RT&A, No 4 (89)
Volume 20, December 2025

IN COMPLEX INTERVAL-VALUED FUZZY METRIC SPACE, COMMON FIXED-POINT THEOREMS CLR/JCLR/E.A PROPERTY

Umashankar Singh[1*], Naval Singh[2], Heera Ahirwar[3]

•

[1*] Technocrats Institute of Technology, Bhopal, Madhya Pradesh, India
[2]Govt. SPM Science and Commerce College Bhopal, Madhya Pradesh, India
[3]Shri Raghvendra Singh Hazari Govt. College Hata, Damoh, Madhya Pradesh, India
[1] umashankar_singh1977@rediffmail.com
[3] heera.education21@gmail.com

Abstract

The work of D. Singh et al. served as inspiration for this study, which established complex interval valued fuzzy metric space and proved several common fixed print theorems for compatible types of (p-1) on the structure of complex interval valued fuzzy metric space. The equivalent findings in the body of existing research are expanded upon and generalized by these findings. Additionally, we create a novel implicit relation that can aid researchers in accelerating common fixed-point existence and uniqueness wherever a pair of self-mappings fits the Clr/Jclr/E.A-Property. Appropriate examples are provided to demonstrate the usefulness of the primary conclusions that are gained.

Keywords: list Complex Interval-valued continues t-norms, complex Interval-valued fuzzy metric Space ($Civfms$), Common fixed-point, compatible type of mappings.

I. Introduction

Lorem Interval valued fuzzy metric spaces are a novel generalization of fuzzy metric spaces created by Shen et al. [26]. Two natural fuzzy metric spaces are induced by each interval-valued fuzzy metric space. The topology generated by the interval-valued fuzzy metric space coincides with the topology produced by the interval-valued fuzzy metric space. three different types of distances between two interval-valued fuzzy sets on real line R were employed by C. Li [16]. The concept of continuous interval-valued t-norm can be used to build an interval-valued fuzzy metric space and to characterize the uncertainty of the distance between two points in such a space. In response to this letter, numerous authors and scholars have thoroughly and in-depthly examined a variety of issues pertaining to this space from different angles, including compatible mapping, weak compatible mapping, and new findings on interval-valued fuzzy metric space. Letters I and $[\wp_n(x)]$ indicate the set of positive integers through this article, where$I = [0,1]$ and $[\wp_n(x)] = [\wp_n{}^-, \wp_n{}^+]$: $0 \leq \wp_n{}^-(x) \leq \wp_n{}^+(x) \leq 1$. Recently in 2023, Some c-fixed-point results for mapping on interval-valued fuzzy metric space using weakly compatible were established by V. Deshmukh et al. [6]. Buckly was the first to introduce fuzzy complex numbers and fuzzy complex analysis [2, 3]. Some scholars were inspired to carry out more research on fuzzy complex numbers by Buckey's discovery. In this series, Ramot et al. [21, 22] generalized fuzzy sets to complex fuzzy sets.

DOI: https://doi.org/10.24412/1932-2321-2025-489-1201-1212

According to Ramot, a membership function that extends beyond [0, 1] to the unit circle in the complex plane defines the complex fuzzy set. The concept of complex valued metric space was first proposed by Azam et al. [1], who also gave enough conditions for the presence of a common f-point of a pair of mappings that meet contractive constraints. The concept of complex valued fuzzy metric spaces was later presented by D. Singh and Kumam [26] using the complex valued continuous t-norm and associated topologies.

In this research, we introduced the concept of complex interval-valued fuzzy metric space to modify the distance function under $Clr/Jclr/E.A$ −Property for common fixed-point theorems in complex interval-valued fuzzy metric space in order to prove several well-known common fixed-point theorems for self mappings.

II. Preliminaries

Definition 2.1[26] A mapping $p_n: \mho \to [I]$ is called an interval-valued fuzzy set on $\mho$ where $\mho$ is non empty set, Collection of all interval-valued fuzzy sets on $\mho$ is denoted by $Ivf(\mho)$. if $p_n \in Ivf(\mho)$. , let $p_n(x) = [p_n^-, p_n^+]$: $0 \le p_n^-(x) \le p_n^+(x) \le 1$ for all $x \in \mho$, then the set $p_n^-: \mho \to [I]$ and $p_n^+: \mho \to [I]$ are called lower fuzzy set and upper fuzzy set of p_n and if $p_n^-(x) = p_n^+(x)$ then is called degenerate fuzzy set for all $x \in \mho$. if $(I] = \{I\} - \{\bar{0}\}$ and $(I) = \{I\} - \{\bar{0}, \bar{1}\}$. For every $\overline{u}_{ivf}, \overline{v}_{ivf} \in [I]$ then the following operations holds:
$(Op-1)$: $\overline{u}_{ivf} \wedge \overline{v}_{ivf} = [u_{inf}^- \wedge v_{inf}^-, u_{inf}^+ \wedge v_{inf}^+]$;
$(Op-2)$: $\overline{u}_{ivf} \vee \overline{v}_{ivf} = [u_{inf}^- \vee v_{inf}^-, u_{inf}^+ \vee v_{inf}^+]$;
$(Op-3)$: $\overline{u}_{ivf}{}^c = \bar{1} - \overline{u}_{ivf} = [1 - u_{inf}^+, 1 - u_{inf}^-]$;

Definition 2.2[26] A binary operation of the form is an interval valued t_{norm} is $*_I: [I] \times [I] \to [I]$ on $[I]$ such that for all $\overline{u}, \overline{v}, \overline{w} \in [I]$ if satisfying following four conditions:
(1) Commutativity : $\bar{a}_{ivf} *_I \bar{b}_{ivf} = \bar{b}_{ivf} *_I \bar{a}_{ivf}$,
(2) Associativity: $[\bar{a}_{ivf} *_I \bar{b}_{ivf}] *_I \bar{c}_{ivf} = \bar{a}_{ivf} *_I [\bar{b}_{ivf} *_I \bar{c}_{ivf}]$,
(3) Monotonicity: $\bar{a}_{ivf} *_I \bar{b}_{ivf} \le \bar{a}_{ivf} *_I \bar{c}_{ivf}$, $whenever$ $\bar{b}_{ivf} *_I \bar{c}_{ivf}$,
(4) Boundary condition: $\bar{a}_{ivf} *_I \bar{1} = \bar{a}_{ivf}$, $\bar{a}_{ivf} *_I \bar{0} = [a^-, a^+] *_I [0,1] = [0, a^+]$.

Note: Each interval valued t_{nrm} satisfies some additional boundary conditions for all $\bar{a} \in [I]$:
$\bar{a} *_I \bar{0} = \bar{0} *_I \overline{u} = \bar{0}$,
$\bar{I} *_I \bar{a} = [0,1] *_I [a^-, a^+] = \bar{0}$,
$\bar{1} *_I \bar{a} = \bar{1}$.

Example 1: (a) $\overline{u} *_I \overline{v} = [u^-.v^-, u^+.v^+]$; (b)$\overline{u} *_I \overline{v} = [u^- \wedge v^-, u^+ \wedge v^+]$;

Definition 2.3[6]: Let$\{\bar{a}_i\}_{i\in\mathbb{N}^+} = \{[a_i^-, a_i^+]\}$be a sequence of interval numbers convergesto $\bar{a} = [a^-, a^+] \in [I]$, if $\lim_{i\to\infty} a_i^- = a^-and\lim_{i\to\infty} a_i^+ = a^+$ then the sequence$\{\bar{a}_i\}$is convergent to$\bar{a}$ and denoted by$\lim_{i\to\infty} \bar{a}_i = \bar{a}$ ($\{\bar{a}_i \to \bar{a}\}$.

Definition 2.4[6]: An interval valuedt_{nrm} $*_I$as continuous iff it is continuous in its first component, i.e. for each $\bar{b} \in [I]$, if $\lim_{i\to\infty} \bar{a}_i = \bar{a}$, then
$\lim_{i\to\infty}(\bar{a}_i *_I \bar{b}) = (\lim_{i\to\infty} \bar{a}_i *_I \bar{b}) = \bar{a} *_I \bar{b}$, Where $\{\bar{a}_i\} \subseteq [\boldsymbol{I}], \bar{a} \in [I]$.

Definition 2.5[10]: A triplet $(X, M, *)$ is called fuzzy metric space*(FMS)* if $\mathfrak{X}$ is an arbitrary set, $*$ is a continuoust-normson [0,∞]and $M$is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:
(1) $M(x, y, t) > 0$;
(2) $M(x, y, t) = 1$ $for\ all\ t > 0$ iff $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, z, t_1 + t_2) \ge T(M(x, y, t_1), M(x, y, t_2))$; $\forall x, y, z \in X$
(5) $M(x, y, *_I): [0, \infty] \to (0,1]$is continuous;

Definition 2.6[29]: A 3-tuple $(\mathbb{X}, \overline{Đ_{Ivfms}}, *_I)$ is called interval valued fuzzy metric space *(Ivfms)* if *Ξ* is an arbitrary set, $*_I$ is a continuous interval valued t_{norm} on *[I]*and $\overline{Đ_{Ivfms}}$is a fuzzy set on $\mathbb{X}^2 \times (0, \infty)$ satisfying the following conditions:

(1) $\overline{Đ_{Ivfms}}(x, y, t_{nrm}) > \bar{0}$;
(2) (2) $\overline{Đ_{Ivfms}}(x, y, t_{nrm}) = \bar{1}$ $for\ all\ t > 0 \quad iff\ \ x = y$;
(3) $\overline{Đ_{Ivfms}}(x, y, t_{nrm}) = \overline{Đ_{Ivfms}}(y, x, t_{nrm})$;
(4) $\overline{Đ_{Ivfms}}(x, y, t_1) *_I \overline{Đ_{Ivfms}}(y, z, t_2) \leq \overline{Đ_{Ivfms}}(x, z, t_1 + t_2)$; $\quad \forall x, y, z \in \mathbb{X}\ and\ t_1, t_2, > 0$
(5) $\overline{Đ_{Ivfms}}(x, y, *_I): [0, \infty] \to (I)$ is continuous;
(6) $\lim_{t\to\infty} \overline{Đ_{Ivfms}}(x, y, t_{nrm}) = \bar{1}$; $\forall\ x, y, z \in \mathbb{X},\ t_{nrm} > 0$

Definition 2.7: A 3-tuple $(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ is called complex interval valued fuzzy metric space *(Civfms),* where $\overline{Đ_{Civfms}}: \mathbb{X}^2 \times (0, \infty) \to r_s e^{i\theta} and \mathbb{X}$ is non-empty set, * is a complex valued continuous, and t-norm. The fuzzy set $\to r_s e^{i\theta}$ has complex values and satisfying the following conditions:
$(CIM - 1)$ $\overline{Đ_{Civfms}}(x, y, t) \succ_i 0$;
$(CIM - 2)$ $\overline{Đ_{Civfms}}(x, y, t) = e^{i\theta}$ $, for\ all\ t > 0\ iff\ \ x = y$;
$(CIM - 3)$ $\overline{Đ_{Civfms}}(x, y, t) = \overline{Đ_{Civfms}}(y, x, t)$;
$(CIM - 4)$ $\overline{Đ_{Civfms}}(x, y, p) * \overline{Đ_{Civfms}}(y, z, q) \gtrsim_i \overline{Đ_{Civfms}}(x, z, p + q)$;
$(CIM - 5)$ $\overline{Đ_{Civfms}}(x, y, .): (0, \infty) \to\ r_s e^{i\theta}$ is continuous;
$For\ all\ x, y, z \in \mathbb{X}, p, q > 0, r_s \in [0,1]\ and\ \ \theta \in [0, \frac{\pi}{2}]$.
Then $(\mathbb{X}, \overline{Đ_{Civfms}}, *)$ is called complex Interval valued fuzzy metric space .

Example 2[31]: Let $(\mathbb{X}, d)$ be a real valued metric space. Let $u * v = min\{u, v\}(or u * v = uv)$, for each$t > 0, u, v \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. Further more $u, v \in \mathbb{X}$, Define $\overline{Đ_{Civfms}}(x, y, t) = e^{i\theta} \frac{t}{t + d(x,y)}$.

Example 3[31]: Let $\mathbb{X} = R$. Let $u * v = min\{u, v\}(or u * v = uv)$, for each $t > 0, u, v \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. Further more $u, v \in \mathbb{X}$, define $\overline{Đ_{Civfms}}(x, y, t) = e^{i\theta} e^{-\frac{|x-y|}{t}}$.

Definition 2.8: An self maps is a mapping that, within its range, has a single value for every point in the domain. As a result, it is either many-to-one or one-to-one.

Definition 2.9: Two self mapping A and B on *Civfms*$(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ are said to be satisfy weakly compatible if $\text{Lim}_{n\to\infty} \overline{Đ_{Civfms}}(Ax_n, Bx_n, t_{nrm}) \to e^{i\theta}$ or $\lim_{n\to\infty} \left|\overline{Đ_{Civfms}}(Ax_n, Bx_n, t_{nrm})\right| \to \bar{1}$ And $\lim_{n\to\infty} \overline{Đ_{Civfms}}(BAx_n, ABx_n, t_{nrm}) \to e^{i\theta}$ or $\lim_{n\to\infty} \left|\overline{Đ_{Civfms}}(BAx_n, ABx_n, t_{nrm})\right| \to \bar{1}$ For all $t_{nrm} > 0, \theta \in \left[0, \frac{\pi}{2}\right]$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some $z \in X\ as\ n \to \infty$.

Definition 2.10: Two single valued mapping A and B on *Civfms*$(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ then compatible if $\lim_{n\to\infty} \overline{Đ_{Ivfms}}(ABx_n, BAx_n, t_{nrm}) = e^{i\theta}$ or $\lim_{n\to\infty} \left|\overline{Đ_{Civfms}}(ABx_n, BAx_n, t_{nrm})\right| \to \bar{1}$ for all $t_{nrm} > 0$, $\theta \in \left[0, \frac{\pi}{2}\right]$ whenever a sequence $\{x_n\}$ in $\mathfrak{X}$ provided $\lim_{n\to\infty} \text{A}x_n = \lim_{n\to\infty} \text{B}x_n = u$, $for\ all\ u \in \mathbb{X}$.

Definition 2.11: Two single valued mapping A and B on *Civfms*$(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ are said to be satisfy compatible of type $(p - 1)$ if $\text{Lim}_{n\to\infty} \overline{Đ_{Civfms}}(ABx_n, BBx_n, t_{nrm}) \to e^{i\theta}$ or $\lim_{n\to\infty} \left|\overline{Đ_{Civfms}}(ABx_n, BBx_n, t_{nrm})\right| \to \bar{1}$ for all $t_{nrm} > 0, \theta \in \left[0, \frac{\pi}{2}\right]$ Whenever $\{x_n\}$ is a sequence in X $such\ that\ Ax_n, Bx_n \to z\ for\ some\ z \in X\ as\ n \to \infty$.

Lemma 2.12 [10]: Let $(\mathbb{X}, \overline{Đ_{Civfms}}, *_I) Civfms$ with $\lim_{t\to\infty} N(x, y, t) = e^{i\theta}$ $, \forall x, y \in X, t \in (0,1)$, $\theta \in \left[0, \frac{\pi}{2}\right]$ for all $x, y \in \mathbb{X}, t_{nrm} > \bar{0}$ with a positive number $k \in (0,1)$ such that $\overline{Đ_{Civfms}}(x, y, kt_{nrm}) \geq \overline{Đ_{Civfms}}(x, y, t_{nrm}) \Rightarrow x = y$.

Definition 2.13: Let $(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ be a *Ivfms,* where * denotes a continuous t-norm and $f, g, h\ and\ k$ be self-mapping on X .The pairs $\{f, g\} and \{h, k\}$ are said to be satisfy the "common limit in the range of g" **(**$CLRg$**)** –Property if and only if there exist sequences $\{\boldsymbol{x_n}\}$ and $\{\boldsymbol{y_n}\}$ and a point $u \in X$ such that

$$\lim_{n\to\infty} \overline{Đ_{Civfms}}(fx_n, gu, t_{nrm}) = \lim_{n\to\infty} \overline{Đ_{Civfms}}(gx_n, gu, t_{nrm})$$
$$= \lim_{n\to\infty} \overline{Đ_{Civfms}}(hy_n, gu, t_{nrm}) = \lim_{n\to\infty} \overline{Đ_{Civfms}}(ky_n, gu, t_{nrm}) \ \ for\ all\ t_{nrm} > 0$$

Similar is the case with $CLRf$, $CLRh$, $CLRk$–Properties where gu is replace by fu, hu , ku in the above equality quantities.

Definition 2.14 Let $(\mathbb{X}, \overline{Đ_{C\iota vfms}}, *_I)$be a Ci*vfms,* where * denotes a continuous $t-nrm$ and f, g, i *and* j be self-mapping on X .The pairs $\{f, g\}$*and* $\{i, j\}$ are said to be satisfy the "joint common limit in the range of g" **(**$JCLRgj$**)** –Property if and only if there exist sequences $\{\boldsymbol{x_n}\}$ and $\{\boldsymbol{y_n}\}$ and a point $u \in X$ such that and $ku = gu$ *and*

$\mathrm{Lim}_{n\to\infty} \overline{Đ_{C\iota vfms}}(fx_n, gu, t_{nrm}) = \mathrm{Lim}_{n\to\infty} \overline{Đ_{C\iota vfms}}(gx_n, gu, t_{nrm}) =$
$\mathrm{Lim}_{n\to\infty} \overline{Đ_{C\iota vfms}}(iy_n, gu, t_{nrm}) = \mathrm{Lim}_{n\to\infty} \overline{Đ_{C\iota vfms}}(jy_n, gu, t_{nrm}) = e^{i\theta}$ *for all* $t_{nrm} > 0, \theta \in \left[0, \frac{\pi}{2}\right]$.

Similar is the case with $CLRf$, $CLRi$, $CLRj$–Properties where gu is replace by fu, iu , ju in the above equality quantities.

Definition 2.15[11] Two self mapping A and B on*Civfms*$(\mathbb{X}, \overline{Đ_{C\iota vfms}}, *_I)$are said to satisfy the Property (E.A) if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

Definition 2.16[11] Two pairs $\{P, Q\}$ *and*$\{C, D\}$ of self-mapping of a *Ivfms*$(\mathbb{X}, \overline{Đ_{C\iota vfms}}, *_I)$are said to be satisfy the Common Property (E.A) if there exist two sequences $\{\boldsymbol{x_n}\}$ and $\{\boldsymbol{y_n}\}$ and a point $u \in X$ such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Cy_n = \lim_{n\to\infty} Dy_n = u$$

III. Main Results

Implicit Relation:

For our theorems first we define a implicit functions which is support our theorems. Let $\boldsymbol{\Phi}$ be the set of all continuous functions $\psi: [(I^+)]^6 \to \mathbf{I}$ satisfying the following conditions:

(ψ_1): $\psi$is non-decreasing in first argument.
(ψ_2): $\psi(a, e^{i\theta}, a, e^{i\theta}, a, e^{i\theta}) \geq 0 \Rightarrow a \geq e^{i\theta}$.
(ψ_3): $\psi(a, e^{i\theta}, e^{i\theta}, a, e^{i\theta}, a) \geq 0 \Rightarrow a \geq e^{i\theta}$.
(ψ_4): $\psi(a, a, e^{i\theta}, e^{i\theta}, a, a) \geq 0 \Rightarrow a \geq e^{i\theta}$.
(ψ_5): $\psi(a, a, e^{i\theta}, e^{i\theta}, e^{i\theta}, e^{i\theta}) \geq 0 \Rightarrow a \geq e^{i\theta}$.

Theorem 3.1 Let$(\mathbb{X}, \overline{Đ_{C\iota vfms}}, *_I)$ be a Civfms with $\lim_{t\to\infty} \overline{Đ_{C\iota vfms}}(x, y, t) = e^{i\theta}$, $\forall x, y \in \mathbb{X}, t \in (0,1), \theta \in \left[0, \frac{\pi}{2}\right], t_{nrm} > \bar{0}$. The maps A and B are two weakly compatible self-maps in with $t_{nrm} *_I t_{nrm} \geq t_{nrm}$, satisfying the following:

(1) $B(X) \subseteq A(X)$;
(2) $A(X)$ or $B(X)$ is complete subspace of $\mathbb{X}$;
(3) A and B share (E. A)property;
(4) For all $t_{nrm} > 0, 0 < k < 1$ *and* $x \neq y \in \mathbb{X}$

$$\overline{Đ_{C\iota vfms}}(Bx, By, kt_{nrm}) \geq \mathrm{Min}\left\{\begin{array}{c}\overline{Đ_{C\iota vfms}}(Ax, Ay, t_{nrm}), [\overline{Đ_{C\iota vfms}}(Ax, By, t_{nrm}) * \overline{Đ_{C\iota vfms}}(Ay, Bx, t_{nrm})], \\ [\overline{Đ_{C\iota vfms}}(Ax, Bx, t_{nrm}) * \overline{Đ_{C\iota vfms}}(Ay, By, t_{nrm})]\end{array}\right\}$$

Then A and Bhave UCFP.

Proof: Since B and A share (E. A) -property. So, by the definition, there exist sequences $\{x_n\}$ in $\mathbb{X}$ such that

$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ax_n = u$, for some $u \in \mathbb{X}$.

Suppose A(X) is complete then $\lim_{n\to\infty} Ax_n = Aa$, for some $a \in \mathbb{X}$. Therefore $\lim_{n\to\infty} Bx_n = Aa$

Now to claim $Ba = Aa$.

Put $x = x_n$ and $y = a$ in inequality (d)

$$\overline{Đ_{C\iota vfms}}(Bx_n, Ba, kt_{nrm}) \geq \mathrm{Min}\left\{\begin{array}{c}\overline{Đ_{Ivf}}(Ax_n, Aa, t_{nrm}), [\overline{Đ_{C\iota vfms}}(Ax_n, Ba, t_{nrm}) * \overline{Đ_{C\iota vfms}}(Aa, Bx_n, t_{nrm})], \\ [\overline{Đ_{C\iota vfms}}(Ax_n, Bx_n, t_{nrm}) * \overline{Đ_{C\iota vfms}}(Aa, Ba, t_{nrm})]\end{array}\right\}$$

As $n \to \infty$, then we get

$\overline{Đ_{Civfms}}(Aa, Ba, kt_{nrm}) \geq \ \text{Min}\left\{\begin{matrix} \overline{Đ_{Civfms}}(Aa, Aa, t_{nrm}), [\overline{Đ_{Civfms}}(Aa, Ba, t_{nrm}) * \overline{Đ_{Civfms}}(Aa, Aa, t_{nrm})], \\ [\overline{Đ_{Civfms}}(Aa, Aa, t_{nrm}) * \overline{Đ_{Civfms}}(Aa, Ba, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(Aa, Ba, kt_{nrm}) \geq \text{Min}\left\{\begin{matrix} e^{i\theta}, [\overline{Đ_{Civfms}}(Aa, Ba, t_{nrm}) *_I e^{i\theta}], \\ [e^{i\theta} *_I \overline{Đ_{Civfms}}(Aa, Ba, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(Aa, Ba, kt_{nrm}) \geq \text{Min}\left\{\begin{matrix} e^{i\theta}, [\overline{Đ_{Civfms}}(Aa, Ba, t_{nrm})], \\ [\overline{Đ_{Civfms}}(Aa, Ba, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(Aa, Ba, kt_{nrm}) \geq \overline{Đ_{Civfms}}(Aa, Ba, t_{nrm})$

By lemma 2.12, we have, $Aa = Ba$.

Now A and B are weakly compatible. i.e. $ABa = BAa = AAa = BBa$ then we show that Ba is the common fixed-point of B and A.

Put $x = a$ and $y = Ba$ in inequality (d)

$\overline{Đ_{Civfms}}(Ba, BBa, kt_{nrm})$
$\geq \text{Min}\left\{\begin{matrix} \overline{Đ_{Civfms}}(Aa, ABa, t_{nrm}), [\overline{Đ_{Civfms}}(Aa, BBa, t_{nrm}) * \overline{Đ_{Civfms}}(ABa, Ba, t_{nrm})], \\ [\overline{Đ_{Civfms}}(Aa, Ba, t_{nrm}) * \overline{Đ_{Civfms}}(ABa, BBa, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(Ba, BBa, kt_{nrm})$
$\geq \text{Min}\left\{\begin{matrix} \overline{Đ_{Civfms}}(Ba, BBa, t_{nrm}), [\overline{Đ_{Civfms}}(Ba, BBa, t_{nrm}) * \overline{Đ_{Civfms}}(BBa, Ba, t_{nrm})], \\ [\overline{Đ_{Civfms}}(Ba, Ba, t_{nrm}) * \overline{Đ_{Civfms}}(BBa, BBa, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(Ba, BBa, kt_{nrm}) \geq \text{Min}\left\{\begin{matrix} Đ_{Ivf}(Ba, BBa, t_{nrm}), [Đ_{Ivf}(Ba, BBa, t_{nrm})], \\ [e^{i\theta} *_I \overline{Đ_{Ivf}}(BBa, BBa, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(Ba, BBa, kt_{nrm}) \geq \text{Min}\left\{\begin{matrix} \overline{Đ_{Civfms}}(Ba, BBa, t_{nrm}), [\overline{Đ_{Civfms}}(Ba, BBa, t_{nrm})], \\ [\overline{Đ_{Civfms}}(BBa, BBa, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(Ba, BBa, kt_{nrm}) \geq \overline{Đ_{Civfms}}(Ba, BBa, t_{nrm})$

By lemma 2.12 we get $Ba = BBa$.

Hence Ba is the common fixed-point of B and A.

In the similar manner if B(X) is complete. The result will be same.

Now to show that fixed-point is unique.

Let α and β be the two c-fixed-point of A, B.

$\overline{Đ_{Civfms}}(\alpha, \beta, kt_{nrm}) = \overline{Đ_{Civfms}}(B\alpha, B\beta, kt_{nrm})$
$\geq \text{Min}\left\{\begin{matrix} \overline{Đ_{Civfms}}(A\alpha, A\beta, t_{nrm}), [\overline{Đ_{Civfms}}(A\alpha, B\beta, t_{nrm}) * \overline{Đ_{Civfms}}(A\beta, B\alpha, t_{nrm})], \\ [\overline{Đ_{Civfms}}(A\alpha, B\alpha, t_{nrm}) * \overline{Đ_{Civfms}}(A\beta, B\beta, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(\alpha, \beta, kt_{nrm}) \geq \text{Min}\left\{\begin{matrix} \overline{Đ_{Civfms}}(\alpha, \beta, t_{nrm}), [\overline{Đ_{Civfms}}(\alpha, \beta, t_{nrm})], \\ [\overline{Đ_{Civfms}}(\alpha, \alpha, t_{nrm}) * \overline{Đ_{Civfms}}(\beta, \beta, t_{nrm})] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(\alpha, \beta, kt_{nrm}) \geq \text{Min}\left\{\begin{matrix} \overline{Đ_{Civfms}}(\alpha, \beta, t_{nrm}), \overline{Đ_{Civfms}}(\alpha, \beta, t_{nrm}), \\ [e^{i\theta} *_I e^{i\theta}] \end{matrix}\right\}$

$\overline{Đ_{Civfms}}(\alpha, \beta, kt_{nrm}) \geq \overline{Đ_{Civfms}}(\alpha, \beta, t_{nrm})$

By lemma 2.12 we have $\alpha = \beta$.

Thus A and B have UCFP.

Example 3.2 Let $\overline{Đ_{Civfms}}(x, y, t_{nrm}) = \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|}$, $t_{nrm} > 0$ and $(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ be a Civfms, where $\mathbb{X} = [1,40]$, and let AandB be two self mappings as follows:

$$Ax = \begin{cases} 1, & x = 1 \text{ or } x > 7 \\ 10, & 1 < x \leq 7 \end{cases}, \quad Bx = \begin{cases} 1, & x = 1 \\ 12, & 1 < x \leq 7 \\ \frac{x+1}{8}, & x > 7 \end{cases}$$

And also a sequence $\{x_n\}$ define as $x_n = 7 + \frac{1}{n}$, $n \geq 1$ then we see $\lim_{n\to\infty} Ax_n = 1 = \lim_{n\to\infty} Bx_n$ then A and B (E.A) property satisfied $\lim_{n\to\infty} \overline{Đ_{Civfms}}(ABx_n, BAx_n, t_{nrm}) = \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |1-1|} = e^{i\theta}$

This shows that A and B compatible property.

Theorem 3.4: Let $(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ be a Civfms with $\lim_{t\to\infty} \overline{Đ_{Civfms}}(x, y, t) = e^{i\theta}$, $\theta \in \left[0, \frac{\pi}{2}\right]$, $t_{nrm} > \bar{0}$.

Let A, B, S, T, G and H be a self mapsof a Civfms($\mathbb{X}, \overline{Đ_{Cıvfms}}, *_I$) satisfying the following:

$$AG(X) \subseteq T(X), \quad BH(X) \subseteq S(X)$$

The pair (AG, S) and (BH, T) share one of the $Clr_{AG}, Clr_{BH}, Clr_S, Clr_T$-property;

$$AG = GA \text{ and either } SA = AS \text{ or } SG = GS$$
$$BH = HB \text{ and either } TB = BT \text{ or } TH = HT.$$

(AG, S) and (BH, T) are Compatible Type of P-1.

$\psi \in \Phi$, for all $x, y \in \mathbb{X}$, $t_{nrm} > 0$

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGx, BHy, t_{nrm}), \overline{Đ_{Cıvfms}}(Sx, Ty, t_{nrm}), \overline{Đ_{Cıvfms}}(AGx, Sx, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(BHy, Ty, t_{nrm}), \overline{Đ_{Cıvfms}}(AGx, Ty, t_{nrm}), \overline{Đ_{Cıvfms}}(Sx, BHy, t_{nrm})\end{Bmatrix} \geq 0$$

Then A, B, S, T, G and H have a UCFP.

Proof: Case (i) Consider (AG, S) and (BH, T)share one of the Clr_{AG} or Clr_S -property. So, by the definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{X}$ such that

$$\lim_{n\to\infty} AGx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} BHy_n = \lim_{n\to\infty} Ty_n = Su \text{, for some } u \in \mathbb{X}$$

Step (i) Putting $x = u$ and $y = y_n$ in (e) then

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGu, BHy_n, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, Ty_n, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(BHy_n, Ty_n, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Ty_n, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, BHy_n, t_{nrm})\end{Bmatrix} \geq 0$$

As $n \to \infty$, then we get

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(Su, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, Su, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}), e^{i\theta}, \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\\ e^{i\theta}, \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}), e^{i\theta}\end{Bmatrix} \geq 0$$

By Implicit relation of (ψ_2). We have

$$\overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}) \geq e^{i\theta} \Rightarrow AGu = Su$$

Step (ii) Since, $AG(X) \subseteq T(X)$ then there is a $v \in X$ such that $AGu = Tv$

Putting $x = x_n$ and $y = v$ in (e) then

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGx_n, BHv, t_{nrm}), \overline{Đ_{Cıvfms}}(Sx_n, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(AGx_n, Sx_n, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(BHv, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(AGx_n, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(Sx_n, BHv, t_{nrm})\end{Bmatrix} \geq 0$$

As $n \to \infty$, then we get

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(Tv, BHv, t_{nrm}), \overline{Đ_{Cıvfms}}(Tv, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(Tv, Tv, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(BHv, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(Tv, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(Tv, BHv, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(Tv, BHv, t_{nrm}), e^{i\theta}, e^{i\theta},\\ \overline{Đ_{Cıvfms}}(BHv, Tv, t_{nrm}), \quad e^{i\theta}, \overline{Đ_{Cıvfms}}(Tv, BHv, t_{nrm})\end{Bmatrix} \geq 0$$

By Implicit relation of (ψ_3). We have

$$\overline{Đ_{Cıvfms}}(BHv, Tv, t_{nrm}) \geq e^{i\theta} \Rightarrow BHv = Tv$$

Thus $AGu = Su = BHv = Tv = z$ (say)

Step (iii) Since(AG, S) and (BH, T) are compatible Type of (P-1)

Then we have, $AGSu = SSu$ and $BHTv = TTv$ i.e $AGz = Sz$ and $BHz = Tz$

Putting $x = z$ and $y = v$ in (e) then

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGz, BHv, t_{nrm}), \overline{Đ_{Cıvfms}}(Sz, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, Sz, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(BHv, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, Tv, t_{nrm}), \overline{Đ_{Cıvfms}}(Sz, BHv, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGz, z, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, z, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, AGz, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(z, z, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, z, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, z, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{Cıvfms}}(AGz, z, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, z, t_{nrm}), \quad e^{i\theta},\\ e^{i\theta}, \overline{Đ_{Cıvfms}}(AGz, z, t_{nrm}), \overline{Đ_{Cıvfms}}(AGz, z, t_{nrm})\end{Bmatrix} \geq 0$$

By Implicit relation of (ψ_4). We have

$$\overline{Đ_{Cıvfms}}(AGz, z, t_{nrm}) \geq e^{i\theta} \Rightarrow AGz = z$$

Thus$AGz = Sz = z$

Similarly by taking $x = u$ and $y = z$ in (e) then

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(AGu, BHz, t_{nrm}), \overline{Đ_{C\imath vfms}}(Su, Tz, t_{nrm}), \overline{Đ_{C\imath vfms}}(AGu, Su, t_{nrm}),\\ \overline{Đ_{C\imath vfms}}(BHz, Tz, t_{nrm}), \overline{Đ_{C\imath vfms}}(AGu, Tz, t_{nrm}), \overline{Đ_{C\imath vfms}}(Sz, BHz, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, z, t_{nrm}),\\ \overline{Đ_{C\imath vfms}}(BHz, BHz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm}), \ e^{i\theta},\\ e^{i\theta}, \overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm})\end{Bmatrix} \geq 0$$

By Implicit relation of (ψ_4). We have

$$\overline{Đ_{C\imath vfms}}(z, BHz, t_{nrm}) \geq \ e^{i\theta} \Rightarrow BHz = z$$

Thus ,$BHz = Tz = z$

Hence ,$AGz = BHz = Sz = Tz = z$

Step (iv) By condition (c)

Suppose $SA = AS$

Since $AG = GA$ then we have $AGAz = AAGz = Az$ and $SAz = ASz = Az$

By taking$x = Az$ and $y = v$ in (e) then

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(AGAz, BHv, t_{nrm}), \overline{Đ_{C\imath vfms}}(SAz, Tv, t_{nrm}), \overline{Đ_{C\imath vfms}}(AGAz, SAz, t_{nrm}),\\ \overline{Đ_{C\imath vfms}}(BHv, Tv, t_{nrm}), \overline{Đ_{C\imath vfms}}(AGAz, Tv, t_{nrm}), \overline{Đ_{C\imath vfms}}(SAz, BHv, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(Az, z, t_{nrm}), \overline{Đ_{C\imath vfms}}(Az, z, t_{nrm}), \overline{Đ_{C\imath vfms}}(Az, Az, t_{nrm}),\\ \overline{Đ_{C\imath vfms}}(z, z, t_{nrm}), \overline{Đ_{C\imath vfms}}(Az, z, t_{nrm}), \overline{Đ_{C\imath vfms}}(Az, z, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(Az, z, t_{nrm}), \overline{Đ_{C\imath vfms}}(Az, z, t_{nrm}), \ e^{i\theta},\\ e^{i\theta}, \overline{Đ_{C\imath vfms}}(Az, z, t_{nrm}), \overline{Đ_{C\imath vfms}}(Az, z, t_{nrm})\end{Bmatrix} \geq 0$$

By Implicit relation of (ψ_4). We have

$$\overline{Đ_{C\imath vfms}}(Az, z, t_{nrm}) \geq \ e^{i\theta} \Rightarrow Az = z$$

Since, $AGz = GAz = Lz$

Thus, $Az = Gz = Sz = z$

And suppose $SG = GS$

Since $G = GA$ then we have $AGGz = GAGz = Gz$ and $SGz = GSz = Gz$

Similarly by taking$x = Gz$ and $y = v$ in (e) thenwe get$Gz = z$

Since, $AGz = Az = z$

Thus, $Az = Gz = Sz = z$

Now suppose $BT = TB$

Since $BH = HB$ then we have $BHBz = BBHz = Bz$ and $TBz = BTz = Bz$

By taking $x = u$ and $y = Bz$ in (e) then

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(AGu, BHBz, t_{nrm}), \overline{Đ_{C\imath vfms}}(Su, TBz, t_{nrm}), \overline{Đ_{C\imath vfms}}(AGu, Su, t_{nrm}),\\ \overline{Đ_{C\imath vfms}}(BHBz, TBz, t_{nrm}), \overline{Đ_{C\imath vfms}}(AGu, TBz, t_{nrm}), \overline{Đ_{C\imath vfms}}(Su, BHBz, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, z, t_{nrm}),\\ \overline{Đ_{C\imath vfms}}(Bz, Bz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm})\end{Bmatrix} \geq 0$$

$$\psi\begin{Bmatrix}\overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm}), \ e^{i\theta},\\ e^{i\theta}, \overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm}), \overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm})\end{Bmatrix} \geq 0$$

By Implicit relation of (ψ_4). We have

$$\overline{Đ_{C\imath vfms}}(z, Bz, t_{nrm}) \geq \ e^{i\theta} \Rightarrow Bz = z$$

Since ,$BHz = HBz = Hz$

Thus, $Bz = Hz = Tz = z$

And suppose $TH = HT$

Since $BH = HB$ then we have $BHHz = HBHz = Hz$ and $THz = HTz = Hz$

Similarly by taking$x = u$ and $y = Hz$ in (e) thenwe get$Hz = z$

Since, $BHz = Bz = z$

Thus, $Bz = Hz = Tz = z$

Combine all results then we get, $Az = Bz = Gz = Hz = Sz = Tz = z$

Case (ii) Suppose (AG, S) and (BH, T) Share one of the Clr_{BH} or lr_T-property. So, by the definition,

there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\to\infty} ALx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} BHy_n = \lim_{n\to\infty} Ty_n = Tv$, for some $v \in X$
Proceeding on similar lines of case (i)
Firstly we get $BHv = Tv$
Since $BH(X) \subseteq S(X)$ then there is a $u \in X$ such that $BHv = Su$
Using step (ii) then we get $AGu = Su$
Thus $AGu = Su = BHv = Tv = z$ (say)
Now using step (iii) we get $AGz = Sz$ and $BHz = Tz$ by using inequality (e) then we get $AGz = BHz = Sz = Tz = z$
From this stage the proof is similar as step (iv) then we get z is a common fixed point of A, B, S, T, G and H in X.
For uniqueness further we consider α and β be the two common fixed-point of A, B, S, T, G and H.
Put $x = \alpha$ and $y = \beta$ in inequality (e) then

$$\psi\left\{\begin{matrix}(\overline{Đ_{Cıvfms}}(AG\alpha, BH\beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(S\alpha, T\beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(AG\beta, S\alpha, t_{nrm}),\\ (\overline{Đ_{Cıvfms}}(BH\beta, T\beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(AG\alpha, T\alpha, t_{nrm}), (\overline{Đ_{Cıvfms}}(S\alpha, BH\alpha, t_{nrm})\end{matrix}\right\} \geq 0$$

$$\psi\left\{\begin{matrix}(\overline{Đ_{Cıvfms}}(\alpha, \beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(\alpha, \beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(\beta, \alpha, t_{nrm}),\\ (\overline{Đ_{Cıvfms}}(\beta, \beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(\alpha, \alpha, t_{nrm}), (\overline{Đ_{Cıvfms}}(\alpha, \alpha, t_{nrm})\end{matrix}\right\} \geq 0$$

$$\psi\left\{\begin{matrix}(\overline{Đ_{Cıvfms}}(\alpha, \beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(\alpha, \beta, t_{nrm}), (\overline{Đ_{Cıvfms}}(\beta, \alpha, t_{nrm}),\\ e^{i\theta},\ e^{i\theta},\ e^{i\theta}\end{matrix}\right\} \geq 0$$

By Implicit relation of (ψ_5). We have

$$(\overline{Đ_{Cıvfms}}(\alpha, \beta, t_{nrm}) \geq e^{i\theta} \Rightarrow \theta = \vartheta$$

Hence z is a unique common fixed point of A, B, S, T, G and H in X.
Now, taking $G = H = I$(Identity mapping)
Remark 3.5: if we apply JClr- property instead of Clr-Proprty then we can ignore condition (a) in theorem (3.4). Now we prove the following:

Theorem 3.6: Let $(\mathbb{X}, \overline{Đ_{Cıvfms}}, *_I)$ be a Civfms with $\lim_{t\to\infty} \overline{Đ_{Cıvfms}}(x, y, t) = e^{i\theta}$, $\theta \in \left[0, \frac{\pi}{2}\right]$, $t_{nrm} > \bar{0}$. Let A, B, S, T, G and H be a self maps of a Civfms$(\mathbb{X}, \overline{Đ_{Cıvfms}}, *_I)$ satisfying the following:

$$AG(X) \subseteq T(X), \qquad BH(X) \subseteq S(X)$$

The pair (AG, S) and (BH, T) share one of the $JClr_{ST}$-property;

$$AG = GA \text{ and either } SA = AS \text{ or } SG = GS$$
$$BH = HB \text{ and either } TB = BT \text{ or } TH = HT.$$

(AG, S) and (BH, T) are Compatible Type of P-1.
$\psi \in \Phi$, for all$x, y \in \mathbb{X}$, $t_{nrm} > 0$

$$\psi\left\{\begin{matrix}\overline{Đ_{Cıvfms}}(AGx, BHy, t_{nrm}), \overline{Đ_{Cıvfms}}(Sx, Ty, t_{nrm}), \overline{Đ_{Cıvfms}}(AGx, Sx, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(BHy, Ty, t_{nrm}), \overline{Đ_{Cıvfms}}(AGx, Ty, t_{nrm}), \overline{Đ_{Cıvfms}}(Sx, BHy, t_{nrm})\end{matrix}\right\} \geq 0$$

Then A, B, S, T, G and H have a UCFP.

Proof: Suppose (AG, S) and (BH, T)share one of the $JClr_{ST}$ -property. So, by the definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} AGx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} BHy_n = \lim_{n\to\infty} Ty_n = Su(= Tu) \text{ , for some } u \in X$$

Step (i) Putting $x = u$ and $y = y_n$ in (e) then

$$\psi\left\{\begin{matrix}\overline{Đ_{Cıvfms}}(AGu, BHy_n, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, Ty_n, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(BHy_n, Ty_n, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Ty_n, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, BHy_n, t_{nrm})\end{matrix}\right\} \geq 0$$

As $n \to \infty$, then we get

$$\psi\left\{\begin{matrix}\overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\\ \overline{Đ_{Cıvfms}}(Su, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}), \overline{Đ_{Cıvfms}}(Su, Su, t_{nrm})\end{matrix}\right\} \geq 0$$

$$\psi\left\{\begin{matrix}\overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\ e^{i\theta}, \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\\ e^{i\theta}, \overline{Đ_{Cıvfms}}(AGu, Su, t_{nrm}),\ e^{i\theta}\end{matrix}\right\} \geq 0$$

by implicit relation (ψ_2) . We have

$$\overline{Đ_{Civfms}}(AGu, Su, t_{nrm}) \geq e^{i\theta} \Rightarrow AGu = Su \ (AGu = Su = Tu).$$

And Putting $x = x_n$ and $y = u$ in (e) thenwe get, $BHu = Tu$

Thus $AGu = Su = BHu = Tu = z$ (say)

Since (AG, S) and(BH, T) are compatible Type of (P-1)

Then we have, $AGSu = SSu$ and $BHTv = TTv$

i.e $AGz = Sz$ and $BHz = Tz$

Now taking $x = z$ and $y = u$ in (e) then we get $AGz = Sz = z$

And taking $x = u$ and $y = z$ in (e) then we get $BHz = Tz = z$

Thus, $AGz = Sz = BHz = Tz = z$.

From this stage, the proof is similar as proof of theorem (3.4).

Remark 3.7: If the Theorem (3.4) shares common properties (E.A) then the proof run on similar steps of the theorem (3.3).

Example 3.9 Let$(\mathbb{X}, \overline{Đ_{Civfms}}, *_I)$ be a Civfms with $\lim_{t\to\infty} \overline{Đ_{Civfms}}(x, y, t) = e^{i\theta}, \theta \in \left[0, \frac{\pi}{2}\right], t_{nrm} > \bar{0}$.

Define $\overline{Đ_{Civfms}}(x, y, t_{nrm}) = \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|}$, where , $\mathbb{X} = [0,40]$

Consider $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = 40t_1 - 12t_2 + 24t_3 - 3t_4 + 18t_5 - 11t_6, \ t \in [0,1]$

Now, $Ax = 2$, $Bx = 2$, $Gx = x$, $Hx = x$, $Sx = \begin{cases} 2, & x \leq 10 \\ \frac{1}{4}, & x > 10 \end{cases}$, $Tx = \begin{cases} 2, & x \leq 10 \\ \frac{1}{2}, & x > 10 \end{cases}$

Case (a): When $x, y \leq 10$ then the LHS of inequality (e) (Theorem 3.4)

$$\psi \begin{Bmatrix} \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}), \\ \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}) \end{Bmatrix} \geq 0$$

$$LHS = \psi \begin{Bmatrix} \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}), \\ \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}) \end{Bmatrix}$$

$$= \psi \begin{Bmatrix} \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|}, \\ \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |x-y|} \end{Bmatrix}$$

$$= \psi \begin{Bmatrix} \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \\ \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|} \end{Bmatrix}$$

$$= \psi \begin{Bmatrix} e^{i\theta}, e^{i\theta}, e^{i\theta}, \\ e^{i\theta}, e^{i\theta}, e^{i\theta} \end{Bmatrix}$$

$$= 40e^{i\theta} - 12e^{i\theta} + 24e^{i\theta} - 3e^{i\theta} + 18e^{i\theta} - 11e^{i\theta}$$

$$= 59e^{i\theta} \geq \bar{0} \ \text{Satisfied.}$$

Case (b): When $x > 10$, $y \geq 10$ then the LHS of inequality (e) (Theorem 3.4)

$$LHS = \psi \begin{Bmatrix} \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}\left(\frac{1}{4}, 2, t_{nrm}\right), \overline{Đ_{Civfms}}\left(2, \frac{1}{4}, t_{nrm}\right), \\ \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}), \overline{Đ_{Civfms}}(2,2, t_{nrm}) \end{Bmatrix}$$

$$= \psi \begin{Bmatrix} \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + \left|\frac{1}{4} - 2\right|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + \left|2 - \frac{1}{4}\right|}, \\ \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|}, \frac{e^{i\theta} t_{nrm}}{t_{nrm} + |2-2|} \end{Bmatrix}$$

$$= \psi \begin{Bmatrix} e^{i\theta}, \frac{4t_{nrm} e^{i\theta}}{4t+7}, \frac{4t_{nrm} e^{i\theta}}{4t+7}, e^{i\theta}, e^{i\theta}, e^{i\theta} \end{Bmatrix}$$

$$= 40e^{i\theta} - 12\frac{4t_{nrm}e^{i\theta}}{4t+7} + 24\frac{4t_{nrm}}{4t+7} - 3e^{i\theta} + 18e^{i\theta} - 11e^{i\theta}$$
$$\geq \bar{0}\ ,\ \text{Satisfied.}$$

Case (c): When $x \leq 10$, $y \geq 10$ then the LHS of inequality (e) (Theorem 3.4)

$$\text{LHS} = \psi\left\{\begin{matrix}\overline{Ð_{Civfms}}(2,2,t_{nrm}),\overline{Ð_{Civfms}}(2,\frac{1}{2},t_{nrm}),\overline{Ð_{Civfms}}(2,2,t_{nrm}),\\ \overline{Ð_{Civfms}}(2,\frac{1}{2},t_{nrm}),\overline{Ð_{Civfms}}(2,\frac{1}{2},t_{nrm}),\overline{Ð_{Civfms}}(2,2,t_{nrm})\end{matrix}\right\}$$

$$= \psi\left\{\begin{matrix}\frac{e^{i\theta}t_{nrm}}{t_{nrm}+|2-2|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|2-\frac{1}{2}\right|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+|2-2|},\\ \frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|2-\frac{1}{2}\right|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|2-\frac{1}{2}\right|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+|2-2|}\end{matrix}\right\}$$
$$= \psi\left\{e^{i\theta},\ \frac{2t_{nrm}e^{i\theta}}{2t+3},\ e^{i\theta},\ \frac{2t_{nrm}e^{i\theta}}{2t+3},e^{i\theta},e^{i\theta}\right\}$$
$$= 40e^{i\theta} - 12\frac{2t_{nrm}e^{i\theta}}{2t+3} + 24e^{i\theta} - 3\frac{2t_{nrm}e^{i\theta}}{2t+3} + 18\frac{2t_{nrm}e^{i\theta}}{2t+3} - 11e^{i\theta}$$
$$\geq \bar{0}\ ,\ \text{Satisfied.}$$

Case (d): When $x > 10$, $y > 10$ then the LHS of inequality (e) (Theorem 3.4)

$$\text{LHS} = \psi\left\{\begin{matrix}\overline{Ð_{Civfms}}(2,2,t_{nrm}),\overline{Ð_{Civfms}}\left(\frac{1}{4},\frac{1}{2},t_{nrm}\right),\overline{Ð_{Civfms}}\left(2,\frac{1}{4},t_{nrm}\right),\\ \overline{Ð_{Civfms}}\left(2,\frac{1}{2},t_{nrm}\right),\overline{Ð_{Civfms}}\left(2,\frac{1}{2},t_{nrm}\right),\overline{Ð_{Civfms}}\left(\frac{1}{4},2,t_{nrm}\right)\end{matrix}\right\}$$

$$= \psi\left\{\begin{matrix}\frac{e^{i\theta}t_{nrm}}{t_{nrm}+|2-2|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|\frac{1}{4}-\frac{1}{2}\right|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|2-\frac{1}{4}\right|},\\ \frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|2-\frac{1}{2}\right|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|2-\frac{1}{2}\right|},\frac{e^{i\theta}t_{nrm}}{t_{nrm}+\left|\frac{1}{4}-2\right|}\end{matrix}\right\}$$
$$= \psi\left\{\begin{matrix}e^{i\theta},\frac{4e^{i\theta}t_{nrm}}{4t_{nrm}+1},\frac{4e^{i\theta}t_{nrm}}{4t_{nrm}+7},\\ \frac{2e^{i\theta}t_{nrm}}{2t_{nrm}+3},\frac{2e^{i\theta}t_{nrm}}{2t_{nrm}+3},\frac{4e^{i\theta}t_{nrm}}{4t_{nrm}+7}\end{matrix}\right\}$$

$$= 40e^{i\theta} - 12\frac{4e^{i\theta}t_{nrm}}{4t_{nrm}+1} + 24\frac{4e^{i\theta}t_{nrm}}{4t_{nrm}+7} - 3\frac{2t_{nrm}e^{i\theta}}{2t+3} + 18\frac{2t_{nrm}e^{i\theta}}{2t+3} - 11\frac{4e^{i\theta}t_{nrm}}{4t_{nrm}+7}$$
$$\geq \bar{0}\ ,\ \text{Satisfied.}$$

Hence all four conditions variables are satisfied. Therefore A, B, S, T, G and H satisfied the hypothesis of theorem 3.4. Hence 2 is a unique c-fixed-point in X.

IV. Conclusion

In many scientific domains, fixed point theory has several applications. In this research, we introduced the concept of complex interval-valued fuzzy metric space to modify the distance function under Clr/Jc;lr/E.A-Property for common fixed-point theorems in complex interval-valued fuzzy metric space in order to prove several well-known common fixed-point theorems for self-mapping. our results enhanced, expanded, and generalized some findings in the literature. These findings can be applied to the solution of LPP in dynamic programming. quantum mechanics, control systems, machine learning, robotics, digital problems, image processing, economics and other fields. These uses demonstrate how complex interval-valued fuzzy metric spaces may be used to handle a wide range of data kinds and uncertainty, making them an invaluable tool in many engineering and scientific fields.

References

[1] Azam, A. (2011). Fuzzy fixed points of fuzzy mappings via a rational inequality. Hacettepe Journal of Mathematics and Statistics, 40(3), 421-431.

[2] Buckley, J. (1989). Fuzzy complex numbers. Fuzzy sets and systems, 33(3), 333-345.

[3] Buckley, J. J., & Eslami, E. (2002). An introduction to fuzzy logic and fuzzy sets (Vol. 13). Springer Science & Business Media.

[4] Azam, A. (2011). Fuzzy fixed points of fuzzy mappings via a rational inequality. Hacettepe Journal of Mathematics and Statistics, 40(3), 421-431.

[5] Buckley, J. (1989). Fuzzy complex numbers. Fuzzy sets and systems, 33(3), 333-345.

[6] Buckley, J. J., & Eslami, E. (2002). An introduction to fuzzy logic and fuzzy sets (Vol. 13). Springer Science & Business Media.

[7] Choi, J., Datta, S. K., Biswas, T., & Islam, M. N. (2017). Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. Honam Mathematical Journal, 39(1), 115-126.

[8] Demir, I. (2021). Fixed Point Theorems in complex valued fuzzy b-metric spaces with application to integral equations. Miskolc Mathematical Notes, 22(1), 153-171.

[9] Deshmukh, V., Jadon, B. P. S., Singh, R., & Malhotra, S. K. (2023). Fixed Point Results on Interval-Valued Fuzzy Metric Space using notation of Pairwise Compatible Maps and Occasionally Weakly Compatible Maps with Application. Asian Research Journal of Mathematics, 19(11), 104-114.

[10] George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. Fuzzy sets and systems, 64(3), 395-399.

[11] Gregori, V., & Sapena, A. (2002). On fixed-point theorems in fuzzy metric spaces. Fuzzy sets and systems, 125(2), 245-252.

[12] Gregori, V., Miñana, J. J., Roig, B., &Sapena, A. (2024). On Completeness and Fixed Point Theorems in Fuzzy Metric Spaces. Mathematics, 12(2), 287.

[13] Gupta, V., Saini, R. K., &Verma, M. (2018). Fixed point theorems for single valued α-ψ-mappings in fuzzy metric spaces. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(1), 392-400.

[14] K. Wadhaw, A. Dubai, Common Fixed point theorems using E.A like property in Fuzzy Metric spaces, International J.of Math. Archive -8(6), 2017, 204-210.

[15] Kanwal, S., Shagari, M. S., Aydi, H., Mukheimer, A., &Abdeljawad, T. (2022). Common fixed-point results of fuzzy mappings and applications on stochastic Volterra integral equations. Journal of Inequalities and Applications, 2022(1), 110.

[16] Kramosil, I., &Michálek, J. (1975). Fuzzy metrics and statistical metric spaces. Kybernetika, 11(5), 336-344.

[17] Kramosil, I., &Michálek, J. (1975). Fuzzy metrics and statistical metric spaces. Kybernetika, 11(5), 336-344.

[18] Kutukcu, S., Sharma, S., &Tokgoz, H. (2007). A fixed point theorem in fuzzy metric spaces. Int. Journal of Math. Analysis, 1(18), 861-872.

[19] Li, C. (2009, June). Distances between interval-valued fuzzy sets. In NAFIPS 2009-2009 Annual Meeting of the North American Fuzzy Information Processing Society (pp. 1-3). IEEE.

[20] Mishra, R. K., & Choudhary, S. (2010). On fixed point theorems in Fuzzy metric spaces. International Magazine on Advances in Computer Science and Telecommunications, 1(1), 45.

[21] Malviya, N. the N-fuzzy metric space and mapping with application, (2015), DOI: 1515/fascmath-0019.

[22] Namdev R., Tiwari R., Singh U., & Bhardwaj, R. (2024). Common Fixed-Point Theorems for Rational Expression in Fuzzy Metric Space using E.A like Property. African Journal of Biological Sciences, 6(10).

[23] Price G. B., "An Introduction to Multi-complex Spaces and Functions", Marcel Dekker, New York, 1991.

[24] Ramot, D., Friedman, M., Langholz, G., Milo, R., & Kandel, A. (2001, December). On complex fuzzy sets. In 10th IEEE International Conference on Fuzzy Systems.(Cat. No. 01CH37297) (Vol. 3, pp. 1160-1163). IEEE.

[25] Ramot, D., Milo, R., Friedman, M., & Kandel, A. (2002). Complex fuzzy sets. IEEE transactions on fuzzy systems, 10(2), 171-186.

[26] Rao, K. P. R., Rao, K. R. K., &Sedghı, S. (2014). Common coupled fixed point theorems in fuzzy metric spaces. Gazi University Journal of Science, 27(2), 739-745.

[27] Segre C., "Le RappresentazioniRealidelleFormeComplesse a GliEntiIperalgebrici, Math. Ann., 40(1892), 413–467.

[28] Sharma, S. (2002). Common fixed point theorems in fuzzy metric spaces. Fuzzy sets and systems, 127(3), 345-352.

[29] Shen, Y., Li, H., & Wang, F. (2012). On interval-valued fuzzy metric spaces. International Journal of Fuzzy Systems, 14(1), 35-44.

[30] ShikaShendeet. al. Some fixed point theorems for asymptotically regular maps in N-Fuzzy metric space, J. of Science and Technological researchers, Vol. 3, Issue N0. 3, (2021).

[31] Singh, D., Joshi, V., Imdad, M., & Kumam, P. (2016). A novel framework of complex valued fuzzy metric spaces and fixed point theorems. Journal of Intelligent & Fuzzy Systems, 30(6), 3227-3238.

[32] Singh, U., Singh, N., Singh, R., & Bhardwaj, R. (2022). Common Invariant Point Theorem for Multi-valued Generalized Fuzzy Mapping in b-Metric Space. In Recent Trends in Design, Materials and Manufacturing: Selected Proceedings of ICRADMM 2020 (pp. 23-35). Singapore: Springer Nature Singapore.

[33] Tiwari, R., & Rajput, S. (2022). On (ψ, ϕ) Contraction in Bicomplex Valued Fuzzy b-Metric Spaces with Application. European Journal of Mathematics and Statistics, 3(4), 43-53.

[34] Umashankar Singh and Naval Singh, Common fixed point theorem for rational expression in fuzzy metric space, Journal of the Gujrat research society, vol-21, issue-14, (2019).

[35] Wadhwa, K., Choudhari, S., & Bohare, P. (2019). Common Fixed Point Theorems using Common EA like Property in Fuzzy Metric Space. International Journal of Mathematics Trends and Technology-IJMTT, 65.

[36] Zadeh, L. A., Klir, G. J., & Yuan, B. (1996). Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers (Vol. 6). World scientific.