

A SURVIVAL WEIGHTED RESILIENCE FRAMEWORK FOR THE MARSHALL-OLKIN DISTRIBUTION WITH APPLICATIONS IN ENVIRONMENTAL AND BIOMEDICAL MODELING

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Abstract

This paper develops a survival weighted hazard interpretation of the Marshall–Olkin extension for lifetime distributions. The resulting formulation, termed the Resilience Distribution (RD), introduces a parameter $\theta > 0$ that modulates the baseline hazard through the survival probability, providing a representation of resilience and hazard damping in reliability systems. The framework allows flexible hazard patterns, including increasing, decreasing, and non-monotonic shapes. Key statistical properties, including the density, survival and hazard functions, quantile function, moments, and order statistics, are derived. Theoretical results on stochastic ordering, tail behavior, and hazard elasticity are established. Parameter estimation is performed using maximum likelihood, supported by a simulation study. A special case based on the Lindley distribution is applied to real datasets, demonstrating the flexibility and interpretability of the proposed framework for modeling lifetime data.

Keywords: Resilience modeling, Marshall–Olkin extension, Hazard rate function, Lindley distribution

1. INTRODUCTION

Survival analysis and reliability modeling rely heavily on classical lifetime distributions—such as the exponential, Weibull, Lindley, and log normal models—to describe time to event behavior. Although these distributions provide mathematically tractable foundations, their hazard functions are often structurally limited and may fail to capture the complex failure patterns observed in modern reliability applications. To improve flexibility, numerous generator families have been developed in the literature, including the exponentiated family ([1]), the Alpha-Power Transformation ([2]), and the Marshall–Olkin extension ([3]). These approaches typically enhance flexibility by modifying the scale, shape, or tail behavior of a baseline distribution.

Among these, the Marshall–Olkin framework ([3]) has become one of the most widely used techniques for generating generalized lifetime models. By introducing an additional parameter through a transformation of the baseline distribution function, it produces a rich class of models capable of exhibiting diverse hazard rate behaviors and has been successfully applied in reliability and survival studies.

Despite these advances, an important class of real-world phenomena remains challenging to model adequately: systems that exhibit *resilience*. In such systems, the instantaneous risk of

failure may depend not only on time but also on the accumulated survival experience of the system itself. This behavior arises in engineering components operating under varying stress conditions, biomedical relapse processes, and environmental systems that exhibit temporary recovery before failure. In these situations, the hazard rate evolves dynamically, responding to the current survival state rather than following a purely monotonic trend.

Motivated by this perspective, this study develops a survival-weighted hazard formulation that provides an alternative interpretation of the Marshall–Olkin transformation. Specifically, a resilience parameter $\theta > 0$ is introduced to modulate the baseline hazard through the survival probability. Under an appropriate parameter mapping, the resulting formulation can be expressed as a reparameterized form of the Marshall–Olkin transformation, while offering a distinct interpretation in which the additional parameter directly represents resilience behavior. In this framework, the hazard rate becomes a survival-dependent function, allowing the model to capture adaptive failure dynamics.

From a reliability engineering perspective, this formulation is particularly relevant for modeling systems whose failure behavior depends on accumulated survival experience. Such systems require models in which hazard rates adapt to the system’s operational state, making resilience-based representations essential for realistic analysis.

The main contribution of this work is to provide a survival-weighted hazard interpretation of the Marshall–Olkin framework. Rather than introducing a new distributional form, the proposed approach offers a conceptually distinct perspective in which the additional parameter quantifies resilience and governs how the hazard rate adapts to the survival level of the system. This interpretation establishes a meaningful link between classical lifetime models and resilience-based mechanisms observed in practice.

The proposed formulation retains the analytical tractability of the Marshall–Olkin family, allowing closed-form expressions for the density, distribution, survival function, quantile function, and hazard rate. It also supports a wide range of hazard shapes, including increasing, decreasing, bathtub, and non-monotonic patterns, while preserving the tail behavior of the baseline distribution.

To illustrate its usefulness, a specialization based on the Lindley distribution, referred to as the Resilience Lindley model, is considered. Its statistical properties are derived, and parameter estimation is performed using the maximum likelihood method. A simulation study is conducted to evaluate estimator performance, and the model is applied to real datasets to demonstrate its practical relevance in modeling lifetime data.

Overall, the proposed formulation provides a useful bridge between classical distributional modeling and resilience-based interpretations of hazard dynamics, with potential applications in reliability engineering, environmental studies, and biomedical analysis.

2. THE RESILIENCE DISTRIBUTION (RD)

Classical generator families modify a baseline distribution through algebraic, power, shock, or rank-based transformations of $F(x)$ or $f(x)$. Consequently, the resulting hazard function $h(x)$ changes only indirectly and does not explicitly incorporate the *resilience effect*—the idea that systems with a high conditional survival probability may experience a reduced instantaneous risk of failure.

The *Resilience-based formulation* introduces a mechanism in which the baseline hazard is modulated according to the magnitude of the survival probability. Systems that have survived well up to time x retain resilience and may exhibit a reduced instantaneous hazard, whereas systems with lower survival probability experience weaker damping and relatively higher hazard. This interpretation embeds resilience directly into the hazard structure and allows the model to capture adaptive failure dynamics observed in many practical survival and reliability contexts.

It is important to note that the resulting cumulative distribution function can be expressed as a reparameterized form of the Marshall–Olkin extension ([3]). Under an appropriate parameter mapping, the functional form of the proposed distribution coincides with the classical

Marshall–Olkin transformation. However, the present formulation provides a conceptually distinct interpretation in which the additional parameter represents a resilience effect through survival-weighted hazard modulation. This perspective explicitly links the hazard rate to the survival state of the system, offering an interpretation that is not directly emphasized in the classical Marshall–Olkin formulation.

2.1. Definition

Let $F(x)$ be a baseline cumulative distribution function with probability density function $f(x)$ and survival function $S(x) = 1 - F(x)$. The *Resilience Distribution (RD)* is defined through the transformation

$$F_{RD}(x; \theta) = \frac{F(x)}{1 + \theta S(x)}, \quad \theta > 0, \quad (1)$$

where θ is the *resilience parameter* controlling the degree of hazard modulation induced by the survival probability.

The corresponding probability density function is obtained by differentiation:

$$f_{RD}(x; \theta) = \frac{(1 + \theta)f(x)}{[1 + \theta S(x)]^2}, \quad (2)$$

where $f(x)$ denotes the baseline density.

A random variable X having density (2) will be denoted by

$$X \sim RD(\theta, F).$$

The survival function of the RD takes the form

$$S_{RD}(x; \theta) = \frac{(1 + \theta)S(x)}{1 + \theta S(x)}, \quad (3)$$

which shows how the resilience parameter modifies the baseline survival probability in a nonlinear manner.

The hazard function is given by

$$h_{RD}(x; \theta) = \frac{h(x)}{1 + \theta S(x)}, \quad (4)$$

where $h(x) = f(x)/S(x)$ is the baseline hazard function. This expression reveals that the parameter θ acts as a damping factor on the baseline hazard, with the magnitude of the effect depending on the survival probability.

Remark 1. Equation (1) can be written as

$$F_{RD}(x; \theta) = \frac{F(x)}{1 + \theta(1 - F(x))}.$$

Let $\alpha = 1 + \theta$. Then

$$F_{RD}(x; \theta) = \frac{F(x)}{\alpha + (1 - \alpha)F(x)},$$

which coincides with the classical Marshall–Olkin transformation ([3]). Hence the RD can be viewed as a reparameterized Marshall–Olkin family with a resilience-based interpretation.

3. SOME STATISTICAL PROPERTIES

This section provides some statistical properties of RD.

3.1. Quantile function

For $X \sim RD(\theta, F)$ with baseline quantile function $Q(p) = F^{-1}(p)$, the quantile function of X is:

$$Q_{RD}(p; \theta) = Q\left(\frac{p(1+\theta)}{1+p\theta}\right), \quad p \in (0, 1), \quad (5)$$

where $Q(\cdot)$ denotes the baseline quantile function. In particular, the first quartile, median, and third quartile are obtained by putting $p=0.25, 0.5$ and 0.75 , respectively, in (5).

3.2. Order statistics

Let $X_1, X_2, \dots, X_n \sim RD(\theta, F)$ be an i.i.d. sample from the resilience distribution with baseline distribution $F(x)$ and resilience parameter $\theta > 0$.

For the k -th order statistic $X_{(k)}$, its CDF is:

$$F_{X_{(k)}}(x) = \sum_{i=k}^n \binom{n}{i} [F_{RD}(x; \theta)]^i [S_{RD}(x; \theta)]^{n-i},$$

simplifying, we get,

$$F_{X_{(k)}}(x) = \sum_{i=k}^n \frac{\binom{n}{i} (\theta + 1)^{n-i} [F(x)]^i [S(x)]^{n-i}}{(1 + \theta S(x))^n}. \quad (6)$$

The cdf of first order statistic is:

$$F_{X_{(1)}}(x) = 1 - [S_{RD}(x; \theta)]^n = 1 - \left(\frac{(1 + \theta)S(x)}{1 + \theta S(x)}\right)^n. \quad (7)$$

The cdf of n^{th} order statistic is:

$$F_{X_{(n)}}(x) = [F_{RD}(x; \theta)]^n = \left(\frac{F(x)}{1 + \theta S(x)}\right)^n. \quad (8)$$

The pdf of k^{th} order statistic is:

$$f_{X_{(k)}}(x) = \binom{n}{k} [F_{RD}(x; \theta)]^{k-1} [S_{RD}(x; \theta)]^{n-k} f_{RD}(x; \theta),$$

simplifying, we get,

$$f_{X_{(k)}}(x) = \frac{\binom{n}{k} (\theta + 1)^{n-k+1} [F(x)]^{k-1} [S(x)]^{n-k} f(x)}{(1 + \theta S(x))^{n+1}}. \quad (9)$$

3.3. Moments

For $X \sim RD(\theta, F)$ with baseline distribution $F(x)$, the r -th raw moment is:

$$\mathbb{E}[X^r] = (1 + \theta) \int_{-\infty}^{\infty} \frac{x^r f(x)}{[1 + \theta S(x)]^2} dx,$$

using binomial expansion of $[1 + \theta S(x)]^{-2} = \sum_{k=0}^{\infty} (k+1)(-\theta)^k S(x)^k$, we get,

$$\mathbb{E}[X^r] = \sum_{k=0}^{\infty} (k+1)(-\theta)^k (1 + \theta) \mathbb{E}_F[X^r S(X)^k]. \quad (10)$$

Where, $\mathbb{E}_F[X^r S(X)^k] = \int_{-\infty}^{\infty} X^r S(X)^k f(x) dx$.

The variance exhibits non-monotonic behavior in θ :

$$\text{Var}(X) = (1 + \theta) \mathbb{E}_F \left[\frac{X^2}{(1 + \theta S(X))^2} \right] - \left((1 + \theta) \mathbb{E}_F \left[\frac{X}{(1 + \theta S(X))^2} \right] \right)^2. \quad (11)$$

3.4. Moment Generating Function

The moment generating function $M_X(t) = \mathbb{E}[e^{tX}]$ satisfies:

$$M_X(t) = (1 + \theta) \int_{-\infty}^{\infty} \frac{e^{tx} f(x)}{[1 + \theta S(x)]^2} dx,$$

when the baseline MGF $M_F(t)$ exists, the RD MGF admits the series representation:

$$M_X(t) = \sum_{k=0}^{\infty} (k+1)(-\theta)^k (1+\theta) \mathbb{E}_F[e^{tX} S(X)^k]. \quad (12)$$

4. SOME IMPORTANT THEOREMS

Theorem 1. Strict Survival-Weighted Hazard Damping:

Let $X \sim \text{RD}(\theta, F)$ with baseline hazard rate $h(x)$ and survival function $S(x)$. For all resilience parameters *strictly positive* ($\theta > 0$):

$$h_{\text{RD}}(x; \theta) < h(x) \quad \forall x \text{ where } S(x) > 0, \quad (13)$$

with the damping factor $(1 + \theta S(x))^{-1}$.

Proof. We have,

$$h_{\text{RD}}(x; \theta) = \frac{h(x)}{1 + \theta S(x)},$$

where $h(x)$ is the baseline hazard rate.

Since $S(x) > 0$ and $\theta > 0$ by definition:

$$1 + \theta S(x) > 1.$$

Thus for all $\theta > 0$ and $S(x) > 0$:

$$h_{\text{RD}}(x; \theta) = \frac{h(x)}{1 + \theta S(x)} < h(x).$$

■

Theorem 2. Resilience-Induced Stochastic Ordering:

For two RD variants $X_1 \sim \text{RD}(\theta_1, F)$ and $X_2 \sim \text{RD}(\theta_2, F)$ with $\theta_1 < \theta_2$:

$$X_1 \leq_{hr} X_2, \quad (14)$$

where $\leq_{hr}$ denotes hazard rate ordering. Higher θ increases resilience by uniformly reducing the hazard rate.

Proof. For $\theta_1 < \theta_2$, compare their hazard rates:

$$\begin{aligned} \frac{h_{\text{RD}}(x; \theta_1)}{h_{\text{RD}}(x; \theta_2)} &= \frac{h(x)/(1 + \theta_1 S(x))}{h(x)/(1 + \theta_2 S(x))} \\ &= \frac{1 + \theta_2 S(x)}{1 + \theta_1 S(x)} \\ &\geq 1 \quad \forall x \text{ since } \theta_2 > \theta_1 \text{ and } S(x) > 0. \end{aligned}$$

Thus $h_{\text{RD}}(x; \theta_1) \geq h_{\text{RD}}(x; \theta_2)$ for all x , which by definition means $X_1 \leq_{hr} X_2$ in hazard rate ordering. ■

Theorem 3. Implications of Hazard Rate Ordering: For $X_1 \sim \text{RD}(\theta_1, F)$ and $X_2 \sim \text{RD}(\theta_2, F)$ with $\theta_1 < \theta_2$, the following hold:

1. Stochastic Dominance: $X_2 \leq_{st} X_1$.
2. Mean Residual Life Ordering: $m_2(x) \geq m_1(x) \forall x$.
3. Quantile Dominance: $F_{RD}^{-1}(p; \theta_2) \geq F_{RD}^{-1}(p; \theta_1) \forall p \in (0, 1)$.

Where $\leq_{st}$ denotes usual stochastic ordering and $m_i(x)$ is the mean residual life function.

Proof.

1. Hazard rate ordering implies stochastic ordering: $X_1 \leq_{hr} X_2 \Rightarrow X_2 \leq_{st} X_1$.
2. The mean residual life $m(x) = \mathbb{E}[X - x | X > x]$ satisfies:

$$m_i(x) = \frac{1}{S_{RD}(x; \theta_i)} \int_x^\infty S_{RD}(t; \theta_i) dt,$$

since $S_{RD}(x; \theta_2) \geq S_{RD}(x; \theta_1)$, the integral dominates. Thus, $m_2(x) \geq m_1(x) \forall x$.

3. For quantiles, note that:

$$F_{RD}^{-1}(p; \theta) = F^{-1} \left(\frac{p(1 + \theta)}{1 + p\theta} \right)$$

■

Theorem 4. Tail Resilience Preservation: The RD asymptotically preserves the baseline's tail behavior while modulating mid-range hazards. For $x \rightarrow \infty$:

$$h_{RD}(x; \theta) \sim h(x),$$

where $\sim$ denotes asymptotic equivalence. Resilience effects are concentrated in regions where $S(x)$ is non-negligible.

Proof. As $x \rightarrow \infty$, $S(x) \rightarrow 0$ by definition of survival functions. Therefore:
For the hazard rate:

$$\begin{aligned} h_{RD}(x; \theta) &= \frac{h(x)}{1 + \theta S(x)}, \\ &\sim h(x) \quad \text{since } S(x) \rightarrow 0. \end{aligned}$$

■

Theorem 5. Adaptive Hazard Elasticity and Its Bounds

The elasticity of $h_{RD}(x; \theta)$ with respect to θ is state-dependent and is given by

$$\eta_{RD}(x; \theta) = -\frac{\theta S(x)}{1 + \theta S(x)}. \quad (15)$$

Furthermore, the following properties hold:

1. **Elasticity Bounds:**

$$-\theta \leq \eta_{RD}(x; \theta) \leq 0. \quad (16)$$

2. **Maximal Sensitivity:** The elasticity attains its minimum value when $S(x) = 1$:

$$\min_x \eta_{RD}(x; \theta) = -\frac{\theta}{1 + \theta}. \quad (17)$$

3. **Risk-Regime Insensitivity:**

$$\lim_{S(x) \rightarrow 0} \eta_{RD}(x; \theta) = 0. \quad (18)$$

4. State-dependent Sensitivity: The magnitude of the elasticity is greater in high-survival regions ($S(x) \approx 1$) and approaches zero in high-risk regions ($S(x) \approx 0$).

Proof. The elasticity of the RD hazard rate with respect to θ is defined as

$$\eta_{RD}(x; \theta) = \frac{\partial h_{RD}(x; \theta)}{\partial \theta} \cdot \frac{\theta}{h_{RD}(x; \theta)}.$$

Since

$$h_{RD}(x; \theta) = \frac{h(x)}{1 + \theta S(x)},$$

we have

$$\begin{aligned} \frac{\partial}{\partial \theta} h_{RD}(x; \theta) &= \frac{\partial}{\partial \theta} \left(\frac{h(x)}{1 + \theta S(x)} \right) \\ &= -\frac{h(x)S(x)}{(1 + \theta S(x))^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \eta_{RD}(x; \theta) &= \left(-\frac{h(x)S(x)}{(1 + \theta S(x))^2} \right) \frac{\theta}{\frac{h(x)}{1 + \theta S(x)}} \\ &= -\frac{\theta S(x)}{1 + \theta S(x)}. \end{aligned}$$

This proves Equation (15).

For the bounds, since $0 \leq S(x) \leq 1$ and $\theta > 0$,

$$0 \leq \frac{\theta S(x)}{1 + \theta S(x)} \leq \theta.$$

Consequently,

$$-\theta \leq \eta_{RD}(x; \theta) \leq 0.$$

For maximal sensitivity, consider

$$g(S) = -\frac{\theta S}{1 + \theta S}.$$

Its derivative with respect to S is

$$g'(S) = -\frac{\theta}{(1 + \theta S)^2} < 0.$$

Hence, $g(S)$ is decreasing in S , and therefore its minimum occurs at $S = 1$. Thus,

$$\min_x \eta_{RD}(x; \theta) = -\frac{\theta}{1 + \theta}.$$

Finally, as $S(x) \rightarrow 0$,

$$\eta_{RD}(x; \theta) = -\frac{\theta S(x)}{1 + \theta S(x)} \rightarrow 0.$$

Thus, the elasticity approaches zero in the high-risk region where the survival probability becomes negligible. This completes the proof. ■

5. SOME SPECIAL MODELS

5.1. Resilience Exponential distribution (RED)

Consider $X \sim \exp(\lambda)$ with baseline functions:

$$F(x) = 1 - e^{-\lambda x}, f(x) = \lambda e^{-\lambda x},$$

and

$$S(x) = e^{-\lambda x}, h(x) = \lambda.$$

Then the pdf, cdf, survival and hazard rate functions of RED are:

$$F_{RED}(x; \theta) = \frac{1 - e^{-\lambda x}}{1 + \theta e^{-\lambda x}},$$

$$f_{RED}(x; \theta) = \frac{(1 + \theta)\lambda e^{-\lambda x}}{[1 + \theta e^{-\lambda x}]^2},$$

$$S_{RED}(x; \theta) = 1 - \frac{1 - e^{-\lambda x}}{1 + \theta e^{-\lambda x}},$$

and

$$h_{RED}(x; \theta) = \frac{\lambda}{1 + \theta e^{-\lambda x}}.$$

5.2. Resilience Weibull distribution (RWD)

Consider $X \sim Weibull(k, \lambda)$ with baseline functions:

$$F(x) = 1 - e^{-(x/\lambda)^k}, f(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k},$$

and

$$S(x) = e^{-(x/\lambda)^k}, h(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1}.$$

For $X \sim RD(\theta, Weibull(k, \lambda))$, the pdf, cdf, survival and hazard rate functions are:

- Cumulative Distribution Function (CDF):

$$F_{RWD}(x; \theta) = \frac{1 - e^{-(x/\lambda)^k}}{1 + \theta e^{-(x/\lambda)^k}}.$$

- Probability Density Function (PDF):

$$f_{RWD}(x; \theta) = \frac{(1 + \theta) \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}}{\left[1 + \theta e^{-(x/\lambda)^k}\right]^2}.$$

- Survival Function:

$$S_{RWD}(x; \theta) = \frac{(1 + \theta) e^{-(x/\lambda)^k}}{1 + \theta e^{-(x/\lambda)^k}}.$$

- Hazard Rate Function:

$$h_{RWD}(x; \theta) = \frac{\frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1}}{1 + \theta e^{-(x/\lambda)^k}}.$$

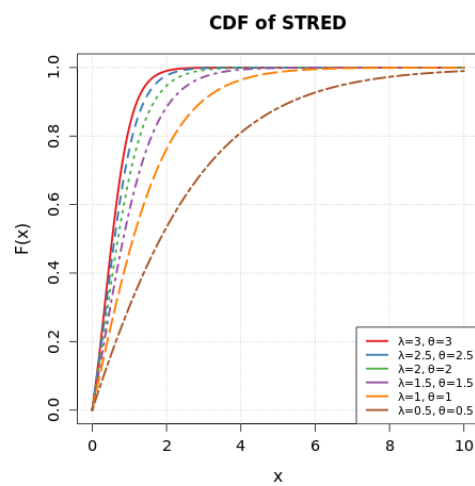


Figure 1: CDF of RED with different values of λ and θ .

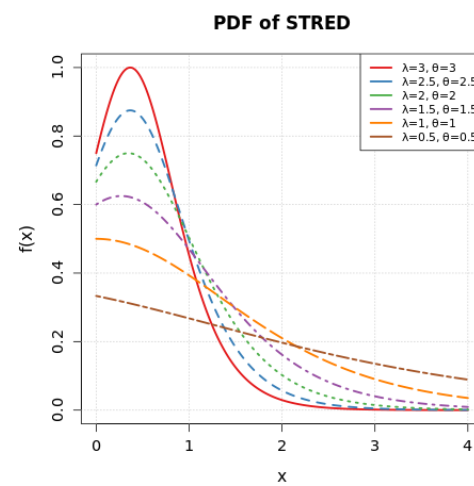


Figure 2: PDF of RED with different values of λ and θ .

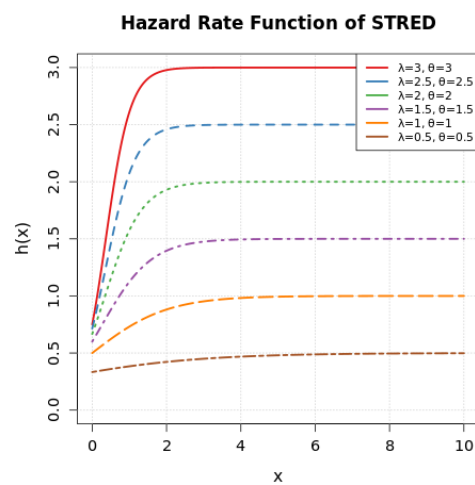


Figure 3: Hazard rate of RED with different values of λ and θ .

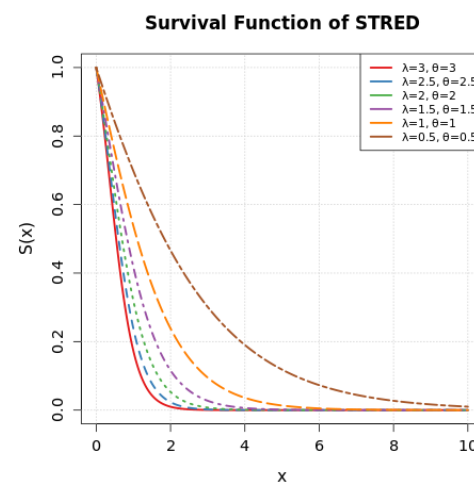


Figure 4: Survival rate of RED with different values of λ and θ .

Remark. When the Weibull shape parameter $k = 1$, the Weibull distribution reduces to the exponential distribution with scale parameter λ . Consequently, the Resilience Weibull Distribution reduces to the Resilience Exponential Distribution for $k = 1$.

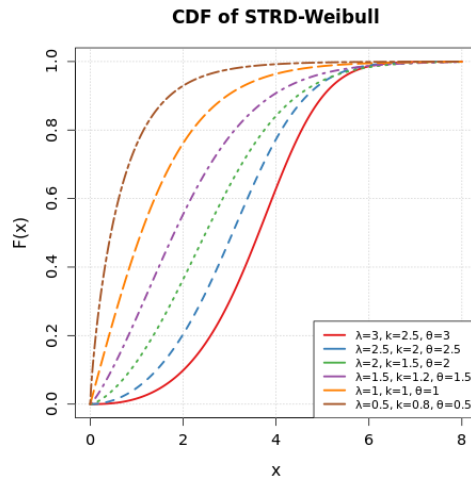


Figure 5: CDF of RWD with different values of λ , k and θ .

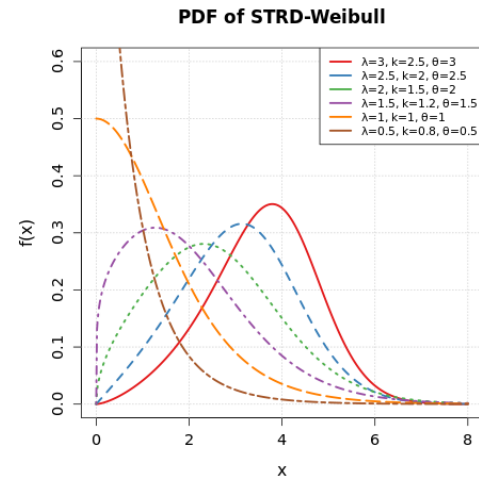


Figure 6: PDF of RWD with different values of λ , k and θ .

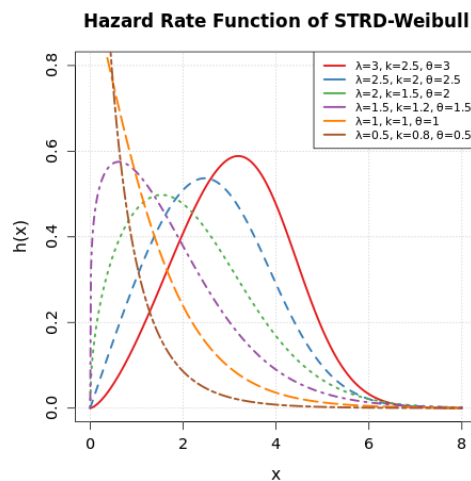


Figure 7: Hazard rate of RWD with different values of λ , k and θ .

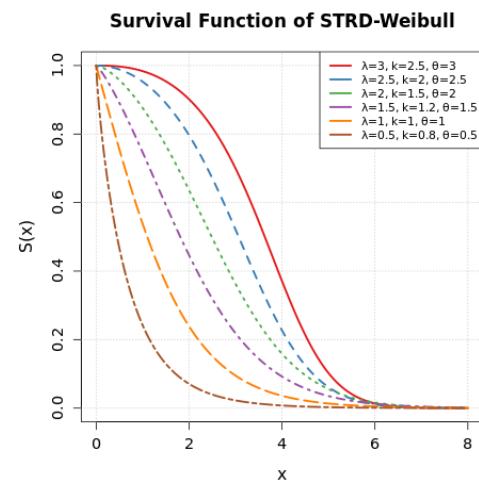


Figure 8: Survival rate of RWD with different values of λ , k and θ .

5.3. Resilience Lindley Distribution(RLD)

For $X \sim \text{Lindley}(\beta)$ with shape parameter $\beta > 0$, the pdf, cdf, survival and hazard rate functions are:

$$f(x; \beta) = \frac{\beta^2}{\beta + 1} (1 + x) e^{-\beta x}, \quad x > 0, F(x; \beta) = 1 - \left(1 + \frac{\beta x}{\beta + 1}\right) e^{-\beta x},$$

and

$$S(x; \beta) = \left(1 + \frac{\beta x}{\beta + 1}\right) e^{-\beta x}, h(x; \beta) = \frac{\beta^2(1 + x)}{\beta + 1 + \beta x}.$$

Then, for $X \sim \text{RD}(\theta, \text{Lindley}(\beta))$ where $\beta > 0$ is the Lindley shape parameter, then the pdf, cdf, survival and hazard rate functions are:

- Cumulative Distribution Function (CDF):

$$F_{RLD}(x; \theta) = \frac{1 - \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}}{1 + \theta \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}}.$$

- Probability Density Function (PDF):

$$f_{RLD}(x; \theta) = \frac{(1 + \theta) \frac{\beta^2}{\beta+1} (1 + x) e^{-\beta x}}{\left[1 + \theta \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}\right]^2}.$$

- Survival Function:

$$S_{RLD}(x; \theta) = \frac{(1 + \theta) \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}}{1 + \theta \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}}.$$

- Hazard Rate Function:

$$h_{RLD}(x; \theta) = \frac{\frac{\beta^2}{\beta+1} (1 + x)}{1 + \theta \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}}.$$

6. MAXIMUM LIKELIHOOD METHOD

Let $X_1, X_2, \dots, X_n$ be a random sample from the RD.

The likelihood function for the parameter θ based on the observed sample is;

$$L(\theta) = \prod_{i=1}^n \frac{(1 + \theta) f(x_i)}{[1 + \theta S(x_i)]^2}.$$

Taking the natural logarithm of the likelihood function, the log-likelihood function becomes,

$$\ell(\theta) = \sum_{i=1}^n [\log(1 + \theta) + \log f(x_i) - 2 \log(1 + \theta S(x_i))].$$

This simplifies to

$$\ell(\theta) = n \log(1 + \theta) + \sum_{i=1}^n \log f(x_i) - 2 \sum_{i=1}^n \log(1 + \theta S(x_i)).$$

Differentiating $\ell(\theta)$ with respect to θ yields the score function:

$$\frac{d\ell}{d\theta} = \frac{n}{1 + \theta} - 2 \sum_{i=1}^n \frac{S(x_i)}{1 + \theta S(x_i)}.$$

The maximum likelihood estimator $\hat{\theta}$ is the value of θ that satisfies the nonlinear equation

$$\frac{n}{1 + \theta} = 2 \sum_{i=1}^n \frac{S(x_i)}{1 + \theta S(x_i)}.$$

Since this equation does not admit a closed-form solution, numerical methods such as the Newton-Raphson algorithm or other iterative procedures must be employed to obtain $\hat{\theta}$. The required calculations depend on the choice of the baseline distribution and its corresponding survival function $S(x)$.

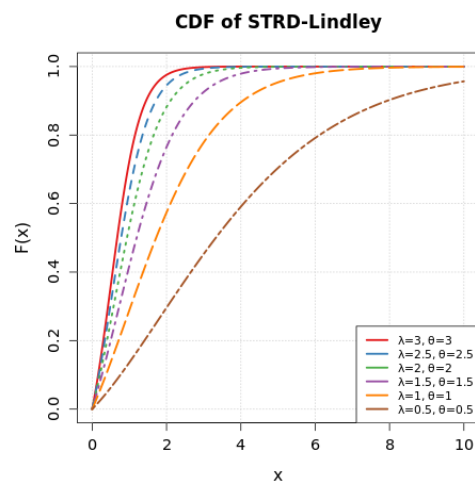


Figure 9: CDF of RLD with different values of β and θ .

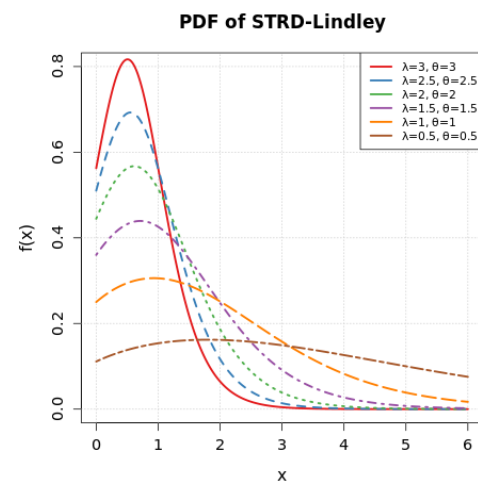


Figure 10: PDF of RLD with different values of β and θ .

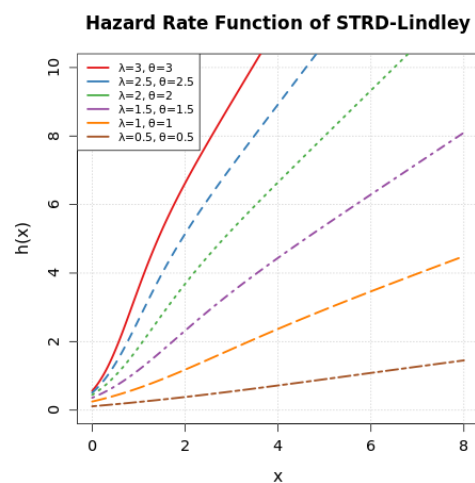


Figure 11: Hazard rate of RLD with different values of β and θ .

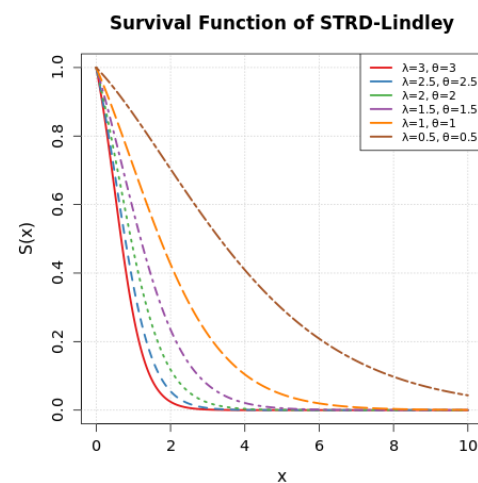


Figure 12: Survival rate of RLD with different values of β and θ .

7. SIMULATION STUDY FOR THE RLD

To investigate the finite-sample performance of the proposed estimation procedure, a Monte Carlo simulation study was conducted for the Resilience Lindley Distribution (RLD) introduced in Section 5.3. The parameters $\beta > 0$ and $\theta > 0$ denote the Lindley parameter and the resilience parameter, respectively.

We considered the following true parameter combinations:

$$(\beta, \theta) \in \{(2, 0.5), (1, 2.5), (0.5, 0.75), (3, 3)\},$$

and the sample sizes

$$n \in \{30, 50, 100, 150, 250, 400, 600, 750, 1000\}.$$

For each pair (β, θ) and each sample size n , the following steps were repeated for $R = 1000$ Monte Carlo replications:

1. Generate a random sample $X_1, \dots, X_n$ from the Resilience Lindley Distribution with true parameter values $(\beta_{\text{true}}, \theta_{\text{true}})$.
2. For each generated sample, compute the maximum likelihood estimators $\hat{\beta}$ and $\hat{\theta}$.
3. Record $(\hat{\beta}_r, \hat{\theta}_r)$ for replication $r = 1, \dots, R$.

After all replications, the following summary measures were computed.

Absolute Error (AE).

$$AE_{\beta} = \frac{1}{R} \sum_{r=1}^R |\hat{\beta}_r - \beta_{\text{true}}|, \quad AE_{\theta} = \frac{1}{R} \sum_{r=1}^R |\hat{\theta}_r - \theta_{\text{true}}|.$$

Absolute Bias (AB).

$$AB_{\beta} = \left| \frac{1}{R} \sum_{r=1}^R \hat{\beta}_r - \beta_{\text{true}} \right|, \quad AB_{\theta} = \left| \frac{1}{R} \sum_{r=1}^R \hat{\theta}_r - \theta_{\text{true}} \right|.$$

Average Variance (AV). Define

$$\bar{\hat{\beta}} = \frac{1}{R} \sum_{r=1}^R \hat{\beta}_r, \quad \bar{\hat{\theta}} = \frac{1}{R} \sum_{r=1}^R \hat{\theta}_r.$$

Then

$$AV_{\beta} = \frac{1}{R} \sum_{r=1}^R (\hat{\beta}_r - \bar{\hat{\beta}})^2, \quad AV_{\theta} = \frac{1}{R} \sum_{r=1}^R (\hat{\theta}_r - \bar{\hat{\theta}})^2.$$

Mean Squared Error (MSE).

$$MSE_{\beta} = \frac{1}{R} \sum_{r=1}^R (\hat{\beta}_r - \beta_{\text{true}})^2, \quad MSE_{\theta} = \frac{1}{R} \sum_{r=1}^R (\hat{\theta}_r - \theta_{\text{true}})^2.$$

Across all four parameter settings, a clear and consistent pattern emerges. As the sample size n increases from 30 to 1000:

- AE_{β} and AE_{θ} decrease steadily;
- AB_{β} and AB_{θ} move closer to zero;
- AV_{β} and AV_{θ} shrink, indicating reduced variability;

Table 1: Simulation results for RLD model

Sample Size	AE		AB		AV		MSE	
	β	θ	β	θ	β	θ	β	θ
$\beta = 2, \theta = 0.5$								
30	0.4177	1.0272	0.2207	0.6939	0.2584	3.2090	0.3068	3.6873
50	0.3247	0.6918	0.1364	0.3765	0.1636	1.1102	0.1820	1.2508
100	0.2375	0.4514	0.0514	0.1558	0.0887	0.3509	0.0913	0.3748
150	0.2049	0.3784	0.0421	0.1242	0.0650	0.2659	0.0667	0.2811
250	0.1547	0.2728	0.0048	0.0396	0.0364	0.1177	0.0364	0.1191
400	0.1204	0.2234	0.0118	0.0397	0.0228	0.0782	0.0229	0.0797
600	0.1032	0.1835	0.0088	0.0273	0.0170	0.0553	0.0170	0.0560
750	0.0890	0.1615	0.0038	0.0164	0.0123	0.0403	0.0123	0.0405
1000	0.0748	0.1354	0.0011	0.0101	0.0089	0.0292	0.0088	0.0293
$\beta = 1, \theta = 2.5$								
30	0.1885	2.6886	0.0596	1.5726	0.0542	20.4704	0.0577	22.9231
50	0.1460	1.8233	0.0367	0.8712	0.0332	7.2886	0.0345	8.0403
100	0.1026	1.1921	0.0196	0.4480	0.0166	2.5271	0.0170	2.7253
150	0.0807	0.9285	0.0110	0.2702	0.0102	1.4780	0.0103	1.5496
250	0.0616	0.6692	0.0064	0.1539	0.0059	0.7287	0.0060	0.7516
400	0.0498	0.5574	0.0014	0.0697	0.0039	0.5063	0.0039	0.5106
600	0.0392	0.4139	0.0014	0.0504	0.0024	0.2725	0.0024	0.2748
750	0.0367	0.3967	0.0035	0.0692	0.0021	0.2509	0.0021	0.2554
1000	0.0304	0.3203	0.0010	0.0345	0.0014	0.1665	0.0014	0.1676
$\beta = 0.5, \theta = 0.75$								
30	0.1038	1.4087	0.0465	0.9523	0.0170	6.3881	0.0192	7.2886
50	0.0836	1.0000	0.0275	0.5702	0.0109	2.0141	0.0116	2.3372
100	0.0592	0.5901	0.0139	0.2207	0.0052	0.6021	0.0053	0.6502
150	0.0468	0.4557	0.0039	0.1124	0.0033	0.3595	0.0033	0.3718
250	0.0365	0.3529	0.0048	0.0830	0.0020	0.2012	0.0021	0.2079
400	0.0274	0.2626	0.0030	0.0527	0.0012	0.1090	0.0012	0.1117
600	0.0243	0.2336	0.0016	0.0295	0.0009	0.0852	0.0009	0.0860
750	0.0219	0.2101	0.0001	0.0251	0.0008	0.0688	0.0008	0.0693
1000	0.0185	0.1791	0.0013	0.0224	0.0005	0.0497	0.0005	0.0501
$\beta = 3, \theta = 3$								
30	0.5617	2.6886	0.1520	1.4058	0.5068	17.7024	0.5294	19.6608
50	0.4242	1.9312	0.1238	1.0081	0.2863	7.9257	0.3013	8.9342
100	0.3029	1.2760	0.0560	0.4432	0.1404	2.8954	0.1434	3.0889
150	0.2353	0.9476	0.0287	0.2591	0.0873	1.5324	0.0880	1.5980
250	0.1925	0.7455	0.0169	0.1392	0.0548	0.8792	0.0550	0.8977
400	0.1404	0.5568	0.0107	0.0867	0.0307	0.4947	0.0308	0.5017
600	0.1206	0.4890	0.0098	0.0695	0.0224	0.3775	0.0224	0.3819
750	0.1025	0.4065	0.0029	0.0521	0.0164	0.2600	0.0164	0.2624
1000	0.0884	0.3605	0.0017	0.0351	0.0121	0.2041	0.0121	0.2051

AE = Absolute Error, AB = Absolute Bias, AV = Average Variance, MSE = Mean Squared Error.

- MSE_{β} and MSE_{θ} also decrease, often quite sharply.

This behavior is characteristic of consistent estimators whose precision improves as the sample size grows. For large n , all entries are small, demonstrating that both $\hat{\beta}$ and $\hat{\theta}$ estimate the true parameters accurately.

In almost all scenarios, θ is somewhat harder to estimate than β : AV_θ and MSE_θ are typically larger than their counterparts for β , especially for small to moderate n . However, this difference becomes much less pronounced as n increases.

8. APPLICATION

In this section, we evaluate the performance of the proposed Resilience Lindley Distribution (RLD) using two real-world datasets. The model comparison includes six established distributions: Lindley, Exponential, Gamma-Lindley ([4]), Generalized Lindley ([5]), DUS Lindley([6]), and DUS Exponential ([7]) distributions.

Model performance is evaluated using standard statistical criteria: the log-likelihood (Log-Lik), Akaike Information Criterion (AIC), corrected AIC (AICc), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), and the Kolmogorov-Smirnov (KS) test with corresponding p-values.

Data set 1: The following data set, studied by [8], represents the successive daily precipitation (in inches) recorded over thirty days in March.

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Data set 2: This data set, provided by [9], consists of blood cancer (leukaemia) measurements from 40 patients treated at Ministry of Health hospitals in Saudi Arabia. The observations are as follows:

0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.37, 2.532, 2.693, 2.805, 2.91, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381.

Model Comparison and Evaluation

The fitting results for the two datasets are summarized in Table(2). The RLD achieves the highest log-likelihood values and the lowest AIC, AICc, BIC, and HQIC scores across both datasets, indicating a consistently better fit compared to alternative models. Additionally, the Kolmogorov-Smirnov (KS) test p-values for the RLD are higher than those of competing distributions, reflecting stronger agreement with the empirical data. The parameter estimates for the RLD are $\beta = 1.791$ and $\theta = 5.995$ for Dataset 1, and $\beta = 1.142$ and $\theta = 10$ for Dataset 2, demonstrating the model's adaptability to varying data structures. These findings confirm the RLD's flexibility and reliability in modeling real-world survival and reliability data, reinforcing its potential as a robust tool for applied statistical analysis in environmental and medical research contexts.

The visual diagnostics in Figures 13–18 confirm these quantitative results. The density plots show that the RLD accurately captures the shape characteristics of both datasets. The CDF plots demonstrate close agreement between theoretical and empirical distributions, while the P-P plots exhibit nearly perfect diagonal alignment.

13. CONCLUSION

This paper develops a resilience-based interpretation of the Marshall-Olkin transformation through a survival-weighted hazard formulation. The resulting framework, referred to as the Resilience Distribution (RD), introduces a parameter that modulates the hazard function via the survival probability, providing a meaningful representation of resilience in survival and reliability systems. Rather than proposing a new distributional form, the study offers an alternative perspective in which the additional parameter governs the adaptive behavior of the hazard rate.

A range of statistical properties of the RD has been derived, including the quantile function, moments, moment generating function, and order statistics. Theoretical results such as hazard damping, stochastic ordering, tail behavior, and elasticity properties establish a solid foundation

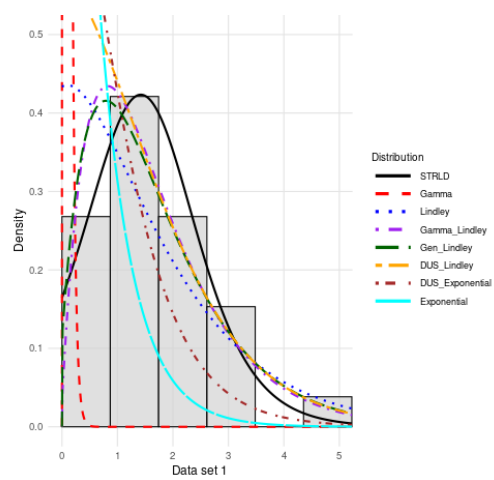


Figure 13: Fitted density curves of data set 1.

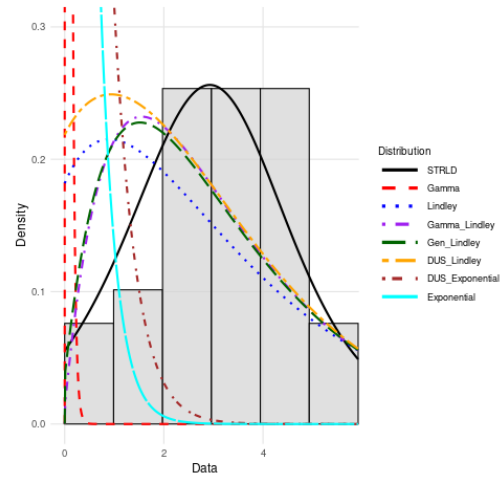


Figure 14: Fitted density curves of data set 2.

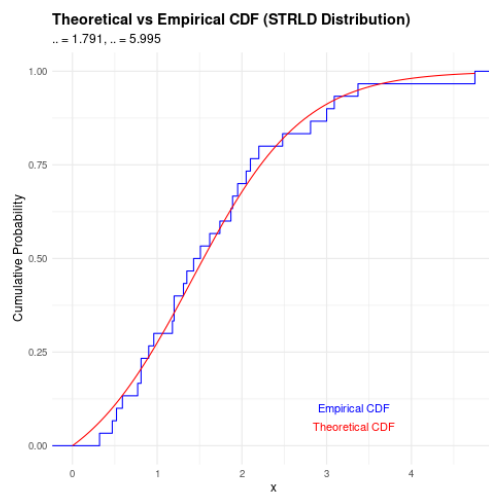


Figure 15: Theoretical vs empirical CDF of data set 1.

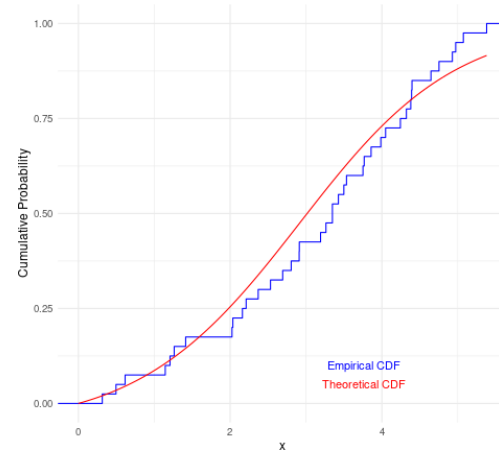


Figure 16: Theoretical vs empirical CDF of data set 2.

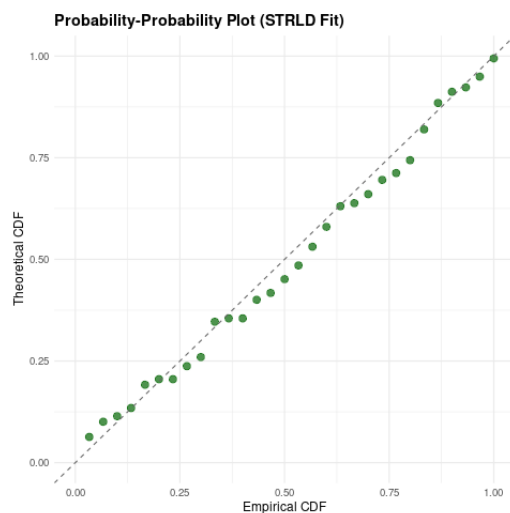


Figure 17: P-P Plot for RLD of data set 1.

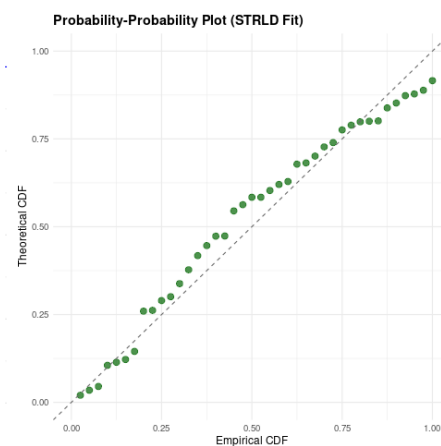


Figure 18: P-P Plot for RLD of data set 2.

Table 2: Comparative Fit Statistics for Lifetime Data Models

Distribution	Parameters	LogL	AIC	AICc	BIC	HQIC	KS Statistic	KSp-value
Dataset 1 (n=60)								
RLD	$\beta = 1.791, \theta = 5.995$	-39.4628	82.9255	83.3700	85.7279	83.8221	0.0673	0.9992
Gamma-Lindley	$\theta = 1.180, \beta = 20$	-39.5559	82.9518	83.3962	85.7442	83.8483	0.1149	0.8229
Generalized Lindley	$\theta = 1.133, \alpha = 10$	-40.0676	84.1353	84.5797	86.9376	85.0318	0.1221	0.7628
Lindley	$\theta = 0.910$	-43.1438	88.2875	88.4303	89.6886	88.7357	0.1882	0.2383
DUS Lindley	$\theta = 1.119$	-41.4765	84.9530	85.0959	86.3542	85.4013	0.1578	0.4436
DUS Exponential	$\lambda = 0.780$	-43.1016	88.2031	88.3459	89.6043	88.6513	0.1988	0.1868
Exponential	$\lambda = 0.597$	-45.4744	92.9488	93.0916	94.3450	93.3971	0.2352	0.0724
Dataset 2 (n=40)								
RLD	$\beta = 1.142, \theta = 10$	-69.5721	143.1443	143.4686	146.5220	144.3656	0.1198	0.6137
Gamma-Lindley	$\theta = 0.631, \beta = 20$	-76.5481	157.0962	157.4205	160.4740	158.3175	0.1973	0.0889
Generalized Lindley	$\theta = 0.618, \alpha = 10$	-76.9203	157.8405	158.1649	161.2183	159.0618	0.2019	0.0767
Lindley	$\theta = 0.527$	-80.2506	162.5012	162.6065	164.1900	163.1118	0.2406	0.0195
DUS Lindley	$\theta = 0.646$	-77.2856	156.5713	156.6766	158.2602	157.1819	0.2168	0.0515
DUS Exponential	$\lambda = 0.417$	-81.7586	165.5171	165.6224	167.2060	166.1277	0.2726	0.0052
Exponential	$\lambda = 0.318$	-85.7782	173.5563	173.6616	175.2452	174.1669	0.3002	0.0015

for the formulation. Parameter estimation based on the maximum likelihood method shows stable and consistent performance, supported by simulation results.

The practical relevance of the approach is demonstrated through a specialization based on the Lindley distribution, applied to real datasets. The results illustrate the flexibility and interpretability of the resilience-based formulation in capturing diverse hazard rate patterns observed in practice.

From a reliability engineering perspective, the proposed framework is particularly useful for modeling systems whose failure behavior depends on accumulated survival experience. By linking hazard dynamics to the survival state, the formulation provides a realistic representation of adaptive failure mechanisms in engineering, environmental, and biomedical contexts.

Overall, the proposed approach provides a useful bridge between classical lifetime models and resilience-based hazard modeling while preserving analytical tractability. Future work may consider extensions incorporating covariate-dependent resilience parameters and regression structures to enhance applicability in complex reliability and risk analysis settings.

ACKNOWLEDGEMENT

The authors acknowledge the use of ChatGPT (OpenAI) for language editing, improving clarity and readability, and assisting with the presentation of the manuscript. The authors reviewed and verified all content and take full responsibility for the accuracy and integrity of the work.

AUTHOR CONTRIBUTIONS

Both authors contributed equally to the conception, methodology development, data analysis, and manuscript preparation. Both authors read and approved the final manuscript.

DATA AVAILABILITY

The datasets used in this study are publicly available and were obtained from previously published sources cited in the article.

FUNDING

The authors received no financial support for the research, authorship, and/or publication of this article.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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