

A FLEXIBLE EXTENSION OF THE POWER FUNCTION DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

This paper introduces a new extension of the Power Function Distribution, referred to as the Exponentiated Gumbel Power Function (EGuPF) distribution derived from the exponentiated Gumbel family of distributions using the power function as the baseline distribution. The EGuPF distribution extends the classical power function distribution by incorporating additional shape parameters, thereby enabling it to capture J-shaped, reverse J-shaped, and bathtub-shaped hazard rate behaviours. The new distribution is expressed as an infinite linear combination of a power function distribution. The quantile function, moments, incomplete moments, Trimmed-L Moments, mode, Moment of Residual Life (MRL), Moment of Reversed Residual Life (MRRL), entropy, and order statistics of EGuPF distribution are derived. Parameter estimation is carried out using the maximum likelihood estimation (MLE) method, and the performance of the estimators is evaluated through Monte Carlo simulation. Simulation results show that the estimators of the parameters of the EGuPF distribution are consistent, efficient, and reliable. The application of the EGuPF distribution is demonstrated using two real datasets: survival times of acute myelogenous leukaemia patients and monthly tax revenue data. In both cases, EGuPF distribution outperforms competing distributions with the same baseline, achieving the lowest AIC, BIC, and goodness-of-fit statistics. These findings establish the EGuPF distribution as a flexible and robust model for analyzing bounded and skewed data.

Keywords: Power Function Distribution, Exponentiated Gumbel-G, Reliability, Estimation, Simulation

1. INTRODUCTION

The power function distribution (PFD) is a bounded continuous probability distribution that has been widely studied due to its simplicity and flexibility in modeling data constrained within a finite range. The PFD is related to the standard Pareto distribution through an inverse transformation [1]. Several authors have investigated its characterization and structural properties; notable contributions include [2],[3], and [4]. The probability density function (pdf) and cumulative distribution function (cdf) of the PFD, as given by [5], are respectively

$$F(x) = \left(\frac{x}{b}\right)^a \quad (1)$$

and

$$f(x) = \left(\frac{a}{b}\right) \left(\frac{x}{b}\right)^{a-1}, 0 < x < b, a > 0 \quad (2)$$

Statistical literature is very rich in classical probability distributions such as PFD, exponential, lognormal etc for modelling time to event data. There are instances where datasets arising from physical phenomenon do not follow known classical probability distributions. Hence researchers have come up with different methods of transformations otherwise known as generated family of distributions aimed at making baseline distributions more flexible. Some examples of generated family of distribution include beta-G [6], Kumaraswamy-G [7], Transformed-Transformer [8], Weibull-G [9], T-Normal family [10], exponentiated weibull-G [11], Topp-Leone family [12], exponentiated gumbel G [13], T-Kumaraswamy family [14], Marshal Olkin gumbel-G [15].

Several extensions of the PFD have been proposed to enhance its flexibility for bounded, lifetime, reliability, and survival data. [16] introduced the Transmuted PFD using the quadratic rank transmutation map, providing greater flexibility in the shape of the distribution. [17] proposed the Weibull PFD, using the Weibull-G class whose hazard rate can assume increasing, decreasing, and bathtub shapes, with improved performance in application to two different bathtub lifetime data. [18] derived the Kumaraswamy PFD and investigated its TL-Moments and L-Moments, while exponentiated Weibull power function [19] proposed the Exponentiated Weibull PFD, studied its properties, and fitted it to three-real life datasets.[20] subsequently introduced the Exponentiated Kumaraswamy PFD and investigated the effect of the extra shape parameters on the skewness and kurtosis of the distribution.

Furthermore, [21] introduced the Exponentiated PFD and studied the limiting properties of the pdf, cdf, sf, and hrf, and applied it to two failure time data, while [22] proposed the Exponentiated Generalized PFD, offering greater flexibility, particularly for Type-II censored data. [23] derived the Another Generalized Transmuted PFD, which showed improved fit over its sub-models for component failure-time data. More recently, [24] proposed a Cubic Transmuted PFD based on a second-order transmutation map, further enhancing the shape flexibility of the PFD, and estimated the parameters of the proposed model using five different estimation methods.

In this paper, we introduce an extension of PFD given (1) and (2) using a Exponentiated Gumbel family of distribution proposed by [13].The motivations of the study are: (a) To improve the characteristics of the PFD (b) To derive an extension of PFD that can capture bathtub shaped , J-shaped and reverse J-Shaped hazard rate behaviours (c) Empirical evidence form modelling the survival times, in weeks, of 33 patients suffering from Acute Myelogenous Leukaemia and monthly actual taxes revenue in Egypt from January 2006 to November 2010 show that the proposed extension of PFD provides a better fit when compared with other distribution with the same baseline. The rest of the paper is organized as follows. After compiling this paper write the outline ie the sections of the paper.

2. METHODS

2.1. The Exponentiated Gumbel Power Function Distribution

A family of distribution known as exponentiated Gumbel family of distribution was proposed by [13], using the logit of the cdf of a random variable. The cdf of the exponentiated gumbel family of distribution is given by (3)

$$G(x) = 1 - \left[1 - \exp \left\{ -B \left(\frac{F(x)}{1 - F(x)} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\alpha} \quad x > 0, \alpha, \sigma > 0, -\infty < \mu < \infty \quad (3)$$

The pdf corresponding to (3) is

$$g(x) = \frac{\alpha B f(x) (F(x))^{-\frac{1}{\sigma}-1}}{\sigma (1-F(x))^{-\left(\frac{1}{\sigma}-1\right)}} \exp \left\{ -B \left(\frac{F(x)}{1-F(x)} \right)^{-\frac{1}{\sigma}} \right\} \left[1 - \exp \left\{ -B \left(\frac{F(x)}{1-F(x)} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\alpha-1} \quad (4)$$

Where, $B = \exp\left(\frac{\mu}{\sigma}\right)$ and $F(x)$ and $f(x)$ are arbitrary cdf and pdf of a random variable X often referred to as the baseline.

Choosing the baseline distribution to be PFD given in (1) and (2) and substituting same in (3) and (4) birthed a new member of the exponentiated gumbel family called exponentiated gumbel power function distribution denoted as EGuPF distribution. The cdf and density function of EGuPF distribution are respectively given by (5) and (6)

$$G(x) = 1 - \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\alpha} \quad (5)$$

$$g(x) = \frac{\alpha B a b^a x^{-\left(\frac{a}{\sigma}+1\right)}}{\sigma (b^a - x^a)^{-\left(\frac{1}{\sigma}-1\right)}} \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\alpha-1} \quad (6)$$

$0 < x < b, a, \alpha, \sigma > 0, -\infty < \mu < \infty$

Note: if $\alpha = 1$ (5) reduces to Gumbel power function by [25]. The survival function $S(x)$ and hazard rate $h(x)$ function associated with (5) are respectively

$$S(x) = \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\alpha}$$

and

$$h(x) = \frac{\alpha B a b^a x^{-\left(\frac{a}{\sigma}+1\right)}}{\sigma (b^a - x^a)^{-\left(\frac{1}{\sigma}-1\right)}} \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]^{-1}$$

While the cumulative hazard rate $H(x)$ and reversed hazard rate $r(x)$ function of EGuPF distribution are

$$H(x) = -\alpha \log \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]$$

and

$$r(x) = \frac{\alpha B a b^a x^{-\left(\frac{a}{\sigma}+1\right)} \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\alpha-1}}{\sigma (b^a - x^a)^{-\left(\frac{1}{\sigma}-1\right)} \left[1 - \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\alpha} \right]}$$

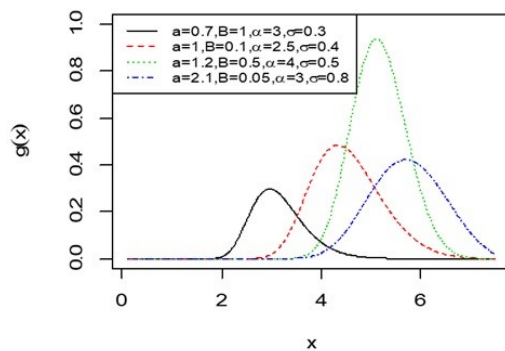


Figure 1: Plot of pdf of EGuPF distribution for selected parameter values.

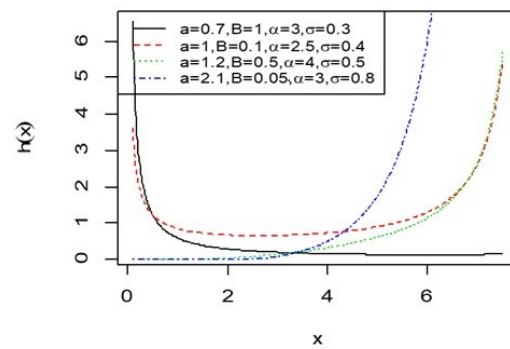


Figure 2: Plot of hrf of EGuPF distribution for selected parameter values.

The plots of the pdf of EGuPF distribution is given Figure 1, which show that EGuPF distribution is unimodal while the plots of its hazard rate function $h(x)$ shown in Figure 2 reveal that it has bathtub shaped, J-shaped and reverse J-shaped. The various shapes assumed by the $h(x)$ makes the proposed distribution fit for modelling data sets whose failure rate assumes the above shapes.

Theorem 1: Given that Y is distributed as exponentiated Gumbel distribution with location parameter μ , scale parameter σ , and shape parameter α , then the random variable $X = b(\exp(-y) + 1)^{-a}$ is distributed as EGuPF distribution with parameters μ, σ, α, a , and b

Proof: Given that

$$r(y) = \frac{\alpha B}{\sigma} \exp\left(-\frac{y}{\sigma}\right) \exp\left\{-B \exp\left(\frac{-y}{\sigma}\right)\right\} \left[1 - \exp\left\{-B \exp\left(\frac{-y}{\sigma}\right)\right\}\right]^{\alpha-1}$$

$$\text{and } x = b(\exp(-y) + 1)^{-a} \text{ implies } y = -\log\left(\left(\frac{b}{x}\right)^a - 1\right)$$

$$\frac{dy}{dx} = \left(\frac{a}{b}\right) \frac{\left(\frac{x}{b}\right)^{-1}}{\left(1 - \left(\frac{x}{b}\right)^a\right)}$$

By transformation,

$$g(x) = r(x) \left| \frac{dy}{dx} \right| \quad (7)$$

Substituting for $\frac{dy}{dx}$ in (7) and simplifying gives

$$g(x) = \frac{\alpha B a \left(\frac{x}{b}\right)^{-\left(\frac{a}{\sigma}+1\right)}}{\sigma b \left(1 - \left(\frac{x}{b}\right)^a\right)^{-\left(\frac{1}{\sigma}-1\right)}} \exp\left\{-B \left(\frac{\left(\frac{x}{b}\right)^a}{1 - \left(\frac{x}{b}\right)^a}\right)^{-\frac{1}{\sigma}}\right\} \left[1 - \exp\left\{-B \left(\frac{\left(\frac{x}{b}\right)^a}{1 - \left(\frac{x}{b}\right)^a}\right)^{-\frac{1}{\sigma}}\right\}\right]^{\alpha-1}$$

This distribution is the same as the one given in (6).

2.2. Quantile Function

The quantile function plays an important role in statistical theory and applications, particularly in simulation studies. Quantile function is defined as

$$u = F(x_u) = P(X \leq x_u), \quad 0 < u < 1.$$

The quantile function $Q(u)$ of the EGuPF distribution is easily obtained by inverting (5) we have

$$Q(u) = b \left[1 + \left\{ -\frac{1}{B} \log \left[1 - (1-u)^{\frac{1}{a}} \right] \right\}^{\sigma} \right]^{-\frac{1}{a}} \quad (8)$$

In particular the median of EGuPF is obtained by substituting $u = \frac{1}{2}$ in (8) as

$$Q(u) = b \left[1 + \left\{ -\frac{1}{B} \log \left[1 - \left(\frac{1}{2} \right)^{\frac{1}{a}} \right] \right\}^{-\sigma} \right]^{-\frac{1}{a}}$$

Random samples from EGuPF distribution can be generated using (8) by assuming that u follows Unifrom distribution

2.3. Important Expansions

Two important representations of the pdf and cdf of EGuPF distribution are derived in this subsection. The general binomial expansion, for $w > 0$ real non integer and $|z| < 1$

$$(1-z)^{w-1} = \sum_{i=1}^{\infty} \binom{w-1}{i} (-1)^i z^i \quad (9)$$

Furthermore, the power series expansion for exponential function $\exp(-z)$ is given by

$$\exp(-z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^j \quad (10)$$

Using (9) and (10) the pdf of EGuPF distribution given in (6) can written as

$$g(x) = \sum_{i,j,k=0}^{\infty} \Psi_{ijk} f_{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}(x) \quad (11)$$

where $\Psi_{ijk} = \frac{(-1)^{i+j+k} \alpha B (B(i+1))^j}{j! \sigma \left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)} \binom{\alpha-1}{i} \binom{\frac{j}{\sigma} + \frac{1}{\sigma} - 1}{k}$ and $f_{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}(x)$ is the pdf of power function distribution.

Hence the EGuPF distribution can be expressed as a linear combination of power function with parameters $a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)$ and b .

Similarly, given that ℓ is a positive integer then using (9) and (10) $G(x)^\ell$ can be expressed as

$$G(x)^\ell = \sum_{m=0}^{\ell} \sum_{p,q,u=0}^{\infty} \omega_{mpqu} F_{a\left(u-\frac{q}{\sigma}\right)} \quad (12)$$

where $\omega_{mpqu} = (-1)^{m+p+q+u} \binom{\ell}{m} \binom{\alpha m}{p} \binom{\frac{q}{\sigma}}{u} (Bq)^q$ and $F_{a\left(u-\frac{q}{\sigma}\right)}$ is the cdf of power function distribution with power parameter $a\left(u-\frac{q}{\sigma}\right)$ and scale parameter b . The expressions for

$g(x)$ and $G(x)^\ell$ given in (11) and (12) respectively are very useful in deriving some properties of EGuPF distribution.

2.4. Properties

In this subsection, we examined some of the statistical and reliability properties of EGuPF distribution.

2.4.1. Moments and Incomplete Moments

The r th moment of a random variable is defined as $\mu_r = \int_{-\infty}^{\infty} x^r g(x) dx$. The r th moment of EGuPF distribution can be obtained using (11) as follows

$$\begin{aligned}\mu_r &= \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \int_0^b x^r f_{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}(x) dx \\ \mu_r &= \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \int_0^b x^r \frac{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}{b} \left(\frac{x}{b}\right)^{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)-1} dx \\ \mu_r &= ab^r \sum_{ijk=0}^{\infty} \Psi_{ijk} \frac{\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}{\left(r+a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)\right)}\end{aligned}\quad (13)$$

The first four central moments can be obtained by setting $r = 1, 2, 3, 4$. The moment generating function (mgf) of EGuPF distribution is given by

$$M_X(t) = E\left(e^{tX}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \quad (14)$$

Substituting (13) in (14)

$$M_X(t) = a \sum_{r=0}^{\infty} \sum_{ijk=0}^{\infty} \Psi_{ijk} \frac{(tb)^r \left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}{r! \left(r+a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)\right)}$$

Second, the incomplete moment, say $\phi_s(z)$ of EGuPF distribution is obtained using (11) as

$$\begin{aligned}\phi_s(z) &= \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \int_0^z x^s f_{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}(x) dx \\ \phi_s(z) &= a \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \frac{\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right) z^{s+a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}}{\left(s+a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)\right) b^{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}}\end{aligned}\quad (15)$$

The upper incomplete moment of the proposed distribution defined using the pdf given in (11) we have

$$\gamma_s(z) = \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \int_z^b x^s f_{a\left(k-\frac{j}{\sigma}-\frac{1}{\sigma}\right)}(x) dx$$

The upper incomplete moment of EGuPF distribution takes the form

$$\phi_s(z) = a \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \frac{\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}{b^{a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}} \left[\frac{b^{s+a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)} - z^{s+a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}}{\left(s + a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)\right)} \right] \quad (16)$$

The mean deviation from mean and the median deviation from the mean can be easily obtained using the following relations $\sigma_1(x) = 2\mu G(\mu) - 2J(\mu)$ and $\sigma_2(x) = \mu - 2J(M)$ where $J(c) = \int_0^c xg(x) dx$. The mean deviation from the mean and median deviation from the mean of EGuPF distribution respectively are given by

$$\sigma_1(x) = 2\mu G(\mu) - 2a \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \frac{\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right) \mu^{s+a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}}{\left(s + a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)\right) b^{a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}}$$

and

$$\sigma_2(x) = \mu - 2a \sum_{i,j,k=0}^{\infty} \Psi_{ijk} \frac{\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right) M^{s+a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}}{\left(s + a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)\right) b^{a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}}$$

2.4.2. Trimmed-L Moments

The trimmed L-moments was introduced by [26] as an extension of L-moments obtained from linear combinations of order statistics after symmetric or asymmetric trimming. The r th Trimmed L-moment $\lambda_r^{t_1, t_2}$ as defined by [26] is

$$\lambda_r^{(t_1, t_2)} = \frac{1}{r} \sum_{k_1=0}^{r-1} (-1)^{k_1} \binom{r-1}{k_1} E(Y_{r+t_1-k_1, r+t_1+t_2}), r = 1, 2 \quad (17)$$

For a symmetric case $t_1 = t_2 = t$, (17) reduces to

$$\lambda_r^{(t)} = \frac{1}{r} \sum_{k_1=0}^{r-1} (-1)^{k_1} \binom{r-1}{k_1} E(Y_{r+t-k_1, r+2t})$$

then using the well-known representation of the expectation of order statistics, we obtain

$$\lambda_r^{(t)} = \frac{1}{r} \sum_{k_1=0}^{r-1} (-1)^{k_1} \binom{r-1}{k_1} \frac{(r+2t)!}{(r+t-k_1-1)!(t+k_1)!} \int_0^b xg(x) G(x)^{r+t+k_1-1} (1-G(x))^{(t+k_1)} dx$$

Expanding the term $(1-G(x))^{t+k_1}$ using the binomial series yields

$$\lambda_r^{(t)} = \frac{1}{r} \sum_{k_1=0}^{r-1} \sum_{d=0}^{t+k_1} (-1)^{k_1+d} \binom{r-1}{k_1} \binom{t+k_1}{d} \frac{(r+2t)!}{(r+t-k_1-1)!(t+k_1)!} \int_0^b xg(x) G(x)^{r+t+k_1+d-1} dx \quad (18)$$

Substituting (11) and (12) in (18) and replacing ℓ in (12) with $r+t+k_1+d-1$, the r th trimmed-L moment of EGuPF distribution takes the following form

$$\lambda_r^{(t)} = \frac{b}{r} \sum_{k_1=0}^{r-1} \sum_{d=0}^{t+k_1} \sum_{m=0}^{r+t+k_1+d-1} \sum_{i,j,k,p,q,u=0}^{\infty} \Psi_{ijk} \omega_{mpqu} (-1)^{k_1+d} \frac{(r+2t)!}{(r+t-k_1-1)!(t+k_1)!} \binom{r-1}{k_1} \binom{t+k_1}{d} \\ \times \left(a \left(u + k - \frac{q}{\sigma} - \frac{j}{\sigma} - \frac{1}{\sigma} \right) + 1 \right)^{-1}$$

The above expression provides the general form of the trimmed-L moments for the EGuPF distribution in terms of convergent infinite series involving the coefficients Ψ_{ijk} and ω_{mpqu} . The population measures of location, scale, skewness and kurtosis can be obtained for $t = 1, r = 1, 2, 3, 4$ respectively.

2.4.3. Mode

The mode of a distribution is the value of x that maximizes the probability density function. It is especially informative for skewed distributions where the mean may be misleading. The mode of EGuPF distribution can be determined by differentiating the (6) with respect to x and equating to zero and its is given as

$$g'(x) = \frac{a-1}{x} - \frac{a}{b} \left(\frac{x}{b}\right)^{a-1} \left[\left(\frac{1}{\sigma} + 1\right) \left(\frac{x}{b}\right)^{-a} + \left(\frac{1}{\sigma} - 1\right) \left(\frac{b^a - x^a}{b^a}\right)^{-1} \right] \\ + \frac{aB}{\sigma b} \frac{\left(\frac{x}{b}\right)^{-\left(\frac{a}{\sigma} + 1\right)}}{\left(\frac{b^a - x^a}{b^a}\right)^{-\left(\frac{1}{\sigma} - 1\right)}} \left[1 - (\alpha - 1) \exp \left\{ -B \left(\frac{b^a - x^a}{x^a}\right)^{\frac{1}{\sigma}} \right\} \left(1 - \exp \left\{ -B \left(\frac{b^a - x^a}{x^a}\right)^{\frac{1}{\sigma}} \right\} \right)^{-1} \right]$$

2.5. Reliability Measures

Reliability measures provide important insights into the ageing properties and failure dynamics of lifetime distributions. This section presents the derivation of some fundamental reliability measures of the EGuPF distribution.

2.5.1. Moment of Residual Life (MRL)

The n th moment of the residual life function denoted by $m_n(t) = E[(x - t)^n / X > t], n = 1, 2, \dots$, completely determines $G(x)$ [27]. The n th moment of residual life of X is given by

$$m_n(t) = \frac{1}{1 - G(t)} \int_t^\infty (x - t)^n dG(x) = E[(X - t)^n / X > t] \quad (19)$$

Using the general binomial expansion, (19) can be written as

$$m_n(t) = \frac{1}{1 - G(t)} \sum_{r=0}^n (-1)^{n-r} \binom{n}{r} t^{n-r} \int_t^\infty x^r dG(x)$$

Hence the MRL of EGuPF takes the following form

$$m_n(t) = \frac{a}{1 - G(t)} \sum_{r=0}^n \sum_{ijk=0}^\infty (-1)^{n-r} \Psi_{ijk} \binom{n}{r} t^{n-r} \frac{\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}{b^a \left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)} \left[\frac{b^{s+a \left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)} - t^{s+a \left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}}{\left(s + a \left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)\right)} \right]$$

2.5.2. Moment of Reversed Residual Life (MRRL)

The n th moment of the reversed residual life, defined as $m_n(t) = E[(t - X)^n | X \leq t]$ for $t > 0$ and $n = 1, 2, \dots$, completely determines $G(x)$.

$$M_n(t) = \frac{1}{G(t)} \int_0^t (t - x)^n dG(x)$$

Using (9) MRRL can be expressed as

$$M_n(t) = \frac{1}{G(t)} \sum_{r=0}^n (-1)^r \binom{n}{r} t^{n-r} \int_0^t x^r dG(x)$$

Therefore the MRRL for EGuPF distribution is

$$M_n(t) = \frac{a}{G(t)} \sum_{r=0}^n \sum_{i,j,k=0}^{\infty} (-1)^r \Psi_{ijk} \binom{n}{r} t^{n-r} \frac{\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right) t^{s+a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)}}{b^{a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)} \left(s + a\left(k - \frac{j}{\sigma} - \frac{1}{\sigma}\right)\right)}$$

2.6. Entropy

In this subsection, we derive the Rényi entropy of order $\gamma > 0, \gamma \neq 1$ for the proposed distribution. Rényi entropy generalizes Shannon entropy and is particularly useful in reliability, information theory, and uncertainty quantification. The Renyi entropy is defined by [28] as

$$I_{R(\gamma)} = \frac{1}{1-\gamma} \log \int_0^{\infty} g^{\gamma}(x) dx, \gamma > 0, \gamma \neq 1 \quad (20)$$

Theorem 2 : The Renyi entropy of a random variable that is distributed as EGuPF distribution is given by,

$$I_{R(\gamma)} = \frac{1}{1-\gamma} \log \left[b \sum_{ijk=0}^{\infty} \eta_{ijk} \left(ak - \gamma \left[\left(\frac{a}{\sigma} + 1 \right) + \frac{j}{\sigma} \right] \right)^{-1} \right]$$

Proof: From (20) we have that

$$g^{\gamma}(x) = \left(\frac{\alpha B a}{\sigma b} \right)^{\gamma} \frac{\left(\frac{x}{b} \right)^{-\gamma \left(\frac{a}{\sigma} + 1 \right)}}{\left(1 - \left(\frac{x}{b} \right)^a \right)^{-\gamma \left(\frac{1}{\sigma} - 1 \right)}} \exp \left\{ -B \gamma \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \left[1 - \exp \left\{ -B \left(\frac{x^a}{b^a - x^a} \right)^{-\frac{1}{\sigma}} \right\} \right]^{\gamma(\alpha-1)}$$

Applying the general binomial expansion and power series expansion for exponential function to $g^{\gamma}(x)$ we obtain

$$g^{\gamma}(x) = \sum_{ijk=0}^{\infty} \eta_{ijk} \left(\frac{x}{b} \right)^{ak - \left[\gamma \left(\frac{a}{\sigma} + 1 \right) + \frac{j}{\sigma} \right]} \quad (21)$$

where $\eta_{ijk} = \left(\frac{\alpha B a}{\sigma b} \right)^{\gamma} \frac{(-1)^{i+j+k}}{j!} \binom{\gamma(\alpha-1)}{i} \binom{\gamma \left(\frac{1}{\sigma} - 1 \right) + \frac{j}{\sigma}}{k} (B(\gamma+i))^j$

substituting (21) in (20) and integrating we have

$$I_{R(\gamma)} = \frac{1}{1-\gamma} \log \left[b \sum_{ijk=0}^{\infty} \eta_{ijk} \left(ak - \gamma \left[\left(\frac{a}{\sigma} + 1 \right) + \frac{j}{\sigma} \right] \right)^{-1} \right]$$

Furthermore, the q-entropy is defined by

$$H_q(X) = \frac{1}{1-q} \log \left(1 - \int_0^{\infty} g^q(x) dx \right); q > 0, q \neq 1,$$

Hence, the q-entropy of EGuPF distribution is given as

$$H_q(X) = \frac{1}{1-q} \log \left[1 - b \sum_{ijk=0}^{\infty} \eta_{ijk} \left(ak - q \left[\left(\frac{a}{\sigma} + 1 \right) + \frac{j}{\sigma} \right] \right)^{-1} \right]$$

2.7. Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the EGuPF distribution with pdf $g(x)$ and cdf $G(x)$, and let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ denote the corresponding order statistics. The r th order statistics is given by

$$g_{r;n}(x) = \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} g(x) G(x)^{v+r-1} \quad (22)$$

where $B(.,.)$ is Beta function, substituting (11) and (12) into (22) and replacing l with $v+r-1$, and simplify, the order statistics of EGuPF distribution takes the form

$$g_{r;n}(x) = \frac{1}{B(r, n-r+1)} \sum_{v=0}^{n-r} \sum_{m=0}^{v+r-1} \sum_{ijk=0}^{\infty} \sum_{pqu=0}^{\infty} \Phi^* \left(\frac{a \left(k - \frac{j}{\sigma} - \frac{1}{\sigma} \right)}{b} \right) \left(\frac{x}{b} \right)^{a \left(k + u - \frac{q}{\sigma} - \frac{j}{\sigma} - \frac{1}{\sigma} \right) - 1}$$

$$\text{where } \Phi^* = (-1)^v \binom{n-r}{v} \Psi_{ijk} \omega_{mpqu}$$

2.8. Maximum Likelihood Estimation

Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the proposed EGuPF distribution with parameter vector $\Lambda = (\mu, \alpha, a, b, \sigma)$ and pdf given in (6); the unknown parameters are estimated using the maximum likelihood method. The log-likelihood function l for the complete sample, corresponding to the parameter vector Λ is given by

$$l = n \log \alpha + n \log B + n \log a - n \log \sigma - n \log b - \left(\frac{a}{\sigma} + 1 \right) \sum_{i=1}^n \log \left(\frac{x_i}{b} \right) + \left(\frac{1}{\sigma} - 1 \right) \times \sum_{i=1}^n \log \left(1 - \left(\frac{x_i}{b} \right)^a \right) - B \sum_{i=1}^n Z_i + (\alpha - 1) \sum_{i=1}^n \log [1 - \exp \{-BZ_i\}] \quad (23)$$

where

$$Z_i = \left(\frac{\left(\frac{x_i}{b} \right)^a}{1 - \left(\frac{x_i}{b} \right)^a} \right)^{-\frac{1}{\sigma}}, \quad \text{The sample maxima is the estimator of } b, \text{ i.e., } \hat{b} = X_{(n)}.$$

The associated score function for the likelihood function in (23) are:

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \log (1 - \exp \{-BZ_i\}) \\ \frac{\partial l}{\partial \mu} &= \frac{n}{\sigma} - \frac{1}{\sigma} B \sum_{i=1}^n Z_i + (\alpha - 1) \sum_{i=1}^n \left[\frac{Z_i B \exp(-BZ_i)}{\sigma [1 - \exp(-BZ_i)]} \right] \\ \frac{\partial l}{\partial a} &= \frac{n}{a} - \frac{1}{\sigma} \sum_{i=1}^n \log \left(\frac{x_i}{b} \right) Z_i - \left(\frac{1}{\sigma} - 1 \right) \sum_{i=1}^n \frac{x_i^a \log \left(\frac{x_i}{b} \right)}{(b^a - x_i^a)} + \frac{b^a B}{\sigma} \sum_{i=1}^n \frac{Z_i \log \left(\frac{x_i}{b} \right)}{(b^a - x_i^a)} \\ &\quad + \frac{b^a B}{\sigma} (\alpha - 1) \sum_{i=1}^n \left\{ \frac{Z_i \log \left(\frac{x_i}{b} \right) \exp \{-BZ_i\}}{(b^a - x_i^a)(1 - \exp \{-BZ_i\})} \right\} \\ \frac{\partial l}{\partial \sigma} &= -\frac{n}{a} + \frac{a}{\sigma^2} \sum_{i=1}^n \log \left(\frac{x_i}{b} \right) - \frac{1}{\sigma^2} \sum_{i=1}^n \log \left(1 - \left(\frac{x_i}{b} \right)^a \right) - \frac{B}{\sigma^2} \sum_{i=1}^n Z_i \log \left(\frac{x_i^a}{b^a - x_i^a} \right) \\ &\quad + \frac{B}{\sigma^2} (\alpha - 1) \sum_{i=1}^n \left\{ \frac{Z_i \exp \{-BZ_i\}}{(1 - \exp \{-BZ_i\})} \log \left(\frac{x_i^a}{b^a - x_i^a} \right) \right\} \end{aligned}$$

The system of equations above obtained from the partial derivatives of (23) with respect to the individual parameters does not admit explicit closed-form solutions. Therefore, the system of nonlinear equations is solved numerically using iterative optimisation algorithms like the BFGS algorithm or the Newton–Raphson method to produce the maximum likelihood estimates (MLEs) of the parameters.

3. RESULTS

In this section, we present results from simulation studies alongside the application of the EGuPF distribution to two real dataset.

3.1. Simulation Studies

A simulation experiment is conducted to ascertain the consistency and stability of the maximum likelihood estimators (MLEs) of the parameters in the proposed distribution. The simulation is repeated $N = 5000$ times, each having sample sizes $n = 200, 500, 800, 1200$ and 1500 for parameter values $\mu = 1.5, a = 1.2, \alpha = 1.2$ and $\sigma = 2$. For the purpose of evaluation, the following five measures were computed: mean estimates (ME), average bias (AVB), root mean squared error (RMSE), coverage probability (CP) and average confidence width (ACW). The estimated MEs, AVBs and RMSEs are given by

$$\begin{aligned} \text{ME} &= \frac{1}{N} \sum_{i=1}^N \Lambda_i, \\ \text{AVB} &= \frac{1}{N} \sum_{i=1}^N (\Lambda_i - \Lambda), \\ \text{RMSE} &= \sqrt{\frac{1}{N} \sum_{i=1}^N (\Lambda_i - \Lambda)^2}, \end{aligned}$$

for $\Lambda = (\mu, \alpha, a, \sigma)$. Coverage probability (CP) is the percentage of intervals that contain the true value of Λ , while average confidence width (ACW) is the mean of the absolute value of the 95% confidence interval for $\hat{\Lambda}$. The Table 1 show the results of simulation studies.

Table 1: Simulation results for parameter values, $\mu = 1.5$, $a = 1.5$, $\alpha = 1.2$ and $\sigma = 2$.

Parameter	Sample size	ME	AVB	RMSE	ACW	CP
μ	200	2.7628	1.2628	3.0306	18.2354	0.9855
	500	1.9946	0.4946	1.8543	7.6529	0.9730
	800	1.7054	0.2054	1.1482	4.5767	0.9510
	1200	1.5120	0.0120	0.6582	2.7127	0.9305
	1500	1.4687	-0.0312	0.6353	2.2169	0.8950
α	200	2.6483	1.4483	2.4793	10.7388	1
	500	2.0852	0.8852	1.6453	5.2920	1
	800	1.8633	0.6632	1.2034	3.5617	0.9995
	1200	1.7302	0.5301	0.8749	2.5347	0.999
	1500	1.6882	0.4882	0.8134	2.1968	0.996
a	200	7.3815	5.8815	17.701	99.6582	0.9935
	500	3.9075	2.4075	9.8804	30.1218	0.985
	800	2.6579	1.1579	5.0372	12.0799	0.975
	1200	2.0629	0.5629	2.3151	5.5437	0.9595
	1500	1.9852	0.4851	2.5876	4.8543	0.9585
σ	200	3.4542	1.4542	2.9918	13.6400	0.9985
	500	2.7634	0.7634	2.0373	8.15045	0.9955
	800	2.4869	0.4869	1.6086	6.1649	0.9900
	1200	2.3228	0.3228	1.2344	4.7983	0.9845
	1500	2.2601	0.2601	1.1479	4.2350	0.977

Table 1 reports the simulation outcomes for the parameters μ , α , a , and σ under increasing sample sizes. The results show a consistent pattern: as sample size increases, the mean estimates approach their true values, the absolute bias and RMSE decrease, and the confidence intervals become narrower, reflecting improved accuracy and efficiency of the estimators.

3.2. Applications

The proposed EGuPF distribution is compared with the following probability distributions that have the same baseline namely, Transmuted PFD [16], Another generalized Transmuted PFD [23], PFD, Weibull PFD [17], and exponentiated weibull PFD [19].

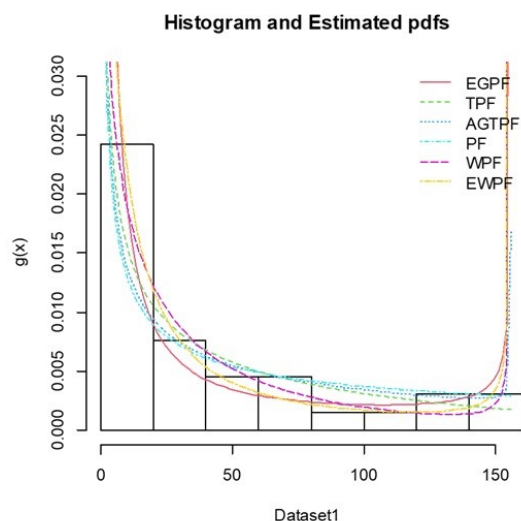
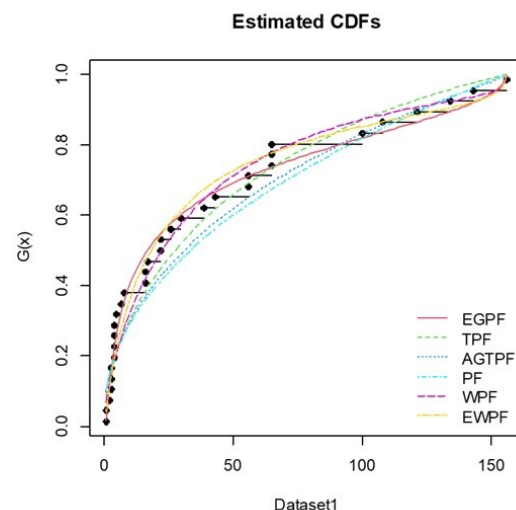
The first data (Dataset 1) were first analyzed by [29]. The data represent the survival times, in weeks, of 33 patients suffering from Acute Myelogenous Leukaemia. The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, and 43. The skewness and Kurtosis of dataset 1 are 1.221 and 3.349 respectively. These values indicates that Dataset 1 is positively skewed and more peaked than the normal distribution. Table 2 gives the ML estimates of parameters of EGuPF distribution and other competing models with the same baseline while Table 3 presents some Statistics measures that facilitates comparions among models. The plots of the estimated pdfs and cdfs for dataset 1 are given in Figures 3 and 4

Table 2: MLEs of EGuPF parameters and other competing models for dataset 1 (standard errors in parenthesis).

Distributions	Estimates				
EGUPF($\mu, \sigma, \alpha, a, b$)	-2.472 (0.625)	0.462 (1.04)	0.214 (0.506)	0.586 (0.345)	156
TPF(λ, a, b)	0.523 (0.287)	0.562 (0.100)	156		
AGTPF(α, λ, a, b)	0.652 (0.18)	0.775 (0.24)	0.484 (0.10)	156	
PF(a, b)	0.450 (0.078)	156			
WPF(α, β, a, b)	10.540 (7.25)	0.068 (0.043)	2.833 (0.618)	156	
EWPF($\theta, \alpha, \beta, a, b$)	11.680 (14.45)	4.708 (1.19)	0.0600 (0.026)	3.898 (2.07)	156

Table 3: Statistics Measures for Dataset1 .

Distributions	AIC	BIC	CVM	AD	KS	-loglik
EGUPF	297.28	303.27	0.043	0.319	0.096	144.6449
TPF	307.34	310.33	0.109	0.892	0.125	151.6718
AGTPF	306.91	311.40	0.155	0.984	0.127	150.4578
PF	307.40	308.89	0.196	1.238	0.144	152.7016
WPF	300.13	304.62	0.074	0.496	0.125	147.0697
EWPF	299.76	305.75	0.065	0.414	0.121	145.8845

**Figure 3:** Plots of Estimated pdf of all competing models for Dataset1.**Figure 4:** Plots of Estimated cdf of all competing models for Dataset1.

The second data (Dataset 2) have been used by [30]. The data represent the monthly actual taxes revenue in Egypt from January 2006 to November 2010. The distribution is highly skewed to the right. The actual taxes revenue data (in 1000 million Egyptian pounds) are: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 9.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8. The values of Skewness and

kurtosis for dataset 2 are 1.08 and 4.25 respectively. The dataset 2 is positively skewed and leptokurtic. Table 4 gives the ML estimates of fitted distributions while Table 5 provides the different statistics measures. Figures 5 and 6 shows the plots of estimated pdfs and cdfs of fitted distributions.

Table 4: MLEs of EGuPF parameters and other competing models for dataset2 (standard errors in parenthesis)

Distributions	Estimates				
EGUPF($\mu, \sigma, \alpha, a, b$)	-10.782 (7.04)	4.299 (4.09)	1.32 (0.93)	8.112 (5.94)	36
TPF(λ, a, b)	0.894 (0.09)	1.168 (0.12)	36		
AGTPF(α, λ, a, b)	1.471 (0.32)	0.812 (0.13)	1.372 (0.17)	36	
PF(a, b)	0.827 (0.11)	36			
WPF(α, β, a, b)	5.405 (0.95)	0.041 (0.03)	44.049 (36.35)	36	
EWPF($\theta, \alpha, \beta, a, b$)	72.392 (81.12)	8.370 (1.11)	0.032 (0.02)	14.713 (10.35)	36

Table 5: Statistics Measures for Dataset2 .

Distributions	AIC	BIC	CVM	AD	KS	-loglik
EGUPF	373.67	381.98	0.033	0.205	0.056	182.8351
TPF	415.48	419.64	0.887	4.793	0.195	205.7399
AGTPF	414.59	420.83	0.602	3.599	0.176	204.2999
PF	432.66	434.74	1.731	8.443	0.298	215.3321
WPF	393.39	399.62	0.291	1.893	0.143	193.6944
EWPF	375.74	384.05	0.071	0.399	0.081	183.8702

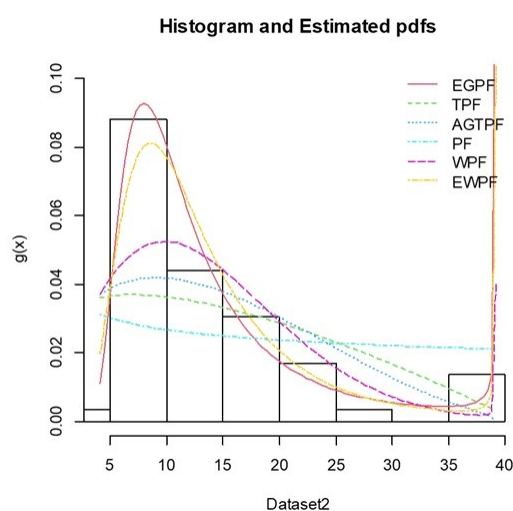


Figure 5: Plots of Estimated pdf of all competing models for Dataset2.

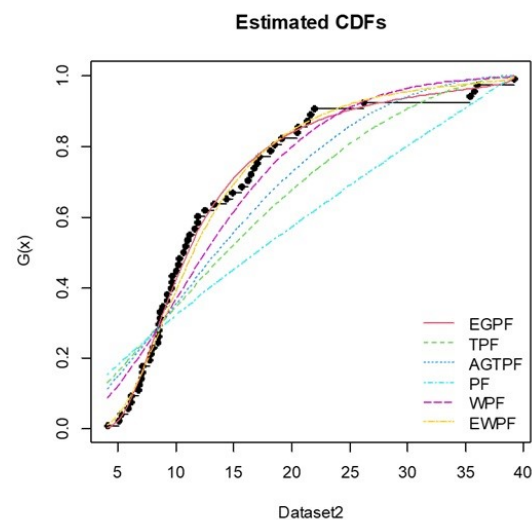


Figure 6: Plots of Estimated pdf of all competing models for Dataset2.

3.3. Discussion

The findings from the simulation analysis provide strong empirical support for the asymptotic behaviour of the maximum likelihood estimators (MLEs) of the parameters of the proposed EGuPF distribution. As the sample size increases from $n = 200$ to $n = 1500$, a clear and systematic pattern is observed across all parameters (μ, α, a, σ). In particular, the mean estimates progressively approach their true parameter values, demonstrating the consistency of the estimators. This convergence behaviour is evident for all parameters, but is especially pronounced for μ , where the estimate decreases from 2.7628 at $n = 200$ to 1.4687 at $n = 1500$, closely approaching the true value of 1.5.

Furthermore, the average bias (AVB) and root mean square error (RMSE) decrease monotonically as the sample size increases, thereby confirming the efficiency and stability of the estimators. The reduction in bias indicates that the estimators become increasingly unbiased in large samples, while the decline in RMSE across all parameters of the proposed model reflects improved estimation precision. In addition, the average confidence width (ACW) decreases consistently with increasing sample size, suggesting enhanced precision of the interval estimates, whereas the coverage probabilities (CP) remain close to the nominal 95% level across different sample sizes. Collectively, these results demonstrate that the estimators of the parameters of the EGuPF distribution are consistent, efficient, and reliable.

The practical applicability and flexibility of the proposed EGuPF distribution are further demonstrated using two real datasets. Dataset 1, which consists of survival times of patients with acute myelogenous leukaemia, exhibits positive skewness and leptokurtic behaviour. The maximum likelihood estimates of the parameters for the EGuPF distribution and competing models are presented in Table 2, with the corresponding standard errors given in parentheses. The estimates of the parameters of EGuPF distribution given in the first row of Table 1 reflect the characteristics of the observed survival pattern, which is concentrated toward shorter survival times with a positively skewed right tail. The negative estimate of $\hat{\mu}$ is consistent with the location of the fitted distribution toward the lower survival-time region, while the relatively small $\hat{\sigma}$ contributes to the concentration and shape of the observation. The estimate $\hat{a} = 0.586$ provides the power-function component needed to describe the shape of the survival times, whereas the relatively small $\hat{\alpha} = 0.214$ indicates a substantial exponentiated-Gumbel modification of the baseline power-function distribution, thus providing flexibility for the observed right-skewed pattern. The estimated upper endpoint $\hat{b} = 156$ coincides with the longest observed survival time, indicating that the fitted distribution accommodates the observed range of survival times.

Model comparison statistics for assessing the performance of the EGuPF distribution and competing models are reported in Table 3. The results indicate that the EGuPF model outperforms all competing distributions, as evidenced by the lowest values of the Akaike Information Criterion (AIC = 297.28), Bayesian Information Criterion (BIC = 303.27), Cramér-von Mises (CVM = 0.043), Anderson-Darling (AD = 0.319), and Kolmogorov-Smirnov (KS = 0.096) statistics, in addition to the highest log-likelihood value. This improved performance can be attributed to the additional flexibility introduced by the extra parameters, which enable the model to better capture skewness and tail behaviour.

A similar pattern is observed for Dataset 2, which represents highly right-skewed tax revenue data. For the Egyptian monthly tax-revenue data, the estimates of parameters of EGuPF model reflect a distribution that is more dispersed and positively skewed, with occasional relatively high revenue values. The negative value of $\hat{\mu}$ reflects the location of the fitted transformed distribution. The large estimate of $\hat{\sigma}$ is consistent with the greater spread of the tax-revenue observations and shape of the distribution. Also, the large value of $\hat{a}$ indicates a substantial power-function shape required to represent the data. The estimate $\hat{\alpha} = 1.32 > 1$ suggests

a stronger exponentiated-Gumbel modification of the power-function baseline, providing additional flexibility for the observed right-skewed and leptokurtic behaviour. The estimated upper endpoint $\hat{b} = 36$ coincides with the largest observed monthly tax revenue, indicating that the fitted EGuPF model captures the observed range of the revenue data.

Once again, the EGuPF distribution provides the best fit among all considered models, yielding the smallest values of AIC (373.6703), BIC (381.9804), and goodness-of-fit statistics (CVM, AD, and KS).

The graphical assessments, including plots of the estimated probability density functions (pdfs) and cumulative distribution functions (cdfs) for both datasets, as presented in Figures 3–6, further corroborate the numerical findings reported in Tables 3 and 5. These figures show that the EGuPF distribution closely follows the empirical pdf and cdf more accurately than the competing models in both cases, thereby reinforcing its adequacy and superior fitting capability.

4. CONCLUSION

This study proposes a new extension of the power function distribution, referred to as the EGuPF distribution, based on the exponentiated Gumbel-G family of distributions introduced by [13]. The probability density function of the EGuPF distribution is unimodal, while its hazard rate function exhibits bathtub-shaped, J-shaped, and reverse J-shaped behaviours, demonstrating its flexibility in modelling diverse lifetime data. The density function is expressed as an infinite linear combination of power function distributions, which enhances its analytical tractability. Several important statistical and reliability properties of the proposed distribution are investigated, including the quantile function, moments, trimmed L-moments, mode, mean residual life (MRL), and mean reversed residual life (MRRL). The maximum likelihood estimators of the model parameters do not admit closed-form expressions; therefore, they are obtained numerically using suitable optimization routines. Results from the simulation study indicate that the estimators are consistent, efficient, and stable across different sample sizes. Applications to two real datasets demonstrate the practical usefulness of the proposed model, as the EGuPF distribution outperforms competing distributions with the same baseline in terms of goodness-of-fit measures.

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