function of the number of failure of rigid items and number of acts of yielding. Let initial cross-section of considered (the weakest) critical volume be

$$f = f_R + f_Y,$$

where f_R, f_Y are cross-sections of rigid and yielding parts of critical volume correspondingly.

If failure of i rigid items in the same cross-section takes place, then cross-section area decreases:

$$f_{Ri} = f_R \cdot (1 - i / R_R)$$

The cross-section of yielding part does not change but the length of it changes as the function of the number of yielding. If both rigid and plastic parts are working within the limits of elasticity then we have two equations for corresponding stress calculation:

$$\begin{cases} S_R \cdot f_R + S_Y \cdot f_Y = S \cdot f, \\ \dfrac{S_R}{E_R} = \dfrac{S_Y}{E_Y} \end{cases},$$

where S is stress, E is Young's modulus, R and Y are subscripts of rigid and yielding parts correspondingly. The first equation is an equation of equilibrium; the second is the condition of compatibility: equality of strains of both parts. If the lengths of both parts are equal, we have the following solution of this equation system:

$$S_R = S \cdot f /(f_R + f_Y \cdot E_Y / E_R),\ S_Y = S \cdot f /(f_Y + f_R \cdot E_R / E_Y).$$

But if we have some yielding of the plastic part and its new length becomes equal to $l_Y = (1 + \varepsilon_Y)$ instead of an initial length $l_Y = 1$ then we should take into account the residual stress, which appears in both parts after an outside load is eliminated. This residual stress can be found as the solution of the equation system:

$$\begin{cases} \Delta S_R \cdot f_R = \Delta S_Y \cdot f_Y, \\ 1 + \dfrac{\Delta S_R}{E_R} = (1 + \varepsilon_Y)\left(1 - \dfrac{\Delta S_Y}{E_Y} \right), \end{cases}$$

Here again the first equation is an equation of equilibrium, the second is an equation of length equality. The solution of this system (in limits of elasticity) is defined by the formulae

$$\Delta S_R = E_R \cdot \varepsilon_Y / \left(1 + (1 + \varepsilon_Y) f_R E_R / f_Y E_Y\right),\ \Delta S_Y = E_Y \cdot \varepsilon_Y / \left(1 + \varepsilon_Y + f_Y E_Y / f_R E_R\right).$$

We make an additional assumption that the value of ε_Y is proportional to j - the number of acts of yielding: $\varepsilon_Y = j\varepsilon_{Y1},\ j = 1,\dots, r_Y$. The value of ε_{Y1} is considered as the parameter of the model.

The vector of probability functions of times to absorption from different initial transient states is defined by formula (4.1) again, but now we should use column vector b of type (0, … 0,…,1,…,0,…,1)' where the order number of units is equal to the ordered number of absorbing states of matrix P.

By renumbering the states the matrix of transition probabilities again can be transformed to the one which is defined by (4.4). Then, if an initial state is the first state ($i_Y = i_R = 1$), c.d.f. of number of MC steps up to absorption, T_S , is defined by the equation

$$F_{T_S}(t,S,\eta) = aP^t b\,,\ t = 1,2,3,....$$

where a = (1, 0, 0, …0), b = (0, … 0,…,1,…,1)', where the number of units is equal to the number of absorbing states. Here S is cycle maximum stress, η - parameter-vector with component equal to the parameters of c.d.f. of strength of an item of elastic part and the yielding point of matrix.

If one step in MC corresponds to k_M cycles in fatigue test, the c.d.f. of the number of cycles, T_C , is defined in a similar equation

$$F_{T_C}(t,S,\eta) = aP^{t/k_M} b\,,\ t = k_M, 2k_M, 3k_M, ...$$

The k_M is the component of η also.

The fatigue life (cycles), corresponding probability of failure p in a test with an initial stress S (the p-quantile fatigue curve) is defined by the equation

$$t_p(S) = k_M F_{T_S}^{-1}(p;S,\eta) = F_{T_C}^{-1}(p;S,\eta)\,.$$

Again the vectors of average and variances of step numbers to absorption from different transient states are defined by the formulae (4.5).

In the considered model we are interested in the matrix $B = \{B_{ij}\}$, which defines the probabilities of absorption in S_j absorbing state if an initial transient state is S_i. This matrix is defined by the formula

$$B = \{B_{ij}\} = NR\,,$$

where N , R are matrices corresponding to (4.4). Using matrix B we can calculate, for example, the probability that just the failure of matrix is the reason of final failure of specimens.

4.5.2. Application to the program loading. Residual fatigue life in two-stage fatigue loading

The offered model can be easily used for fatigue life calculation for program fatigue test. Let us assume that k_M =1 but instead of T_S we shall use just T . For any arbitrary stress cycle consequence $\{S_1, S_2, ...\}$ the probability distribution function of time to failure from the first initial state is defined by the formula

$$F_T(t) = a(\prod_{i=1}^{t} P_i)b\,,$$

where matrix P_i is the matrix of transition probabilities, corresponding to the parameters of i-th cycle (for example, maximum value of stress S_i , $i = 1,2,3,...$, in ith cycle); a, b are the same as in the previous formula. At our disposal we have experimental data [11] corresponding to the program which is shown in Fig 4.5.

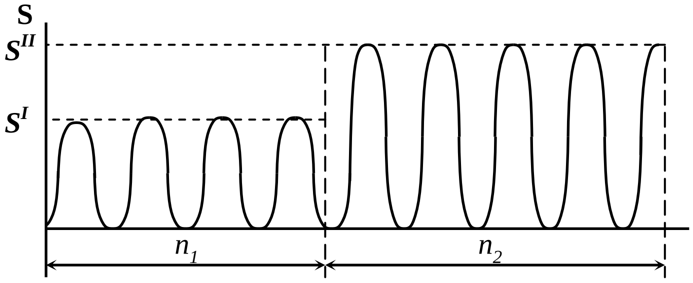


Fig.4.5. Example of the program of cyclic loading with a single change in the level of the maximum cycle stress.

For this program the distribution function of time to failure is defined by the formula

$$F_T(t) = \begin{cases} a(\prod_{i=1}^{t} P_{S^I}^{i})b, & \text{if } t \le n_1, \\ a(\prod_{i=1}^{n_1} P_{S^I}^{i} \prod_{j=1}^{t-n_1} P_{S^{II}}^{j})b, & \text{if } t > n_1. \end{cases} \quad (4.9)$$

Conditional distribution function of residual fatigue life, $\boldsymbol{T_2}$, of the specimen without failure in the first stage test after n_1 cycles with S^I, as usually, is defined by the equation

$$F_{T_2}(t) = \frac{F_T(n_1+t) - F_T(n_1)}{1 - F_T(n_1)}.$$

Here we limit ourselves by checking the offered model forecasting the expectation value of T_2 if fatigue curve and n_1 are known. The calculation of expectation value of T_2 is similar to the calculation of expectation value, $E(T)$, and the variance, $V(T)$, for the case without stress change. But now we should take into account that the order number of the initial state at the end of n_1 cycles with stress $S=S^I$ is a random variable with distribution

$$\pi^{II} = (1,0,0...)P_{S^I}{}^{n_1}. \quad (4.10a)$$

Then expectation value and variance of time to failure at the second stage with $S=S^{II}$ will be defined by the formulae [10]

$$E(T_2) = \pi^{II} \cdot \tau^{II}, \; V(T_2) = E(T_2^2) - (E(T))^2 = \pi^{II} \cdot (\tau_2^{II} + \tau_{sq}^{II}) - (\pi^{II} \tau^{II})^2. \quad (4.10b)$$

4.5.3. Processing of residual fatigue life data set in two-stage fatigue loading

In [11], the results of fatigue tests at an approximately pulsed ($S_{\min}/S_{\max} = 0.1$) load on CFRP specimens with an average static strength of 356 MPa are represented. The purpose of these tests was to construct the fatigue curve (Table 2.11 in [11]) and to examine residual durability at a single change in the loading mode (Table 4.3 in [11] and Fig.4.5 of the present study).

The maximum stresses were 290 and 323 MPa; an average fatigue life was 1,200,000 and 116,000 cycles respectively (Tables 4.2 and 4.3 in [11]). Figure 4.6 of the present study shows experimental data for constructing the fatigue curve and presents the calculation results for mathematical expectations of order statistics for the fatigue life (the sample size is equal to the number of specimens tested at the corresponding loading level).

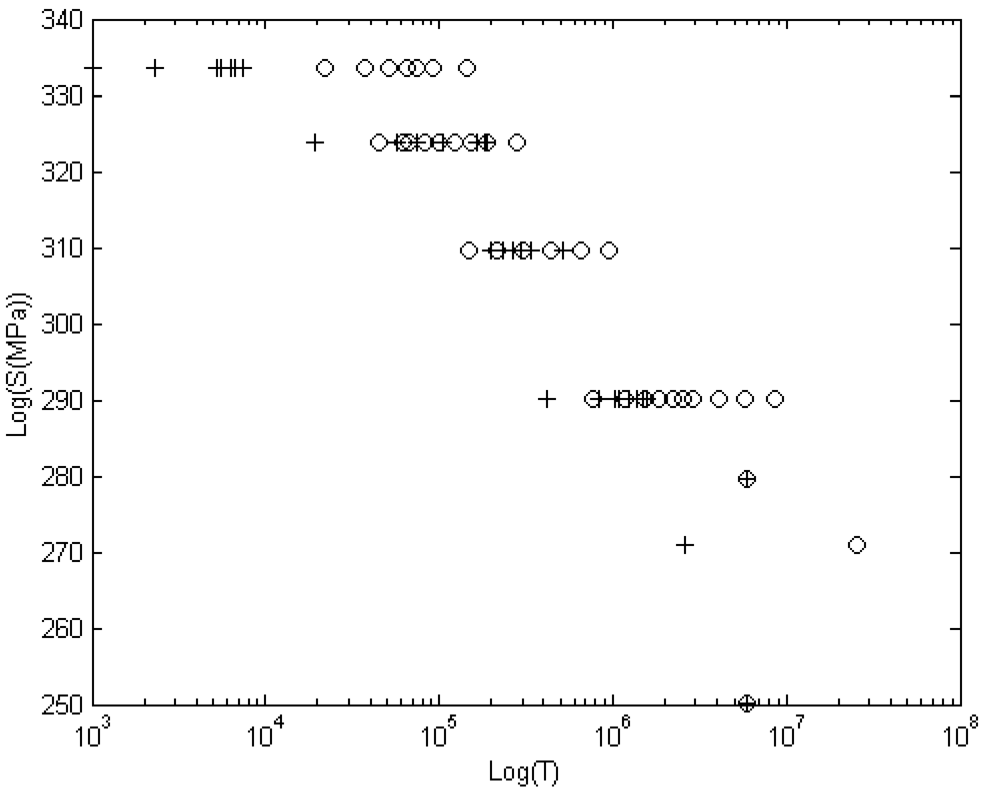
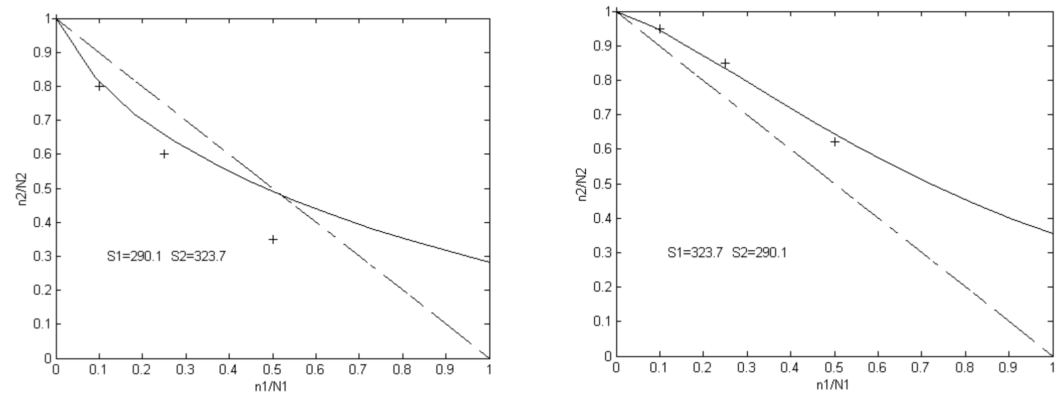


Fig.4.6. Results of fatigue life tests at a pulsating loading (+) according to [11] and the calculated estimates of mathematical expectations of the respective order statistics (O).

The calculation results for a relative residual durability in the tests with a single change in the loading mode namely n_2/N_2 (the conditional relative residual durability $E(T|T>n_1)/E(T^{II})$ of specimens not destroyed at the first stage of loading) as functions of n_1/N_1 (more clearly, $n_1/E(T^I)$), in comparison with experimental data and calculations according to the Palmgren-Miner linear hypothesis, are illustrated in Fig.4.7.



a b

Fig.4.7. Calculation results for the relative residual durability n_2/N_2 according to the model suggested (—), the Palmgren-Miner hypothesis (- - -), and experimental data(+) from [11] at S^I = 290 MPa, S^{II} = 323.7 MPa (a) and S^I =323.7 Mpa, S^{II} =290 Mpa (b).

Similar information based on the experimental data published in [3] is given in Fig.4.8 and Fig.4.9.

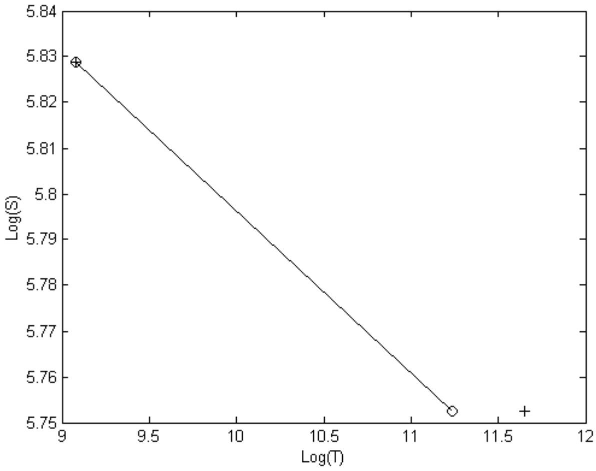
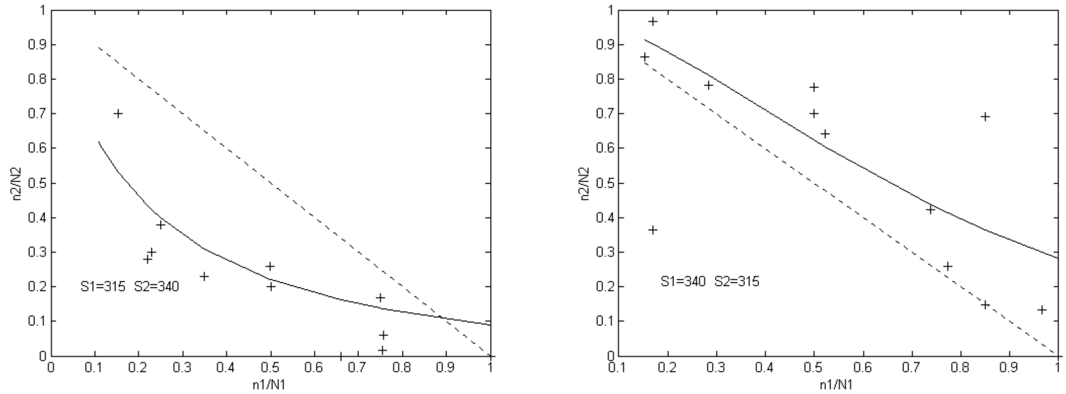


Fig.4.8. Experimental data (+) (according to [3]) and the corresponding calculated values (o) of the average fatigue life.



a b

Fig.4.9 Calculation results for the relative residual durability n_2/N_2 according to the model suggested (—), the Palmgren-Miner hypothesis (- - -), and experimental data (+) at S^I = 315 MPa, S^{II} = 340 MPa (a) and S^I = 340 MPa, S^{II} = 315 MPa (b). (Experimental results for n_2/N_2 = 2.73, 1.63, and 4.33 at n_1/N_1 = 0. 1086, 0.1129, and 0.1476 are not shown.)

Unfortunately, the data on the fatigue curve in [3] are very scanty: only average values of strength at two stress levels are given. Therefore, Fig.4.8 actually has no independent importance - it is given here only to better understand Fig.4.9. Table 1 and Figures 2 and 3 given in [3] represent data on the residual durability of CFRP specimens with a static strength of 422 MPa. The cycle of fatigue loading is close to a pulsating one (with a ratio between the minimum and maximum stresses, *R,* equal to 0.05).The residual durability was investigated at two loading levels with maximum stresses 315 and 340 MPa, and average durability was found to be 115,150 and 8800 cycles, respectively. The data on the residual durability are given for each specimen separately.

The parameters of the model (of the critical volume) used in calculations are shown in Table 4.2.

Table 4.2. Parameters of the model.

Parameter	Data	
	[11]	[3]
Elastic modulus of longitudinal elastic elements, E_R, MPa.	207,972	21,0000
Elastic modulus of the plastic part, E_Y, MPa.	27,946	28,000
Ultimate total relative elongation of the plastic part, ε_Y.	0,00255	0,00255
Relative area of the elastic part, f_R (f_Y =1- f_R)	0,286	0,14
Number of longitudinal elements in the elastic part of critical microvolume, r_R	2	9
Number of elongation "steps" in the plastic part of a critical microvolume, r_Y.	3	9
Average value of the (natural) logarithm of the strength of longitudinal elastic elements, θ_{0R} (exp(θ_{0R})).	7,5 (1808 MPa)	7,55
Standard deviation of the (natural) logarithm of the strength of longitudinal elastic elements, θ_{1R}	0,15	0,053
Average value of the (natural) logarithm of yield point of the plastic part of critical microvolume, θ_{0Y} (exp(θ_{0Y})).	5,8579 (350 MPa)	5,7038
Srandard deviation of the (natural) logarithm of yield point of the plastic part of critical microvolume, θ_{1Y} .	0,3	0,2
Number of cycles equivalent to one step of Markov chain, k_M	1	1,945
Minimum stress for the two-level loading program, $S_{\min}$, Mpa	290,1	315
Maximum stress for the two-level loading program, $S_{\max}$, Mpa	323,7	340
Probability that the destruction of the specimen is caused by destruction of the plastic part of a microvolume under loading with a constant amplitude equal to $S_{\min}$, b_Y ($S_{\min}$)	0,10087	0,057306
Probability that the destruction of the specimen is caused by destruction of the plastic part of a microvolume under loading with a constant amplitude equal to $S_{\max}$, b_Y ($S_{\max}$)	0,35015	0,76965

Note. In parentheses, values according to notural scale are given.

The first four parameters are regarded as known constants of the CMV of material, while the subsequent seven are the parameters of nonlinear regression and are chosen upon processing the experimental data. Clearly, to better describe the results of fatigue tests, the first four parameters can be chosen as distinct from the "theoretical" values. For processing the data presented in [11], the values of these parameters were taken directly from the study. Almost the same values of the parameters (slightly rounded) were also used for processing the results of [3] (because there was not necessary detailed information in [3]). We remind that in the considered model the "plastic" part is the entire structure of CMV of the composite except the longitudinal elastic elements (and not only the corresponding volume of the "physical" matrix of the composite).

4.5.4. Residual strength in two-stage fatigue loading

Investigation of degradation of residual strength after fatigue is of vital importance to the reliability of aircraft structures. A lot of papers are devoted to this problem. Fine discussion of the state-of-the-art phenomenological residual strength models is provided in [13] referencing to 49 papers.

All the considered models give deterministic phenomenological description of degradation of residual strength. For example considered in [13], some (nunlinear) modification of linear Bourtman and Sahu [14] model for description of residual strength $X(n)$ after n fatigue cycles the following equation is offered:

$$X(n) = X(0) - (X(0) - \sigma_{\max})(n/N)^k .$$

Corresponding c.d.f. of $X(n)$ was developed using Weibull distribution for $X(0)$

$$F_{X(n)}(x) = 1 - \exp\left(-\left(\frac{x - (n/N)^k \sigma_{\max}}{\beta(1-(n/N)^k}\right)^{\alpha} + \left(\frac{\sigma_{\max}}{\beta}\right)^{\alpha}\right)$$

with a, β being parameters of Weibull c.d.f.: $F(x) = 1 - \exp(-(x/\beta)^{\alpha})$. For offered in [13] model, which is referred to as OM model, $k = k_1 \exp(k_2 n / N)$, k_1 and k_2 are some parameters. But in conclusion of [13] it is said that "even though the OM … in most cases predict satisfactory the residual strength … it requires a considerable experimental effort for implementation and do not consistently produce safe prediction"

The stress amplitude (at a symmetric reversed loading cycle) or, in the general case, the maximum stress (at an arbitrary fixed stress ratio) at which no fatigue failure takes place after a fixed number of cycles, N, is regarded as the restricted fatigue limit on the fixed base (duration) of tests. A certain development of the statistical theory of the restricted fatigue limit and fatigue curve is given in [15] and [16], the joint cumulative distribution function, $F(S,N)$, which is equal to the probability of fatigue failure at a number of cycles smaller than N and a stress lower than S, has the form

$$F(S,N) = 1 - \exp\left(-\frac{V}{V_0}\frac{(S - S_0)^{\alpha}(N - N_0)^{\beta}}{C^{\alpha\beta}}\right)$$

where V_0, S_0, N_0, α and β are some material parameters

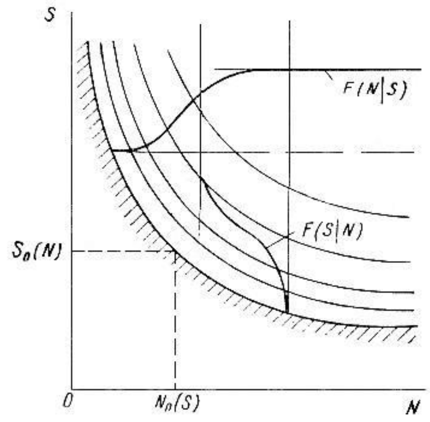


Fig. 7.10. Schematic of a family of quantile fatigue curves, of c.d.f. of the conditional fatigue limit at a fixed limited duration of tests F(S | N) and of c.d.f. of the fatigue life at a constant stress level F(N | S).

Fig. 7.10, taken from [16], schematically shows a family of quantile fatigue curves, as well as the c.d.f. of the conditional fatigue limit at a fixed ultimate number of cycles and the c.d.f. of service life at a fixed level of stresses.

It seems that without a deep probabilistical analysis we can not get the solution of the problem. One Step version of a model is considered in [17]. Here we try to use the binomial version of transition probability matrix of a considered model for processing data set reported in special OPTIMAT BLADES testing program documents [19-22]. It is shown that this model gives some reasonable result.

First, let us consider distribution of the restricted fatigue limit at a fixed number of cycles of fatigue loading. The matrix of transition probabilities, P, is a function of S. Therefore, the function

$$F_{\sigma_t}(x) = aP^t b\,, \tag{4.11}$$

where $x = S$, a, b are the same as in the previous section, determines the probability of failure of specimens in t steps ((k_M t) cycles) at a stress equal to S, i.e., it determines the distribution function of conditional fatigue limit at a fixed restricted number of fatigue $k_M\, t$ cycles.

Now let us consider a distribution of the residual strength. The vector of probabilities on the states of the Markov chain after loading (S_1, n_1), i.e., after n_1 steps with a stress S_1, is defined as

$$\pi_{S_1 n_1} = (1,0,...)P_1^{n_1}\,, \tag{4.12}$$

where P_1 is P for S_1.

The last $(r_Y + 1 + r_R)$ components of this vector define the absorption probabilities of the Markov process at the states corresponding to the failure of the specimen.

The residual strength $\sigma_{S_1 n_1}$ after loading (S_1, n_1), i.e., after n_1 steps with a stress S_1 is measured, of course, only on intact specimens. The corresponding components of the vector of distribution of probabilities on the irreversible states of the Markov chain are

$$\pi^*_{S_1 n_1}(k) = \pi_{S_1 n_1}(k) / \sum_{m=1}^{m^*} \pi_{S_1 n_1}(m)\,, \tag{4.13}$$

where $\pi_{S_1 n_1}(k)$, and $k = 1,...,m^*$,- are components of the vector $\pi_{S_1 n_1}$; $m^* = (r_Y + 1)(r_R + 1)$ - $(r_Y + 1 + r_R)$ - is the total number of non absorption (irreversible) states. The last $(r_Y + 1 + r_R)$ components of the vector $\pi^*_{S_1 n_1}$, corresponding to the absorbing states, are obviously equal to zero, since here we consider only the specimens not failed upon the preliminary loading.

For such specimens, the distribution function of the stress $\sigma^{II}_{n_1}$ at which absorption in one step in the Markov chain occurs (which corresponds to the failure of a specimen in k_M cycles), has the form

$$F_{\sigma_{S_1 n_1}}(x) = \pi^*_{S_1 n_1} P(x) b\,, \tag{4.14}$$

where $x \geq S_1$, $P(x)$ - is the matrix of transition probabilities at $S = x$. If k_M is equal to unity or is relatively small, the relation (4.14) determines the distribution of the residual strength. The words “relatively small” have to be defined more exactly during the accumulation of practical experience in using the model for processing experimental data. In general case, the function $F_{\sigma_{S_1 n_1}}(x)$ determines the distribution of a conservative estimate of residual strength since, obviously, the destruction in one cycle requires a greater load than in k_M cycles.

The formulae (4.12)-(4.14) are easily generalized for a case where the residual strength is investigated after some arbitrary sequence of loadings (S_1, n_1), (S_2, n_2),..., (S_r, n_r):

$$\pi_{(S_1,n_1),\ldots,(S_r,n_r)} = (1,0,\ldots)P_1^{n_1}\ldots P_r^{n_r}\,, \tag{4.15}$$

$$\pi^*_{(S_1,n_1),\ldots,(S_r,n_r)}(k) = \pi_{(S_1,n_1),\ldots,(S_r,n_r)}(k) / \sum_{m=1}^{m^*} \pi_{(S_1,n_1),\ldots,(S_r,n_r)}(m)\,, \tag{4.16}$$

$$F_{\sigma_{(S_1,n1),\ldots,(S_r,n_r)}}(x) = \pi^*_{(S_1,n_1),\ldots,(S_r,n_r)} P(x) b\,. \tag{4.17}$$

For example, after the application of a load ($S_{2,}$ n_2), subsequent to a "normal" preliminary load (S_1, n_1), we have to assume that r = 2 in Eqs. (4.15) – (4.17).

4.5.5. Processing of residual strength data set in two-stage fatigue loading

In [19-22], the results of 17 fatigue tests at R=0.1 and the results of 33 residual strength tests for 3 different stress levels (48.5; 63.6 and 78.3 MPa) of a preliminary loading in the framework of OPTIMAL BLADES testing program of the OB UD material are reported. ISO standard specimens, [+45/-45]s, were therefore used. We see that in this case there are no straight longitudinal items (fibers or strands). But if we consider the described model just as a nonlinear regression model we can try to make fitting of fatigue and residual strength test data using this model.

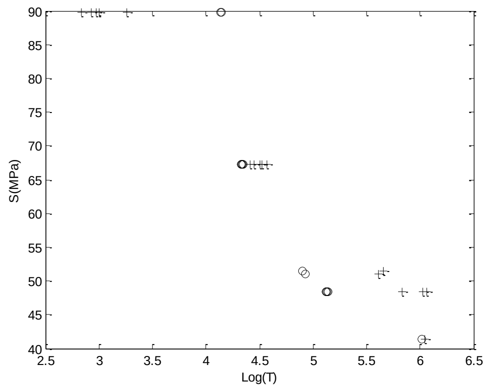


Fig.4.11. Results of fatigue life tests at approximatelly pulsating (R=0.1) loading (+) according to [19] and the calculated estimates of mathematical expectations of the respective order statistics (O).

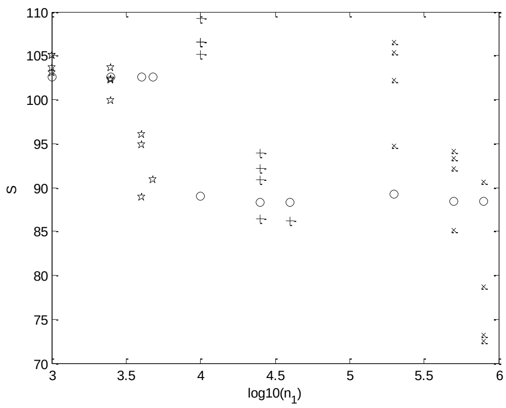


Fig. 4.12. Test results (+) [19] and estimates of the average residual strength vs the duration of preliminary loading with three stress levels (78.3 (☆); 63.6 (+) and 48.5 (x) MPa).

In Fig. 4.11. we see the fitting of the fatigue data and in Fig. 4.12. the residual strength and corresponding "prediction" using the considered model and fatigue parameter estimates are shown. In Table 4.3 we see parameters of the model, which was used for these calculations.

Table 4.3. Parameters of the model

Parameters	
Elastic modulus of longitudinal elastic elements, E_R, MPa.	79, 000
Elastic modulus of the plastic part, E_Y, MPa.	5,000
Ultimate total relative elongation of the plastic part, ε_Y.	0,2
Relative area of the elastic part, f_R (f_Y =1- f_R)	0,21

Number of longitudinal elements in the elastic part of a critical microvolume, r_R	20
Number of elongation "steps" in the plastic part of a critical microvolume, r_Y.	10
Average value of the (natural) logarithm of the strength of longitudinal elastic elements, θ_{0R} (exp(θ_{0R})).	5.7038 (300 MPa)
Standard deviation of the (natural) logarithm of the strength of longitudinal elastic elements, θ_{1R}	0,2
Average value of the (natural) logarithm of yield point of the plastic part of a critical microvolume, θ_{0Y} (exp(θ_{0Y})).	4.0943 (60 MPa)
Srandard deviation of the (natural) logarithm of yield point of the plastic part of a critical microvolume, θ_{1Y}.	0,2
Number of cycles equivalent to one step of Markov chain, k_M	6934

Note. In parentheses, values according to natural scale are given.

4.6. Conclusions and areas for further research

An analysis of the calculation results shows that the considered model satisfactorily describes the fatigue curve; however, in some cases predicted fatigue life is greater than actual for large stress (see Fig.4.6). The model examined also allows to predict the probabilities that, upon loading with a constant amplitude, the failure of a specimen is caused by the destruction of the plastic part (but not by the destruction of the elastic elements). These probabilities are illustrated in the last two rows of Table 4.2. As it is seen, the model correctly reflects the expected effect: with an increasing stress, this probability increases.

The residual durability calculated by the model is much closer to the experimental data than that predicted by the Palmgren-Miner linear hypothesis. The model predicts the inequality $\frac{n_1}{N_1}+\frac{n_2}{N_2}>1$ if $S^I > S^{II}$ and $\frac{n_1}{N_1}+\frac{n_2}{N_2}<1$ if $S^I < S^{II}$. This phenomenon is observed in the experiment, too. We should note that, in describing similar test results, it is usually assumed (see, for example, [3]) that $n_2/N_2=0$ for $n_1/N_1=1$. However, if $n_1/N_1=1$ about half of the specimens did not fail at the first stage of loading, and their residual durability at the second stage is not equal to zero, as it is shown by the calculations presented.

Comparing experimental data on residual static strength after fatigue load with the calculation results, we can conclude that with the model parameters found, it is possible to simultaneously describe both the data used to construct the fatigue curve and the data on the residual static strength of the composite after some preliminary loading (see Fig. 4.11 and Fig. 4.12).

The mathematical model suggested, from unified positions, gives a tool for consistently analyzing the distribution of static strength, the data for constructing the fatigue curve, the restricted fatigue limit, and the prediction of both the distribution of fatigue life under a program loading and residual strength after some preliminary loading. The model is too simple to be able to exactly predict the fatigue life based only on the data of static strength. In essence, it is a model of nonlinear regression, but as distinct from, for example, the Weibull model of fatigue curve, the parameters of the model can be interpreted as parameters of distribution of the local static strength.

The model has a rather general "modular" structure. Its "moduli", namely the distribution law for the static strength, the laws of accumulation of residual stresses and residual permanent strain in the critical micro volume, and the size of the critical micro volume can vary depending on the structure of a particular composite.

The determination of parameters of the model of nonlinear regression (the parameter k_M can be easily found by using the methods of linear regression) is a serious problem. The search for its efficient solution is the subject of a special investigation. Therefore, at present, it seems that the model considered here cannot be recommended for "engineering" applications yet. However, it is of a great interest not only for student studies, but also for a serious scientific investigation, since it gives a sufficiently informative "translation of mathematics into physics".

And it deserves to be studied more circumstantially.

There are several possible extentsions that deserve to be explored further:

- The model considered is most suitable for describing the processes with pulsating loading cycles. But using some method of "converting" of some cycle with arbitrary $R=\sigma_{\min}/\sigma_{\max}$ into "equivalent" pulsating loading cycles (see, for example, [23]) this model can be extended to arbitrary loading cycles.
- In the preceding section we considered the influence of the size of the specimens on its static strength. The same investigation can be done in exploring the fatigue life. If specimen has stress uniformly distributed along its length then just as in the study of static strength the specimen can be parted in n_L parts and it can be assumed that a fatigue process takes place in K_L, $1 \le K_L \le n_L$, items but fatigue life of specimens is the fatigue life of the weakest item. Conditional binomial (under condition: $P(K_L=0)=0$) or its Poisson's approximation (under conditions: $P(K_L=0)=0$ and $P(K_L \ge n_L+1)=0$) distributions of r.v. K_L can be explored. The value of $l_1 = L/n_L$, where L is the length of specimens, appears in this case as an additional parameter of the model."Direct Poisson approach" can be tried out also. In this case it is assumed that r.v. K_L has a conditional Poisson distribution under condition: $P(K_L=0)=0$.
- In the general case, we should take into account that fatigue process in several cross sections (weak sites) does not begin simultaneously. We may assume that the instants of origination of weak sites form a Poisson process with intensity $\mu = cF(S)$, where c is the factor of proportionality, $F(.)$ is the c.d.f. of static strength, and S is the maximum cycle stress. Then, the time intervals between the occurrences of the weak sites, $X_1, X_2, X_3, \dots$ have an exponential distribution with an average 1/μ . Let T_1, T_2, T_3, ... be a fatigue lives of weak sites. Then the c.d.f. of the random variable Y (the time before failure of the "weakest" element),

$$Y = \min(T_1, T_2 + X_1, T_3 + X_1 + X_2, \dots),$$

is determined by the formula [12]

$$F_Y(y) = 1-(1-F_T(y))\exp(-\mu\int_0^y F_T(t)dt).$$

For example, if $F_T(t)$ can be approximated by a lognormal distribution [12], we have

$$F_Y(y) = 1-\Phi(-z)\exp(-\mu(y(\Phi(z)-\exp(\mu+\theta_{1L}^2/2)*\Phi(z-\theta_{1L}))),$$

where $z=(\log(y)-\theta_{0L})/\theta_{1L}$; θ_{0L}, θ_{1L} are the parameters of the lognormal distribution. However, the estimation of the parameters of the model becomes a very serious problem.

4.7. References

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5. Appendices

5.1. Aircraft fatigue. History and current trends

Compiled by Yu. M. Paramonov

5.1.1. Introduction

This is a short survey of aircraft fatigue fractures. The safe-life concept, the fail-safe concept, damage tolerance and the DTR analysis are discussed.

As it was told in " A survey of serious aircraft accidents involving fatigue fracture " prepared by G. S. Campbell and R. Lahey in 1981 [1]: “A total of 1885 accidents since 1927 to 1981 where identified as having fatigue fractures as a related cause, and these accidents resulted in 2240 deaths;... the earliest reported accident was the wing-failure of a Dornier Merkur on 23 september 1927,... recently, there has been an average of about 69 fixed - wing fatigue accidents per year”.

C. Torkingtion in his paper [2] reminds: “In a two year period from 1942, about 20 Vickers Weillington bombers were lost in the UK as a result of fatigue failures of the wing main spar joints. In the war situation, if 20 failures in the UK were identified as fatigue, one can only guess that at least a similar number were lost over the sea or enemy territory.

The Wellington wing utilised a rather too-clever serrated wing joint which was subsequently found to have a mean fatigue life of between 200 and 300 flying hours. Following fatigue testing, a new joint was designed which achieved a life of 1200 hours.

The Wellington case is well-documented, but many other aircraft of that time had similar problems. The post-war Martin 202 had a similar four-step wing spar splice fitting which resulted in fatigue failure on two aircraft at 1542 and 1400 hours. Fatigue cracks were found in a further 13 aircraft, with total times as low as 335 hours.”

The most significant accidents were the catastrophic failure of Comet (1953, 1954), Fokker F-27 (1968), F-111 (1969), Hawker Siddeley (1976) and Boeing - 707 - 321C (1977). The most massive structural failure ever survived by an airliner was a geriatric (89,000 - flight) failure of B - 737 (1988).

5.1.2. The de Havilland crashes. Safe-life principle

The crash of three Comets was the most significant event, which has a very strong influence on the next aircraft airframe development. The first comet accident happened at Calcutta on May 2nd. 1953. As it would be told later by Lord Brabazon of Tara, chairman of the Air Registration Board [2], that was a strange accident, occurring at 9,000ft. Great experts had gone into it, and they told us that the aeroplane had met weather condition - I think it was a downward gust, they said, of 90 ft/ sec on the tail so extraordinary and so unprecedented as to be capable of breaking any machine in the air...

The second accident happened during the regular flight from Singapore to London on January 10th. 1954."On leaving Rome the aircraft climbed rapidly in accordance with the flight plan, and at 0950 hours, when the last message was received at Ciampino the pilot reported that he was at 26,500 ft. over the Orbetello beacon, and intended to continue to climb to 36,000ft. as planed...The absence of any further message to the second stations provided a negative indications, after which the usual emergency services were soon alerted. Meanwhile, positive indication of an accident was received. The eye-witnesses had seen pieces of the aircraft falling, in flames, into the sea...

Chief officer of the Harbour Authority of Portoferraio had immediately arranged for a search by aircraft and ships, and the search for wreckage was taken up by Italian and British Navies... Between February and September 80 percent of aircraft structure was recovered... "The Accident Branch of the Ministry of Transport and Civil Aviation had gone into action at once as soon as the accident was reported, and they were making investigations in all directions. They arranged for the wreckage as it came up to be sent to England, and the Royal Aircraft Establishment agreed to carry out an examination of it; but of course, this took time and mean while other events occurred..."

The third accident “took place on April 8th, 1954, when the machine had done 2,704 hours. It was on a flight from London to Johannesburg... Again the take off was perfectly normal. The last message was received at 1905 hours that is to say, 33 minutes after the aircraft took off. The pilot said that he was climbing to 35,000ft, which was the cruising height called for by the flight plan, and he was asked to report when he reached that height, but no more was heard". There were no eye-witnesses, but emergency service were put into operation that night, and the following morning some wreckage and five bodies (out of 14 passengers and seven crew) were found". [3].

Of course, already after the Elba accident, the Air Safety Board was asked by the Minister of Transport and Civil Aviation for advice concerning the resumption of Comet services. On March 2nd. a meeting was held, at which the A.R.B.(Air Registration Board) described their investigation, the modification they proposed, and the consultation they had had with the Havilland, B.O.A.C.(British Overseas Airways Corporation) and outside experts. As a result of the meeting, a minute was sent from the Board of the Minister, which read as follows: "The Board... realizes that no cause has yet been found that would satisfactorilly account for the Elba disaster... the Board realizes that every thing humanly possible has been done to insure that the desired standard of safety shall be maintained... and... recommends that Comet aircraft should return to normal operational use after all the current modifications have been incorporated and the aircraft have been flight tested." [3].

And indeed after the Elba accident, the B.O.A.C. ceased Comet operations and a programme of investigations was arranged together with the de Havilland Company. Firstly, the remaining Comet aircraft were to be inspected.

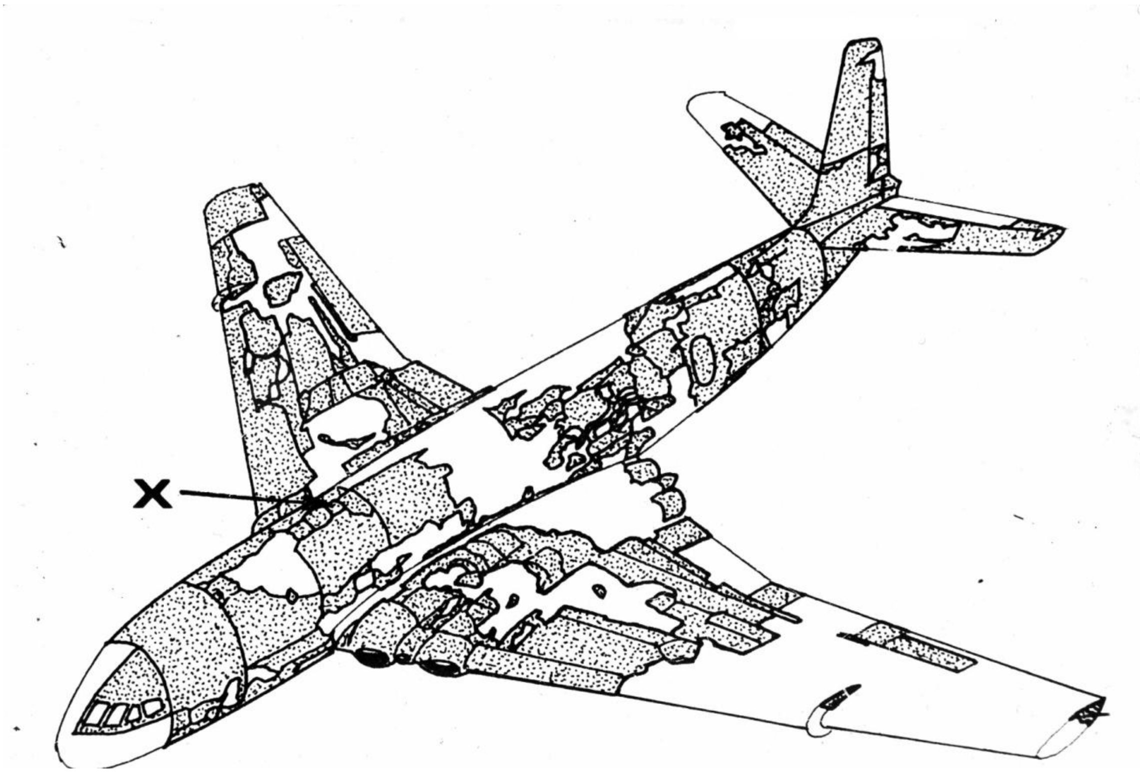

Fig. 5.1. The remaining Comet aircraft.

And secondly, consideration was given to all possible features which might have resulted in the accident. The possible causes were: 1, control flutter; 2, primary structure failure; 3, flying control failure; 4, fatigue, and particularly wing fatigue; 5, explosive decompression of pressure cabin; and 6, the possibility of malfunction of either the engines or the installation. Inspection and modification were introduced to guard against these contingencies. Preliminary medical evidence had indicated an explosion.

Modification after the Elba accident included the addition of steel plate protection against turbine-blade penetration, for the fuel tank and the main aircraft structure, and precautions against fatigue failure of the impeller and diffuser vanes. All these modifications have been devised to cover the worst possible combinations of circumstances which it was thought might arise.

"...About six Comets were given proof tests on their pressure cabins of 11 lb/sq. Later the veritable cause of the crash has been found, -the fatigue crack - but it was subsequent to the Naples accident.

The cracks in fact were formed during manufacture and were drilled to prevent their propagation. But it appears that if you do stop a crack with a location hole, there is quite a good chance of the crack spreading beyond that hole.

The source of the pressure cabin failure was a point near the rear automatic direction finder window: fracture analysis showed that all fractures in the fuselage in the area ran away from this point, and none ran into the area. The sources of information apart from the wreckage, included Comet G-ALYU which was subjected to fatigue tests, and Comet G-ANAV which was used for flight test. The Farnborough report described test on a one tenth scale perspex model of a Comet cabin, fitted with seats and dummy passengers. This had been placed in a pressure chamber evacuated in effect to 40,000ft, with the cabin at 8 1/4 lb/sq in differential pressure, and had been caused to burst near the point at which Elba Comet G-ALYP was beloved to have failed.

A number of photographs showed the effect of sudden decompression: after the equivalent of 0.03 sec, the seat-backs in the aft end of the cabin were moving forward; after 0.07sec, seats were flying about in all directions and one dummy passenger was hitting the roof with considerable violence. Although the timing could not be considered exact, these pictures gave a qualitative idea of what probably happened and again gave agreement with conclusions from all others sources.

There was an evidence of decompression of the lungs in the bodies of the victims consistent with sudden loss of pressure in the cabin. "It is clear on general principles that if a fracture of any substantial size occurs in the wall of a tube or vessel which in under 8 lb pressure, a large hole will immediately open up and the tube will at once become what the layman might describe as a compressed air-gun. A terrific blast of air will force anything and everything out of the hole and will tend to throw the aircraft into violent contortions and so tear the whole of the fuselage to pieces."

This experiment "indicated the sequence in which, according to the R.A.E. (Royal Aircraft Establishment) report the Elba Comet had broken up. The first thing that happened was a violent disruption of the centre part of the pressure cabin. The next thing that happened was that the fuselage aft of the rear spar, the nose and the outer port wing fell away under what are called downward force. Thirdly, the main part of the wing separated and caught fire. Next, the fuselage aft of the rear spar with the tail unit still attached, fell into sea with the open end first and the tail plane last. Last of all, the main part of the wing, still on fire, hit the water in an inverted position..."[3].

Additional evidence was given by full scale Farnborough fatigue tests. The fuselage pressurization was tested in a water tank, while fluctuating gust loads were applied to the wings (which projected out of the tank) by means of hydraulic rams.

The Elba Comet failure bore a marked similarity to that of the Comet tested to failure at Farnborough. The failure occurred after 5,546 total pressurization, i.e. after a total equivalent life (including actual flying) of 9,000 flying hours. The 3,861 hours of the Elba Comet and these 9,000 hours cannot be regarded as outside the realm of the possible spread of the fatigue trouble....3,681 is within what is called the reasonable scatter, and the 2,701 (of the Naples Comet) is also said to be within that area.

The apparent cause of the disruption was obtained by a process of elimination. The possibility of an internal explosion, abnormally high tail-loads, insufficient tail strength, an abnormal increase in atmospheric pressure, failure of the powered control system or the pressurization control, and inefficiency of the pilot had in turn been eliminated. There was only one thing left - metal fatigue, and nothing contradicts it.

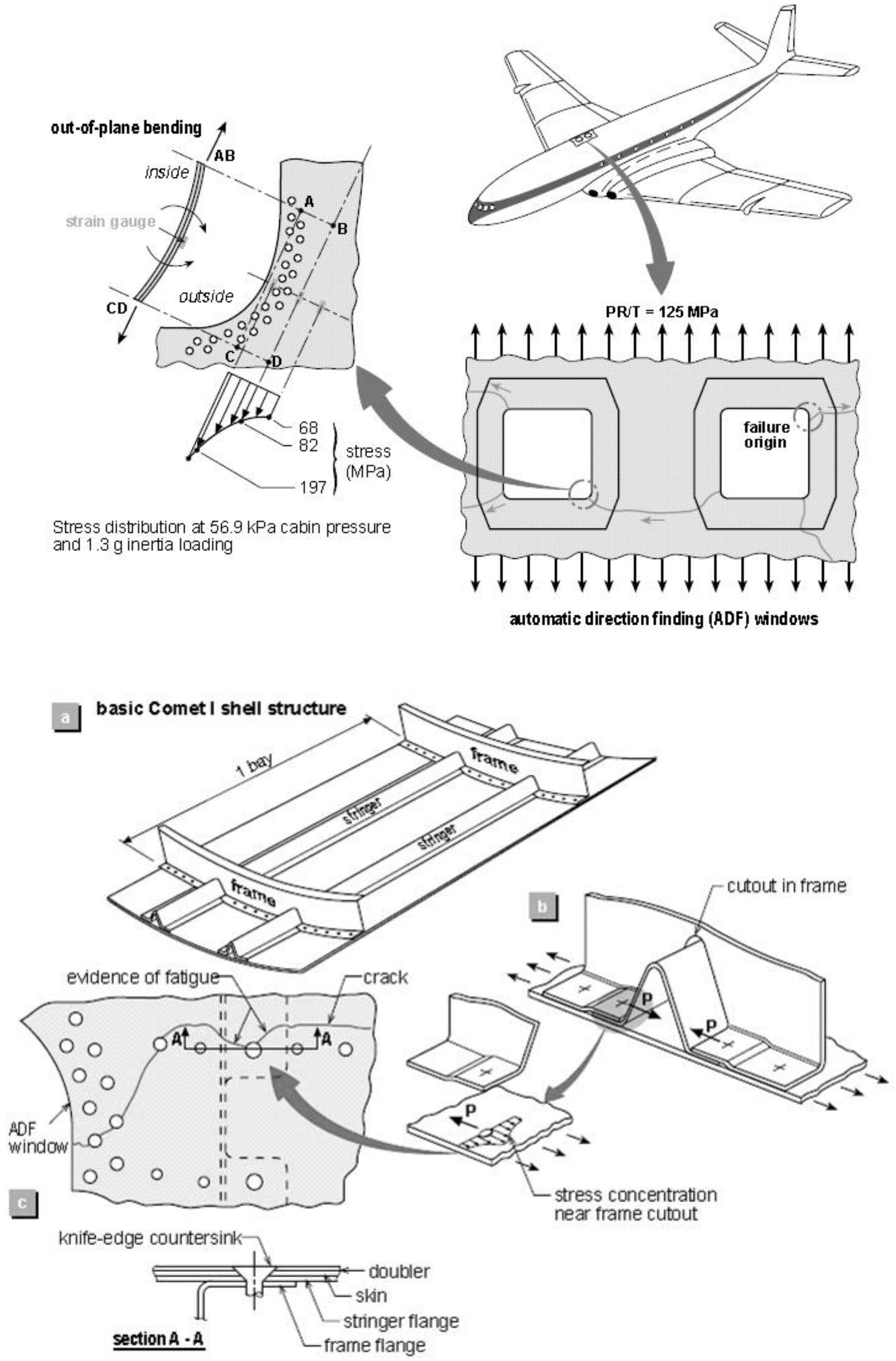

Fig. 5.2. Probable failure origin of service aircraft Comet G-ALYP [4].

So the conclusion that the accident to Comet G-ALYP near Elba in January, 1954, was caused by structural failure of the pressure cabin brought about by fatigue was put forward by the Crown in the opening stages of the Comet inquiry in London. This conclusion was based on a detailed investigation by the staff of the Royal Aircraft Establishment into all possible causes of the disaster, using evidence from the wreckage, practical tests and theoretical analysis. In the absence of wreckage from the accident to G-ALYP near Naples in April, 1954, no definite cause could be suggested, although the same explanation appeared to be applicable [3].

Although there was earlier theoretical knowledge of fatigue, it did not begin to be seriously considered as a practical problem until about the beginning of the 1952. Although already in 1947 it had been decided that the wings should be submitted to repeated loading tests, but that static testing would suffice for the fuselage. Fatigue problem of pressure cabin was indeed not fully appreciated at that time.

In the whole of the Comet project the de Havilland Aircraft Company had carried out 500 test investigations, involving some thousands of actual tests. Even before the Comet went to Farnborough for the tank test, it was the most tested civil aircraft.

But it was not enough, that special philosophy and system of aircraft development should have been created in order to prevent aircraft fatigue failure.

The first main ideas of the system were offered just during the Comet inquiry [3] in October - November 1954. Air Registration Board proposed to take certain practical steps. Board indicated its intention that complete cabins of pressurized aircraft should be submitted to tank tests similar to those used at Farnborogh. At least two airframes of each prototype would have to be made available, one for static testing and the other for fatigue testing.

Much attention was paid to the scatter of fatigue life. This is the opinion of director of the Royal Aircraft Establishment: ".... I would have the whole aeroplane tests carried on until the next failure took place, and then take half a dozen specimens and get a safety life, we would then put variation as 3:1 on either side of the average. Whereas, if you only work on a single specimen, you would have to give a safety life of about one ninth of what the specimen comes to, because the specimen might by chance have been the strongest..."[3]. “Нормы летной годности” of SU defines the figures as 3:1 and 5:1 correspondingly).

The approach to the fatigue problem, which developed from these ideas, was called a SAFE-LIFE approach. Basically this requires that all the parts of the structure, the failure of which could result in loss of the aircraft, are to be able to remain safely in use for a predetermined retirement life. Safe - life components are not normally subject to inspections related to fatigue, though they may be examined periodically for such problems as wear or corrosion.

5.1.3. Fail-safe concept

Now the fail - safe concept will be discussed, which was initially developed in U.S.A., but until now most general aviation aircraft have safe-life structures; all helicopters have a mandatory retirement schedule which commonly includes almost every component that moves and even some that do not.

U.S.rules in fatigue evaluation for fixed wing transport aircraft later would serve as the framework for common international standards. This new standard resulted from the U.S. FAA Transport Category Airplane Fatigue Regulatory Review Conference held in March, 1977. The European position, primarily advocated by the United Kingdom, was that transport category aircraft should meet two standards, the fail-safe and the safe-life method, for certification of fixed wing aircraft.

U.S. industry officials opposing the safe-life concept cite the services experience of the 8,000 U.S. aircraft manufactured under the damage tolerance practice, the cost and time involved in conducting fatigue tests, difficulty in simulating corrosion damage after 10 years in a laboratory test, and the extra weight an aircraft would have to carry around to take care of a "what if" situation.

The damage tolerance evaluation of structure is intended to insure that should serious fatigue cracks or damage occur, the remaining structure can withstand reasonable loads without excessive structural deformations until the damage is detected.

Design features which may be used in attaining a damage tolerance structure are:

* Use of multipath construction and the provision of crack stoppers to limit the growth of cracks.

* Use of duplicate structures so that a fatigue failure occurring in one-half of the member will be confined to that half and the remaining structure will still posess appreciable load carrying ability.

* Use of a backup structure where one member carries the entire load, with the second member available and capable of assuming the extra load if the primary member fails.

* Selection of materials and stress levels that provide a controlled slow rate of crack propagation combined with high residual strength after initiation of cracks.

* Arrangement of design details to permit easy detection of failures in all critical structural elements before the failures can result in appreciable strength loss, and to permit replacement and repair [5].

"Fail-Safe structures are generally more efficient for large modern aircraft with low thickness/chord ratio wings and integral fuel tanks. However, the substantiation of such designs can be difficult and expensive. A structure can only be accepted as fail-safe or damage-tolerant if it is inspected using a substantiated technique at repetitive inspection intervals. Hence there is a greatly increased maintenance cost. It is surprising how often some manufactures have to be reminded about inspection capability in their designs.

There is a general tendency to adopt a complacent attitude towards a defect in a fail-safe design. After all, is not the structure designed to accept a failure of any part in a safe manner with the remaining structure caring full loads? A few benchmark case histories may assist in answering these questions.

The Fokker F27. At the time of certification, Fokker carried out a comprehensive full-scale fatigue test of the F27 which resulted in important structural modifications to production aircraft and formed the basis for the structural inspection system used in service.

One of the problems identified during the fatigue testing was cracking of the structure surrounding the outer wing lower surface fuel tank access door opening. Cracking of the tank door surround was identified and propagation rates established. Modification actions were taken on Fokker-built aircraft to vary the skin thickness and improve the tank doors. Additionally, a radiographic inspection technique was devised, capable of detecting fatigue cracking before it reached a critical stage.

In December, 1968, an accident occurred to a Fairchild F27B aircraft in Alaska, when the right outer wing separated from the aircraft during decent. The outer wing failed at a tank access door location in an area weakened by fatigue cracking.

Of great significance was the fact that from their fatigue test Fokker had identified the tank door area as a fatigue crack location and the radiographic inspection technique had been specifically developed for aircraft in service. Unfortunately, although the accident aircraft had been so inspected and cracking was visible on radiographs taken over a year before the accident, this cracking was not identified at the time and no action was taken to repair the wing. It is worth noting that, of 67 other aircraft of same type reinspected after the accident, eight were found to have cracks not previously identified.

The Hawker Siddeley HS748. The HS748 is another example of fail-safe structure which had been thoroughly fatigue-tested by the manufacturer but which nevertheless suffered a fatal fatigue accident. The accident occurred in Argentina in April, 1976, as a result of an in-flight wing failure.

Fatigue cracking occurred in the wing lower surface in a reinforced area in the region of the engine outer rib. The total crack length at the time of final separation was approximately 36in.[90cm.] subsequently, cracks approximately 27 in (68.6cm) and 10in. [25.4cm.] long were found on two other aircraft of the same fleet.

The Argentine investigators stated that the detectable cracking progressed rapidly, in less than a year. They based this statement on the fact that the three aircraft with large cracks had been inspected at between three and a half and 10 months previously and no cracks had been found. The inspection schedule included a 500 hour repeat inspection for cracks in the failure area. The main conclusion of the accident reports reads:

"The cracks remained undetected and became critical because the manufacturer's inspection program for the area concerned was insufficiently precise and made it possible for the operator not to detect and correct them in time."

The manufactures position was that this particular crack location had been discovered in the fatigue test and propagation rates measured. Full-scale structural tests showed that a 36in. [90cm.] crack would take 10,000 hours, or five years flying and 20 inspections, to grow from detectable size to such a length as outsiders, all we can say is that the fail-safe, system failed, either because inspections were not sufficiently precise, or because they were not properly carried out" [2].

In 1977, the British Airways fleet of Trident 3's was temporarily grounded by the discovery of a fatigue crack in the wing root joint. This fortuitous discovery prevented an accident, but the disruption of schedules caused British Airways considerable loss [6].

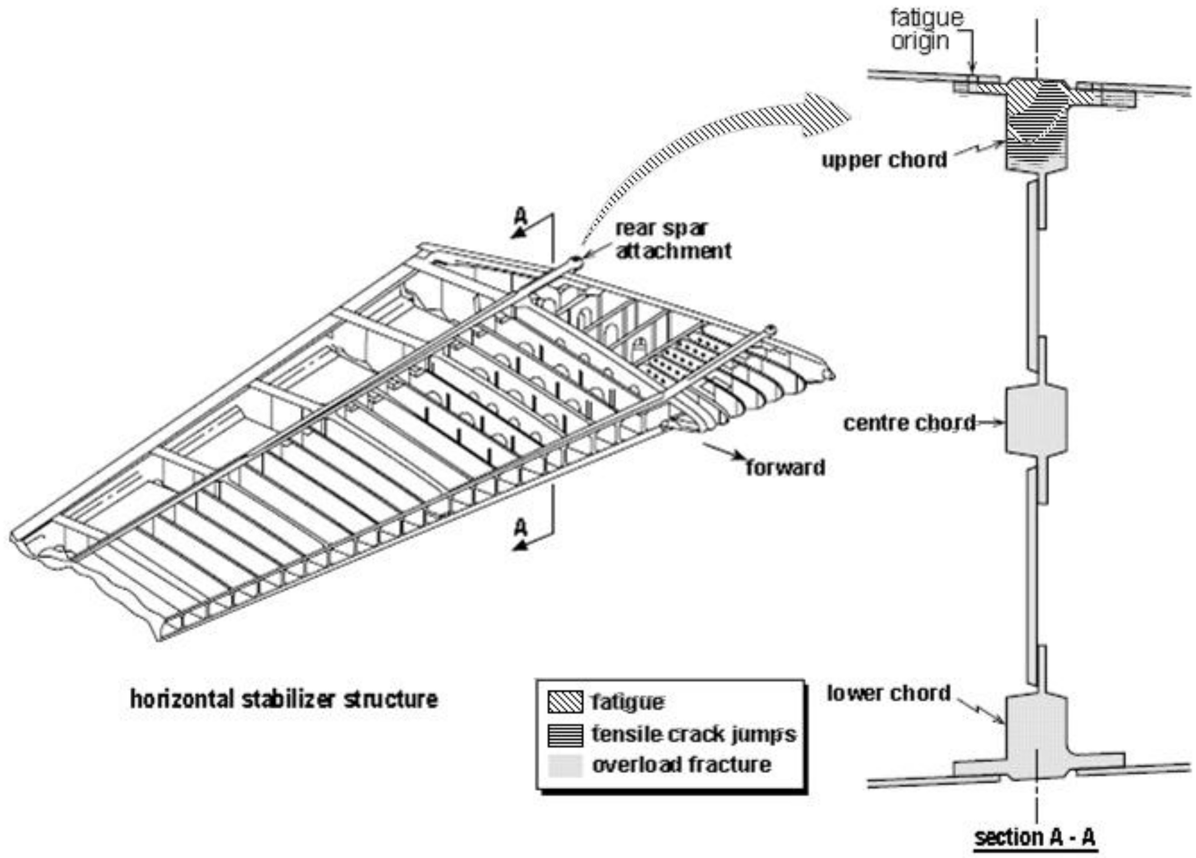

Fig.5.3. Failure origin of Boeing 707-321C right hand horizontal stabilizer [4].

"The Boeing 707: On May 14, 1977, an accident occurred to a Boeing 707-321C at Lusaka when the right hand horizontal stabilizer failed in flight.

Examination of the detached stabilizer revealed a fatigue failure of the top chord of the rear spar, about 36cm outboard of the root attachment pin.

The Boeing 707 was designed and certified against the fail-safe option of the airworthiness requirements and the -100 series underwent fatigue and fail-safe tests before certification.

These included tests specific to the horizontal stabilizer structure. As a result of the fatigue test program, cracks were found in the horizontal stabilizer rear spar upper chord rear flange. Fail-safe tests demonstrated that the structural integrity of the horizontal stabilizer was maintained.

When the 707-300/400 series was developed, the stabilizer assembly was extensively redesigned. No fatigue tests of the redesigned stabilizer structure were carried out.

The airworthiness criteria were met by calculations which were deemed to show that the static and fail-safe strengths of the 300 series horizontal stabilizer were adequate for the design.

In order to comply with fail-safe requirements it would be necessary to define inspections adequate to ensure that small cracks were detected in the spar chord, or to establish that the full fail-safe load could be supported with a complete chord failure.

In the event, neither of these conditions was met. Post accident tests showed that the structure did not have the anticipated residual strength, and neither the inspections detailed in the approved maintenance schedule, nor those recommended by the manufacturer, were adequate to detect partial cracks in the horizontal stabilizer rear spar top chord.

In the immediate post-accident inspections carried out around the world, a further 38 aircraft with horizontal stabilisers were discovered with rear spar fatigue cracks.

The Douglas Dc-10. The accident to a DC-10 at Chicago in May,1979, is covered briefly because, though not resulting from a fatigue problem, it did highlight some important fail-safe issues. Following the accident, the structure concerned was not only re-assessed against the FAR 25.571 rules to which it had been designed, but also was evaluated against the new fatigue rules of FAR 25.571 at amendment 25-45.

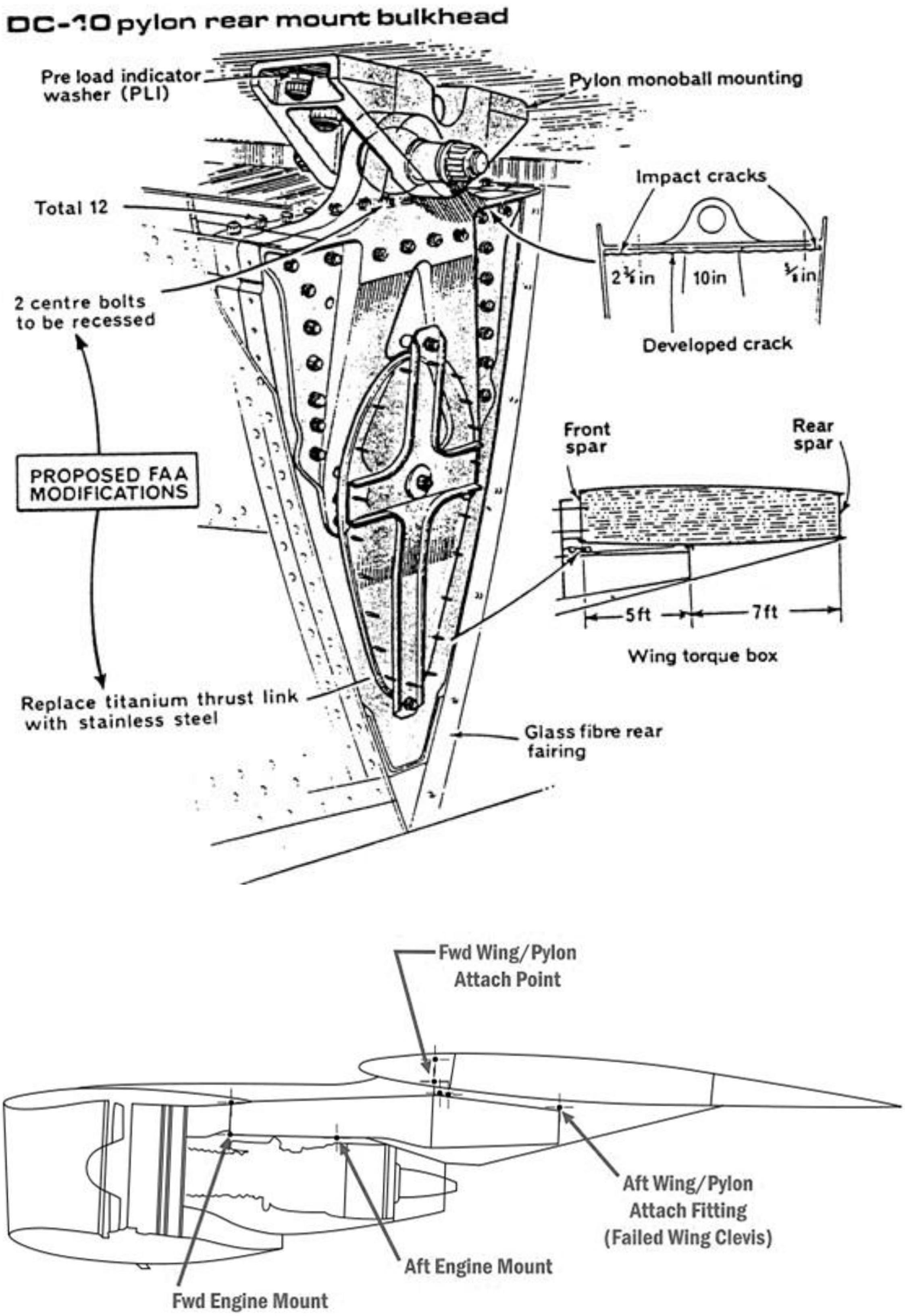

Fig. 5.4. DC-10 Pylon-to-wing attachment [7].

The aircraft crashed following the loss of the left engine and pylon assembly during take-off. The separation of the assembly from the wing damaged the leading edge slat system, causing retraction of the outboard slats and a consequential asymmetric stall of the aircraft.

Following exhaustive tests, the NTSB concluded that the only pre-existing damage to the pylon and its attachment to the wing were a 13in. cracks in the pylon aft bulkhead upper flange. This flange connected the aft bulkhead to the pylon upper spar. Application of take-off thrust resulted in failure of the aft bulkhead allowing the pylon to rotate upwards, fail the other attachments, and separate from the aircraft.

It was determined that 10in. [25.4cm.] of the crack in the bulkhead flange was caused during aircraft maintenance such that, when the pylon was being installed, the wing clevis fitting contacted the flange and its fasteners. A further 3in. propagated in service as a result of fatigue.

During post-accident inspections, six other DC-10 aircraft were found to have cracked upper flanges, though only one had started any fatigue propagation.

Some serious fatigue defects were found in other aircraft which, though not related to the accident, did influence the airworthiness action at the time.

One aircraft was found to have suffered extensive damage to upper spar. This aircraft was certainly not airworthy and maintenance practices were not involved. Because this area was subject only to a sampling inspection program, the fastener failures and web cracking would not have been detected on this particular aircraft had it not been for the inspection required as a result of the accident. Another 30 aircraft was found with fastener defects" [2].

The Aloha Airlines Boeing 737 accident of April 1988 drew popular attention to the aging aircraft question although, of course, the problem has been with us for many years.

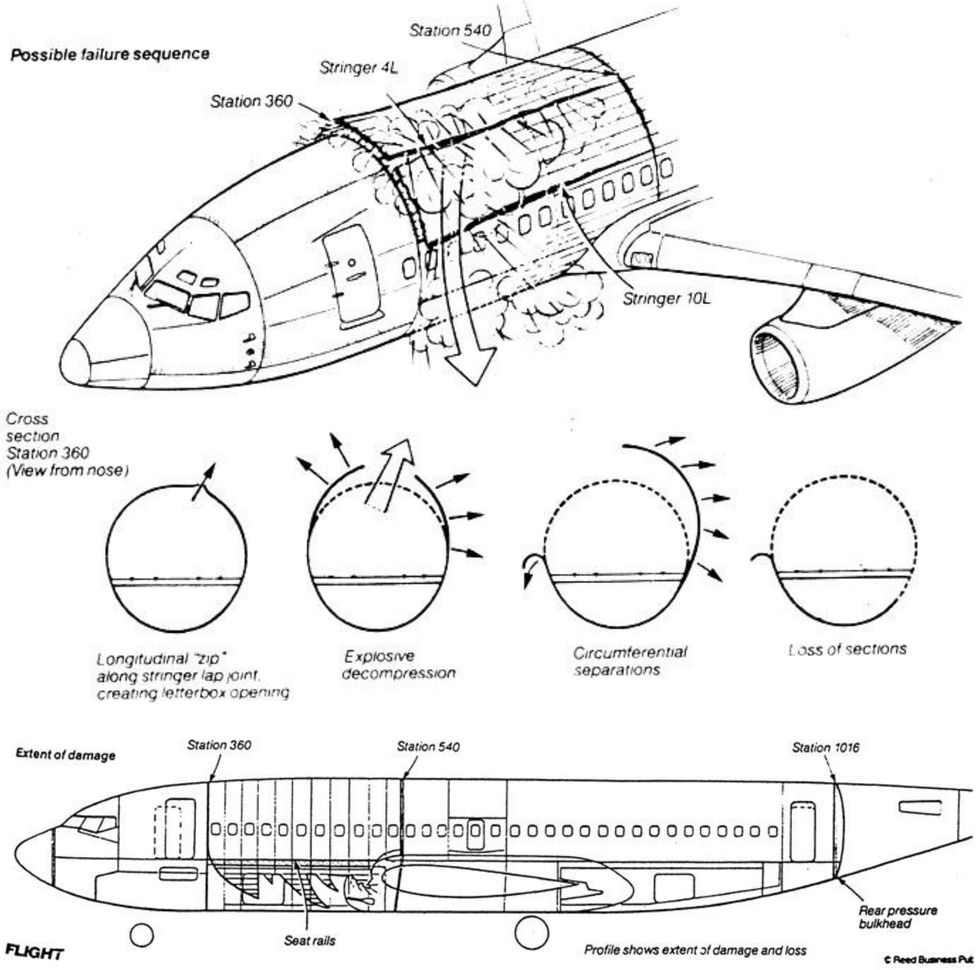

Fig. 5.5. The Aloha Airlines Boeing 737 accident [4, 9].

The U S National Transportation Safety Board (NTSB) findings as to the probable cause of the Aloha accident, released in May 1989, are worth some study as they illustrate the breadth of the problem.

In part, the findings read as follows. " The National transportation Safety Board determines that the probable cause of this accident was the failure of the Aloha Airlines' maintenance programme to detect the presence of significant disbonding and fatigue damage which ultimately led to failure of the lap joint at 5-10L and the separation of the fuselage upper lobe.

Contributing to the accident was:

* the failure of Aloha Airlines’ management to supervise properly its maintenance force;

* the failure of the Federal Aviation Administration (FAA) to evaluate property Aloha Airlines maintenance programme and to assess the airline's inspection and quality control deficiencies;

* the failure of FAA Airworthiness Directive 87-21-08 to require inspection of all the lap joints proposed by Boeing Alert Service Bulletin 737-53A1039;

* and, the lack of a complete terminating action (nether generated by Boeing nor required by the FAA) after the discovery of early production difficulties in the Boeing 737 cold bond lap joint which resulted in low level durability, corrosion and premature fatigue cracking."

The NTSB report clearly allocates blame to all parties including the operator of the aircraft, the manufacturer and the regulatory authority, and covers almost every aspect from initial design to maintenance supervision." [8].

5.1.4. Structural Damage Tolerance. MSG-3

The F-111 is an unusual aircraft: it is variable geometry "swing-wing" fighter-bomber. On December 22, 1969, just over a year after entering service, F-111 #94 lost the left wing during a low-level training flight. The aircraft had accumulated only 107 flight hours and a failure occurred while it was pulling about 3.5g (less than half the design limit load factor). An immediate on-site investigation revealed a flaw in the lower plate of the left-hand wing pivot fitting. This flaw had developed during manufacture and remained undetected despite its considerable size: 23.4 mm x 5.9 mm.

The USAF reconsiders and abandons a SAFE-LIFE approach and provided new guidelines: DAMAGE TOLERANCE philosophy.

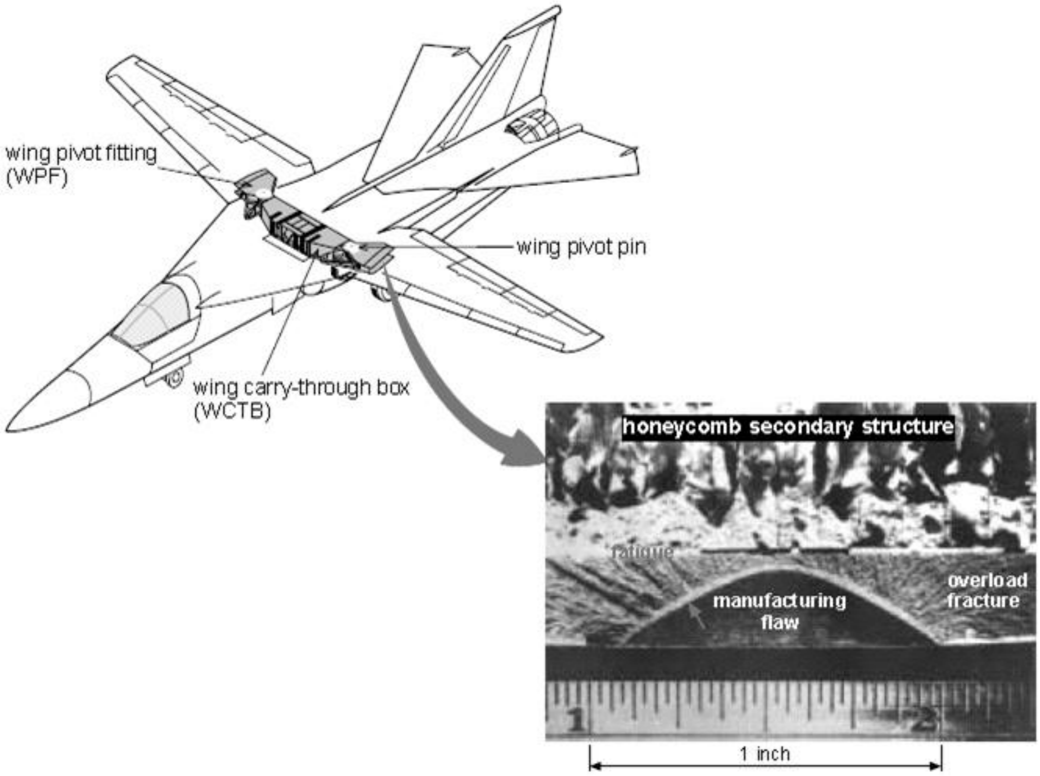

Fig.5. 6. F-111 lost the left wing during a low-level training flight. An immediate on-site investigation revealed a flaw in the lower plate of the left-hand wing pivot fitting [4].

This philosophy is in many ways similar to the fail-safe approach but it goes somewhat further in that consideration is given to crack growth from flaws which may be present in the structure as manufactured. Such flaws may arise from inherent metallurgical imperfections in the material used, or from manufacturing imperfections. The size of the flaws which must be assumed to exist are laid down in the appropriate specifications-typically, they have dimensions lying between 0.51 mm and 2.54 mm and they are assumed to be located at fastener holes and in other critical areas [10].

Fail-Safe concept makes emphasis on the design and test. During strength analysis we have to provide strength under the limit load factor for structure, in which there is so called tolerant damages (usually, fatigue cracks, length of which is equal two interval between stringers on the wing skin or between two frames on the fuselage skin).

During fail-safe test we have to prove, that the requirements are met. But it appears that "fail-safe" structure is not safe, if it is not timely inspected and repaired. So "far better title would be "inspection dependent". This clearly puts the emphasis for safety on the inspector, and implies that, without inspection, things may well be dangerous" [6].

The most clearly the idea was given in the ATA documents, "Airline/Manufacturer Maintenance Program Planning Document, MSG-3", in which described the Damage Tolerance Concept. The document gives guidance for detection inspection program planning. Unlike MSG-1 and MSG-2, used on U S wide body jet transports, MSG-3 consider a quantitative rating system, which is based on the evaluation of probability of timely detection fatigue damage. It is assumed, that the probability is a function of three independent probabilities : (1) the probability, P1, of inspecting an airplane with a damaged structural significant item (SSI), whose failure could affect the structural integrity necessary for the safety of the airplane; (2) the probability, P2, of inspecting the SSI; (3) the probability, P3, of crack detection - a function of crack length, inspection frequency and method, and many variables that defy precise analysis, but may be classified (inspector skill, lighting condition, etc.).

If we have n independent inspections, then the probability of not detecting damage

1-P(D) = (1-P(D, 1))... (1-P(D, n),

where P(D, 1),..., P(D, n) are probabilities of detections for 1,..., n-th inspections.

If we put

DTR = log(1-P(D))/log(1/2),

then

DTR = DTR(1) +... + DTR(n)

-is a such way of representing the cumulative probability of detection, that (1) instead of products of probabilities of non-detections we can use addition of DTR of different inspections; (2) the value of DTR can be interpreted as equivalent number of inspections, when each inspection has a true 50/50 chance of detecting a crack.

DTR increases, when P(D) increases. (P(D) =.96875 corresponds to DTR = 5; P(D) =.99902 corresponds to DTR = 10), and choosing the number of inspection we can get required value of P(D) or DTR.

The DTR system was used by Boeing and operations in order to develop the 727/737/747 Supplemental Structural Inspection programs, which provides credit for existing maintenance programs and options for individual operations to select the most convenient combination of method and frequency for supplemental inspections. Later by Boeing and customer airlines was developed the 757/765 Structural Inspection Programs.

The DTR system is described, for example in [11], in 1984, but fatigue failure of Aloha Airlines Boeing 737 in 1988 has shown, that we have not found a solution to the problem. In fact, there are at least three problems : (1) the assurance, that the designed inspection program will be accomplished by inspector; (2) required value of P(D) or DTR; (3) initial information and method of calculation of DTR;

The first problem is a problem of "human factor", and we do not touch upon it. The second problem in the first approximation can be resolved by the comparison of the value DTR for the fleet of new airplanes with the value for the fleet of similar already discarded airplanes, the level of reliability acceptable for us. And, the last, the third problem, in fact, is "virgin soil". Information is required about the crack lengths detected and associated inspection intervals and methods. A key element is operator feedback to manufacturer, and execution of this task is very difficult. The method of DTR calculation, which is offered by Boeing, is only "very-very approximate approach" (more serious approach was made in the works [12-13]. In fact special worldwide system is needed in order to collect the initial information and to create computer program system for simulation of maintenance process and corresponding calculation. The work is worth to be done.

5.1.5. References

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5.2. Extensions and techniques

5.2.1. A connection of the c.d.f. of the strength of the specimens and the c.d.f. of the strength of a single LI in series system (n_C =1)

In accordance with the structures shown in the Table 3.1. we have the following equations.

For the structures A1, A2, A3 we have respectively

$$F(x)=1-\sum_{k=0}^{n_L} p_k(1-F_Y(x))^k(1-F_Z(x))^{n_L-k}, \qquad (5.1)$$

$$F(x)=1-\left(1-F_Z(x)\right)\sum_{k=0}^{n_L} p_k\left(1-F_Y\left(x\right)\right)^k, \qquad (5.2)$$

$$F(x)=1-\left(1-F_Y\left(x\right)\right)\left(1-F_Z(x)\right). \qquad (5.3)$$

For the structures B1, B2, B3

$$F(x) = 1 - \sum_{k=1}^{n_L} p_k \left(1 - F_Y\left(x\right)\right)^k - p_0\left(1 - F_Z(x)\right)^{n_L} , \tag{5.4 }$$

$$F(x) = 1 - \sum_{k=1}^{n_L} p_k \left(1 - F_Y\left(x\right)\right)^k - p_0(1 - F_Z(x)) , \tag{5.5}$$

$$F(x) = (1 - p_0)F_Y(x) + p_0 F_Z(x) , \tag{5.6}$$

where $\{p_k, k = 0,1,...,n_L\}$ is the probability distribution for the r.v. K_L.

For binomial distribution

$$p_k = b(k; p_L, n_L) , \tag{5.7}$$

where $b(k; p_L, n_L) = p_L^k (1 - p_L)^{n_L - k} n_L !/ k!(n_L - k)!$, p_L is the distribution parameter.

Let us note that in this case

$$p_0 = (1 - p_L)^{n_L} . \tag{5.8}$$

If we approximate a binomial distribution by the right-censored conditional Poisson distribution (under condition that $K_L \le n_L$) then :

$$p_k = (\exp(-\lambda)\lambda^k / k!) / (\exp(-\lambda)\sum_{r=0}^{n_L} \lambda^r / r!) = (\lambda^k / k!) / (\sum_{r=0}^{n_L} \lambda^r / r!) \tag{5.9}$$

and

$$p_0 = 1 / (\sum_{r=0}^{n_L} \lambda^r / r!) . \tag{5.10}$$

If n_L is large enough, then using usual Poisson distribution, instead of equations (5.1) and (5.2), the following equations can be used

$$F(x) = 1 - (1 - F_Z(x))^{n_L} \exp(-\lambda(1 - \delta(x)) , \delta(x) = (1 - F_Y(x))/(1 - F_Z(x)) , \tag{5.11}$$

$$F(x) = 1 - (1 - F_Z(x)) \exp(-\lambda F_Y(x)) , \tag{5.12}$$

where $\lambda = n_L p_L$ or it is an independent parameter of the Poisson distribution. If the defects appear during the process of loading, it may be assumed that $p_L = F_K(x)$, where $F_K(x)$ is a c.d.f. of stress of initiation of a link of Y-type.

We should pay a special attention to the case when r.v. K_L can take only two values. For example $K_L = n_L$ with probability p_{KL} and K_L = 0 with probability (1- p_{KL}). Then, for example, for the structure B2 we have

$$F(x) = p_{KL}(1 - \left(1 - F_Y\left(x\right)\right)^{n_L}) + (1 - p_{KL})F_Z(x) . \tag{5.13}$$

In the numerical examples the Weibull distribution is used for a single LI strength, S. Then the smallest extreme value distribution takes place for $\log(S)$ with c.d.f.

$$F(x) = 1 - \exp(-\exp((x - \theta_0) / \theta_1)) . \tag{5.14}$$

The same type of distribution (but with specific parameters) was used also for defect initiation stress of Y - type link and also for $F_Y(x)$ and $F_Z(x)$ in case of n_C =1. Assumption that the strength of LI without damage is very large is equivalent to the assumption that $F_Z(x) = 0$ for all $-\infty < x < \infty$. In this case it should be simultaneously assumed that $p_0 = 0$.

5.2.2. Connection of the c.d.f. of the strength of the specimens and the c.d.f. of the strength of a single LI using the MC theory (n_C =1)

We consider two cases:

1) general definition of $F_Z(x)$; and

2) $F_Z(x)$ is defined by equation (3.19).

General definition of $F_Z(x)$

As examples, the specifying of the matrix P for p.s. M A1 and MB3, for n_C =1, are considered. The probability that in some link a defect appears at the stress x_t under the condition that it has not appeared at the stress x_{t-1} is

$$b(t) = (F_K(x_t) - F_K(x_{(t-1)})) / (1 - F_K(x_{(t-1)})) .$$

Consider the case of s defects present. The probability that r new defects appear, $0 \le r \le k = n - s$, and the total number of defects is equal to $m=s+r$

$$\tilde{p}_{sm}(t) = (b(t))^r (1 - b(t))^{k-r} k!/r!(k-r)!$$

Conditional probability of Y-type link fracture at the nominal stress x_t

$$q_Y(t) = (F_Y(x_t) - F_Y(x_{(t-1)})) / (1 - F_Y(x_{(t-1)})) .$$

Conditional probability of Z-type link fracture at the nominal stress x_t

$$q_Z(t) = (F_Z(x_t) - F_Z(x_{(t-1)})) / (1 - F_Z(x_{(t-1)})) .$$

Corresponding probability that none of the links (of both types) fail when there are defects in m links for probability structure MA1 is

$$u_m(t) = (1 - q_Y(t))^m (1 - q_Z(t))^{n_L - m} ,$$

The probability of coincidence of these events, which we consider as independent, and the probability of transition from state $i=s+1$ to state $j=i+r$

$$p_{ij}(t) = \tilde{p}_{(i-1)(j-1)}(t) u_{j-1}(t) ,$$

where $i \le j \le (n+1)$.

Conditional fracture probability for the structure MA1 at state i

$$p_{i(n+2)}(t) = 1 - \sum_{j=i}^{n+1} p_{ij}(t) .$$

Of course, $p_{ij}(t) = 0$, if $j < i$, and $p_{(n+2)(n+2)}(t) = 1$.

The corresponding Markov chain for probability structures MB3 has only three states. The first state corresponds to the absence of damaged links, the second one means the presence of at least one damaged link, and the third, an absorbing one, means the failure of the specimen. Corresponding probabilities at a t-th step are determined by the formulae

$$p_{11}(t) = [1 - b(t)]^{n_L} ,\ \ p_{12}(t) = (1 - p_{11}(t))(1 - q_Y(t))(1 - q_Z) ,\ \ p_{13}(t) = 1 - p_{11}(t) - p_{12}(t) ,$$

$$p_{21}(t) = 0 ,\ \ p_{22}(t) = (1 - q_Y(t))(1 - q_Z(t)) ,\ \ p_{23}(t) = 1 - p_{22}(t) ,\ \ p_{31}(t) = p_{32}(t) = 0 ,\ \ p_{33}(t) = 1 .$$

$F_Z(x)$ is defined by equation (3.19).

Specification of the transition probability matrix for a model in framework of probability structures MA.

If $F_Z(x)$ is defined by equation (3.19) and $C=\infty$ then the probability that in some element a defect appears at the stress x_t under the condition that it has not appeared at the stress x_{t-1} is

$$b(t)=(F_K(x_t)-F_K(x_{(t-1)}))/(1-F_K(x_{(t-1)})).$$

Consider the case of *s* defects present. The probability that *r* new defects appear, $0\le r\le k=n-s$, and now the total number of defects is equal to *m=s+r*

$$\tilde{p}_{sm}(t)=(b(t))^r(1-b(t))^{k-r}k!/r!(k-r)!$$

In case when $F_Z(x)$ is defined by (3.19) and parameter *C* is very large (the ‘theoretical’ strength is much higher than the real strength) conditional probability of one element fracture at the nominal stress x_t

$$q(t)=(F_Y(x_t)-F_Y(x_{(t-1)}))/(1-F_Y(x_{(t-1)})).$$

Corresponding probability that there are defects in *m* elements but none of the elements fails is

$$u_m(t)=(1-q(t))^m.$$

The probability of coincidence of these events, which we consider as independent, is the probability of transition from state *i=s+1* to state *j=i+r*

$$p_{ij}(t)=\tilde{p}_{(i-1)(j-1)}(t)u_{j-1}(t),$$

where $i\le j\le(n+1)$.

Conditional probability of fracture at state *i*

$$p_{i(n+2)}(t)=1-\sum_{j=i}^{n+1}p_{ij}(t).$$

Of course, $p_{ij}(t)=0$, if $j<i$, and $p_{(n+2)(n+2)}(t)=1$.

Specification of the transition probability matrix for a model in framework of probability structures MB

The corresponding Markov chain has only three states. The first state corresponds to the absence of defective elements, the second one means the presence of at least one defective element, and the third, an absorbing one, means failure of the specimen. The corresponding probabilities at an *t*th step are determined by the formulae

$$p_{11}(t)=[1-b(t)]^n,\ p_{12}(t)=(1-p_{11}(t))(1-q(t)),\ p_{13}(t)=(1-p_{11}(t))q(t),$$

$$p_{21}(t)=0,\ p_{22}(t)=1-q(t),\ p_{23}(t)=q(t),\ p_{31}(t)=p_{32}(t)=0,\ p_{33}(t)=1.$$

Here $b(t)$ and $q(t)$ are the same as in previous section.

5.2.3. Hypothesis testing

In accordance with goodness-of-fit OSPPTest (Test based on Probability Plot of Ordered Statistics) the ordered observations, $x_{(1)},x_{(2),}...x_{(n)}$, from population with c.d.f. of the type $F(x)=F_0((x-\theta_0)/\theta_1)$, are plotted versus expected values of standard order statistics corresponding to θ_0=0, θ_1=1, $E(\overset{o}{X}_{(i)})$. Such a plot can be used for preliminary visual evaluation of the applicability of the model.

The critical region of the goodness-of-fit test of the hypothesis under consideration is defined by the inequality

$$OSPPt = \left(\sum_{i=1}^{n} (\hat{x}_{(i)} - x_{(i)})^2 / ns^2 \right)^{1/2} > C_{alfa},$$

where $OSPPt$ is the statistic of OSPPTest, $\hat{x}_{(i)} = \hat{\theta}_0 + \hat{\theta}_1 E(\overset{o}{X}_{(i)})$, $\hat{\theta}_0$, $\hat{\theta}_1$ are estimates of θ_0, θ_1, $\bar{x} = \sum_{i=1}^{n} x_i / n$, $s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 / n$.

If linear regression analysis is used for estimation of θ_0 and θ_1, then

$$OSPPt = \bar{R}_{LR} = (1 - R^2)^{1/2} = \left(\sum_{i=1}^{n} (\hat{x}_{(i)} - x_{(i)})^2 / ns^2 \right)^{1/2} > C_{alfa},$$

where $\bar{R}_{LR}$ is the specific notation for $OSPPt, R^2$ is the standard statistic of linear regression analysis (coefficient of determination). An example of a plot ($x_{(i)}$ vs $E(\overset{o}{X}_{(i)})$) for s.e.v. distribution of *X* =log (*Y*), where *Y* is fiber tensile strength for glassfiber with *l*=10 mm (see [5]), is shown in Fig. 5.7.a, and the statistic amounts to *OSPPt*=0.184. The same data are presented in Fig. 5.7.b for normal distribution of *X*, where *OSPPt*=0.321.

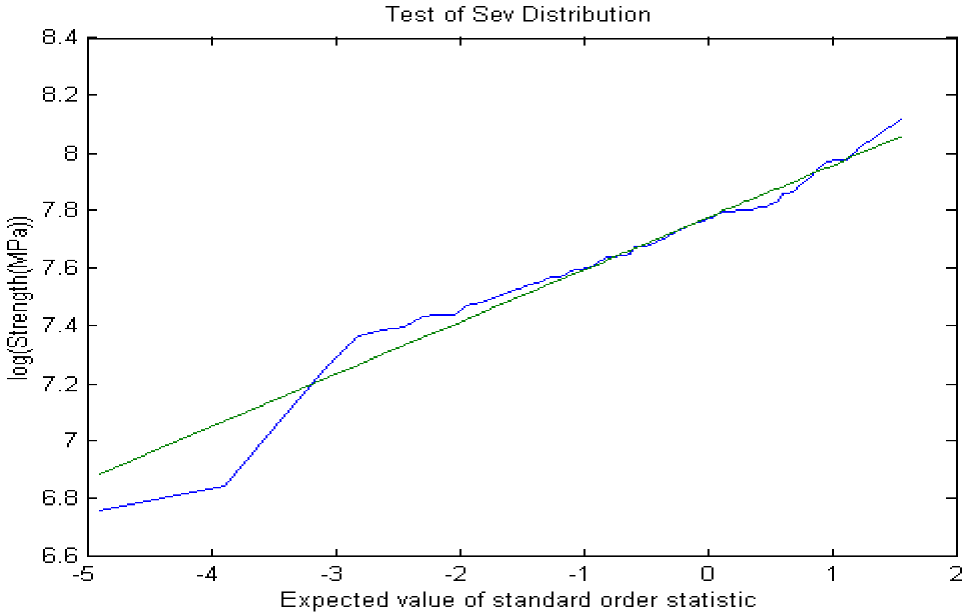

Fig. 5.7.a. Plot of $x_{(i)}$ versus $E(\overset{o}{X}_{(i)})$ for sev distribution of *X* =log (*Y*), where *Y* is fiber tensile strength for specimens with *l*=10 mm; OSPPT=0.184.

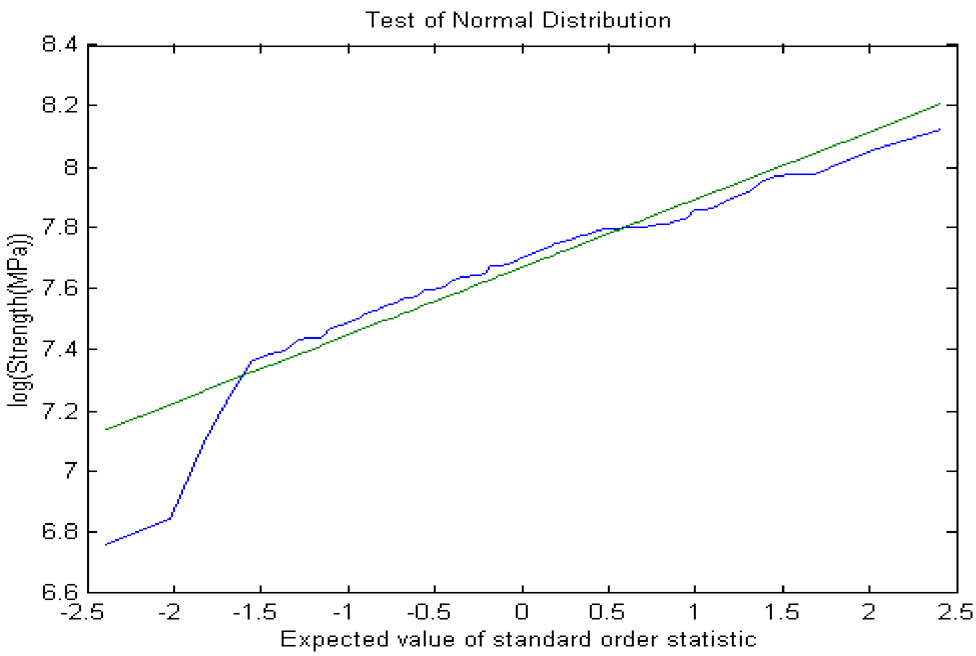

Fig. 5.7.b. Plot of $x_{(i)}$ versus $E(\overset{o}{X}_{(i)})$ for normal distribution of X =log (Y), where Y is fiber tensile strength for specimens with l=10 mm; OSPPt=0.321.

In Fig. 5.8.a, the histograms of OSPPT statistics for sev distribution hypothesis and normal alternative, and in Fig. 5.8.b, the histograms of OSPPT statistics for normal distribution hypothesis and s.e.v. alternative are shown. Numerical values of the boundaries of the critical region, C_{alfa}, and powers of the test for OSPPTest are also given in Fig.5.8: vector (C_{alfa}, power) is equal to (0.23, 0.625) and to (0.179, 0.85) for s.e.v. and normal hypotheses respectively. The Monte Carlo modeling was used for the necessary calculations. Number of Monte Carlo trials, N_{MC}, is equal to 1000. We see that *OSPPt*=0.184< C_{alfa} = 0.23 and *OSPPt*=0.321> C_{alfa} = 0.179 so the s.e.v distribution is more appropriate than a normal distribution. The same conclusion is reached also for glass fibers with l = 20, 40, 80 mm [5]. Nevertheless, it should be noted that the power of OSPPTest is rather limited. Therefore additional analysis is performed using an approximation of the uniformly most powerful invariant test.

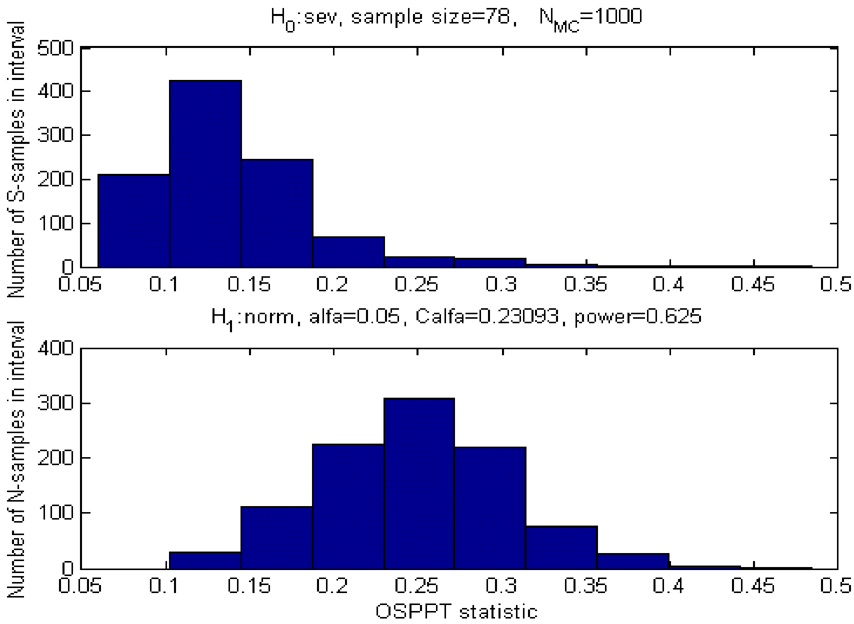

Fig. 5.8.a. Histograms of OSPPT statistics for sev distribution hypothesis and normal alternative

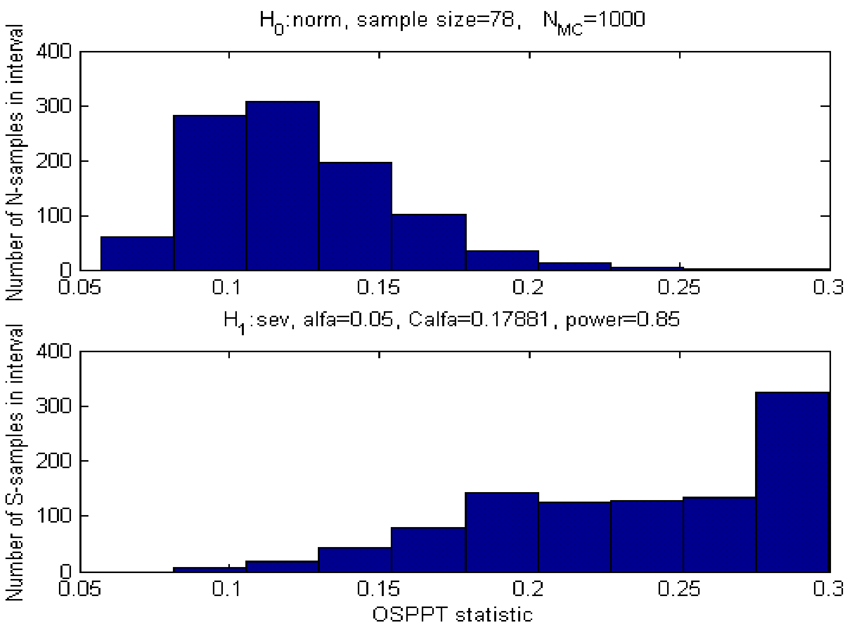

Fig. 5.8.b. Histograms of OSPPT statistics for normal distribution hypothesis and sev alternative

But for the case when we need to make a choice of one of two c.d.f. with unknown location and scale parameters, that is we need to make testing of the hypothesis H_0: $f(x) = (1/\theta_1) g((x-\theta_0)/\theta_1)$, against H_1 - $f(x) = (1/\theta_1) h((x-\theta_0)/\theta_1)$, as it has been already told in section 1.6, there is uniformly most powerful invariant test (UMPIT) [18,19]. For the case when hypothesis H_0 is sev distribution, and alternative H_1 is normal distribution the statistic of this criterion is

$\aleph_{SN} = f_N \,/\, f_S$;

where $f_N = \Gamma((n-1)/2)/2n^{n/2}(\pi D_Z)^{(n-1)/2}$, $f_S = \Gamma(n)\int_0^{\infty} t^{n-2}dt/(\sum_{i=1}^{n}\exp(t(z_i-\bar{z})))^n$,

$z_i = (x_i - \bar{x})/s$, $\bar{x} = \sum_{i=1}^{n} x_i / n$, $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 / n$.

Calculation of c.d.f. for the statistics of the corresponding test is difficult enough. More simple test which has nearly the same power is offered in [19]. The critical region in this case is described by inequality

$$\rho_{SN}(x) = \sum_{i=1}^{n}\exp(\pi(x_i - \bar{x})/s\sqrt{6})/n > C,$$

Conversely, when H_0 is normal c.d.f. and H_1 is sev c.d.f., statistic ρ_{NS} =1/ ρ_{SN} should be used.

In Fig. 5.9.a the histograms of ρ_{SN} statistics for the sev hypothesis and the normal alternative and in Fig. 5.9.b the histograms of ρ_{NS} statistics for the normal hypothesis and the sev alternative are given. In the same figures, the boundary values of critical regions and the test power are shown. For the considered sample of processing the data [5] of test of glass fibers (*l* =10, *n*=78) ρ_{SN} = 1.7652, ρ_{NS} = 0.5665, hence:

ρ_{SN} = 1.7652< C_{alfa} = 1.96,

ρ_{NS} = 0.5665 > C_{alfa} = 0.51.

Therefore, we arrive at the same conclusion as with OSPPTest: the s.e.v. hypothesis can be accepted while normal hypothesis should be rejected. But now the test power is equal to 0.95, clearly higher than power of OSPPTest. The same conclusion again holds true for *l* = 20, 40, 80 mm.

Note that it would be of a considerable interest to apply the OSPPTest to the whole strength data sample (i.e. strength data at all four length values of *l*=10, 20, 40, 80 mm combined); this is an area for further research.

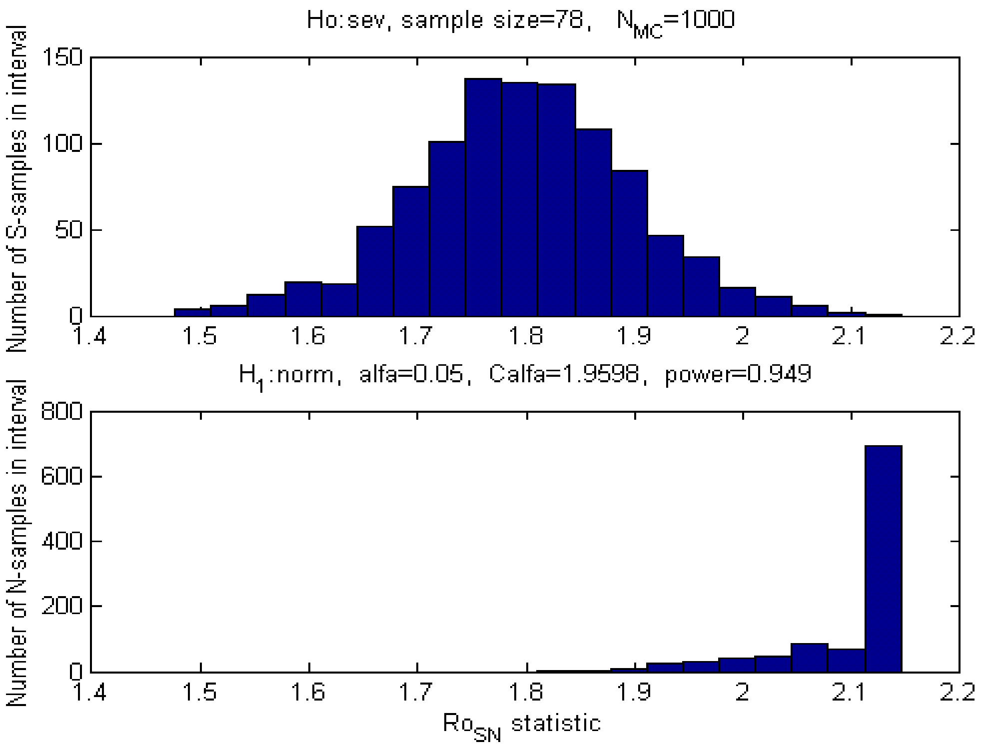

Fig. 5.9.a. Histograms of ρ_{SN} statistics for sev hypothesis and for normal alternative.

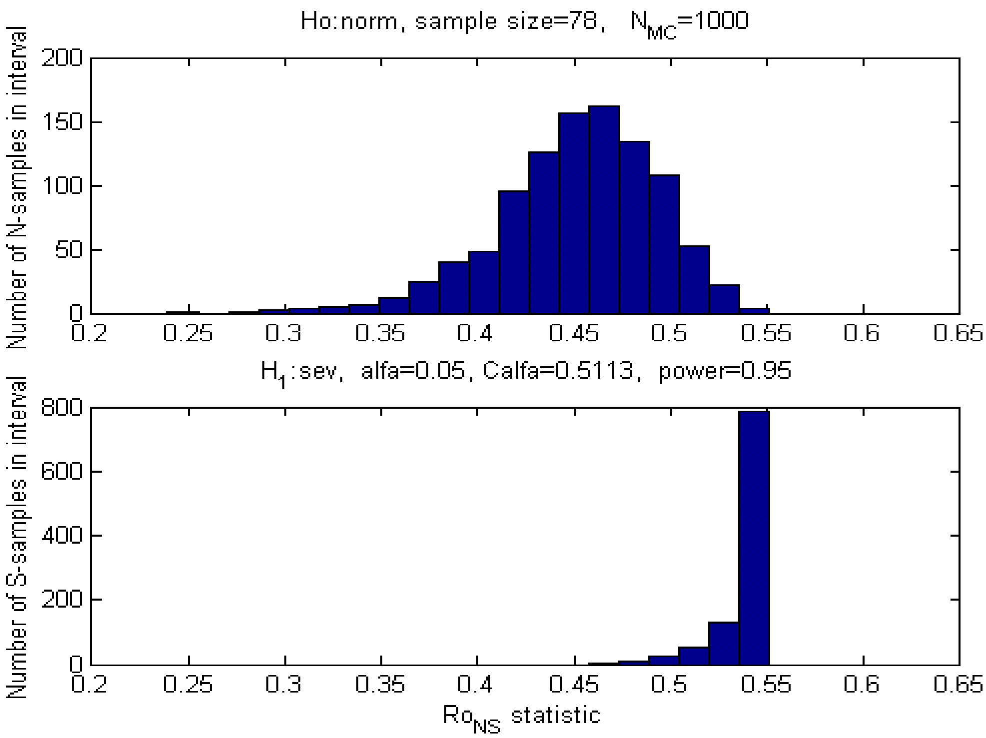

Fig. 5.9.b. Histograms of ρ_{NS} statistics for normal hypothesis and for sev alternative.

5.2.4. Four main versions (hypotheses) of the structure of matrix *P*

Four main versions (hypotheses) of the structure of matrix *P*, denoted as P_a, P_{an_C}, P_b and P_c are considered. In the simplest version we assume that in one step of MC failure of only one

LI can take place. It is convenient (but not necessary) to think that the first failure appears at the boundary of the link and all the following failures can appear only in the adjacent LI. This version corresponds to a transverse crack growth in the monolayer. The stress concentration is supposed to be negligibly small in the so called Global Load Sharing [10-12], leading to uniform distribution of load between the intact LI. We assume that a very small stress concentration is present at a break and neglect it.

For the corresponding matrix P_a we define $p_{ii} = 1 - F_C(x_t(i))$, where $x_t(i) = (x_t n_C / (n_C - i + 1))$, $F_C(x_t(i)) = (F_0(x_t(i) - F_0(x_{t-1}(i)) / (1 - F_0(x_{t-1}(i)))$ is the conditional c.d.f. of strength of a LI, the failure of which does not take place under load x_{t-1}, $F_0(x)$ is the initial c.d.f. of strength of a LI ; $p_{i(i+1)} = 1 - p_{ii}$, $i = 1,...,n_C$, $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

In the second hypothesis we suppose again that in one step of MC a failure of only one LI can take place but now it is the weakest intact LI in the link. Then for the matrix P_{an_C} $p_{ii} = (1 - F_C(x_t(i)))^{n_C+1-i}$ and, again, $p_{i(i+1)} = 1 - p_{ii}$, $i = 1,...,n_C$, $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

In the third hypothesis it is assumed also that the number of failures in one step of MC has a binomial distribution. Then for the corresponding matrix P_b we have $p_{i(i+r)} = b(r; p, k) = p^r (1-p)^{k-r} k!/ r!(k-r)!$, $p = F_C(x_t(i))$, $k = n_c + 1 - i$, $r = 0,...,k$, $i = 1,...,n_C$; and again $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

For the previous versions of P, denoted by P_a, P_{an_C} and P_b, we suppose a uniform load distribution between intact LI. In the fourth hypothesis it is assumed again that the matrix P_C corresponds to a transverse crack growth in the monolayer but this time we take into account the stress concentration at the tip of the crack. Let us denote by *j* the order number of LI in a link (j=1 for the boundary LI) and let the redistribution of load $x(t)$ between the intact LI be defined by a "stress concentration" function $h(j; i, n_C)$. Then in the corresponding P_c matrix we have $p_{ij} = \prod_{i+1}^{j} F_C(x_{ij}(t)) \prod_{j+1}^{n_C+1} (1 - F_C(x_{ij}(t))$ for $j = i+1,...,n_C$; $p_{i(n_C+1)} = \prod_{i+1}^{n_C+1} F_C(x_{ij}(t))$ for $j = n_C$; $p_{ii} = 1 - \sum_{i+1}^{n_C+1} p_{ij}$, $p_{ij} = 0$ for $j < i$, $i = 1,...,n_C$; where $x_{ij}(t) = h(j; i, n_C) x(t) n_C / (n_C + 1 - i)$ describes stress in *j*-th order LI after failure of *i*-th order LI.

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