

RELIABILITY MODELING USING AN EQUILIBRIUM-BASED XGAMMA DISTRIBUTION: PROPERTIES, ESTIMATION, AND APPLICATIONS

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Abstract

The equilibrium distribution approach is applied to the existing Xgamma model in this work to develop a new lifetime model known as the Induced Xgamma (i-Xg) distribution. Because of its adaptable shape properties, the suggested distribution can be used to model heterogeneous and positively skewed lifetime datasets. Probability density and distribution functions, moments, the moment generating function, reliability measures (mean residual life, hazard rate, and reversed hazard), stochastic dominance, entropy measures, inequality curves, and order statistics, including extremes, are among the important analytical properties that we derive. The model's identifiability is rigorously established. The model parameter is estimated using seven traditional estimation methods: Maximum Likelihood (ML), Least Squares (LS), Weighted LSE (WLS), Cramer-von-Mises (CvM), Maximum Product of Spacings (MPS), Anderson-Darling (AD), and Right-tail AD (RAD). Their performance is assessed in terms of bias and MSE for varying sample sizes using a Monte-Carlo simulation study. The suggested i-Xg model provides a better fit than competing distributions, as shown by applications to a variety of real-life datasets, confirming its value in statistical modelling of survival-type and reliability data.

Keywords: Lorenz curve, Entropy, Generating function, Hazard function, MLE, MRLF, Order Statistics

1. INTRODUCTION

The field of data analytics is significant in the age of data technology. The data set's level of uncertainty is determined using the probability distribution. A class of probability distributions that fits all the data sets is improbable. When a class of probability distributions adequately matches a data collection, one can look for another class that does so. Despite the abundance of probability distributions described in the literature, there is always room to investigate new ones that might fit a data set more effectively than those previously used. No probability distribution can claim to be the best match for a specific data set, and what is the best fit now may not be the best fit tomorrow.

The primary motivation behind this research stems from the mathematical elegance and practical utility of induced or equilibrium distributions in modelling lifetime phenomena. For any

continuous, non-negative random variable X representing a baseline lifetime model with survival function $1 - F(x)$ and a finite mean μ , the integral of its survival function equals its mean:

$$\mu = \int_0^{\infty} (1 - F(x)) dx \quad (1)$$

This mathematical identity guarantees that $g(x) = \frac{1-F(x)}{\mu}$ constitutes a valid probability density function (PDF). By incorporating the failure rate function, $r(x) = \frac{f(x)}{1-F(x)}$, this equilibrium transformation can also be structured as a weighted distribution:

$$g(x) = \frac{w(x)f(x)}{E[w(X)]}, \quad \text{where } w(x) = \frac{1}{r(x)}, \quad (2)$$

which provides a weighted interpretation of the baseline model studied by Patil et al. (1977) and Gupta and Kirmani (1990) [1, 2]. This weighted mechanism offers an intuitive, physics- and actuarial-based framework to analyse waiting times and ageing mechanisms, making it highly valuable across queueing systems, transportation engineering, actuarial modelling, and survival analysis.

Historically, generating new families of continuous distributions relied on classical methodologies, such as the transformation-based systems proposed by Johnson (1949) [3] or the differential-equation frameworks introduced by Pearson (1894) [4]. Modern advancements have yielded highly flexible generators, including the beta-generated framework by Eugene et al. (2002) [5], the Kumaraswamy-generalisation by Cordeiro and de Castro (2011)[6], and the Exponentiated T-X family by Alzaghal et al. (2013) [7]. Recently, equilibrium theory has emerged as an alternate frontier for model generation. Notably, Singh and Das (2023) [9] applied this transformation to the Lindley distribution to derive the induced-Lindley (i-Lindley) model, which features the cumulative distribution function (CDF):

$$F(x; \xi) = 1 - \frac{1 + \xi + \xi x}{1 + \xi} e^{-\xi x} \quad (3)$$

The above mathematical formulation in equation (3) was found to be identical to the Garima distribution proposed by Shanker (2016) [8], which later prompted subsequent studies on induced Garima (i-Garima) properties. Driven by these foundational advancements, the novelty of this paper lies in extending this exact line of reasoning beyond the Lindley family to the Xgamma distribution introduced by Sen et al. (2016) [10].

While existing literature provides a vast array of generalised families (such as the beta and Kumaraswamy-G classes), these traditional approaches often introduce many parameters, making estimation mathematically cumbersome. Furthermore, while the equilibrium transformation has been successfully applied to the Lindley and Garima frameworks to capture specific hazard structures, a distinct gap remains in the literature regarding its application to other flexible baseline models. Specifically, the standard Xgamma distribution, as described by Sen et al. (2016) [10], has emerged as a powerful model for lifetime data, yet the literature has not explored how its unique mathematical properties behave under an equilibrium transformation. This leaves a gap in available models that can natively inherit the beneficial traits of the Xgamma baseline while utilising an induced, weighted interpretation to model complex survival structures.

To bridge this gap, this paper explicitly contributes the derivation and introduction of a novel induced distribution known as the Induced Xgamma (i-Xg) model. Based on the framework of the Induced Xgamma (i-Xg) model, our contributions are as follows:

- **Derivation of a Novel Model:** Formulates and introduces the Induced Xgamma (*i-Xg*) distribution as a new mathematical model within lifetime and waiting-time statistics.
- **Methodological Expansion:** Successfully applies equilibrium theory and weighted distribution transformations to the standard Xgamma baseline, extending the application of induced distributions beyond classical Lindley and Garima frameworks.

- **Enhanced Analytical Flexibility:** Contributes a robust, highly flexible alternative that does not belong to the standard exponential family, making it uniquely suited to model complex ageing mechanisms.
- **Improved Practical Data Application:** Provides data practitioners in queueing systems, actuarial science, and survival analysis with an optimised statistical tool engineered specifically for fitting right-skewed lifetime data.

The rest of the article is framed as follows. Section 2 develops the proposed i-Xg distribution and its properties, such as moments, reliability, stochastic ordering, order statistics and their extreme domains, stress-strength reliability, entropy, and inequality curves. Section 3 is devoted to checking the identifiability condition of the proposed distribution. Seven estimation procedures are described in Section 4. In Section 5, simulation study results are presented and compared in terms of bias and mean squared error. Five data sets have been analysed to validate the proposed distribution in Section 6. Section 7 makes some concluding remarks.

2. THE PROPOSED DISTRIBUTION

By applying the equilibrium transformation to the Xgamma distribution, we obtain the induced Xgamma (i - Xg) model as

$$g(x; \xi) = \frac{\xi}{3 + \xi} \left(1 + \xi + \xi x + \frac{\xi^2 x^2}{2} \right) e^{-\xi x}, \quad x > 0, \xi > 0. \quad (4)$$

$$G(x; \xi) = 1 - \left[1 + \frac{\xi x}{\xi + 3} \left(2 + \frac{\xi x}{2} \right) \right] e^{-\xi x}. \quad (5)$$

It is important to note that the i - Xg distribution can be expressed as a mixture of three well-known distributions such as the exponential distribution with rate ξ , the gamma distribution with shape 2 and rate ξ , and the gamma distribution with shape 3 and rate ξ . Specifically,

$$g(x; \xi) = p_1 g_1(x) + p_2 g_2(x) + p_3 g_3(x),$$

where $g_1(x) = \text{Exp}(\xi)$, $g_2(x) = \text{Gamma}(2, \xi)$, $g_3(x) = \text{Gamma}(3, \xi)$, and the mixing weights are

$$p_1 = \frac{1 + \xi}{3 + \xi}, \quad p_2 = \frac{1}{3 + \xi}, \quad p_3 = \frac{1}{3 + \xi}.$$

Although the i-Xg admits a finite mixture representation, it is not reducible to the baseline Xgamma or any known special case. The plot of some PDF and CDF is given in Figure 1.

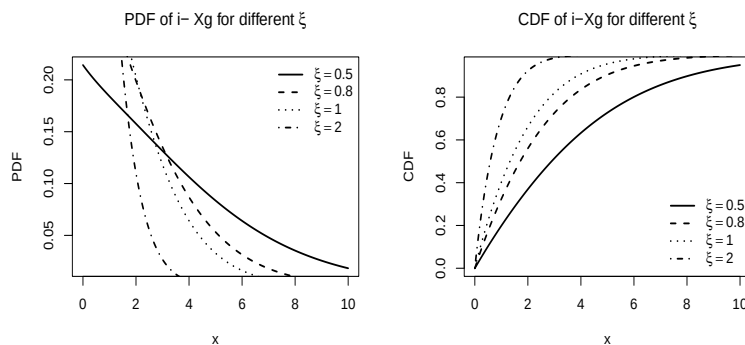


Figure 1: Plot of i-Xg for different values of ξ

2.1. Moments

The r^{th} moment about the origin is

$$\mu'_r = E[X^r] = \frac{r!}{\xi^r(\xi+3)} \left[(1+\xi) + \frac{(r+1)(r+4)}{2} \right].$$

For $r = 1, 2$ and 3 we get,

$$\begin{aligned}\mu'_1 &= \frac{\xi+6}{\xi(\xi+3)} = E(X) = \mu, \\ \mu'_2 &= \frac{2(\xi+10)}{\xi^2(\xi+3)}, \\ \mu'_3 &= \frac{6(\xi+15)}{\xi^3(\xi+3)}.\end{aligned}$$

and from above the variance can be obtained as,

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = \frac{\xi^2 + 14\xi + 24}{\xi^2(\xi+3)^2}.$$

The index of dispersion (ID), $ID(X) = \frac{\mu_2}{\mu} = \frac{\xi^2 + 14\xi + 24}{\xi(3+\xi)(6+\xi)}$ indicates the suitability of the distribution for under or over-dispersed data sets. $ID(X) \geq 1$ implies $\xi \geq 1.399$. Therefore, for $0 < \xi < 1.399$, the distribution is suitable for fitting over-dispersed data, whereas for $\xi > 1.399$, under-dispersed data fitting is appropriate. Here, the coefficient of variation (CV) is less than 1 for all values of ξ , also coefficient of skewness ($\sqrt{\beta_1}$) and kurtosis (β_2) are found to be more than 1 and 3 respectively for different values of ξ ; therefore the proposed distribution is positively skewed and leptokurtic (see, Figure 2).

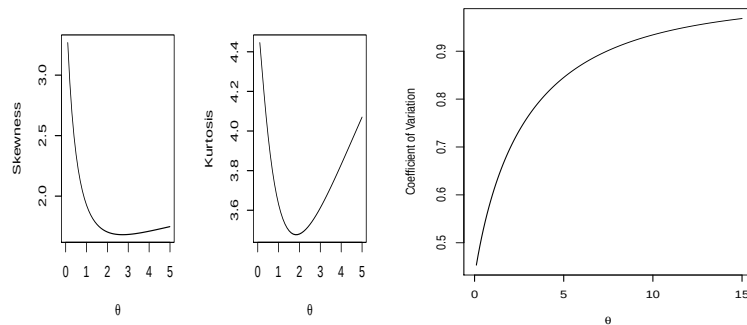


Figure 2: The graph of skewness, kurtosis and Coefficient of variation of i -Xg for different values of ξ

- The harmonic mean is obtained from

$$E\left(\frac{1}{X}\right) = \frac{\xi}{\xi+3}$$

The moment generating function is

$$M_X(t) = \frac{\xi(3\xi^2 + t^2 - 3\xi t)}{(\xi+3)(\xi-t)^3}$$

The central moments of i -Xg distribution is given by

$$E(X - \mu)^k = \frac{\Gamma(k+1)}{\xi^k(3+\xi)} \left(1 + \xi + \frac{(k+1)(k+4)}{2} \right) \sum_{j=0}^k \binom{k}{j} (-1)^{j-k} \mu^{j-k}$$

2.2. Mean Deviation about mean and median

• Mean Deviation about mean

$$\begin{aligned} MD_{mean} &= 2\mu F(\mu) - 2 \int_0^\mu xf(x)dx \\ &= 2\mu \left[1 - \left(1 + \frac{\xi}{\xi+3} (2\mu + \frac{\xi^2\mu^2}{2}) e^{-\xi\mu} \right) \right] \\ &\quad - \frac{2}{\xi(\xi+3)} \left[(6+\xi) \left((1+\mu\xi)(6+\xi) + \frac{5\mu^2\xi^2}{2} + \frac{\xi^3\mu^3}{2} \right) e^{-\mu\xi} \right] \end{aligned}$$

• Mean Deviation about median

$$\begin{aligned} MD_{median} &= \mu - 2 \int_0^\mu xf(x)dx \\ &= \mu - \frac{2}{\xi(\xi+3)} \left[(6+\xi) \left((1+\mu\xi)(6+\xi) + \frac{5\mu^2\xi^2}{2} + \frac{\mu^3\xi^3}{2} \right) e^{-\mu\xi} \right] \end{aligned}$$

2.3. Reliability properties

The reliability function is given by

$$R(x) = 1 - F(x) = \left[1 + \frac{\xi}{\xi+3} (2x + \frac{\xi x^2}{2}) \right] e^{-\xi x}.$$

The hazard rate function is given by

$$r(x) = \frac{f(x)}{R(x)} = \frac{\xi \left(1 + \xi + \xi x + \frac{\xi^2 x^2}{2} \right)}{\left(3 + \xi + 2\xi x + \frac{\xi^2 x^2}{2} \right)}.$$

It is to be noted that $\frac{\xi(1+\xi)}{3+\xi} \leq r(x) < \xi$.

The MRL function is given by

$$mrl(x) = P(X - x | X > x) = \frac{1}{\xi} \left(\frac{6 + \xi + 3\xi x + \frac{\xi^2 x^2}{2}}{3 + \xi + 2\xi x + \frac{\xi^2 x^2}{2}} \right)$$

Since

$$6 + \xi + 3\xi x + \frac{\xi^2 x^2}{2} > 3 + \xi + 2\xi x + \frac{\xi^2 x^2}{2}$$

we found that $mrl(x) > \frac{1}{\xi}$ and $\frac{1}{\xi} < mrl(x) \leq \frac{6+\xi}{\xi(3+\xi)}$. Note that $mrl(0) = \frac{\xi+6}{\xi(\xi+3)} = \mu'_1$.

3. STOCHASTIC ORDERING

Theorem 1. Let $X_i \sim \text{i-Xg}(\xi_i)$, $i = 1, 2$. If $\xi_1 \geq \xi_2$, then

$$X_1 \leq_{lr} X_2 \Rightarrow X_1 \leq_{hr} X_2 \Rightarrow X_1 \leq_{st} X_2 \text{ and } X_1 \leq_{cx} X_2.$$

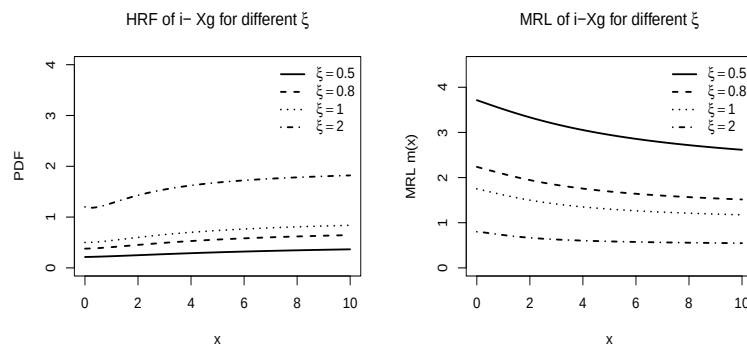


Figure 3: The graph of hazard rate and mean remaining life function of i -Xg for different values of ξ

Proof. Consider the likelihood ratio

$$\begin{aligned}\phi(x) &= \frac{g(x; \xi_1)}{g(x; \xi_2)} \\ &= \frac{1 + \xi_1 + \xi_1 x + \frac{\xi_1^2 x^2}{2}}{1 + \xi_2 + \xi_2 x + \frac{\xi_2^2 x^2}{2}} e^{-(\xi_1 - \xi_2)x}.\end{aligned}$$

Now, differentiating the $\log \phi(x)$ w.r.t. x , we get

$$\begin{aligned}\frac{d}{dx} \log \phi(x) &= \frac{\xi_1 + \xi_1^2 x}{1 + \xi_1 + \xi_1 x + \frac{\xi_1^2 x^2}{2}} - \frac{\xi_2 + \xi_2^2 x}{1 + \xi_2 + \xi_2 x + \frac{\xi_2^2 x^2}{2}} - (\xi_1 - \xi_2). \\ &= \frac{\xi_2^2 \left(1 + \frac{\xi_2 x}{2}\right)}{1 + \xi_2 + \xi_2 x + \frac{\xi_2^2 x^2}{2}} - \frac{\xi_1^2 \left(1 + \frac{\xi_1 x}{2}\right)}{1 + \xi_1 + \xi_1 x + \frac{\xi_1^2 x^2}{2}}.\end{aligned}$$

Let $A(\xi; x) = \frac{\xi^2 \left(1 + \frac{\xi x}{2}\right)}{1 + \xi + \xi x + \frac{\xi^2 x^2}{2}}$. A calculus argument shows $A(\xi; x)$ increases in ξ for $x \geq 0$.

Thus, if $\xi_1 \geq \xi_2$ then $A(\xi_1; x) \geq A(\xi_2; x)$ and

$$\frac{d}{dx} \log \phi(x) = A(\xi_2; x) - A(\xi_1; x) \leq 0.$$

Hence $\phi(x)$ is decreasing in x , proving $X_1 \leq_{lr} X_2$. Standard results give $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$, and for this model the convex order also follows. ■

From the above it is found that ξ acts as a failure-intensity control. A larger value of ξ corresponds to stochastically shorter lifetimes. When $\xi_1 > \xi_2$, the system modeled by X_1 tends to fail earlier, exhibits a uniformly higher hazard rate, and has lower survival probability than the system modeled by X_2 . Thus, ξ governs the rate of aging and reliability behavior of the i -Xg model.

3.1. Distribution of order statistics

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the corresponding order statistics, then the PDF and CDF of r^{th} order statistics say, $Y = X_{(r)}$ is given by,

$$f_{r,n}(y) = \frac{n! \xi \left(1 + \xi + \xi y + \frac{\xi^2 y^2}{2}\right) e^{-\xi y}}{(\xi + 3)(r-1)!(n-r)!} \sum_{l=0}^{n-r} \binom{n-r}{l} \left[1 - \left(1 + \frac{\xi}{\xi + 3} \left(\xi y + \frac{\xi^2 y^2}{2}\right)\right) e^{-\xi y}\right]^{r+l-1}$$

$$F_{r,n}(y) = \sum_{j=r}^n \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-j}{l} (-1)^l \left[1 - \left(1 + \frac{\xi}{\xi+3} (\xi y + \frac{\xi y^2}{2}) \right) e^{-\xi y} \right]^{j+l}$$

3.2. Extreme domain of order statistics

Let the random observations $(X_1, X_2, \dots, X_n)$ come from the proposed distribution. According to the central limit theorem, the $\frac{\bar{X}_n - \mu}{\sqrt{\mu_2}}$ approaches to standard normal distribution as $n \rightarrow \infty$. The asymptotic distributions of order statistics may be of interest. The highest and smallest order statistics are $X_{(n)} = \max(X_1, X_2, \dots, X_n)$ and $X_{(1)} = \min(X_1, X_2, \dots, X_n)$, respectively. These two extreme order statistics serve as representations of the lives of parallel and series systems and are important tools in reliability analysis. For the CDF in (5), it is noticed that

$$\lim_{t \rightarrow \infty} \frac{1 - G(t+x)}{1 - G(t)} = e^{-\xi x}$$

and

$$\lim_{t \rightarrow 0} \frac{G(tx)}{G(t)} = x.$$

So,

$$\lim_{n \rightarrow \infty} P[a_n(X_{(n)} - b_n) \leq x] = e^{-e^{-\xi x}}, \quad (6)$$

where, $a_n = 1$ and $b_n = G^{-1}(1 - \frac{1}{n})$ and

$$\lim_{n \rightarrow \infty} P[c_n(X_{(1)} - d_n) \leq x] = 1 - e^{-x}, \quad (7)$$

where, $c_n = 1$ and $d_n = G^{-1}(\frac{1}{n})$. It is to be noted that for the exponential distribution, the asymptotic distributions of $X_{(n)}$ and $X_{(1)}$ also take the forms given by (6) and (7). The largest order statistic converges to the extreme value distribution, and the smallest order statistic converges to the exponential distribution.

3.3. Stress- strength Reliability

Let the stress $X \sim i-Xg(\xi_1)$ and strain $Y \sim i-Xg(\xi_2)$ be independent to each other. The stress-strength index (R) is

$$R = P(X < Y) = \int_0^\infty F_X(y; \xi_1) g_Y(y; \xi_2) dy.$$

Let,

$$\begin{aligned} g_Y(y; \xi_2) &= K (1 + \xi_2 + \xi_2 y + \frac{\xi_2^2}{2} y^2) e^{-\xi_2 y} \\ F_X(y; \xi_1) &= 1 - \left(1 + a_1(2y + b_1 y^2) \right) e^{-\xi_1 y} \end{aligned}$$

where $K = \xi_2 / (\xi_2 + 3)$, $a_1 = \frac{\xi_1}{\xi_1 + 3}$ and $b_1 = \frac{\xi_1}{2}$. Hence

$$R = \int_0^\infty g_Y(y; \xi_2) dy - K \int_0^\infty \left(1 + a_1(2y + b_1 y^2) \right) \left(1 + \xi_2 + \xi_2 y + \frac{\xi_2^2}{2} y^2 \right) e^{-(\xi_1 + \xi_2)y} dy.$$

Since $\int_0^\infty g_Y(y; \xi_2) dy = 1$, we have $R = 1 - I$ with

$$I = K \int_0^\infty \underbrace{\left(1 + 2a_1 y + \frac{a_1 \xi_1}{2} y^2 \right)}_{A(y)} \underbrace{\left(1 + \xi_2 + \xi_2 y + \frac{\xi_2^2}{2} y^2 \right)}_{B(y)} e^{-\lambda y} dy, \quad \lambda = \xi_1 + \xi_2.$$

Define, $A(y) = 1 + c_1y + c_2y^2$ with $c_1 = 2a_1 = \frac{2\zeta_1}{\zeta_1 + 3}$ where, $c_2 = \frac{a_1\zeta_1}{2} = \frac{\zeta_1^2}{2(\zeta_1 + 3)}$ and $B(y) = b_0 + b_1y + b_2y^2$ where $b_0 = 1 + \zeta_2$, $b_1 = \zeta_2$, $b_2 = \frac{\zeta_2^2}{2}$. Then

$$A(y)B(y) = p_0 + p_1y + p_2y^2 + p_3y^3 + p_4y^4,$$

where

$$\begin{aligned} p_0 &= b_0 = 1 + \zeta_2, \\ p_1 &= b_0c_1 + b_1 = \frac{2\zeta_1(1 + \zeta_2)}{\zeta_1 + 3} + \zeta_2, \\ p_2 &= b_0c_2 + b_1c_1 + b_2 = \frac{(1 + \zeta_2)\zeta_1^2}{2(\zeta_1 + 3)} + \frac{2\zeta_1\zeta_2}{\zeta_1 + 3} + \frac{\zeta_2^2}{2}, \\ p_3 &= b_1c_2 + b_2c_1 = \frac{\zeta_2\zeta_1^2}{2(\zeta_1 + 3)} + \frac{\zeta_1\zeta_2^2}{\zeta_1 + 3}, \\ p_4 &= b_2c_2 = \frac{\zeta_1^2\zeta_2^2}{4(\zeta_1 + 3)}. \end{aligned}$$

Using $\int_0^\infty y^n e^{-\lambda y} dy = \frac{n!}{\lambda^{n+1}}$ for $\lambda > 0$, we get

$$I = K \left(\frac{p_0}{\lambda} + \frac{p_1}{\lambda^2} + \frac{2p_2}{\lambda^3} + \frac{6p_3}{\lambda^4} + \frac{24p_4}{\lambda^5} \right), \quad K = \frac{\zeta_2}{\zeta_2 + 3}, \quad \lambda = \zeta_1 + \zeta_2.$$

Therefore,

$$R = 1 - \frac{\zeta_2}{\zeta_2 + 3} \left(\frac{p_0}{\lambda} + \frac{p_1}{\lambda^2} + \frac{2p_2}{\lambda^3} + \frac{6p_3}{\lambda^4} + \frac{24p_4}{\lambda^5} \right), \quad \lambda = \zeta_1 + \zeta_2,$$

with $p_0, \dots, p_4$ given above. This closed form follows directly from the polynomial expansion and elementary gamma-integrals.

Since larger ζ yields stochastically smaller lifetimes for i-Xg, R is increasing in ζ_1 (stress becomes smaller) and decreasing in ζ_2 (strength becomes smaller). Hence designs with *large* ζ_1 and *small* ζ_2 achieve higher stress–strength reliability.

3.4. Inequality curves

The Lorenz curve is given by

$$\begin{aligned} L_F(x) &= \frac{1}{E(x)} \int_0^x t f(t) dt \\ &= \frac{1}{\zeta + 9} \left[(\zeta + 6) - \left((\zeta + 6)(x + 1) + \frac{(5 + \zeta x)}{2} \zeta^2 x^2 \right) e^{-\zeta x} \right] \end{aligned}$$

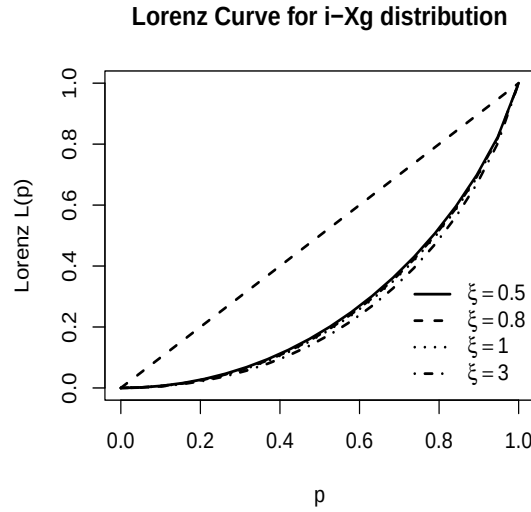


Figure 4: The Lorenz curve of i -Xg for different values of ξ

4. IDENTIFIABILITY CONDITION

We have to check whether the i -xgamma model is identifiable. Assume for two different parameter ξ_1 and ξ_2 we have the same distribution i.e. $f(x; \xi_1) = f(x; \xi_2)$. The distributions can be shown as,

$$f(x; \xi_1) = \frac{\xi_1}{3 + \xi_1} \left(1 + \xi_1 + \xi_1 x + \frac{\xi_1^2 x^2}{2} \right) e^{-\xi_1 x}; \quad x > 0, \xi_1 > 0$$

$$f(x; \xi_2) = \frac{\xi_2}{3 + \xi_2} \left(1 + \xi_2 + \xi_2 x + \frac{\xi_2^2 x^2}{2} \right) e^{-\xi_2 x}; \quad x > 0, \xi_2 > 0.$$

Now, from our assumption, we get

$$f(x; \xi_1) = f(x; \xi_2)$$

$$\text{or, } \frac{\xi_1}{3 + \xi_1} \left(1 + \xi_1 + \xi_1 x + \frac{\xi_1^2 x^2}{2} \right) e^{-\xi_1 x} = \frac{\xi_2}{3 + \xi_2} \left(1 + \xi_2 + \xi_2 x + \frac{\xi_2^2 x^2}{2} \right) e^{-\xi_2 x}$$

Taking "log" on both sides, we get

$$\log \left(\frac{\xi_1}{\xi_1 + 3} \right) + \log \left(1 + \xi_1 + \xi_1 x + \frac{\xi_1^2 x^2}{2} \right) - \xi_1 x = \log \left(\frac{\xi_2}{\xi_2 + 3} \right) +$$

$$\log \left(1 + \xi_2 + \xi_2 x + \frac{\xi_2^2 x^2}{2} \right) - \xi_2 x$$

$$\text{or, } \log \left(\frac{\xi_1(\xi_2 + 3)}{\xi_2(\xi_1 + 3)} \right) + \log \left(\frac{1 + \xi_1 + \xi_1 x + \frac{\xi_1^2 x^2}{2}}{1 + \xi_2 + \xi_2 x + \frac{\xi_2^2 x^2}{2}} \right) - (\xi_1 - \xi_2)x = 0$$

Since the expression equals zero for all $x > 0$, the coefficient of x must be zero, which implies $\xi_1 = \xi_2$. Thus, the parameters ξ_1 and ξ_2 are equal, which contradicts our assumption that they are different. This proves that the parameter ξ of the given statistical distribution i -Xg is identifiable, and that we can obtain precise estimates of its value.

5. ESTIMATION OF ξ USING DIFFERENT METHODS

Let $x_1, \dots, x_n$ be an i.i.d. sample from $i\text{-Xg}(\xi)$. Here we have used seven methods of estimation, namely,

1. Maximum Likelihood Estimation (MLE),
2. Ordinary Least Squares Estimation (OLSE),
3. Weighted Least Squares Estimation (WLSE),
4. Cramér-von Mises Estimation (CvME),
5. Maximum Product of Spacings Estimation (MPSE),
6. Anderson-Darling Estimation (ADE) and
7. Right-Tail Anderson Darling Estimation (RADE)

5.1. MLE

The likelihood and log-likelihood are

$$L(\xi) = \prod_{i=1}^n g(x_i; \xi) = \left(\frac{\xi}{\xi+3}\right)^n \prod_{i=1}^n \left(1 + \xi + \xi x_i + \frac{\xi^2 x_i^2}{2}\right) e^{-\xi x_i},$$

$$\ell(\xi) = n\{\log \xi - \log(\xi + 3)\} + \sum_{i=1}^n \log\left(1 + \xi + \xi x_i + \frac{\xi^2 x_i^2}{2}\right) - \xi \sum_{i=1}^n x_i.$$

Differentiating *w.r.t* ξ ,

$$\begin{aligned} \ell'(\xi) &= \frac{n}{\xi} - \frac{n}{\xi+3} + \sum_{i=1}^n \frac{1 + x_i + \xi x_i^2}{1 + \xi + \xi x_i + \frac{\xi^2 x_i^2}{2}} - \sum_{i=1}^n x_i \\ &= \frac{3n}{\xi(\xi+3)} + \sum_{i=1}^n \frac{1 + x_i + \xi x_i^2}{1 + \xi + \xi x_i + \frac{\xi^2 x_i^2}{2}} - n\bar{x}. \end{aligned}$$

The MLE $\hat{\xi}_{\text{MLE}}$ solves $\ell'(\xi) = 0$. A closed form is unavailable; use a 1-D root finder (e.g., Newton, BFGS) starting from a positive initial value.

5.2. OLSE

Let the ordered sample as $x_{1:n} < \dots < x_{n:n}$. All minimum-distance estimators below minimize a sum of squared (or weighted) differences between model CDF values $F(x_{i:n}; \xi)$. The normal equations use the CDF derivative

$$\eta_1(x | \xi) := \frac{\partial}{\partial \xi} F(x; \xi).$$

From $F(x; \xi) = 1 - A(\xi)e^{-\xi x}$ where $A(\xi) = 1 + \frac{\xi}{\xi+3}\left(2x + \frac{\xi x^2}{2}\right)$,

$$\eta_1(x | \xi) = -A'(\xi)e^{-\xi x} + A(\xi)x e^{-\xi x},$$

with

$$A'(\xi) = \frac{3}{(\xi+3)^2} \left(2x + \frac{\xi x^2}{2}\right) + \frac{\xi}{\xi+3} \cdot \frac{x^2}{2}.$$

So,

$$\eta_1(x | \xi) = \left[x \left(1 + \frac{\xi}{\xi + 3} \left(2x + \frac{\xi x^2}{2} \right) \right) - \frac{3}{(\xi + 3)^2} \left(2x + \frac{\xi x^2}{2} \right) - \frac{\xi}{\xi + 3} \cdot \frac{x^2}{2} \right] e^{-\xi x} \quad (8)$$

Define

$$S(\xi) = \sum_{i=1}^n \left[F(x_{i:n}; \xi) - \frac{i}{n+1} \right]^2.$$

The LSE $\hat{\xi}_{\text{LSE}}$ minimizes $S(\xi)$ and solves the normal equation

$$\sum_{i=1}^n \left[F(x_{i:n}; \xi) - \frac{i}{n+1} \right] \eta_1(x_{i:n} | \xi) = 0.$$

where $\eta_1(x_{i:n} | \xi)$ is given in equation 8.

5.3. WLSE

With Beta-variance weights $w_i = \frac{(n+1)^2}{(n+2)i(n-i+1)}$,

$$W(\xi) = \sum_{i=1}^n w_i \left[F(x_{i:n}; \xi) - \frac{i}{n+1} \right]^2,$$

and the WLSE $\hat{\xi}_{\text{WLSE}}$ satisfies

$$\sum_{i=1}^n w_i \left[F(x_{i:n}; \xi) - \frac{i}{n+1} \right] \eta_1(x_{i:n} | \xi) = 0.$$

where $\eta_1(x_{i:n} | \xi)$ is given in equation 8.

5.4. CvME

Using the standard plotting positions $\frac{2i-1}{2n}$,

$$C(\xi) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}; \xi) - \frac{2i-1}{2n} \right]^2,$$

and the CvM estimator $\hat{\xi}_{\text{CvME}}$ solves

$$\sum_{i=1}^n \left[F(x_{i:n}; \xi) - \frac{2i-1}{2n} \right] \eta_1(x_{i:n} | \xi) = 0.$$

where $\eta_1(x_{i:n} | \xi)$ is given in equation 8.

5.5. MPSE

Define spacings on the transformed scale

$$D_i(\xi) = F(x_{i:n}; \xi) - F(x_{i-1:n}; \xi), \quad F(x_{0:n}; \xi) = 0, \quad F(x_{n+1:n}; \xi) = 1.$$

The MPS estimator maximizes the geometric mean

$$G(\xi) = \left(\prod_{i=1}^{n+1} D_i(\xi) \right)^{1/(n+1)} \iff H(\xi) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\xi) \text{ is maximized.}$$

Differentiating $H(\xi)$ w.r.t. ξ gives

$$\frac{\partial H}{\partial \xi} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{\eta_1(x_{i:n} | \xi) - \eta_1(x_{i-1:n} | \xi)}{D_i(\xi)} = 0.$$

5.6. ADE

Define,

$$A(\xi) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left\{ \log F(x_{i:n}; \xi) + \log [1 - F(x_{n+1-i:n}; \xi)] \right\},$$

We obtain ADE by minimising $A(\xi)$ w.r.t. ξ and the normal equation as

$$\sum_{i=1}^n (2i-1) \left[\frac{\eta_1(x_{i:n} | \xi)}{F(x_{i:n}; \xi)} - \frac{\eta_1(x_{n+1-i:n} | \xi)}{1 - F(x_{n+1-i:n}; \xi)} \right] = 0.$$

5.7. RADE

The RADE uses

$$R(\xi) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n}; \xi) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log [1 - F(x_{n+1-i:n}; \xi)],$$

and satisfies

$$-2 \sum_{i=1}^n \eta_1(x_{i:n} | \xi) + \frac{1}{n} \sum_{i=1}^n (2i-1) \frac{\eta_1(x_{n+1-i:n} | \xi)}{1 - F(x_{n+1-i:n}; \xi)} = 0.$$

Here all estimating equations are one-dimensional in ξ and can be solved by a bracketed method (e.g., Brent) in latest R-4.5.2 which enforces $\xi > 0$. We take,

$$\xi_{\text{MLE}}^{(0)} = \min \left\{ \frac{1}{\bar{x}}, \frac{3}{\bar{x}} \right\}, \quad \xi_{\text{MD}}^{(0)} : \text{minimize the objective on a coarse grid (say, } [10^{-3}, 10]).$$

For stability in MPS/AD/RAD, clip F to $[\epsilon, 1 - \epsilon]$ with $\epsilon \approx 10^{-12}$ to avoid $\log 0$, and regularize tiny spacings $D_i(\xi)$ similarly.

6. SIMULATION STUDY

To evaluate the performance of the proposed estimation methods (MLE, LSE, WLSE, CvME, MPS, ADE, and RADE) for the i-Xg distribution, a Monte Carlo simulation experiment is conducted. The primary objective of this study is to compare the bias and mean squared error (MSE) of each estimator across different sample sizes and parameter values.

Random samples from the i-Xg distribution are generated using its mixture representation. For a given parameter $\xi > 0$, the distribution can be expressed as a mixture of three components:

$$p_1 = \frac{1 + \xi}{\xi + 3}, \quad p_2 = \frac{1}{\xi + 3}, \quad p_3 = \frac{1}{\xi + 3},$$

corresponding respectively to an exponential(ξ) distribution, a gamma(2, ξ) distribution, and a gamma(3, ξ) distribution. Thus, a simulated observation from the i-Xg distribution is obtained by

1. Generate $U \sim \text{Uniform}(0, 1)$.
2. If $U \leq p_1$, set $X \sim \text{Exponential}(\xi)$.
3. Else if $p_1 < U \leq p_1 + p_2$, set $X \sim \text{Gamma}(2, \xi)$.
4. Otherwise, set $X \sim \text{Gamma}(3, \xi)$.

This algorithm produces exact samples from the i-Xg model without relying on numerical inversion or approximation techniques.

The simulation is carried out for sample sizes

$$n = 10, 20, 40, 70, 100,$$

and parameter values

$$\zeta = 0.1, 0.5, 1.0, 1.5.$$

For each (n, ζ) combination, $N = 5000$ independent datasets are generated. Each dataset is analyzed using all seven estimation techniques implemented in R, ensuring that the parameter space restriction $\zeta > 0$ is enforced during optimization and two evaluation metrics are computed as

$$\text{Bias}(\hat{\zeta}) = E[\hat{\zeta}] - \zeta, \quad \text{MSE}(\hat{\zeta}) = E[(\hat{\zeta} - \zeta)^2].$$

The Bias and MSE plot for different ζ are shown in Figure 5.

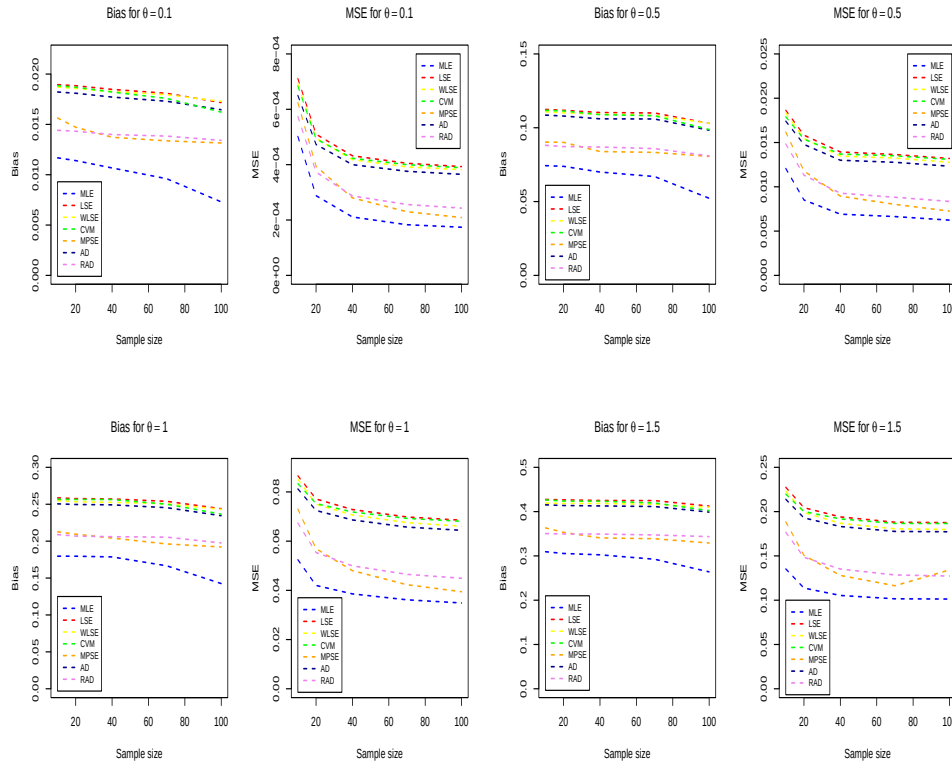


Figure 5: Bias and MSE plot for different ζ

From the Table 1 and Table 2 it is observed that lower values of both bias and MSE indicate superior estimator performance. All estimators are compared to determine their convergence behaviour as the sample size increases. In general, consistent estimators are expected to exhibit decreasing bias and MSE as n increases, confirming desirable asymptotic properties. Though, the minor increase observed for ADE and RADE at $n=100$ relative to $n=70$ is attributable to Monte Carlo variability and does not indicate estimator inconsistency, since the overall trend remains decreasing.

Table 1: True value of ξ , average bias and std. error (in parentheses) of the different estimation procedures for i -the Xg distribution

ξ	n	MLE	LSE	WLSE	CVM	MPSE	AD	RAD
0.1	10	0.007179 (0.021762)	0.017169 (0.025945)	0.017319 (0.025674)	0.016201 (0.025425)	0.015536 (0.024456)	0.016349 (0.024942)	0.013307 (0.023437)
	20	0.009822 (0.017093)	0.018036 (0.022530)	0.018014 (0.022286)	0.017531 (0.022166)	0.014954 (0.020020)	0.017309 (0.021735)	0.013917 (0.019335)
	40	0.010687 (0.014472)	0.018472 (0.020766)	0.018224 (0.020405)	0.018213 (0.020547)	0.013738 (0.016686)	0.017703 (0.019986)	0.014001 (0.016922)
	70	0.011344 (0.013455)	0.018718 (0.019995)	0.018477 (0.019678)	0.018569 (0.019859)	0.013314 (0.015090)	0.017980 (0.019254)	0.014222 (0.015884)
	100	0.011490 (0.012979)	0.018755 (0.019654)	0.018535 (0.019382)	0.018650 (0.019556)	0.012974 (0.014274)	0.018024 (0.018923)	0.014210 (0.015394)
	10	0.056032 (0.110675)	0.105953 (0.138717)	0.105772 (0.137173)	0.101781 (0.135948)	0.093744 (0.128869)	0.101434 (0.133955)	0.084459 (0.123437)
0.5	20	0.065629 (0.093265)	0.108759 (0.125887)	0.108006 (0.124361)	0.106598 (0.124140)	0.088872 (0.109063)	0.104644 (0.121699)	0.085509 (0.106873)
	40	0.071072 (0.084370)	0.110747 (0.118904)	0.109299 (0.117141)	0.109636 (0.117902)	0.084927 (0.095776)	0.106809 (0.114990)	0.086766 (0.097357)
	70	0.073885 (0.081264)	0.112102 (0.116606)	0.110621 (0.114993)	0.111463 (0.116003)	0.082866 (0.089290)	0.108224 (0.112772)	0.087687 (0.093691)
	100	0.074492 (0.079493)	0.112324 (0.115408)	0.110623 (0.113586)	0.111874 (0.114975)	0.081251 (0.085745)	0.108439 (0.111545)	0.087702 (0.091832)
	10	0.148040 (0.231379)	0.247865 (0.296582)	0.247289 (0.294643)	0.240555 (0.291037)	0.217815 (0.273609)	0.238763 (0.287509)	0.202601 (0.262300)
	20	0.165271 (0.205865)	0.252654 (0.277219)	0.249990 (0.274027)	0.248808 (0.273896)	0.208287 (0.239175)	0.244093 (0.268959)	0.203821 (0.235294)
1	40	0.173788 (0.192446)	0.255933 (0.267270)	0.251542 (0.262755)	0.253951 (0.265415)	0.199349 (0.215060)	0.247569 (0.259085)	0.205316 (0.220271)
	70	0.179196 (0.190050)	0.257952 (0.264772)	0.253069 (0.259756)	0.256809 (0.263674)	0.195831 (0.205500)	0.249809 (0.256724)	0.206894 (0.215931)
	100	0.182646 (0.189699)	0.260250 (0.264776)	0.255288 (0.259718)	0.259445 (0.263993)	0.195054 (0.201513)	0.252224 (0.256801)	0.209211 (0.215159)
	10	0.267098 (0.365240)	0.416412 (0.475352)	0.413645 (0.471172)	0.406140 (0.467127)	0.366197 (0.433115)	0.401970 (0.461382)	0.346370 (0.418505)
	20	0.288168 (0.333633)	0.419473 (0.448210)	0.414442 (0.442480)	0.414138 (0.443416)	0.349526 (0.384698)	0.406525 (0.435454)	0.344970 (0.381363)
	40	0.302127 (0.323665)	0.425893 (0.439539)	0.418370 (0.431905)	0.423143 (0.436928)	0.338568 (0.356992)	0.413160 (0.427010)	0.348785 (0.366494)
1.5	70	0.310101 (0.322629)	0.428870 (0.436937)	0.420506 (0.428518)	0.427278 (0.435392)	0.333651 (0.344970)	0.416635 (0.424826)	0.351497 (0.362039)
	100	0.308388 (0.316863)	0.427658 (0.4329708)	0.418520 (0.4238929)	0.426545 (0.4318791)	0.339384 (0.3645649)	0.415331 (0.4207491)	0.348979 (0.3560576)

7. APPLICATION TO LIFETIME DATA

The i -Xg distribution has been used to fit five data sets, and it has been compared to the i -garima and xgamma distributions. Dataset I is taken from Lawless (2003), pp. 204 [12], which contains the failure times (mm) for a sample of 15 electronic components in an accelerated life test. Dataset II is obtained from Lawless (2003), pp. 263 [12], which reports the number of cycles to failure for 25 (100 cm) yarn specimens tested at a particular strain level. Dataset III contains remission times (in weeks) for 20 leukaemia patients randomly assigned to a specific treatment, taken from Atia et al. (2023) [13]. Dataset IV is about the survival time (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal, T. (1960) [14], and Dataset V is about the time to

Table 2: True value of ξ , MSE and std. error (in parentheses) of the different estimation procedures for the i -Xg distribution

ξ	n	MLE	LSE	WLSE	CVM	MPSE	AD	RAD
0.1	10	0.000474	0.000673	0.000659	0.000646	0.000598	0.000622	0.000549
		(0.021762)	(0.025945)	(0.025674)	(0.025425)	(0.024456)	(0.024942)	(0.023437)
	20	0.000292	0.000508	0.000497	0.000491	0.000401	0.000472	0.000374
		(0.017093)	(0.022530)	(0.022286)	(0.022166)	(0.020020)	(0.021735)	(0.019335)
	40	0.000209	0.000431	0.000416	0.000422	0.000278	0.000399	0.000286
		(0.014472)	(0.020766)	(0.020405)	(0.020547)	(0.016686)	(0.019986)	(0.016922)
	70	0.000181	0.000400	0.000387	0.000394	0.000228	0.000371	0.000252
		(0.013455)	(0.019995)	(0.019678)	(0.019859)	(0.015090)	(0.019254)	(0.015884)
	100	0.000168	0.000386	0.000376	0.000382	0.000204	0.000358	0.000237
		(0.012979)	(0.019654)	(0.019382)	(0.019556)	(0.014274)	(0.018923)	(0.015394)
0.5	10	0.012249	0.019242	0.018816	0.018482	0.016607	0.017944	0.015237
		(0.110675)	(0.138717)	(0.137173)	(0.135948)	(0.128869)	(0.133955)	(0.123437)
	20	0.008698	0.015847	0.015466	0.015411	0.011895	0.014811	0.011422
		(0.093265)	(0.125887)	(0.124361)	(0.124140)	(0.109063)	(0.121698)	(0.106872)
	40	0.007118	0.014138	0.013722	0.013901	0.009173	0.013223	0.009478
		(0.084370)	(0.118904)	(0.117141)	(0.117902)	(0.095776)	(0.114990)	(0.097357)
	70	0.006604	0.013597	0.013223	0.013457	0.007973	0.012717	0.008778
		(0.081264)	(0.116606)	(0.114993)	(0.116003)	(0.089290)	(0.112772)	(0.093691)
	100	0.006319	0.013319	0.012902	0.013219	0.007352	0.012442	0.008433
		(0.079493)	(0.115408)	(0.113586)	(0.114975)	(0.085745)	(0.111545)	(0.091832)
1	10	0.053536	0.087961	0.086814	0.084702	0.074862	0.082661	0.068801
		(0.231379)	(0.296582)	(0.294643)	(0.291037)	(0.273609)	(0.287509)	(0.262300)
	20	0.042380	0.076850	0.075091	0.075019	0.057205	0.072339	0.055363
		(0.205865)	(0.277219)	(0.274027)	(0.273896)	(0.239175)	(0.268959)	(0.235294)
	40	0.037035	0.071433	0.069040	0.070445	0.046251	0.067125	0.048519
		(0.192446)	(0.267270)	(0.262755)	(0.265415)	(0.215060)	(0.259085)	(0.220271)
	70	0.036119	0.070104	0.067473	0.069524	0.042230	0.065907	0.046626
		(0.190050)	(0.264772)	(0.259756)	(0.263674)	(0.205500)	(0.256724)	(0.215931)
	100	0.035986	0.070106	0.067454	0.069692	0.040608	0.065947	0.046293
		(0.189699)	(0.264776)	(0.259718)	(0.263993)	(0.201513)	(0.256801)	(0.215159)
1.5	10	0.133400	0.225959	0.222003	0.218208	0.187589	0.212874	0.175146
		(0.365240)	(0.475352)	(0.471172)	(0.467127)	(0.433115)	(0.461382)	(0.418505)
	20	0.111311	0.200892	0.195789	0.196617	0.147992	0.189620	0.145438
		(0.333633)	(0.448210)	(0.442480)	(0.443416)	(0.384698)	(0.435454)	(0.381363)
	40	0.104759	0.193195	0.186542	0.190906	0.127443	0.182337	0.134318
		(0.323664)	(0.439539)	(0.431905)	(0.436928)	(0.356992)	(0.427010)	(0.366494)
	70	0.104089	0.190914	0.183628	0.189566	0.119005	0.180477	0.131072
		(0.322629)	(0.436937)	(0.428518)	(0.435391)	(0.344970)	(0.424826)	(0.362039)
	100	0.100402	0.187464	0.179685	0.186520	0.132908	0.177030	0.126777
		(0.316863)	(0.432971)	(0.423893)	(0.431879)	(0.364565)	(0.420749)	(0.356058)

failure of 18 electronic devices used in Wang (2000) [15].

We can observe that the means of all the datasets are higher than their medians, indicating that the data are positively (right) skewed with positive support. All of these datasets can be used as lifetime data is evidence from Table 3. The first four datasets are nearly concave, indicating that the data are drawn from a distribution with an increasing failure rate (IFR). Consequently, it has a decreasing mean remaining life (DMRL). For Dataset V, it is approximately convex-concave, indicating that the data come from a distribution with a bathtub failure rate. For each data set, the probability model is chosen using Akaike's Information Criterion ($AIC = -2 \log\text{-likelihood} +$

2k, where k is the number of parameters in the model). A lower AIC indicates a better fit for the model. Table 3 presents some fundamental summary statistics, and calculated AICs are shown in Table 4 and demonstrate that the i-Xg is a better fit. For each data set, the histogram and the three fitted distributions such as xgamma, i-garima, and i-Xg, are plotted in Figures 6 for comparison.

Table 3: Summary of the Datasets

	Min	Q_1	Q_2 (Median)	Mean	Q_3	Max
Dataset I	1.40	11.45	22.20	27.55	41.80	66.20
Dataset II	15.0	86.0	175.0	178.3	224.0	653.0
Dataset III	1.00	7.00	16.50	19.55	28.25	49.00
Dataset IV	12.00	54.75	70.00	99.82	112.75	376.00
Dataset V	5.00	53.25	155.00	172.11	281.00	420.00

Table 4: Fitting of different distributions and corresponding negative log-likelihood and AIC values

Dataset	Xgamma			I-Garima			I-Xg		
	$\hat{\xi}$	$-\log L$	AIC	$\hat{\xi}$	$-\log L$	AIC	$\hat{\xi}$	$-\log L$	AIC
Dataset I	0.0066	65.4170	134.834	0.0489	64.4600	132.920	0.0725	64.0600	132.120
Dataset II	0.0189	154.8800	313.760	0.0076	154.0100	312.020	0.0115	152.9850	309.970
Dataset III	0.1620	80.1163	164.233	0.0687	79.0970	162.194	0.1017	78.5750	161.150
Dataset IV	0.3167	325.8400	655.680	0.1360	326.8540	657.708	0.2017	323.0910	650.182
Dataset V	0.0189	116.0758	236.152	0.0078	110.3970	224.794	0.0115	110.0940	224.188

8. CONCLUDING REMARKS

This study presents an innovative one-parameter lifetime model, the Induced Xgamma (i-Xg) distribution, formulated through an equilibrium (induced) transformation of the existing Xgamma distribution. The model is very flexible and can accurately capture both under-dispersed and over-dispersed behaviour in reliability data. Its capacity to depict various hazard shapes, including increasing, decreasing, bathtub, and inverted-bathtub patterns, highlights its practical utility in reliability engineering and longevity studies. We examine important distributional properties, including moments, stochastic ordering, inequality measures, and reliability characteristics. We also show that the model is identifiable. We examine several traditional estimation methods and assess their performance in simulations. The empirical findings underscore the robustness and efficacy of the estimation methodologies. The i-Xg model is a desirable tool for practitioners evaluating duration and survival-type data, offering greater flexibility and better goodness-of-fit than competing models, as demonstrated in applications across several real-world datasets. All things considered, the suggested distribution adds something useful to the family of continuous reliability models and might encourage further developments and applications in related statistical fields.

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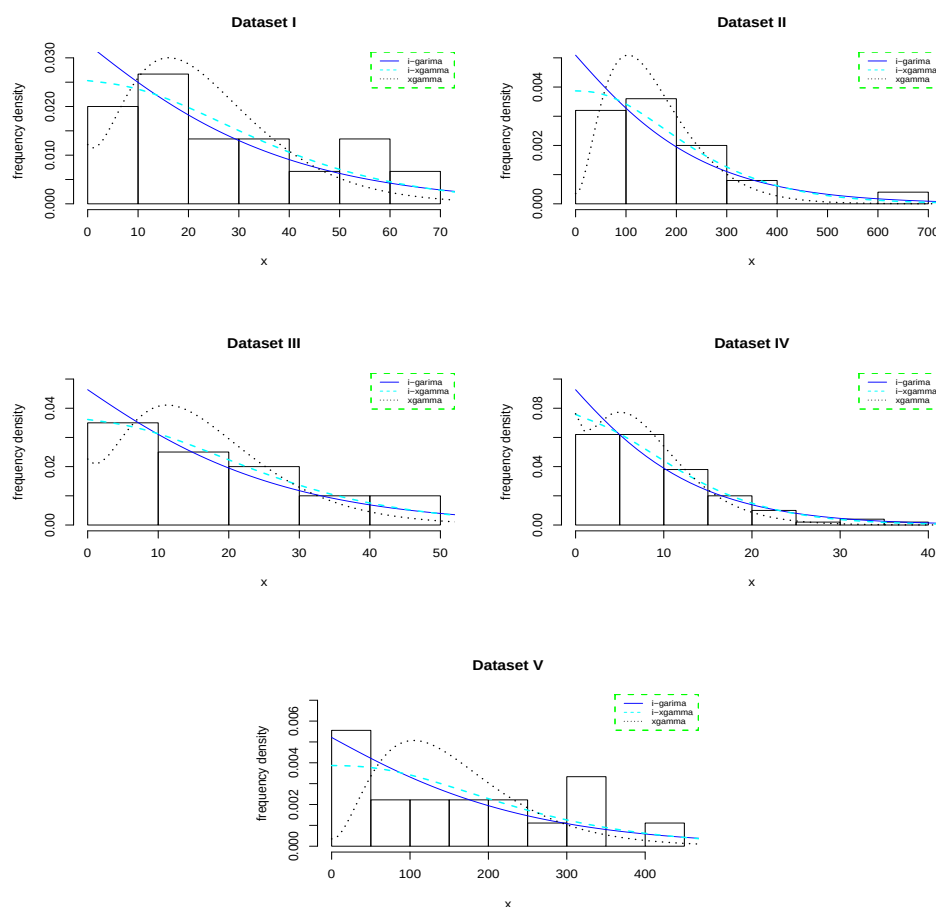


Figure 6: Histogram and fitted distributions for Data set I-V

CONFLICT OF INTEREST

All authors declare no conflicts of interest.

DISCLAIMER

Use of Gen AI: During the preparation of this work, the author(s) utilised ChatGPT from OpenAI to improve language and readability in the Introduction section. Following the use of this tool, the author(s) reviewed and edited the content as necessary and take full responsibility for the entire publication's content.

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