STATISTICAL APPLICATIONS OF THE KUMARASWAMY ALPHA-POWER LOG-LOGISTIC DISTRIBUTION: EVIDENCE FROM HYPERTENSION AND CO2 EMISSIONS IN INDIA

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Abstract

This article introduces a novel statistical model, the Kumaraswamy Alpha-Power Log-Logistic (KAPLL) distribution, as a flexible extension of the traditional log-logistic model. The KAPLL distribution is especially well-suited for examining a variety of ageing and failure events as it is made to improve modelling capabilities by supporting both symmetric and asymmetric forms. Reliability analysis and lifespan data modelling are made more flexible by the KAPLL model's capacity to represent its probability density function as a combination of log-logistic densities. The hazard rate function (HRF), survival function, moments, quantile function (QF), and moment-generating function (MGF) are among the statistical and mathematical characteristics of the KAPLL distribution that are comprehensively examined. The estimation of KAPLL parameters is achieved through maximum likelihood estimation (MLE), a widely used statistical method. We use the KAPLL distribution to measures of hypertension and carbon dioxide (CO_2) emissions in order to demonstrate the usefulness of the suggested model. The findings demonstrate the KAPLL distribution's adaptability and better fit when compared to other well-known log-logistic model extensions, confirming its promise as a useful tool for modelling complicated behaviours in environmental and biological data.

Keywords: Reliability Analysis, KAPLL Distribution, Lifetime Data, Log-Logistic Distribution

I. Introduction

The assumed probability distribution has a major impact on the processes' quality in a statistical study. Therefore, a lot of work has gone into creating generalized distribution classes and the

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associated statistical techniques. Among other domains, probability distributions are used in insurance and actuarial science, risk analysis, investing, market, economic and business research, reliability-engineering, and demography. Numerous new classes of univariate continuous distributions have been created in the statistical literature by accumulation one or more shape parameters to a baseline model. The extended distributions' versatility and capacity to represent both monotone and non-monotone real-world data have drawn the attention of several statisticians who are creating novel models. Marshall Olkin-G (MO-G) [1], Kumaraswamy Marshal-Olkin-G (KMO-G) [2], Weibull Marshall Olkin-G [3], Marshall-Olkin alpha power-G (MOAP-G) [4], and Kumaraswamy alpha power-G (KAP-G) [5] are a few noteworthy classes. The literature on income distribution also refers to the log-logistic (LL) distribution as the Fisk distribution [6,7, 8, 24]. It also has an extra location parameter, which Arnold [9] added and called the Pareto type III distribution. Additionally, when it comes to stream flow and precipitation data, the LL distribution is a particular instance of Burr XII distribution [10] and Kappa distribution [11]. More information regarding the LL model is available in [12]. Numerous writers have examined various generalised versions of the LL distribution in an effort to increase its functionality and adaptability. Since the LL model provides a hazard rate function that increases early and falls subsequently, it may be thought of as the probability-distribution of a random flexible whose logarithm has a logistic-distribution. It is an another to the log-normal distribution. The extended-LL [13], MO-LL [14], odd Lomax LL distributions [15], alpha power transformed-LL [16], beta-LL [17] , and Zografos Balakrishnan LL [18] are a few enhanced forms of the LL model. In this context, the present study introduces the KAPLL distribution, which integrates the alpha-power and Kumaraswamy transformation frameworks. The proposed model is shown to outperform traditional and extended log-logistic variants in capturing skewness, tail behavior, and diverse hazard patterns across two real-world datasets: CO_2 emissions and hypertension measures.

The present paper explores the KAPLL distribution, a novel extended variation of the LL distribution that may offer greater modelling flexibility for real-world data than existing rival LL models. To achieve the suggested model, the KAp-G family is utilized [5]. The KAPLL distribution has several reasons for being used, including: First, the model can be used to model increasing, J-shape, decreasing, reversed J-shape, bathtub, modified bathtub, and unimodal HRF shapes; Second, the KAPLL distribution can be considered a good model for modelling skewed real-life data that may not be adequately modelled by other known distributions; Third, it can be used in a variety of fields, such as survival analysis, public health, industrial reliability, biomedical studies, reliability, and engineering; and finally the KAPLL distribution performs better than many well-known LL distributions on real-world data examples.

The remainder of the paper is divided into seven sections. Section 2 examines the KAPLL distribution. Some important characteristics of the KAPLL distribution are examined in Section 3. An inference on the KAPLL parameters is provided. within Section 4. Section 5 presents a real-world data application. Several findings are presented in Section 6.

II. Methods

I. The KAPLL Distribution

The following section shows the KAPLL model and some of its particular instances. The two parametric LL framework's cumulative distribution function (CDF) indicates this:

$$G(y) = \left(1 + \frac{\varphi}{y^{\delta}}\right)^{-1}, y > 0, \varphi, \delta > 0 \tag{1}$$

where δ represents the form parameter and φ the scale parameter. The probability density function (PDF) of LL simplifies to:

$$g(y) = \varphi\delta y^{-\delta-1}\left(1+\frac{\varphi}{y^{\delta}}\right)^{-2} \tag{2}$$

The CDF specifies the KAP-G family, upon which the KAPLL distribution is built.

$$F(y) = \begin{cases} 1-\left\{1-\left[\frac{\sigma^{G(y)}-1}{\sigma-1}\right]^{c}\right\}^{d} & if\sigma, c, d > 0, \sigma \neq 1 \\ G(y) & if\sigma = 1 \end{cases} \tag{3}$$

where σ, c and d stand shape parameters. The appropriate KAP-G class PDF is represented by

$$f(y) = \frac{cdln\ \sigma}{\sigma-1} g(y)\sigma^{G(y)}\left[\frac{\sigma^{G(y)}-1}{\sigma-1}\right]^{c-1}\left\{1-\left[\frac{\sigma^{G(y)}-1}{\sigma-1}\right]^{c}\right\}^{d-1} \tag{4}$$

Further data concerning the KAP-G family are described in [17]. The CDF of KAPLL distribution is obtained by inserting 1 into Equation 3 as

$$F(y) = 1-\left[1-\left(\frac{\sigma^{\left(1+\frac{\varphi}{y^{\delta}}\right)^{-1}}-1}{\sigma-1}\right)^{c}\right]^{d} \tag{5}$$

The PDF that goes with Equation 5 corresponds to following:

$$f(y) = \frac{cd\varphi\delta\ln\sigma}{\sigma-1}x^{-\delta-1}\left(1+\frac{\varphi}{y^{\delta}}\right)^{-2}\sigma^{\left(1+\frac{\varphi}{y^{\delta}}\right)^{-1}}\left(\frac{\sigma^{\left(1+\frac{\varphi}{y^{\delta}}\right)^{-1}}-1}{\sigma-1}\right)^{c-1}\times\left[1-\left(\frac{\sigma^{\left(1+\frac{\varphi}{y^{\delta}}\right)^{-1}}-1}{\sigma-1}\right)^{c}\right]^{d-1} \tag{6}$$

The KAPLL distribution's HRF decreases to

$$h(y) = \frac{cd\varphi\delta\ln\sigma}{\sigma-1}x^{-\delta-1}\left(1+\frac{\varphi}{y^{\delta}}\right)^{-2}\sigma^{\left(1+\frac{\varphi}{y^{\delta}}\right)^{-1}}\left(\frac{\sigma^{\left(1+\frac{\varphi}{y^{\delta}}\right)^{-1}}-1}{\sigma-1}\right)\times\left[1-\left(\frac{\sigma^{\left(1+\frac{\varphi}{y^{\delta}}\right)^{-1}}-1}{\sigma-1}\right)^{c}\right]^{-1c-1} \tag{7}$$

III. Characteristics of the KAPLL Distribution

I. The Linear Representation

An effective mixed illustration of the PDF of the KAP-G class was given by Mead et al. [5]. As stated in [5], the KAP-G density decreases to

$$f(y) = cd\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j}(\ln\ \sigma)^{m+1}}{m!\,[c(1+i)-j]^{-m}(\sigma-1)^{c(1+i)}}g(y)(G(y))^{m}\binom{d-1}{i}\binom{c(1+i)-1}{j}$$

Following some algebra and using the LL model's PDF and CDF, the KAPLL density has the following form:

$$f(y) = cd \sum_{i,j,m=0}^{\infty} \frac{(-1)^{i+j}(ln\ \sigma)^{m+1}[c(1+i)-j]^m}{m!\,(\sigma-1)^{c(1+i)}} \varphi\delta y^{-\delta-1}\left(1+\frac{\varphi}{y^{\delta}}\right)^{-2} \times \left(1+\frac{\varphi}{y^{\delta}}\right)^{-m} \binom{d-1}{i}\binom{c(1+i)-1}{j}$$

Additionally, it can be rewritten simply as the following:

$$f(y) = \sum_{m=0}^{\infty} f_m \zeta_{m+1}(y) \tag{8}$$

where $\zeta_{m+1}(y) = (m+1)g(y)(G(y))^m$ is the power parameter for the exponentiated-LL density and $(m+1) > 0$, and

$$f_m = cd \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(ln\ \sigma)^{m+1}[c(1+i)-j]^m}{(m+1)!\,(\sigma-1)^{c(1+i)}} \binom{d-1}{i}\binom{c(1+i)-1}{j}$$

II. Quantile Function

Equation 5 can be inverted to obtain the quantile function of the KAPLL distribution as

$$Q(v) = \left[\frac{-1}{\varphi}\left(1-\frac{ln\ \sigma}{ln\ (1+\xi)}\right)\right]^{\frac{-1}{\delta}} \tag{9}$$

where $\xi = \left\{(\sigma-1)\left[1-(1-v)^{\frac{1}{d}}\right]^{\frac{1}{c}}\right\}$, v is distributed uniformly (0,1).

III. Moment

Equation 8 can be used to determine kth moment of Y as

$$\omega_k' = E(Y^k) = \sum_{m=0}^{\infty} f_m \int y^r \zeta_{m+1}(y)dy$$

Hence,

$$\sum_{m=0}^{\infty} f_m \varphi\delta \int_0^{\infty} y^{k-\delta-1}\left(1+\frac{\varphi}{y^{\delta}}\right)^{-m-2} dx$$

subsequently calculating the combination, you get ω_k' as follows

$$\omega_k' = \sum_{m=0}^{\infty} \frac{f_m \varphi^{\frac{k}{\delta}} \Gamma\left(1-\frac{k}{\delta}\right)\Gamma\left(\frac{k}{\delta}+m+1\right)}{\Gamma(m+2)}, \frac{k}{\delta} < 1 \tag{10}$$

IV. Order test

Let $y_1, y_2, \dots \dots y_n$ stand a random illustration of size n and let $y_{1:n}, \dots, y_{n:n}$ represent this connected order indicator. Then, $y_{i:n}$, which represents the PDF of the i[th] order statistics, say $f_{y_{i:n}}(y)$ can be found using:

$$f_{y_{j:n}}(y) = \frac{n! f(y)}{(n-j)!(j-1)!}[F(y)]^{j-1}[1-F(y)]^{n-j}. \quad (11)$$

Equation 11's i th order statistic of the KAPLL distribution decreases if $\sigma \neq 1$ when Equations 5 and 6 are substituted.

$$f_{y_{j:n}}(y) = \frac{cd\varphi\delta n! \ln\ \sigma(\sigma-1)}{(n-j)!\,(j-1)!} x^{-\delta-1} d_j^2 \sigma^{d_j} \eta_j^{c-1}\left[1-\eta_j^c\right]^{d(n-j+1)-1},$$

$$\text{where } d_j = \left(1+\frac{\varphi}{y_j^\delta}\right)^{-1} \text{ and } \eta_j = \left(\sigma^{d_j}-1\right)/(\sigma-1)$$

$$f_{y_{j:n}}(y) = \sum_{p=0}^{j-1}\sum_{j,m=0}^{\infty} \frac{cd(\ln\ \sigma)^{z+1}(-1)^{p+j+m}[c(j+1)-m]^z}{\delta_{(i,n-i+1)} z!\,(\sigma-1)^{c(j+1)}} \varphi\delta y^{-\delta-1} d_i^{z+2} \times \binom{i-1}{p}\binom{d(p+n-i+1)-1}{j}\binom{c(j+1)-1}{m}$$

Basically, in the formula

$$f_{j:n}(y) = \sum_{z=0}^{\infty} d_z h_{z+1}(y)$$

where $h_{z+1}(y) = (z+1)g(y)[G(y)]^z$ is the power parameter for the exponentiated-LL density $(z+1) > 0$, and

$$d_z = \sum_{p=0}^{i-1}\sum_{j,m=0}^{\infty} \frac{cd(ln\ \sigma)^{z+1}(-1)^{p+j+m}[c(j+1)-m]^z}{\delta_{(i,n-i+1)}(z+1)!\,(\sigma-1)^{c(j+1)}}\binom{i-1}{p} \times \binom{d(n+p-i+1)-1}{j}\binom{c(j+1)-1}{m}$$

IV. Assessment of the Parameters

Assuming $y_1, y_2, \ldots, y_n$ are random samples drawn from the KAPLL distribution, the likelihood function's logarithm (ℓf)becomes

$$\ell f = n[ln\ c + ln\ d + ln\ \varphi + ln\ \delta] + nln\left(\frac{ln\ \sigma}{\sigma-1}\right) + (\delta-1)\sum_{j=1}^{n} ln\left(y_j\right) - \varphi\sum_{j=1}^{n} y_j^\delta + ln\left(\sigma\right)\sum_{j=1}^{n} d_j + (\sigma-1)\sum_{j=1}^{n} ln\left(\eta_j\right) + (d-1)\sum_{j=1}^{n} ln\left(1-\eta_j^c\right)$$

To gain the MLE of c, d, σ, φ and δ, the initial derivatives of ℓf are attained through respect to c, d, σ, φ and δ.

I. Akaike Information Criteria

In order to assess the relative accuracy of statistical approaches for a particular set of data, the Akaike information criteria (AIC), a measurement of the out-of-sample error in forecasting, are used. AIC calculates the proportionate quantity of data that a given model loses. Remember that the more accurate a model is, the more information it retains. Both over-fitting and under-fitting risks are taken into consideration by AIC when calculating the amount of information the system loses. To evaluate the parametric structure with the AIC formula that was used is

$$AIC = -2LogL + 2q$$

where AIC principles that are lesser suggest better models for information adaption, LogL is the log-likelihood, and q is the number of parameters in the model.

II. Kolmogorov-Simron Test

To predict product reliability, reliability analysis uses the Kolmogorov-Smirnov (K-S) assessment to see if a sample fits into a specific distribution. The smaller KS Statistic indicates that the real information is closer to fitted-distribution.

$$D = maximum \ | Fn(y) - F(y) |$$

The cumulative distribution function of the suggested model is denoted by $F(y)$, while the empirical distribution function is denoted by $Fn\ (y)$.

V. Results and Discussion

I. Real Life Applications

Application to two datasets of the KAPLL distribution is demonstrated in this section. The KAPLL distributions' fits will be contrasted with those of a few competing distributions, including the log-logistic (LL), the added Weibull log logistic (AWLL) by [23], the Kumaraswamy Marshall Olkin log-logistic (KMOLL) by [21], the McDonald log-logistic (McLL) by [22], and the alpha power transformed log-logistic (APTLL) by [16]. We take into consideration a few metrics, including the p value (pv), KS, and AIC, in order to compare the fitted distributions. In this part, the numerical findings are obtained using the R program.

II. Carbon emissions Data

The first application of KAPLL distribution, alongside competing models and their submodels, is demonstrated using carbon emission data for India spanning the period from 1990 to 2023. This dataset, obtained from our world data, comprises annual carbon dioxide (CO_2) emissions (in metric tons per capita or total national emissions) and reflects the evolving environmental footprint of India over more than three decades.

Table 2 reports the estimated parameters and key goodness-of-fit information for several statistical models applied to the CO_2 emissions data set. The competing models include the LL, Kumaraswamy Modified Odd Log-Logistic, McLL, Alpha-Weibull Log-Logistic, and the newly proposed Kumaraswamy Alpha Power Log-Logistic distribution. Among these models, the KAPLL distribution exhibits the most favorable performance across all fit criteria. It achieves the maximum log-likelihood value and the minimum AIC, indicating strong model parsimony and an improved ability to capture the underlying distributional structure of the CO_2 emissions data. Furthermore, the KS statistic for the KAPLL model is the lowest among all competitors (KS = 0.07229), and the corresponding p-value (0.96578) is well above conventional significance thresholds, suggesting that the model does not significantly deviate from the empirical distribution.

Table 2: *MLEs and Goodness-of-Fit Statistics for CO_2 Emission Data*

Parameter/Statistic	LL	KAPLL	KMOLL	McLL	AWLL
α	-	14.3050	1.6880	6.5417	8.470
β	3.56431	9.3353	1.8637	56.591	-
λ	2367.782	25.4088	1.3077	1.9573	45.546
a	-	0.1183	3.2901	42.4707	0.1448
b	-	22.1002	0.9319	24.1273	5.4822
c	-	-	-	-	854.599
d	-	-	-	-	3.0543
AIC	172.678	-1681.976	-238.553	46.678	52.334
KS Statistic	0.1582	0.07229	0.1765	0.1291	0.1301
KS p-value	0.4643	0.96578	0.8674	0.5832	0.7649

In contrast, the alternative models particularly the LL, McLL, and AWLL distributions display higher AIC values and less favorable KS statistics, indicating suboptimal performance in representing the variability and distributional behavior of CO_2 emissions. These results underscore the flexibility and robustness of the KAPLL distribution in modeling environmental data. Therefore, the KAPLL distribution is recommended as the most appropriate model for the given CO_2 emissions dataset, providing more accurate and reliable statistical inference for environmental and sustainability analyses.

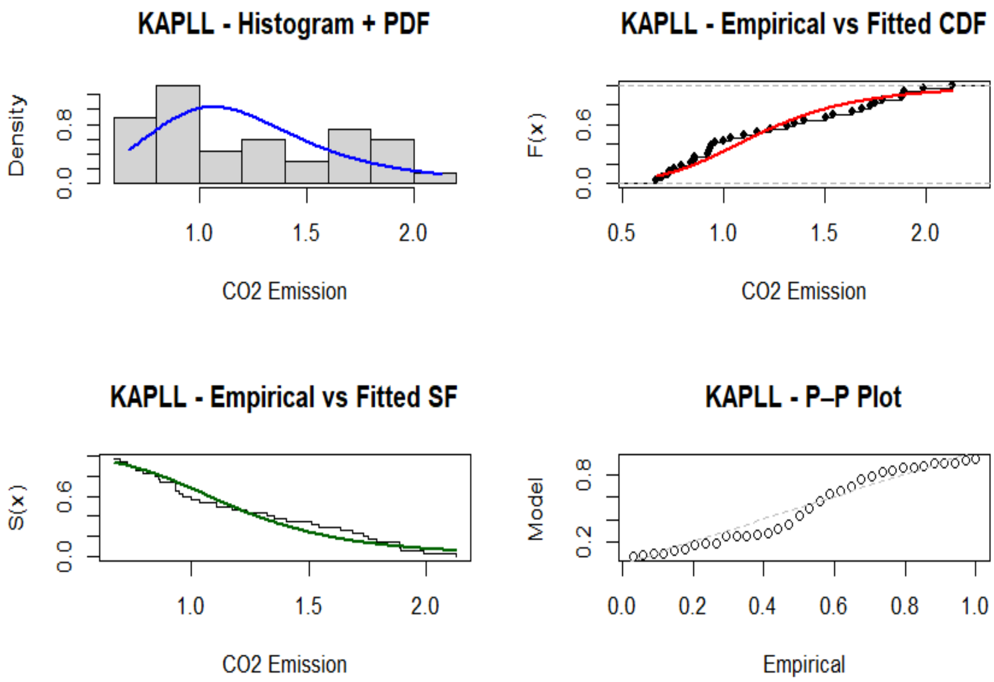

Figure 1: *The fitted functions of the KAPLL model for CO2 emissions data.*

Figure 1 shows diagnostic plots for the KAPLL distribution fitted to CO_2 emission data, including the histogram with PDF, empirical vs. fitted CDF, survival function, and P–P plot. The plots indicate that the KAPLL model fits information well, with close alignment in the density, CDF, and survival curves. The P–P plot also shows minimal deviation, confirming the model's adequacy.

III. Hypertension Data

This dataset contains survival times (in years) before the onset of hypertension in India, covering the period from 1990 to 2021. Sourced from Our World in Data, it provides valuable insights into the progression of hypertension within the Indian population.

Table 3 reports the maximum likelihood estimations (MLEs), standard errors (SEs), and goodness-of-fit statistics for the hypertension data using five different models: LL, KAPLL, KMOLL, McLL, and AWLL. The results show that the KAPLL model offers the best fit, as indicated by its lowest AIC (-1505.1303) and highest KS p-value (0.99874), along with a low KS statistic (0.8739). These values suggest that the KAPLL model closely aligns with the observed data. In comparison, the other models demonstrate higher AIC values and less favorable fit statistics, confirming the superior performance of the KAPLL distribution for this dataset.

Table 3: *MLEs and Goodness-of-Fit Statistics for Hypertension Data*

Parameter/Statistic	LL	KAPLL	KMOLL	McLL	AWLL
α	-	74.1409	1.5269	3.8746	1.984
β	0.8793	2.4510	0.9939	5.6352	-
λ	7654.746	0.0264	0.0097	0.8638	57.683
a	-	24.0998	1.8250	65.29673	0.0136
b	-	74.1846	0.9698	0.2487	6.9367
c	-	-	-	-	7756.60
d	-	-	-	-	4.0678
AIC	186.7931	-1505.130	–208.726	199.3959	176.736
KS Statistic	0.1436	0.8739	1778.3907	0.4852	0.1357
KS p-value	0.567	0.998	0.867	0.663	0.7708

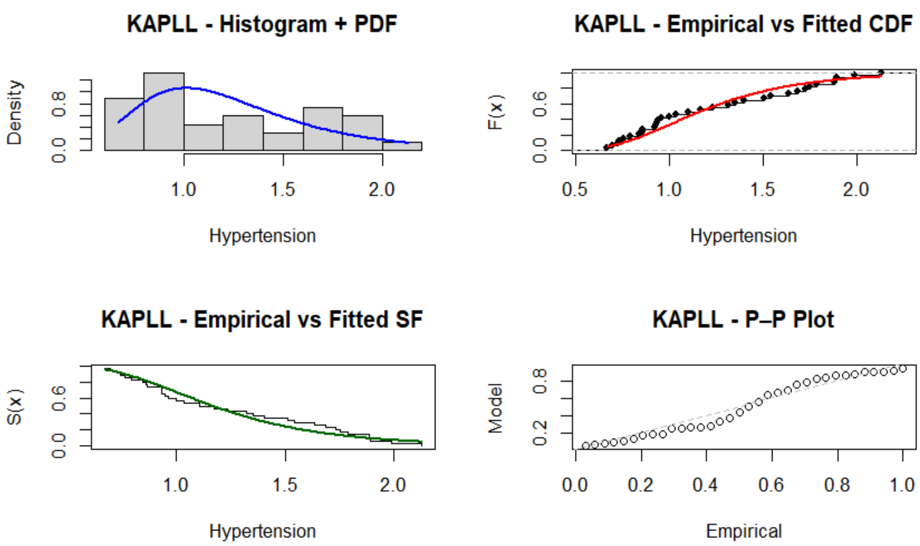

Figure 2: *The diagnostic plots of the KAPLL model fitted to the hypertension data.*

Figure 2 demonstrates four diagnostic plots for the KAPLL distribution fitted to the hypertension data: the histogram with the fitted PDF, the empirical versus fitted CDF, the survival function, and the P–P plot. These plots confirm that the KAPLL model fits the hypertension statistics well. The PDF closely follows the histogram, the fitted CDF aligns with empirical-distribution, and the survival function accurately reflects the observed survival trend. The P–P plot shows points clustering near the 45-degree line, indicating a strong agreement between the model and the empirical data.

VI. Conclusion

This paper introduced the KAPLL distribution, a new five-parameter model developed to enhance flexibility in modelling complex data behaviors. We derived its key mathematical properties, as well as the probability density function, survival, moments, hazard rate functions, and quantile function. Parameters were efficiently estimated using the MLE method, and their performance was confirmed through simulation. To determine the practical value of the proposed model, we utilized it to two distinct real-world datasets: carbon dioxide emissions and hypertension measurements. These datasets, representing environmental and biomedical domains respectively, exhibit different distributional characteristics, such as skewness and non-monotonic hazard behavior. Comparative analyses using goodness-of-fit criteria revealed that the KAPLL model provided a superior fit over several competing log-logistic-based distributions, including APTLL, LL, KMOLL, McLL, and AWLL. These findings highlight the adaptability and robustness of the KAPLL distribution in handling heterogeneous data types. As such, the model presents a valuable tool for researchers and practitioners working in fields such as environmental statistics, public health, and reliability analysis. We expect that the suggested distribution will be used in a wider range of domains, such as reliability analysis, medicine, engineering, and economics [25,26,27], among others.

References

[1] Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3): 641-652.

[2] Alizadeh, M. Tahir, M. H. Cordeiro, G. M. Mansoor, M. Zubair, M. and Hamedani, G. (2015). The Kumaraswamy marshal-Olkin family of distributions. *Journal of the Egyptian Mathematical Society*, 23(3): 546-557.

[3] Korkmaz, M. Ç. Cordeiro, G. M. Yousof, H. M. Pescim, R. R. Afify, A. Z. and Nadarajah, S. (2019). The Weibull Marshall–Olkin family: Regression model and application to censored data. *Communications in Statistics-Theory and Methods*, 48(16): 4171-4194.

[4] Nassar, M. Kumar, D. Dey, S. Cordeiro, G. M. and Afify, A. Z. (2019). The Marshall–Olkin alpha power family of distributions with applications. *Journal of Computational and Applied Mathematics,* 351: 41-53.

[5] Mead, M. E. Afify, A. Z. and Butt, N. S. (2020). The Modified Kumaraswamy Weibull Distribution: Properties and Applications in Reliability and Engineering Sciences. *Pakistan Journal of Statistics and Operation Research*, 16(3).

[6] Dagum, C. (1975). A model of income distribution and the conditions of existence of moments of finite order. *Bulletin of the International Statistical Institute,* 46(1): 199-205.

[7] Fisk, P. R. (1961). The graduation of income distributions. *Econometrica: journal of the Econometric Society,* 171-185.

[8] Shoukri, M. M. Mian, I. U. H. and Tracy, D. S. (1988). Sampling properties of estimators of the log-logistic distribution with application to Canadian precipitation data. *Canadian Journal of Statistics,* 16(3): 223-236.

[9] Arnold, B. C. (2008). Pareto and generalized Pareto distributions. *In Modeling income*

distributions and Lorenz curves (pp. 119-145). New York, NY: Springer New York.

[10] Burr, I. W. (1942). Cumulative frequency functions. *The Annals of mathematical statistics*, 13(2): 215-232.

[11] Mielke JR, P. W. and Johnson, E. S. (1973). Three-parameter kappa distribution maximum likelihood estimates and likelihood ratio tests. *Monthly Weather Review*, 101(9): 701-707.

[12] Kleiber, C. and Kotz, S. (2003). Statistical size distributions in economics and actuarial sciences. John Wiley & Sons.

[13] Alfaer, N. M. Gemeay, A. M. Aljohani, H. M. and Afify, A. Z. (2021). The extended log-logistic distribution: inference and actuarial applications. *Mathematics,* 9(12): 1386.

[14] Gui, W. (2013). Marshall-Olkin extended log-logistic distribution and its application in minification processes. *Applied Mathematical Sciences,* 7(77-80): 3947-3961.

[15] Cordeiro, G. M. Afify, A. Z. Ortega, E. M. Suzuki, A. K. and Mead, M. E. (2019). The odd Lomax generator of distributions: Properties, estimation and applications. *Journal of Computational and Applied Mathematics,* 347: 222-237.

[16] Aldahlan, M. A. (2020). Alpha Power Transformed Log-Logistic Distribution with Application to Breaking Stress Data. *Advances in Mathematical Physics, 2020*(1): 2193787.

[17] Lemonte, A. J. (2014). The beta log-logistic distribution. *Brazilian journal of probability and statistics*, 313-332.

[18] Ramos, M. W. A. Cordeiro, G. M. Marinho, P. R. D. Dias, C. R. B. and Hamedani, G. G. (2013). The Zografos-Balakrishnan log-logistic distribution: Properties and applications. *Journal of Statistical Theory and Applications*, *12*(3): 225-244.

[19] De Santana, T. V. F. Ortega, E. M. Cordeiro, G. M. and Silva, G. O. (2012). The Kumaraswamy-log-logistic distribution. *Journal of Statistical Theory and Applications, 11*(3): 265-291.

[20] Rosaiah, K. Kantam, R. R. L. and Kumar, S. (2006). Reliability test plans for exponentiated log-logistic distribution.

[21] Cakmakyapan, S. Ozel, G. Gebaly, Y. M. H. E. and Hamedani, G. G. (2018). The Kumaraswamy Marshall-Olkin log-logistic distribution with application. *Journal of Statistical Theory and Applications*, *17*(1): 59-76.

[22] Tahir, M. H. Mansoor, M. Zubair, M. and Hamedani, G. G. (2014). McDonald log-logistic distribution with an application to breast cancer data. *Journal of Statistical Theory and Applications*, *13*(1): 65-82.

[23] Hemeda, S. (2018). Additive Weibull log logistic distribution: Properties and application. *Journal of Advanced Research in Applied Mathematics and Statistics*, *3*(4): 8-15.

[24] Alabdulhadi, M. H. and Elgarhy, M. (2024). Different Estimation Methods for A New Extension Three Parameter Log-Logistic Distribution with Applications. *Journal of Statistics Applications and Probability,* 13(4), 1181-1202.

[25] Duraisamy, P. Alam, M. S. Uddin, M. A. Kuppusamy, A. and Palanisamy, M. (2022). Modelling Volatility of Petroleum Productions by Using Markov-Switching GARCH Models. *International Journal of Agricultural and Statistical Sciences,* 18(1): 1657-1667.

[26] Kavitha, S. Rayalu, G. M. Pachiyappan, D. and Manigandan, P. (2023). Forecasting of Extreme Risk Using Markovswitching GARCH Models: Evidence From Global Energy Markets. Reliability: Theory and Applications, 18(4 (76)): 325-337.

[27] Alam, M. S. Manigandan, P. Kisswani, K. M. and Baig, I. A. (2025). Achieving goals of the 2030 sustainable development agenda through renewable energy utilization: Comparing the environmental sustainability effects of economic growth and financial development. *Sustainable Futures*, *9*: 100534.