

A NEW EXTENDED EXPONENTIAL RAYLEIGH DISTRIBUTION: ESTIMATION AND APPLICATION

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Abstract

In the present paper, we have introduced a new extended exponential Rayleigh distribution with the help of the Transformed-Transformer technique using traditional exponential and Rayleigh distributions, and some important statistical properties of the newly developed distribution have been studied. In addition, the parameters of the distribution have been estimated through the maximum likelihood, least squares, and maximum product of spacing methods of estimation on the basis of complete samples of it. Two real data sets have been considered in order to show the applicability of the newly developed distribution to the real life problems.

Keywords: Transformed-Transformer, Exponential distribution, Rayleigh distribution, Estimation, Simulation Study.

1. INTRODUCTION

In the field of survival and reliability analysis to capture realistic situations, the Rayleigh distribution plays a key role to analyze the uncertainty involved in the data. The Rayleigh distribution was originally introduced by [18]. The probability density function (pdf) and cumulative distribution function (cdf) of the Rayleigh distribution are given by equations (1) and (2) respectively.

$$\psi(t) = 2\theta t e^{-\theta t^2}; t > 0, \theta > 0. \quad (1)$$

$$\Psi(t) = 1 - e^{-\theta t^2}; t > 0, \theta > 0. \quad (2)$$

Its hazard rate function, i.e., $h(t) = 2\theta t$, is an increasing function, linearly with a variate value. It is clearly a rarely attainable situation. A number of authors have motivated to generalize or transform Rayleigh distribution in order to get a new distribution which has different shapes of their hazard rate function such as monotonic decreasing or bath tub or inverted bath tub etc.

For example, [7] have introduced a new Rayleigh-Rayleigh distribution (RRD) and studied its various statistical properties and compared its performance over existing Rayleigh distribution on real data sets. [15] introduced the generalized Rayleigh distribution and various estimation techniques have been used to estimate its unknown parameters. [20] introduced a new generalization of the Rayleigh distribution using the conservability approach. [8] have introduced the power exponentiated Rayleigh distribution, and studied its various statistical properties, also demonstrates its effectiveness through real data sets and simulated data using goodness-of-fit measures. [17] have introduced the transmuted generalized Rayleigh distribution using the

quadratic rank transmutation map (QRTM) and demonstrated its important statistical properties. [13] obtained Bayesian estimators for the Rayleigh distribution parameter using squared error and LINEX loss functions. [2] analyzes the impact of various censored sampling schemes on the estimation of the parameter of the Rayleigh distribution. [1] proposed the unit generalized Rayleigh distribution, establish its statistical and entropy properties, and demonstrates its effectiveness through simulations and also carried out real data analysis. [16] discussed the estimation of parameters of the exponentiated Rayleigh distribution using Type II censored data. [22] compared maximum likelihood and Bayesian estimation methods for the inverted exponentiated Rayleigh distribution under step-stress type-I hybrid censoring.

Motivated by the review of the above literature on the generalizations and transformations of the Rayleigh distribution, we introduce a new extended exponential Rayleigh (NEER) distribution using the Transformed-Transformer technique proposed by [5]. Let $\psi(t)$ be the pdf of a random variable $T \in [a, b]$, for $-\infty < a \leq t \leq b < \infty$. Let X be another random variable with pdf $g(x)$ and cdf $G(x)$. Let $R[G(x)]$ be a function of the cdf $G(x)$ of the random variable X , such that $R[G(x)]$ satisfies the following conditions:

$$\left. \begin{aligned} R[G(x)] &\in [a, b], \\ R[G(x)] &\text{ is differentiable and monotonically non-decreasing,} \\ R[G(x)] &\rightarrow a \text{ as } X \rightarrow -\infty \text{ and } R[G(x)] \rightarrow b \text{ as } X \rightarrow \infty. \end{aligned} \right\} \quad (3)$$

Here, $R[G(x)]$ is called a transformer.

Now, according to the Transformed-Transformer technique or T-X technique, the cdf $F(x)$ of the new family of distributions is defined as follows,

$$F(x) = \int_a^{R[G(x)]} \psi(t) dt = \Psi\{R[G(x)]\}, \quad (4)$$

where, $\Psi(t)$ is the cdf corresponding to the pdf $\psi(t)$ of the random variable T . Also, the pdf $f(x)$, say corresponding to the cdf $F(x)$ is obtained as follows,

$$f(x) = \left\{ \frac{d}{dx} R[G(x)] \right\} \psi[R[G(x)]]. \quad (5)$$

Consider the support of T is $[a, \infty)$ with $a \geq 0$, without loss of generality, we can assume that $a = 0$. $R[G(x)]$ can be defined in various forms, such as $-\ln(1 - G(x))$, $\frac{G(x)}{1 - G(x)}$, $-\ln(1 - G^\alpha(x))$ and $\frac{G^\alpha(x)}{1 - G^\alpha(x)}$, where $\alpha > 0$.

In particular, if we take the support of random variable T as $[0, \infty)$, then the functional form of $R[G(x)]$ may be defined as

$$R[G(x)] = \frac{G(x)}{1 - G(x)},$$

which satisfies all three conditions of equation (3). Then the cdf $F(x)$ and the pdf $f(x)$ of newly generalized distribution using Transformed-Transformer technique or T-X technique are given by equations (6) and (7) respectively.

$$F(x) = \int_0^{\frac{G(x)}{1 - G(x)}} \psi(t; \theta) dt = \Psi\left\{ \frac{G(x)}{1 - G(x)} \right\}, \quad (6)$$

$$f(x) = \frac{g(x)}{[1 - G(x)]^2} \psi\left\{ \frac{G(x)}{1 - G(x)} \right\}, \quad (7)$$

where, $g(x) = \frac{d}{dx} G(x)$ is the pdf corresponding to cdf $G(x)$.

In the literature, numerous generalized family of distributions have been proposed and studied using the Transformed-Transformer technique or T-X technique. Notable examples include exponentiated Weibull-G family by [14], Weibull-G family by [9], T-geometric family of

discrete distribution by [4], Gompertz-G family of distributions by [3], Exponentiated Half-Logistic Family of Distributions by [12], The type I half-logistic family of distributions by [11], T-normal family of distributions by [6] etc. These developments reflect a rising interest in generalizing traditional distributions to better representative of complex data sets.

Motivated by the need for a flexible lifetime distribution, this article proposes a new statistical model and investigates its theoretical properties, estimation methods, and practical applications. The remainder of this article is organized as follows: Section 2 describe the pdf and cdf of the proposed distribution. The statistical properties are studies in Section 3. The parameter estimation procedures are described in Section 4. A Monte Carlo simulation study to evaluate the performance of the estimator is explained in Section 5. Applications of the proposed model to two real data sets are presented in Section 6. Finally, Section 7 summarizes the main findings and concludes the article.

2. NEW EXTENDED EXPONENTIAL RAYLEIGH DISTRIBUTION

The pdf $g(x)$ and the cdf $G(x)$ of the exponential distribution using the rate parameter λ is given by equations (8) and (9) respectively.

$$g(x) = \lambda e^{-\lambda x}; x > 0, \lambda > 0. \quad (8)$$

$$G(x) = 1 - e^{-\lambda x}; x > 0, \lambda > 0. \quad (9)$$

If the random variable T follow the Rayleigh distribution having pdf defined in equation (1), consider, Rayleigh distribution as baseline distribution and $G(x)$ be the cdf of exponential distribution defined in equation (9), then the functional form of $R[G(x)]$ is as

$$R[G(x)] = \frac{G(x)}{1 - G(x)} = \frac{1 - e^{-\lambda x}}{1 - (1 - e^{-\lambda x})} = e^{\lambda x} - 1, \quad (10)$$

and consequently the cdf of New Extended Exponential Rayleigh distribution is given as follows:

$$F(x; \lambda, \theta) = 1 - e^{-\theta(e^{\lambda x} - 1)^2}; x > 0, \lambda > 0, \theta > 0. \quad (11)$$

Also, the pdf $f(x; \lambda, \theta)$ corresponding to the cdf $F(x; \lambda, \theta)$ of our proposed distribution is given as follows:

$$f(x; \lambda, \theta) = 2\theta\lambda e^{\lambda x}(e^{\lambda x} - 1)e^{-\theta(e^{\lambda x} - 1)^2}; x > 0, \lambda > 0, \theta > 0. \quad (12)$$

We will denote new extended exponential Rayleigh distribution having pdf as in (12) by NEER(λ, θ) distribution. The shapes of pdf $f(x; \lambda, \theta)$ and cdf $F(x; \lambda, \theta)$ of NEER(λ, θ) distribution for various combination of values of its parameters λ and θ are shown in Figure 1 and 2 respectively.

3. STATISTICAL PROPERTIES

In this section, we examine the statistical properties of the NEER(λ, θ) distribution, with particular emphasis on the survival function, hazard rate function, quantile function, and order statistics.

3.1. Survival and Hazard Rate Function

The survival function and hazard rate function of the NEER(λ, θ) distribution are given as follows:

$$S(x) = e^{-\theta(e^{\lambda x} - 1)^2}, \quad (13)$$

$$h(x) = 2\theta\lambda e^{\lambda x}(e^{\lambda x} - 1). \quad (14)$$

From Figure 3, it is evident that the hazard rate function is strictly increasing for various combinations of the parameters λ and θ . This monotonic behavior indicates that the proposed

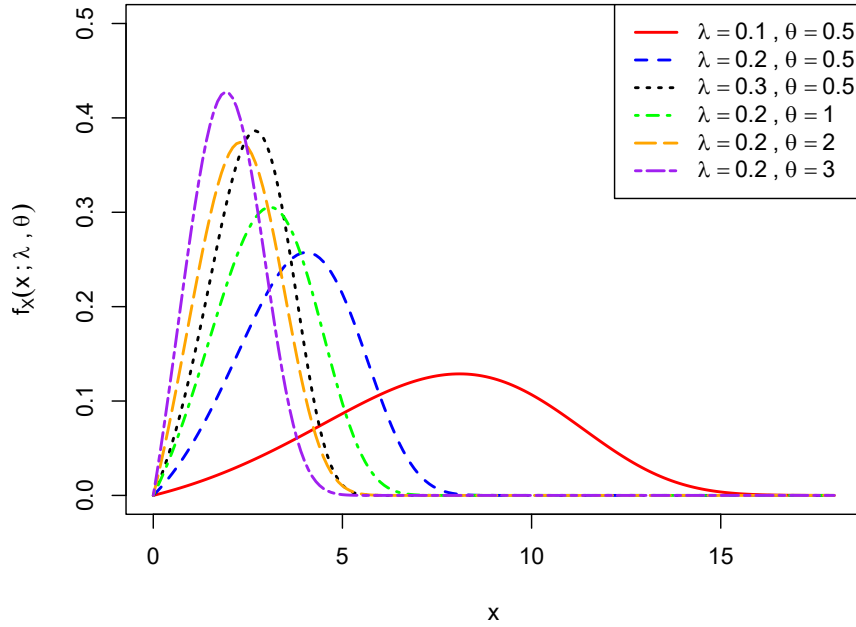


Figure 1: The pdf of NEER(λ, θ) distribution for different combinations values of λ and θ .

distribution is well suited for modeling lifetime data exhibiting progressive aging and wear-out mechanisms. Consequently, it has broad potential applications in reliability engineering, survival analysis, medical research, actuarial science, and preventive maintenance, where the risk of failure increases continuously over time.

3.2. Quantile function

If $Q(u)$ be the quantile function of NEER(λ, θ) distribution, then it can be obtained as the solution of

$$\begin{aligned} F(Q(u)) &= u, \\ \Rightarrow 1 - e^{-\theta(e^{\lambda Q(u)} - 1)^2} &= u, \end{aligned}$$

where $u \sim U[0, 1]$ is a uniformly distributed random variable. On solving the above the equation for $Q(u)$, we obtained

$$Q(u) = \frac{1}{\lambda} \ln \left[1 + \left(-\frac{1}{\theta} \ln(1 - u) \right)^{\frac{1}{2}} \right]. \quad (15)$$

If Q_i be the i^{th} quantile function of NEER(λ, θ) distribution, then the first quantile Q_1 , second quantile Q_2 , and third quantile Q_3 can be obtained by putting $u = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$ respectively in equation (15) as follows:

$$\begin{aligned} Q_1 &= \frac{1}{\lambda} \ln \left[1 + \left(\frac{1}{\theta} \ln \left(\frac{4}{3} \right) \right)^{\frac{1}{2}} \right], \\ Q_2 &= \frac{1}{\lambda} \ln \left[1 + \left(\frac{1}{\theta} \ln(2) \right)^{\frac{1}{2}} \right], \end{aligned}$$

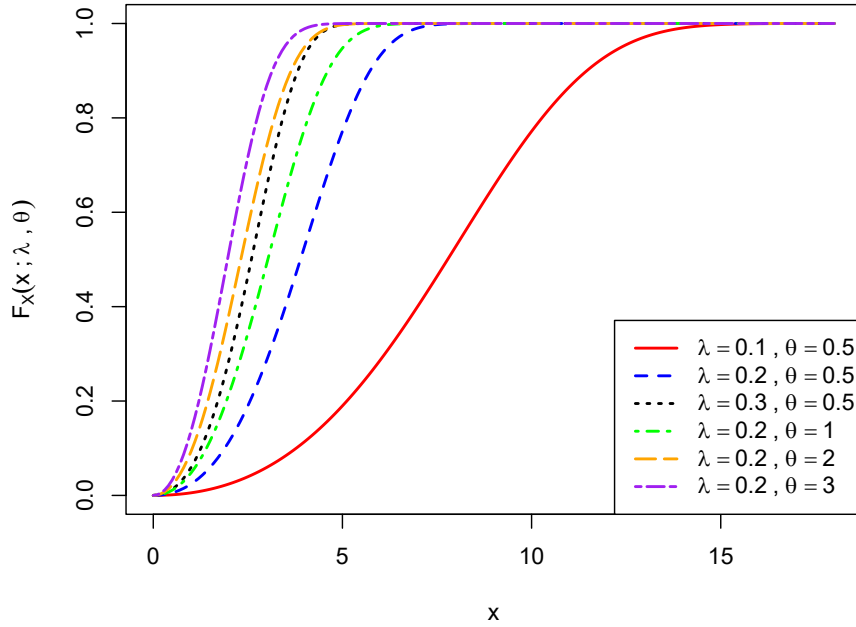


Figure 2: The cdf of NEER(λ, θ) distribution for different combinations values of λ and θ .

$$Q_3 = \frac{1}{\lambda} \ln \left[1 + \left(\frac{1}{\theta} \ln(4) \right)^{\frac{1}{2}} \right].$$

3.3. Order Statistics

Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics of a random sample $\underline{X} = (X_1, X_2, \dots, X_n)$ of size n from NEER(λ, θ) distribution. Then the pdf of r^{th} order statistics is given by

$$f_{X_{(r)}}(x_{(r)}; \lambda, \theta) = \frac{1}{B(r, n-r+1)} [F(x_{(r)}; \lambda, \theta)]^{r-1} f(x_{(r)}; \lambda, \theta) [1 - F(x_{(r)}; \lambda, \theta)]^{n-r}. \quad (16)$$

By using equations (11) and (12) in equation (16), the pdf of the r^{th} order statistics $x_{(r)}$ is obtained as follows:

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} 2\lambda\theta e^{\lambda x} (e^{\lambda x} - 1) e^{-(n-r+1)\theta(e^{\lambda x}-1)^2} \left[1 - e^{-\theta(e^{\lambda x}-1)^2} \right]^{r-1}. \quad (17)$$

For $r = 1$ and n , we obtained the pdfs of $x_{(1)} = \min(x_1, x_2, \dots, x_n)$ and $x_{(n)} = \max(x_1, x_2, \dots, x_n)$ respectively as;

$$f_{X_{(1)}}(x; \lambda, \theta) = 2n\lambda\theta e^{\lambda x} (e^{\lambda x} - 1) e^{-n\theta(e^{\lambda x}-1)^2}, \quad (18)$$

$$f_{X_{(n)}}(x; \lambda, \theta) = 2n\lambda\theta e^{\lambda x} (e^{\lambda x} - 1) e^{-\theta(e^{\lambda x}-1)^2} \left[1 - e^{-\theta(e^{\lambda x}-1)^2} \right]^{n-1}. \quad (19)$$

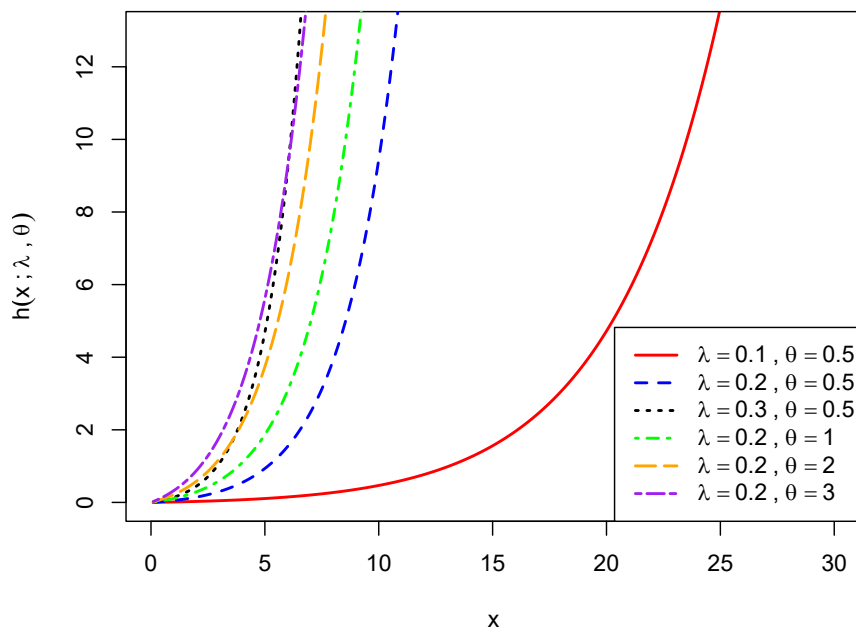


Figure 3: The hazard rate functions of NEER(λ, θ) distribution for different combinations values of λ and θ .

4. ESTIMATION PROCEDURES

4.1. Maximum Likelihood Estimation (MLE)

Let $\underline{X}=(X_1, X_2, \dots, X_n)$ be a random sample of size n and let X_i NEER(λ, θ) distribution $\forall i = 1, 2, \dots, n$. Then the likelihood function of the random sample $\underline{X}$ is given by

$$L = \prod_{i=1}^n f(x_i; \lambda, \theta), \quad (20)$$

and consequently the log-likelihood function of the random sample $\underline{X}$ is given by

$$\ln L = n \ln(2) + n \ln(\lambda) + n \ln(\theta) + \lambda \sum_{i=1}^n x_i - \theta \sum_{i=1}^n (e^{\lambda x_i} - 1)^2 + \sum_{i=1}^n \ln(e^{\lambda x_i} - 1). \quad (21)$$

Now, the log-likelihood equations or normal equations for estimating the parameters λ and θ can be obtained by differentiating the equation (22) with respect to λ and θ , we get

$$\begin{aligned} \frac{\partial \ln L}{\partial \lambda} &= 0, \\ \Rightarrow \frac{n}{\lambda} + \sum_{i=1}^n x_i - 2\theta \sum_{i=1}^n (e^{\lambda x_i} - 1)x_i e^{\lambda x_i} + \sum_{i=1}^n \frac{x_i e^{\lambda x_i}}{e^{\lambda x_i} - 1} &= 0, \end{aligned} \quad (22)$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta} &= 0, \\ \Rightarrow \frac{n}{\theta} - \sum_{i=1}^n (e^{\lambda x_i} - 1)^2 &= 0. \end{aligned} \quad (23)$$

Since the equations (22) and (23) are two non-linear likelihood equations. Solving these equations provides the maximum likelihood estimates (MLEs) $\hat{\lambda}_{ML}$ and $\hat{\theta}_{ML}$ of the parameters λ and θ . However, since these equations are not in closed form, they cannot be solved analytically. Therefore, some numerical approximation technique can be used for their approximate solutions, particularly the Newton-Raphson method is used to obtained the MLE's of the parameters λ and θ .

4.2. Least Squares Estimation (LSE)

Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics of the chosen random sample $X = (X_1, X_2, \dots, X_n)$. Then the least squares estimators of parameters λ and θ can be obtained by minimizing the following w.r.t. λ and θ .

$$\begin{aligned} z(\lambda, \theta) &= \sum_{i=1}^n \left(F(x; \lambda, \theta) - \frac{i}{n+1} \right)^2, \\ \Rightarrow z(\lambda, \theta) &= \sum_{i=1}^n \left(1 - e^{-\theta(e^{\lambda x(i)} - 1)^2} - \frac{i}{n+1} \right)^2. \end{aligned} \quad (24)$$

To minimize $z(\lambda, \theta)$ as defined in equation (24), we take partial derivatives of equation (24) with respect to λ and θ , and equating them to zero, we get the following normal equations.

$$\begin{aligned} \frac{\partial z(\lambda, \theta)}{\partial \lambda} &= 0, \\ \Rightarrow 4\theta \sum_{i=1}^n \left(1 - e^{-\theta(e^{\lambda x(i)} - 1)^2} - \frac{i}{n+1} \right) e^{-\theta(e^{\lambda x(i)} - 1)^2} (e^{\lambda x(i)} - 1) e^{\lambda x(i)} x(i) &= 0, \end{aligned} \quad (25)$$

and

$$\begin{aligned} \frac{\partial z(\lambda, \theta)}{\partial \theta} &= 0, \\ \Rightarrow 2 \sum_{i=1}^n \left(1 - e^{-\theta(e^{\lambda x(i)} - 1)^2} - \frac{i}{n+1} \right) e^{-\theta(e^{\lambda x(i)} - 1)^2} (e^{\lambda x(i)} - 1)^2 &= 0. \end{aligned} \quad (26)$$

The above two equations (25) and (26) are non-linear, which can not be solved analytically for λ and θ . Therefore, as mentioned above numerical approximation technique namely, Newton-Raphson method is used for solving these two equations for the parameters λ and θ respectively and we will abbreviate these estimates by $\hat{\lambda}_{LSE}$ and $\hat{\theta}_{LSE}$ respectively.

4.3. Maximum Product Spacing (MPS) estimation

Now a days, a popular method of estimation technique is the maximum product of sapcing method of estimation and it is introduced by [10]. Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics of chosen random sample $\underline{X} = (X_1, X_2, \dots, X_n)$. Then the spacing is defined as follow;

$$\begin{aligned} D_i &= \int_{x_{(i-1)}}^{x_{(i)}} f(x; \lambda, \theta) dx, \\ &= F(x_{(i)}; \lambda, \theta) - F(x_{(i-1)}; \lambda, \theta) \quad ; i = 1, 2, \dots, (n+1). \end{aligned}$$

Where $F(x_{(0)}; \lambda, \theta) = 0$ and $F(x_{(n+1)}; \lambda, \theta) = 1$. Therefore,

$$D_i = \begin{cases} F(x_{(1)}; \lambda, \theta) & \text{for } i = 1, \\ F(x_{(i)}; \lambda, \theta) - F(x_{(i-1)}; \lambda, \theta) & \text{for } i = 2, 3, \dots, n, \\ 1 - F(x_{(n)}) & \text{for } i = n + 1. \end{cases}$$

After substituting the values of $F(x_1)$, $F(x_i)$, $F(x_{i-1})$ and $F(x_n)$ in the above equation, we get;

$$D_i = \begin{cases} 1 - e^{-\theta(e^{\lambda x(1)} - 1)^2} & \text{for } i = 1, \\ e^{-\theta(e^{\lambda x(i-1)} - 1)^2} - e^{-\theta(e^{\lambda x(i)} - 1)^2} & \text{for } i = 2, 3, \dots, n, \\ e^{-\theta(e^{\lambda x(n)} - 1)^2} & \text{for } i = n + 1. \end{cases} \quad (27)$$

Let G be the geometric mean of the spacings $D_1, D_2, \dots, D_{n+1}$. Then The MPS estimates $\hat{\lambda}_{MS}$ and $\hat{\theta}_{MS}$ of λ and θ respectively can be obtained by maximizing the geometric mean of spacing as follow:

$$G = \left(\prod_{i=1}^{n+1} D_i \right)^{\frac{1}{n+1}}, \quad (28)$$

and hence the natural log of geometric mean G is obtained as follows

$$\begin{aligned} \ln G &= \frac{1}{n+1} \sum_{i=1}^{n+1} \ln(D_i), \\ &= \frac{1}{n+1} \left[\ln \left(1 - e^{-\theta(e^{\lambda x(1)} - 1)^2} \right) + \sum_{i=2}^n \ln \left(e^{-\theta(e^{\lambda x(i-1)} - 1)^2} - e^{-\theta(e^{\lambda x(i)} - 1)^2} \right) - \theta(e^{\lambda x(n)} - 1)^2 \right]. \end{aligned} \quad (29)$$

The MPS estimates $\hat{\lambda}_{MS}$ and $\hat{\theta}_{MS}$ of λ and θ can be obtained as simultaneous solution of the following equations

$$\begin{aligned} \frac{\partial \ln G}{\partial \lambda} &= 0, \\ \Rightarrow \frac{1}{n+1} &\left[\frac{2\theta e^{-\theta(e^{\lambda x(1)} - 1)^2} (e^{\lambda x(1)} - 1) e^{\lambda x(1)} x(1)}{1 - e^{-\theta(e^{\lambda x(1)} - 1)^2}} - 2\theta(e^{\lambda x(n)} - 1) e^{\lambda x(n)} x(n) \right. \\ &\left. + \sum_{i=2}^n \left(\frac{-2\theta e^{-\theta(e^{\lambda x(i-1)} - 1)^2} (e^{\lambda x(i-1)} - 1) e^{\lambda x(i-1)} x(i-1) + 2\theta e^{-\theta(e^{\lambda x(i)} - 1)^2} (e^{\lambda x(i)} - 1) e^{\lambda x(i)} x(i)}{e^{-\theta(e^{\lambda x(i-1)} - 1)^2} - e^{-\theta(e^{\lambda x(i)} - 1)^2}} \right) \right] = 0, \end{aligned} \quad (30)$$

and,

$$\begin{aligned} \frac{\partial \ln G}{\partial \theta} &= 0, \\ \Rightarrow \frac{1}{n+1} &\left[\frac{e^{-\theta(e^{\lambda x(1)} - 1)^2} (e^{\lambda x(1)} - 1)^2}{1 - e^{-\theta(e^{\lambda x(1)} - 1)^2}} - (e^{\lambda x(n)} - 1)^2 \right. \\ &\left. + \sum_{i=2}^n \left(\frac{-e^{-\theta(e^{\lambda x(i-1)} - 1)^2} (e^{\lambda x(i-1)} - 1)^2 + e^{-\theta(e^{\lambda x(i)} - 1)^2} (e^{\lambda x(i)} - 1)^2}{e^{-\theta(e^{\lambda x(i-1)} - 1)^2} - e^{-\theta(e^{\lambda x(i)} - 1)^2}} \right) \right] = 0, \end{aligned} \quad (31)$$

Since equation 30 and 31 can not be expressed in nice close form, therefor in order to get the solution for λ and θ , use of numerical technique is suggested.

5. SIMULATION STUDY

In the present section, we have carried simulation study in order to know the performance of the various considered estimators of the parameters λ and θ for their long-run behavior. The estimator have been compared in terms of their mean squared errors (MSEs), To calculate MSEs, we have generated 10000 different samples from NEER(λ, θ) distribution, in order to get convergence of

the result. The MSEs of the estimators are the functions of n , λ , and θ . For calculation of MSEs, we have arbitrarily considered the values $\lambda = 0.2, 0.8, 1.2$ and $\theta = 0.2, 0.8, 1.2$ and generated 10000 different samples of sizes $n = 10, 20, 30, 50$, and 120 using R software. We have also calculated average estimates (AEs), absolute bias (AB), confidence interval (CI), and width of the confidence interval (W_CI) of the considered estimators of the parameters λ and θ . Since the parameter space is restricted to non-negative values, so negative lower confidence limits were truncated at zero. The results of simulation are summarised in Tables [1-3].

Table 1: Average estimates, mean square error (MSE), absolute bias (AB), confidence interval (CI), and width of confidence interval (W_CI) for the parameters λ and θ obtained using the Maximum Likelihood Estimation (MLE) method.

λ	θ	n	$\hat{\lambda}$	$\hat{\theta}$	MSE_{λ}	MSE_{θ}	AB_{λ}	AB_{θ}	CI_{λ}		CI_{θ}		W_CI_{λ}	W_CI_{θ}
									Lower	Upper	Lower	Upper		
0.2	0.2	10	0.2115	0.2672	0.0034	0.0762	0.0376	0.1398	0.1333	0.2896	0.0000	0.6018	0.1563	0.6692
		20	0.2047	0.2355	0.0015	0.0453	0.0244	0.0935	0.1677	0.2418	0.0903	0.3807	0.0740	0.2904
		30	0.2021	0.2249	0.0009	0.0166	0.0183	0.0633	0.1786	0.2256	0.1361	0.3138	0.0470	0.1777
		50	0.2018	0.2107	0.0004	0.0072	0.0132	0.0456	0.1869	0.2168	0.1554	0.2660	0.0300	0.1106
		120	0.2002	0.2039	0.0001	0.0013	0.0057	0.0178	0.1948	0.2057	0.1837	0.2240	0.0109	0.0403
0.2	0.8	10	0.2018	1.0313	0.0023	0.4612	0.0302	0.4060	0.1011	0.3025	0.0000	2.3990	0.2013	2.7353
		20	0.1990	0.9383	0.0011	0.2871	0.0215	0.2694	0.1448	0.2532	0.2287	1.6480	0.1084	1.4193
		30	0.2005	0.9219	0.0010	0.2433	0.0206	0.2565	0.1702	0.2307	0.5334	1.3104	0.0605	0.7769
		50	0.1961	0.8823	0.0004	0.1175	0.0120	0.1640	0.1755	0.2167	0.6116	1.1530	0.0412	0.5414
		120	0.1960	0.8055	0.0001	0.0108	0.0074	0.0352	0.1891	0.2028	0.7213	0.8897	0.0137	0.1684
0.2	1.2	10	0.2021	1.5355	0.0021	0.8953	0.0321	0.5553	0.0872	0.3170	0.0000	3.6963	0.2298	4.3215
		20	0.1987	1.3960	0.0010	0.4115	0.0216	0.3873	0.1425	0.2549	0.3635	2.4285	0.1125	2.0650
		30	0.2000	1.3150	0.0008	0.3016	0.0180	0.3029	0.1602	0.2398	0.6186	2.0114	0.0796	1.3929
		50	0.1953	1.3292	0.0004	0.1645	0.0129	0.2320	0.1737	0.2168	0.9275	1.7309	0.0431	0.8033
		120	0.1984	1.2018	0.0002	0.0425	0.0086	0.0953	0.1899	0.2069	1.0550	1.3486	0.0170	0.2936
0.8	0.2	10	0.9897	0.2746	0.1726	0.2495	0.2955	0.2318	0.7048	1.2746	0.0000	0.6055	0.5698	0.6619
		20	0.8632	0.2691	0.0610	0.1236	0.1689	0.1715	0.7133	1.0130	0.0856	0.4526	0.2997	0.3670
		30	0.8338	0.2301	0.0314	0.0355	0.1155	0.1084	0.7312	0.9364	0.1223	0.3379	0.2052	0.2157
		50	0.8202	0.2197	0.0171	0.0259	0.0866	0.0843	0.7627	0.8776	0.1641	0.2753	0.1149	0.1112
		120	0.7960	0.2095	0.0051	0.0063	0.0426	0.0437	0.7709	0.8212	0.1850	0.2339	0.0502	0.0489
0.8	0.8	10	1.0059	0.9169	0.2337	1.1836	0.3239	0.6507	0.5710	1.4409	0.0000	2.2909	0.8699	2.7480
		20	0.8779	1.0680	0.1022	0.9098	0.2071	0.6221	0.6856	1.0702	0.3710	1.7650	0.3846	1.3940
		30	0.8459	0.9582	0.0557	0.5884	0.1577	0.4643	0.7115	0.9804	0.4962	1.4201	0.2688	0.9239
		50	0.8205	0.9579	0.0304	0.4914	0.1235	0.4118	0.7378	0.9033	0.6664	1.2494	0.1655	0.5830
		120	0.8053	0.8780	0.0123	0.1670	0.0760	0.2416	0.7725	0.8381	0.7711	0.9849	0.0656	0.2138
0.8	1.2	10	0.9422	1.4993	0.1581	2.7643	0.2723	0.9789	0.4787	1.4057	0.0000	3.6214	0.9270	4.2441
		20	0.8568	1.6382	0.0823	2.5911	0.2042	0.9449	0.6369	1.0767	0.5038	2.7726	0.4398	2.2687
		30	0.8390	1.4626	0.0554	1.1283	0.1668	0.7118	0.6977	0.9804	0.7693	2.1560	0.2826	1.3867
		50	0.8337	1.3430	0.0355	0.6433	0.1335	0.5522	0.7457	0.9217	0.9156	1.7705	0.1760	0.8549
		120	0.8214	1.2931	0.0186	0.3939	0.0947	0.4047	0.7859	0.8570	1.1296	1.4566	0.0711	0.3270
1.2	0.2	10	1.4841	0.3057	0.3922	0.2820	0.4768	0.2746	1.0592	1.9090	0.0000	0.7121	0.8499	0.8129
		20	1.3813	0.2670	0.1942	0.2157	0.3277	0.2078	1.1687	1.5938	0.0889	0.4450	0.4251	0.3560
		30	1.3068	0.2524	0.1200	0.1049	0.2570	0.1654	1.1692	1.4444	0.1447	0.3601	0.2753	0.2154
		50	1.2608	0.2289	0.0638	0.0404	0.1840	0.1168	1.1710	1.3506	0.1631	0.2946	0.1796	0.1314
		120	1.2243	0.2085	0.0226	0.0107	0.1088	0.0674	1.1906	1.2580	0.1870	0.2299	0.0674	0.0429
1.2	0.8	10	1.3665	1.4520	0.3363	5.4327	0.4257	1.1421	0.7632	1.9699	0.0000	3.3436	1.2066	3.7832
		20	1.2371	1.3634	0.1483	3.9112	0.2851	0.8986	0.9115	1.5626	0.2717	2.4552	0.6510	2.1836
		30	1.2089	1.3269	0.1167	3.4793	0.2373	0.7900	1.0011	1.4166	0.6212	2.0327	0.4155	1.4114
		50	1.2239	1.0734	0.0876	1.3402	0.2213	0.5839	1.1051	1.3428	0.7372	1.4097	0.2376	0.6725
		120	1.1855	0.9252	0.0281	0.2693	0.1207	0.2969	1.1290	1.2421	0.7814	1.0691	0.1131	0.2876
1.2	1.2	10	1.2842	2.0482	0.2234	7.5080	0.3460	1.3692	0.6454	1.9230	0.0000	4.6598	1.2776	5.2231
		20	1.2246	1.7966	0.1117	4.9470	0.2536	1.0271	0.8746	1.5747	0.3931	3.2002	0.7001	2.8071
		30	1.1609	1.7484	0.0603	2.9393	0.1828	0.7988	0.9057	1.4160	0.7612	2.7356	0.5103	1.9745
		50	1.1702	1.6502	0.0542	1.7794	0.1619	0.6833	1.0360	1.3045	1.1335	2.1669	0.2685	1.0334
		120	1.1884	1.3602	0.0243	0.4384	0.1092	0.3790	1.1297	1.2470	1.1693	1.5511	0.1173	0.3818

Table 2: Average estimates, mean square error (MSE), absolute bias (AB), confidence interval (CI), and width of confidence interval (W_CI) for the parameters λ and θ obtained using the Least Square Estimation (LSE) method.

λ	θ	n	$\hat{\lambda}$	$\hat{\theta}$	MSE_{λ}	MSE_{θ}	AB_{λ}	AB_{θ}	CI_{λ}		CI_{θ}		W_CI_{λ}	W_CI_{θ}
									Lower	Upper	Lower	Upper		
0.2	0.2	10	0.1784	0.5717	0.0044	0.5083	0.0510	0.4123	0.0000	0.5032	0.0000	3.3382	0.6497	5.5330
		20	0.1819	0.4971	0.0035	0.4010	0.0456	0.3416	0.0000	0.3771	0.0000	2.0809	0.3904	3.1675
		30	0.1887	0.4105	0.0028	0.2749	0.0408	0.2622	0.0684	0.3089	0.0000	1.2271	0.2405	1.6331
		50	0.1915	0.3366	0.0019	0.1389	0.0336	0.1902	0.1201	0.2629	0.0000	0.7513	0.1428	0.8294
		120	0.1953	0.2649	0.0010	0.0351	0.0239	0.1097	0.1622	0.2284	0.1141	0.4157	0.0662	0.3016
0.2	0.8	10	0.1797	1.4546	0.0036	1.4647	0.0436	0.8015	0.0000	0.6104	0.0000	9.2854	0.8615	15.6614
		20	0.1800	1.4762	0.0029	1.2418	0.0397	0.7660	0.0000	0.4059	0.0000	5.7666	0.4518	8.5809
		30	0.1897	1.4319	0.0026	1.0032	0.0388	0.6808	0.0110	0.3684	0.0000	4.7232	0.3574	6.5826
		50	0.1850	1.2880	0.0020	0.9579	0.0343	0.6232	0.0798	0.2901	0.0000	3.1396	0.2103	3.7033
		120	0.1991	0.9916	0.0016	0.5488	0.0307	0.4397	0.1536	0.2446	0.3037	1.6795	0.0910	1.3758
0.2	1.2	10	0.1905	1.7497	0.0029	1.6851	0.0399	0.9282	0.0000	0.6549	0.0000	11.4134	0.9288	19.3274
		20	0.1806	1.9288	0.0023	1.4207	0.0355	0.7902	0.0000	0.4304	0.0000	7.8464	0.4997	11.8353
		30	0.1892	1.9177	0.0021	1.3853	0.0334	0.7698	0.0170	0.3615	0.0000	5.9334	0.3445	8.0313
		50	0.1848	1.8382	0.0020	1.2674	0.0323	0.7690	0.0681	0.3014	0.0000	4.5854	0.2332	5.4945
		120	0.1951	1.6308	0.0018	0.7459	0.0317	0.5792	0.1457	0.2445	0.5159	2.7457	0.0989	2.2298
0.8	0.2	10	0.7645	0.5904	0.0889	0.6895	0.2347	0.4796	0.0000	2.3034	0.0000	4.0321	3.0778	6.8833
		20	0.7568	0.5075	0.0657	0.5150	0.2041	0.3827	0.0411	1.4726	0.0000	2.0157	1.4315	3.0165
		30	0.7540	0.4692	0.0545	0.4321	0.1844	0.3400	0.2664	1.2417	0.0000	1.4580	0.9753	1.9776
		50	0.7741	0.3407	0.0340	0.1603	0.1449	0.2105	0.4428	1.1055	0.0000	0.8713	0.6628	1.0611
		120	0.7777	0.2742	0.0202	0.0494	0.1120	0.1329	0.6558	0.8997	0.1259	0.4226	0.2439	0.2967
0.8	0.8	10	0.7735	1.3329	0.0756	1.4366	0.1896	0.8229	0.0000	2.7119	0.0000	8.8517	3.8767	15.0375
		20	0.7648	1.4503	0.0748	1.1746	0.1857	0.7017	0.0000	1.7855	0.0000	6.0017	2.0413	9.1028
		30	0.7736	1.2945	0.0569	1.1420	0.1725	0.6947	0.0820	1.4652	0.0000	4.2582	1.3832	5.9274
		50	0.7764	1.2804	0.0495	1.0096	0.1673	0.6909	0.3705	1.1823	0.0000	3.0481	0.8118	3.5356
		120	0.7923	1.0480	0.0271	0.5853	0.1204	0.4491	0.5797	1.0049	0.2705	1.8255	0.4252	1.5550
0.8	1.2	10	0.7653	1.7078	0.0599	1.5002	0.1578	0.8056	0.0000	2.8344	0.0000	12.0770	4.1381	20.7383
		20	0.7811	1.7755	0.0428	1.4833	0.1463	0.7836	0.0000	1.8579	0.0000	7.2130	2.1536	10.8751
		30	0.7518	1.8425	0.0398	1.2026	0.1400	0.7791	0.0000	1.5538	0.0000	6.4397	1.6040	9.1944
		50	0.7758	1.7425	0.0396	1.0350	0.1385	0.6365	0.2991	1.2526	0.0000	4.3349	0.9535	5.1847
		120	0.8072	1.5099	0.0317	0.7403	0.1287	0.5874	0.5542	1.0602	0.2439	2.7758	0.5059	2.5319
1.2	0.2	10	1.0703	0.6763	0.1864	0.8049	0.3363	0.5488	0.0000	3.2678	0.0000	4.3595	4.3950	7.3664
		20	1.1829	0.4522	0.1442	0.4613	0.2910	0.3391	0.0000	2.3939	0.0000	2.0413	2.4220	3.1781
		30	1.1553	0.4237	0.1135	0.3536	0.2630	0.2971	0.4029	1.9078	0.0000	1.3637	1.5049	1.8799
		50	1.1689	0.3583	0.0850	0.2145	0.2272	0.2313	0.7266	1.6112	0.0000	0.8320	0.8846	0.9474
		120	1.1794	0.2588	0.0379	0.0354	0.1526	0.1156	0.9881	1.3707	0.1090	0.4086	0.3827	0.2996
1.2	0.8	10	1.1073	1.4034	0.1369	1.4798	0.2812	0.8380	0.0000	4.0592	0.0000	9.5947	5.9038	16.3826
		20	1.1224	1.4047	0.1241	1.2673	0.2597	0.7701	0.0000	2.6533	0.0000	5.8017	3.0617	8.7940
		30	1.1346	1.4306	0.1163	1.1457	0.2534	0.7083	0.0988	2.1703	0.0000	4.4486	2.0715	6.0360
		50	1.1846	1.2006	0.1132	1.0317	0.2443	0.6648	0.5114	1.8577	0.0000	3.1039	1.3462	3.8066
		120	1.1702	1.1272	0.0734	0.8294	0.2109	0.5658	0.9002	1.4401	0.3901	1.8642	0.5398	1.4741
1.2	1.2	10	1.1241	1.7816	0.1006	1.9270	0.2281	0.9067	0.0000	4.1963	0.0000	12.3563	6.1444	21.1494
		20	1.1265	1.8310	0.0994	1.6022	0.2229	0.8935	0.0000	2.8540	0.0000	8.1551	3.4549	12.6482
		30	1.1187	1.9161	0.0931	1.4531	0.2101	0.7760	0.0430	2.1943	0.0000	5.9817	2.1513	8.1312
		50	1.1408	1.8962	0.0705	1.2250	0.2057	0.7709	0.3883	1.8934	0.0000	4.9097	1.5051	6.0271
		120	1.1673	1.6609	0.0658	0.8569	0.1779	0.6340	0.8741	1.4606	0.5534	2.7684	0.5864	2.2151

Table 3: Average estimates, mean square error (MSE), absolute bias (AB), confidence interval (CI), and width of confidence interval (W_CI) for the parameters λ and θ obtained using the Maximum Product of Spacing (MPS) estimation method

λ	θ	n	$\hat{\lambda}$	$\hat{\theta}$	MSE_{λ}	MSE_{θ}	AB_{λ}	AB_{θ}	CI_{λ}		CI_{θ}		W_CI_{λ}	W_CI_{θ}
									Lower	Upper	Lower	Upper		
0.2	0.2	10	0.1787	0.3537	0.0019	0.1066	0.0316	0.1682	0.0000	0.3700	0.0000	1.4388	0.3827	2.1700
		20	0.1853	0.2808	0.0009	0.0351	0.0217	0.0940	0.0451	0.3255	0.0000	0.9478	0.2803	1.3339
		30	0.1891	0.2495	0.0005	0.0123	0.0163	0.0591	0.0775	0.3007	0.0000	0.7394	0.2232	0.9797
		50	0.1933	0.2235	0.0003	0.0039	0.0104	0.0341	0.1080	0.2786	0.0000	0.5672	0.1706	0.6874
		120	0.1976	0.2067	0.0001	0.0008	0.0048	0.0118	0.1320	0.2633	0.0000	0.4544	0.1312	0.4953
0.2	0.8	10	0.1831	1.0031	0.0010	0.7454	0.0253	0.8033	0.0000	0.3841	0.0000	3.9862	0.4019	5.9662
		20	0.1896	0.8650	0.0005	0.4651	0.0177	0.6651	0.0446	0.3346	0.0000	2.7741	0.2900	3.8182
		30	0.1936	0.8444	0.0003	0.4271	0.0136	0.6445	0.0611	0.3261	0.0000	2.5206	0.2650	3.3523
		50	0.1931	0.8199	0.0002	0.3870	0.0107	0.6199	0.0849	0.3013	0.0000	2.1795	0.2164	2.7191
		120	0.1936	0.8030	0.0001	0.3638	0.0077	0.6030	0.1363	0.2510	0.0921	1.5139	0.1146	1.4218
0.2	1.2	10	0.1898	1.3486	0.0008	1.4581	0.0229	1.1486	0.0000	0.4168	0.0000	5.6518	0.4541	8.6063
		20	0.1926	1.2409	0.0004	1.1040	0.0163	1.0409	0.0199	0.3653	0.0000	4.3408	0.3454	6.2000
		30	0.1930	1.2207	0.0003	1.0474	0.0161	1.0207	0.0469	0.3390	0.0000	3.7942	0.2921	5.1471
		50	0.1924	1.2043	0.0002	1.0090	0.0128	1.0043	0.0941	0.2908	0.0000	2.9100	0.1966	3.4115
		120	0.1930	1.2001	0.0001	1.0003	0.0095	1.0001	0.1381	0.2479	0.2450	2.1552	0.1098	1.9102
0.8	0.2	10	0.7417	0.3294	0.0250	0.3656	0.1094	0.6040	0.0000	1.5560	0.0000	1.4554	1.6286	2.2519
		20	0.7595	0.2503	0.0094	0.3550	0.0621	0.5931	0.2392	1.2798	0.0000	0.8486	1.0406	1.1965
		30	0.7707	0.2239	0.0054	0.3450	0.0475	0.5800	0.3516	1.1899	0.0000	0.6584	0.8383	0.8689
		50	0.7797	0.2074	0.0023	0.3298	0.0311	0.5604	0.4492	1.1102	0.0000	0.5275	0.6610	0.6403
		120	0.7874	0.1960	0.0007	0.3058	0.0171	0.5209	0.5882	0.9866	0.0114	0.3806	0.3984	0.3692
0.8	0.8	10	0.7544	0.9447	0.0170	0.1195	0.0924	0.1813	0.0000	1.9492	0.0000	4.8215	2.3896	7.7536
		20	0.7709	0.8818	0.0074	0.0509	0.0612	0.1018	0.0000	1.5666	0.0000	3.4134	1.5915	5.0631
		30	0.7794	0.8339	0.0034	0.0146	0.0423	0.0433	0.0709	1.4880	0.0000	3.0356	1.4172	4.4034
		50	0.7895	0.8129	0.0015	0.0034	0.0286	0.0188	0.2454	1.3335	0.0000	2.4875	1.0882	3.3493
		120	0.7954	0.8020	0.0005	0.0001	0.0162	0.0061	0.4294	1.1613	0.0000	1.9246	0.7319	2.2452
0.8	1.2	10	0.7656	1.2951	0.0123	0.3115	0.0829	0.4969	0.0000	1.8846	0.0000	6.3602	2.2379	10.1302
		20	0.7782	1.2456	0.0047	0.2190	0.0530	0.4456	0.0000	1.6007	0.0000	4.8956	1.6450	7.2999
		30	0.7836	1.2215	0.0030	0.1839	0.0417	0.4215	0.1001	1.4672	0.0000	4.2308	1.3671	6.0185
		50	0.7899	1.2074	0.0016	0.1666	0.0295	0.4074	0.2120	1.3679	0.0000	3.7448	1.1559	5.0749
		120	0.7977	1.2019	0.0006	0.1616	0.0161	0.4019	0.4144	1.1809	0.0000	2.8747	0.7666	3.3455
1.2	0.2	10	1.0979	0.3681	0.0716	0.9974	0.1833	0.9985	0.0000	2.4706	0.0000	1.6792	2.7454	2.6222
		20	1.1382	0.2737	0.0291	0.9649	0.1045	0.9796	0.1866	2.0898	0.0000	1.0082	1.9032	1.4689
		30	1.1517	0.2456	0.0166	0.9307	0.0736	0.9552	0.3509	1.9525	0.0000	0.8232	1.6016	1.1551
		50	1.1698	0.2205	0.0071	0.9000	0.0445	0.9293	0.5554	1.7841	0.0000	0.6329	1.2287	0.8249
		120	1.1895	0.2015	0.0011	0.8088	0.0165	0.8523	0.7850	1.5939	0.0000	0.4566	0.8089	0.5102
1.2	0.8	10	1.1236	0.9709	0.0345	0.1851	0.1362	0.4074	0.0000	2.9337	0.0000	5.1695	3.6203	8.3973
		20	1.1525	0.8715	0.0131	0.1664	0.0845	0.3997	0.0000	2.3854	0.0000	3.5494	2.4657	5.3558
		30	1.1655	0.8375	0.0078	0.1610	0.0661	0.3867	0.1808	2.1501	0.0000	2.9161	1.9693	4.1573
		50	1.1694	0.8025	0.0039	0.1544	0.0509	0.3858	0.4400	1.8987	0.0000	2.2974	1.4586	2.9898
		120	1.1768	0.7926	0.0019	0.1528	0.0355	0.3825	0.7545	1.5990	0.0000	1.6490	0.8444	1.7129
1.2	1.2	10	1.1429	1.3411	0.0237	0.1563	0.1180	0.1590	0.0000	2.8424	0.0000	6.6513	3.3992	10.6205
		20	1.1703	1.2367	0.0094	0.0278	0.0760	0.0514	0.0000	2.4791	0.0000	5.1150	2.6177	7.7565
		30	1.1749	1.2173	0.0059	0.0080	0.0615	0.0291	0.1814	2.1684	0.0000	4.1433	1.9870	5.8521
		50	1.1889	1.2064	0.0032	0.0007	0.0466	0.0174	0.3466	2.0312	0.0000	3.6562	1.6845	4.8996
		120	1.1866	1.1981	0.0015	0.0002	0.0323	0.0109	0.6141	1.7592	0.0000	2.8606	1.1451	3.3250

The simulation results indicate that the mean squared error (MSE), Absolute bias (AB), and width of confidence interval (W_CI) decreases with increasing sample size for all estimation methods, confirming their consistency. The MPS estimator consistently produces the smallest MSE across most parameter configurations, demonstrating superior efficiency and robustness. Although the MLE exhibits relatively high MSE for small samples, its performance improves substantially with increasing sample size. The LSE shows moderate performance but generally remains less efficient than MPS. Therefore, based on the MSE criterion, the MPS estimator is the preferred estimation method for the considered distribution.

6. REAL DATA APPLICATION

In this section, we have considered two real data sets to illustrate the usefulness of the proposed distribution in terms of fitting to the real characteristic. The first data set (Table 4) refer to the failure times of 84 aircraft windshields. This data set studied by [19] and fitted the Weibull-Lomax distribution. The second data set (Table 5) represent the lifetime (in years) of 40 blood cancer (leukemia) patients. Recently, the data set is used by [21] and fitted a new extended Weibull distribution. The TTT plots of the first and second data sets are shown in Figures 4 & 5 respectively. Here, we have compared the performance of the proposed NEER(λ, θ) distribution over some existing well known distributions such as Weibull distribution (WD), Gamma distribution (GD), and Exponentiated Exponential (EE) distribution. For the purpose, the tool used are -log-likelihood, AIC, BIC, K-S test statistic and its p-value. These criteria are defined as follows.

$$\begin{aligned} AIC &= 2k - 2 \ln \hat{L}, \\ BIC &= k \ln n - 2 \ln \hat{L}, \\ KS \text{ test statistic} &= \sup_x |F(x) - F_0(x)|, \end{aligned}$$

where k is the number of parameters and n is the number of units in the data set, $\hat{L}$ is the maximum likelihood, $F(x)$ is the theoretical cdf and $F_0(x)$ is the empirical cdf.

Table 6 & 7 present the estimated parameters for the NEER(λ, θ) distribution along with those for the alternative models. These estimates are obtained through appropriate fitting techniques, providing a basis for comparing how effectively each distribution represents the observed data and captures its underlying patterns.

Table 4: Real Data Set of failure times of 84 Aircraft Windshield

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610	3.478
0.557	1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914	2.646	3.699
1.124	1.981	2.661	3.779	1.248	2.010	2.688	3.924	1.281	2.038	2.820	4.035
1.281	2.085	2.890	4.121	1.303	2.089	2.902	4.167	1.432	2.097	2.934	4.240
1.480	2.135	2.962	4.255	1.505	2.154	2.964	4.278	1.506	2.190	3.000	4.305
1.568	2.194	3.103	4.376	1.615	2.223	3.114	4.449	1.619	2.224	3.117	4.485
1.652	2.229	3.166	4.570	1.652	2.300	3.344	4.602	1.757	2.324	3.376	4.663

Table 5: The lifetime (in years) of 40 blood cancer (leukemia) patients.

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162	2.211	2.370
2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348	3.348	3.427	3.499	3.534
3.767	3.751	3.858	3.986	4.049	4.244	4.323	4.381	4.392	4.397	4.647	4.753
4.929	4.973	5.074	5.381								

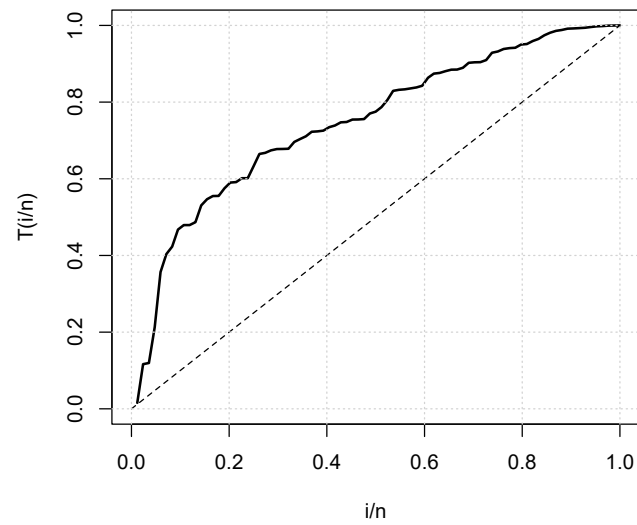


Figure 4: TTT plots for the failure times of 84 Aircraft Windshield

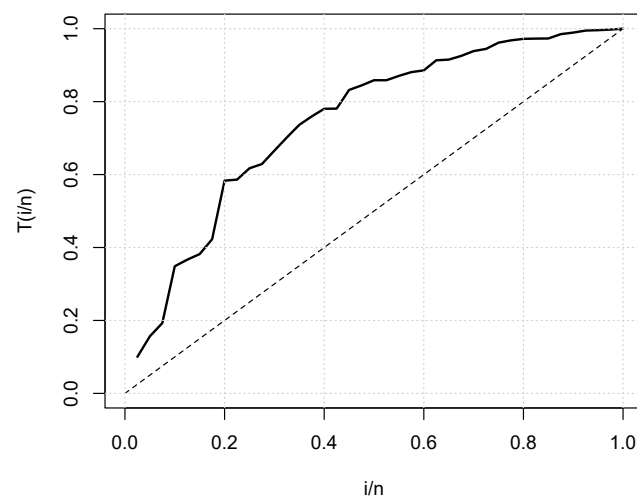


Figure 5: TTT plots for the lifetime (in years) of 40 blood cancer (leukemia) patients.

Table 6: Estimates, $-\log \hat{L}$, AIC, BIC, KS-statistic and p-value for all the considered distributions based on the data of failure times of 84 Aircraft Windshield.

Models	Estimates		$-\log(\hat{L})$	AIC	BIC	KS-statistic	p-value
	$\hat{\lambda}$	$\hat{\theta}$					
NEER(λ, θ)	0.17729	2.13650	128.376	260.751	265.613	0.08159	0.60181
WD(λ, θ)	2.86286	2.37437	130.052	264.105	268.966	0.05368	0.95815
GD(λ, θ)	3.49222	1.36553	136.935	277.871	282.732	0.10351	0.30759
EE(λ, θ)	0.75793	3.56051	139.839	283.678	288.540	0.12097	0.15750

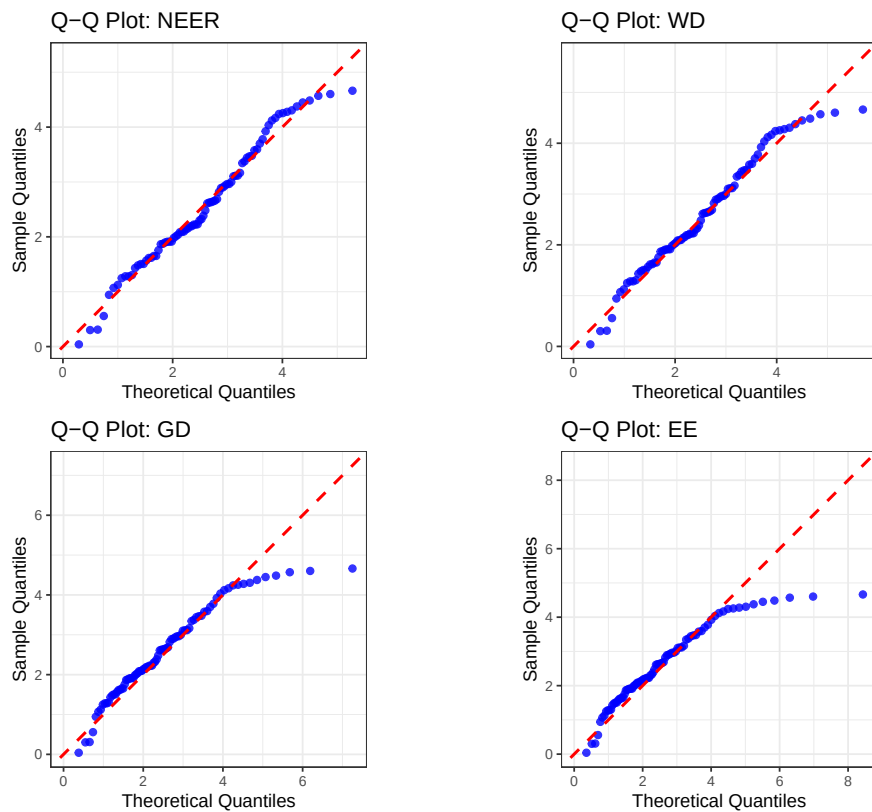


Figure 6: Q-Q plots for the fitted distributions of data set 4.

Table 7: Estimates, $-\log \hat{L}$, AIC, BIC, KS-statistic and p-value for all the considered distributions based on the data of the lifetime (in years) of 40 blood cancer (leukemia) patients.

Models	Estimates		$-\log(\hat{L})$	AIC	BIC	KS-statistic	p-value
	$\hat{\lambda}$	$\hat{\theta}$					
NEER(λ, θ)	0.23253	0.56066	67.5249	139.050	142.428	0.09352	0.84330
WD(λ, θ)	3.51846	2.49949	69.5580	143.116	146.494	0.11834	0.58839
GD(λ, θ)	3.46475	1.10317	73.5487	151.097	154.475	0.15815	0.24283
EE(λ, θ)	0.61415	3.51889	74.9624	153.925	157.303	0.16116	0.22440

From Table 6 & 7, it is clear that the proposed distribution fits better as compared to the other considered distributions in terms of the considered model selections criterion $-\log \hat{L}$, AIC and

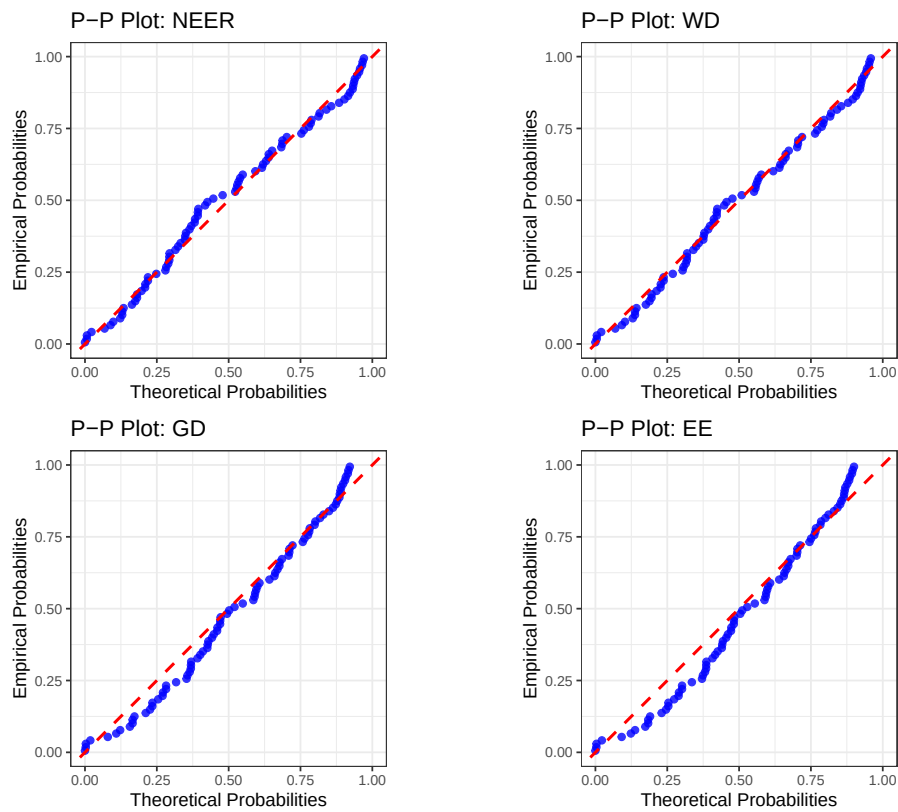


Figure 7: *P–P plots for the fitted distributions of data set 4.*

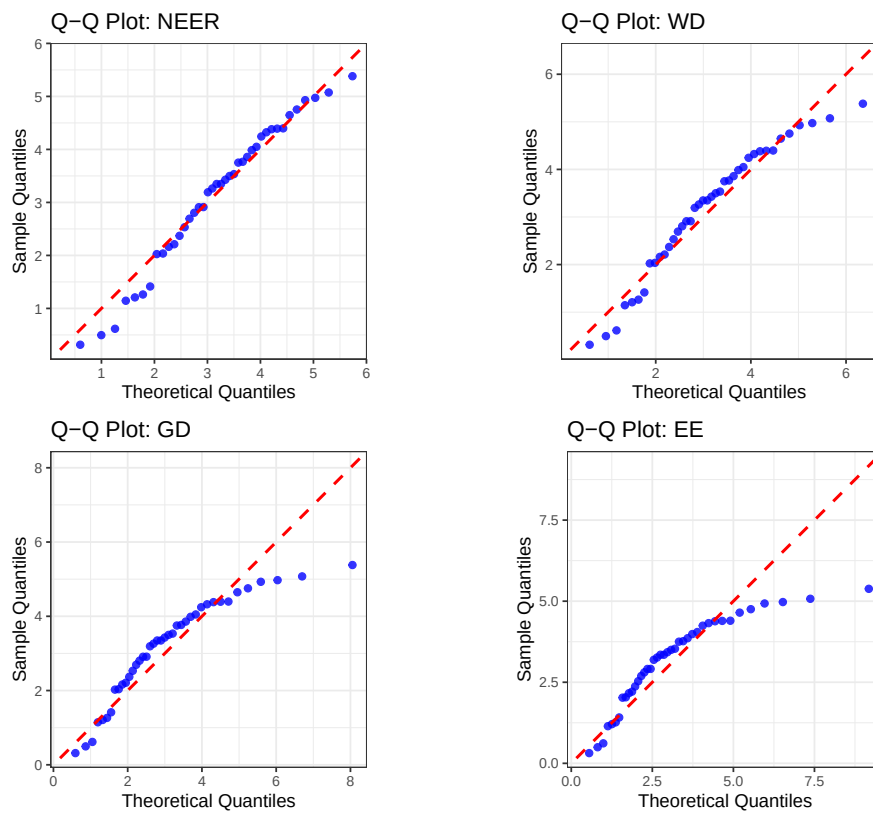


Figure 8: *Q–Q plots for the fitted distributions of data set 5.*

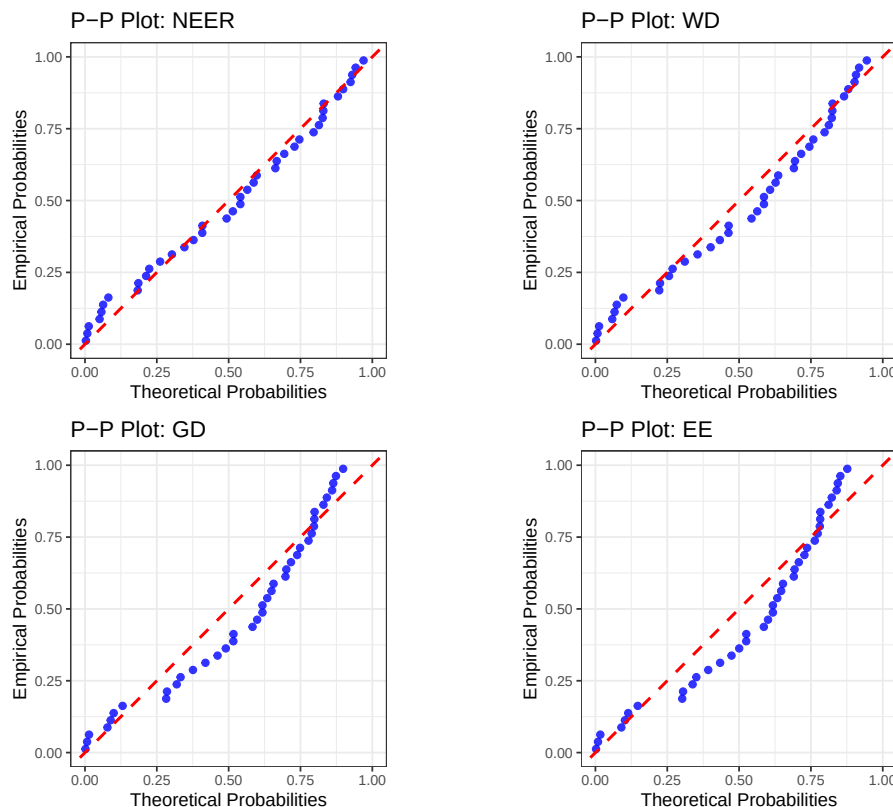


Figure 9: *P-P plots for the fitted distributions of data set 5.*

BIC, as the criterion values are smaller for the proposed $NEER(\lambda, \theta)$ distribution compared to those of for the other considered distributions.

7. CONCLUSIONS

In this study, we have introduced new extended exponential Rayleigh distribution with the parameters λ and θ , i.e., $NEER(\lambda, \theta)$ and we have derived its several key statistical properties such as survival function, hazard rate function, quantile function and order statistics. The distributional behavior is shown by plotting its pdf, cdf and hazard rate function for various combination values of the parameters λ and θ . The parameters λ and θ were estimated using MLE, LSE, and MPS method of estimation on the basis of complete sample from it. The simulation study is carried out to know the performance of these estimations in terms of MSEs. From Tables [1- 3], it is clear that as the sample size n increases the MSEs of these considered estimation methods decreases for all the considered values of the parameters λ and θ . It was noted that the MPS estimates out performs the corresponding MLE and LSE. Finally the two real data applications depicts that $NEER(\lambda, \theta)$ distribution provides a superior fit as compared to the three considered rival probability distributions showing the effectiveness and practical relevance of our proposed $NEER(\lambda, \theta)$ distribution to the real problems.

STATEMENTS AND DECLARATIONS

Data availability

No primary data are used.

Ethics approval

The authors of this article have not conducted any studies with human participants.

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Conflict of Interest

The authors declare that there is no conflicts of interest regarding the publication of this manuscripts.

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