DOUBLE ORBIT RETRIAL QUEUEING-INVENTORY SYSTEM WITH ORBITAL SEARCH UNDER (s, S) POLICY

M. THILAKAVATHY [1], AGASSI MELIKOV [2,3,*] G. AYYAPPAN [4]

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[1,4] Department of Mathematics, Puducherry Technological University, Puducherry, India
[2] Department of Mathematics, Baku Engineering University, Khirdalan, Azerbaijan
[3]Institute of control systems, Baku, Azerbaijan
[1]thilakavathy5184@ptuniv.edu.in [2,3]amelikov@beu.edu.az
[4]ayyappan@ptuniv.edu.in
[*] Corresponding author: amelikov@beu.edu.az

Abstract

We consider two-orbit retrial queueing inventory system with a single server. Service time follows phase-type distribution, while arrivals follow the Markovian arrival process. During busy period of the server, the arriving customer may either balk or move to the limitless orbit. Following service completion, the server takes a vacation if there are no customers in the system. When a breakdown happens during busy times, the server undergoes phase-type repair, and the current customer enters finite orbit. Once the repair is finished, the server either stays idle or searches the finite orbit. The replenishment period is exponential and we use (s, S) policy. The matrix analytic approach has been used to analyze steady state probability. We assess this retrial queueing inventory model both numerically and graphically.

Keywords: Markovian arrival process, phase-type service, phase-type repair, Breakdown, Two orbits, Retrial customers, Orbital search, (s, S) policy
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1. Introduction

The retrial queueing theory is commonly used to explain how real-world systems work. When a customer arrives during the busy period of the server, then the customer may join the queue to get the service. This virtual waiting area for customers is orbit. At the moment of retrial when the server is free, they begin to get the service immediately; if not, they return to the orbit and go through the entire procedure again. In this case, it is also possible for the new arrival to get service directly without joining the orbit, provided that the server is available. Falin et al. [11] released the first literature on retrial queues and it includes a full description of basic methodologies and results on retrial queues. A regenerative approach algorithm was used by Artalejo [3] to determine the steady state probability for the $M/G/l$ retrial queue with breakdowns. Using the matrix analytic method, Ayyappan and Gowthami [5] investigated the effect of retrial rate against the expected number of customers in the system while analyzing the retrial queueing model with a single server.

Retrial queueing model with re-service was researched by Melikov et al [15]. This model assumes that the distribution service time is exponential and analyzes the input flow of calls using a Markovian Arrival Process (MAP). Melikov et al. [16] measured performance metrics for the average number of retrial customers in the orbit, the mean values of both failed and busy servers, and the loss probabilities of both primary and retrial customers. Dudin et al. [4] analyzed

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a single-server retrial queueing model with two types of customer, exponential service times, arrivals governed by a Marked Markovian process, and a joint distribution for the number of customers in the orbits.

Abdul Reiyas and Jeganathan [1] examined a classical retrial queueing inventory system with two component demand rate. Singla and Kaur [20] developed a method for solving difference-differential equations to get the transient state probability for the precise number of arrivals and departures while the server is busy or idle. We experience many situations in our daily lives where customer service is disrupted by server failure. For instance, the server can break due to internal or external issues in manufacturing systems, computers, or electronic systems; in packet transferring of communication channels; or even in the service process of everything in one's life. Customers whose service is interrupted must wait for service to come back once the server's repair is finished. The server may be disrupted during a customer's service by a vital interruption (emergency circumstance) under the Poisson process. Nithya et al. [17] state that due to breakdown, the current customer join in the queue and waits to be served.

Using inventory management and queuing systems together has many benefits for improving customer satisfaction and operational efficiency. With a comprehensive evaluation and management of these systems, firms may better balance maintaining appropriate stock levels with reducing waiting times. For M/M/1 Queueing-Inventory System(QIS) with lost sales, under various inventory management policies, including (r,Q) and (r,S) policies, Schwarz et al. [19] discovered steady state distributions of inventory processes and combined queue length in a particular product form. Ayyappan and Arulmozhi [6] examined the retrial QIS using a constant retrial rate, orbital client collision, working vacation, flush out, balking, breakdown, and repair. Ayyappan and Meena [7] looked at a retry QIS with a phase-type vacation and an inconsistent server.

Two separate types of customers arrive in two different Poisson streams when they are sent to a single server system. The service station can only handle one client at a time and if blockage occurs, the two types of consumers are sent to a different type. Two orbit queues with infinite capacity were assumed and the theory of Riemann (–Hilbert) boundary value problems by Dimitriou [10] can be used to find how the joint stationary orbit queue length distribution generates probabilities. Sangha and Jain [18] have investigated the best course of action for a double-orbit retrial queue with erratic service and vacation interruption . Retrial QIS with an isolated server and feedback for consumers in which the orbit size is finite was covered by Amirthakodi and Sivakumar [2].

The idea of orbital search has been presented in order to make effective use of the server's idle time in a retry queueing system. This idea states that the server will seek for customers in the orbit during idle time rather than waiting for a direct or repeat customer to arrive. Falin [11] has researched the Markovian model that includes dissatisfied customers on a single server. The retrial queueing system and the orbit's search for priority customers have been studied by Krishnamoorthy and Joshua [13]. According to Dhanya et al. [8], the server either resumes orbital search for the interrupted clients in the finite orbit or stays idle once each service and repair is completed. According to Sivakumar [21], retrial demands compete for their exponentially distributed necessity by sending out signals. The steady state example produces the combined likelihood distribution of the number of orbital needs and the inventory level.

The remaining parts of the paper are organized as follows: Our model is described in section 2. Matrix creation and several notations are explained in section 3. The invariant probability vector, the rate matrix R and the stability condition are all obtained in section 4. Section 5 describes the examination of the busy period. Section 6 lists the performance metrics. Sectionn 7 presents cost analysis. There are some graphical and numerical results in section 8. Section 9 presents the conclusion.

2. Mathematical model

We examine a double-orbit, unreliable single server retrial QIS. The model's fundamental function can be characterized as follows:

- **Arrival process:** Primary customers with parameter matrices of order m (D_0, D_1) come in accordance with MAP. If the server is available when a customer arrives, he will receive service directly; if the server is active, the customer has the chance of either joining an infinite orbit with probability (1-b) or may balk the system with probability b.
- **Service Process:** The first-come, first-served approach applies to each customer. The service period is determined by the PH distribution, which is represented as (γ, U) of order n, such that $U^0 + Ue = 0$ and the rate is $\mu = [\gamma(-U)^{-1}e_n]$.
- **Breakdown state:** During busy times, the server is prone to breakdowns. Breakdown time follows an exponential distribution with parameter ψ. Interruptions can happen often, and it is predicted that a customer's service would repeat in the same way. When a server breaks down, the customer goes to a finite orbit N and the server undergoes repair.
- **Retrial process:** Retrial of the customer is successful when the server is free with positive inventory. An exponential distribution governs the interval between retrials, with a rate of σ_2 for a finite orbit and σ_1 for an unlimited orbit.
- **Vacation state:** After service is finished, the server takes a break if there are no clients within the system. When the vacation is over, the server remains in an idle state. The rate of vacation is exponential distribution and is denoted as η.
- **Repair state:** During the busy period, the server may breakdown, the repair process starts immediately. It is represented as a phase-type distribution with the notation (α, T) of order l such that $T^0 + Te = 0$ and the rate is $\tau = [\alpha(-T)^{-1}e_l]$.
- **Orbital search** Once the repair is finished, the server has the option of staying idle with probability q or searching for an orbital customer from the finite orbit with probability p. The search time is negligible.

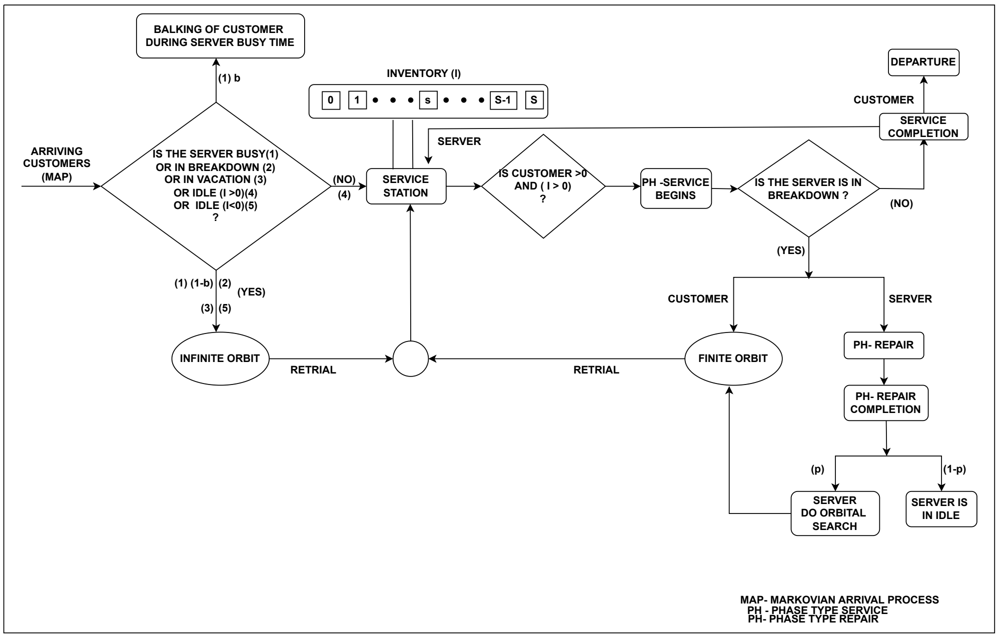

Figure 1: *Schematic representation*

- **Replenishment:** When a customer finishes service, one item is removed from stock. Under the (s, S) policy, an order of S items is placed when inventory drops to level s. The supply lead time follows an exponential distribution with rate β.

The lead time, customer arrival time, retrial time and service times (for both primary and retrial customers) are all considered to be mutually independent. Refer to figure 1 for a schematic representation of the model.

3. Generation Matrix

Abbreviation and Notations

- CTMC- Continuous Time Markov Chain
- QBD process - Quasi-Birth-Death Process
- MAM - Matrix Analytic Method
- e - entries in a column's vector are one;
- I_n - Identity matrix of order n
- 0 - Matrix whose entries are 0, of appropriate size
- $\otimes$ - A block matrix is created by the Kronecker product of two matrices of different dimensions;
- $\oplus$ - The Kronecker addition creates a block matrix from two matrices of varying sizes;
- $N_1(t)$ indicates the number of customers in an infinite orbit;
- $N_2(t)$ indicates the number of customers in the finite orbit;
- $J(t)$ represents the server status;
- $I(t)$ represents the stock level for commodity;
- $S(t)$ represents the service phases;
- $R(t)$ represents the repair phases;
- A(t) represents the arrival phases;
  at time t, where

$J(t)$ represents the status of the server, defined as

$$J_(t) = \begin{cases} 0, & \text{server is in vacation,} \\ 1, & \text{server is in idle,} \\ 2, & \text{server is in busy ,} \\ 3, & \text{server is in repair.} \end{cases}$$

It is obvious that $\{N_1(t), N_2(t), J(t), I(t), S(t), R(t), \text{A(t)} : t \geq 0\}$ is a CTMC with the state space is as follows:

$$\omega = \bigcup_{c_1=0}^{\infty} \chi(c_1),$$

$$\begin{aligned}\chi(c_1) = &\{(c_1, 0, c_3, c_4, c_7) : 0 \le c_3 \le N, 0 \le c_4 \le S, 1 \le c_7 \le m\}\\ &\cup \{(c_1, 1, c_3, c_4, c_7) : 0 \le c_3 \le N, 0 \le c_4 \le S, 1 \le c_7 \le m\}\\ &\cup \{(c_1, 2, c_3, c_4, c_5, c_7) : 0 \le c_3 \le N, 1 \le c_4 \le S, 1 \le c_5 \le n, 1 \le c_7 \le m\}\\ &\cup \{(c_1, 3, c_3, c_4, c_6, c_7) : 1 \le c_3 \le N, 0 \le c_4 \le S, 1 \le c_5 \le l, 1 \le c_7 \le m\}.\end{aligned}$$

The QBD procedure using the (s, S) policy generates an infinitesimal matrix, as provided by

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & 0 & \dots \\ A_2 & A_1 & A_0 & 0 & 0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The following defines each entry in Q's block matrices:,

$$B_0 = \begin{bmatrix} B_0^{11} & B_0^{12} & 0 & 0 \\ 0 & B_0^{22} & B_0^{23} & 0 \\ B_0^{31} & B_0^{32} & B_0^{33} & B_0^{34} \\ 0 & B_0^{42} & B_0^{43} & B_0^{44} \end{bmatrix}, \qquad B_0^{11} = I_{N+1} \otimes E_{001},$$

$$E_{001} = \begin{bmatrix} V_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_2 \\ \mathbf{0} & V_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_2 \\ \mathbf{0} & \mathbf{0} & V_1 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_1 & \mathbf{0} & \dots & \mathbf{0} & V_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_3 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_3 \end{bmatrix},$$

where, $V_1 = D_0 - (\eta + \beta)I_m, \quad V_2 = \beta I_m, \quad V_3 = D_0 - \eta I_m,$

$$B_0^{12} = I_{N+1} \otimes I_{(S+1)}\eta I_m, \qquad B_0^{22} = \begin{bmatrix} E_{002} & 0 \\ 0 & I_N \otimes E_{003} \end{bmatrix},$$

$$E_{002} = \begin{bmatrix} V_4 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_5 \\ \mathbf{0} & V_4 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_5 \\ \mathbf{0} & \mathbf{0} & V_4 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_4 & \mathbf{0} & \dots & \mathbf{0} & V_5 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_6 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_6 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_6 \end{bmatrix},$$

where, $V_4 = D_0 - \beta I_m, \quad V_5 = \beta I_m, \quad V_6 = D_0,$

$$E_{003} = \begin{bmatrix} V_7 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_8 \\ \mathbf{0} & V_9 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_8 \\ \mathbf{0} & \mathbf{0} & V_9 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_8 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_9 & \mathbf{0} & \dots & \mathbf{0} & V_8 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{10} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{10} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{10} \end{bmatrix},$$

$$V_7 = D_0 - \beta I_m, \quad V_8 = \beta I_m, \quad V_9 = D_0 - (\sigma_2 + \beta) I_m, \quad V_{10} = D_0 - \sigma_2 I_m,$$

$$B_0^{23} = \begin{bmatrix} E_{004} & 0 & \dots & 0 & 0 & 0 \\ E_{005} & E_{004} & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & E_{005} & E_{004} & 0 \\ 0 & 0 & \dots & 0 & E_{004} & 0 \end{bmatrix},$$

$$E_{004} = \begin{bmatrix} 0 \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix}, \quad E_{005} = \begin{bmatrix} 0 \\ I_S \otimes \gamma \otimes \sigma_2 I_m \end{bmatrix}, \quad B_0^{31} = \begin{bmatrix} E_{006} & 0 \\ 0 & 0 \end{bmatrix},$$

$$E_{006} = \begin{bmatrix} I_S \otimes u^0 \otimes I_m & 0 \end{bmatrix}, \quad B_0^{32} = \begin{bmatrix} 0 & 0 \\ 0 & I_N \otimes E_{006} \end{bmatrix}, \quad B_0^{33} = I_{N+1} \otimes E_{007},$$

$$E_{007} = \begin{bmatrix} V_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{12} \\ \mathbf{0} & V_{11} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{12} \\ \mathbf{0} & \mathbf{0} & V_{11} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_{11} & \mathbf{0} & \dots & \mathbf{0} & V_{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{13} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{13} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{13} \end{bmatrix},$$

where, $V_{11} = U \oplus D_0 + bD_1 - (\psi + \beta) I_{nm}$, $V_{12} = \beta I_{nm}$, $V_{13} = U \oplus D_0 + bD_1 - \psi I_{nm}$,

$$B_0^{34} = \begin{bmatrix} I_N \otimes E_{008} & 0 \\ 0 & E_{008} \end{bmatrix}, \quad E_{008} = 0 \quad I_S \otimes I_l \otimes \psi I_m, \quad B_0^{42} = \begin{bmatrix} 0 & I_N \otimes E_{009} \end{bmatrix},$$

$$E_{009} = \begin{bmatrix} T^0 \otimes I_m & 0 \\ 0 & I_S \otimes qT^0 \otimes I_m \end{bmatrix}, \quad B_0^{43} = \begin{bmatrix} I_N \otimes E_{010} & 0 \end{bmatrix}, \quad E_{010} = \begin{bmatrix} 0 \\ I_S \otimes pT^0 \gamma \otimes I_m \end{bmatrix},$$

$$B_0^{44} = \begin{bmatrix} I_N \otimes E_{011} \end{bmatrix}, \quad E_{011} = \begin{bmatrix} V_{14} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{15} \\ \mathbf{0} & V_{14} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{15} \\ \mathbf{0} & \mathbf{0} & V_{14} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{15} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_{14} & \mathbf{0} & \dots & \mathbf{0} & V_{15} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{16} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{16} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{16} \end{bmatrix},$$

where, $V_{14} = I_{s+1} \otimes T \oplus D_0 - \beta I_{lm}$, $V_{15} = \beta I_{lm}$, $V_{16} = T \oplus D_0$,

$$A_0 = \begin{bmatrix} A_0^{11} & 0 & 0 & 0 \\ 0 & A_0^{22} & 0 & 0 \\ 0 & 0 & A_0^{33} & 0 \\ 0 & 0 & 0 & A_0^{44} \end{bmatrix},$$

$$A_0^{11} = I_{N+1} \otimes I_{S+1} \otimes D_1, \quad A_0^{22} = I_{N+1} \otimes E_{012}, \quad E_{012} = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_0^{33} = I_{N+1} \otimes I_S \otimes I_n \otimes (1-b) D_1, \quad A_0^{44} = I_N \otimes I_{S+1} \otimes I_l \otimes D_1,$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_2^{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2^{23} = \begin{bmatrix} 0 & 0 \\ 0 & I_N \otimes E_{013} \end{bmatrix}, \quad E_{013} = \begin{bmatrix} 0 \\ I_S \otimes \gamma \otimes \sigma_1 I_m \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} & 0 & 0 \\ 0 & A_1^{22} & A_1^{23} & 0 \\ 0 & A_1^{32} & A_0^{33} & A_1^{34} \\ 0 & A_1^{42} & A_1^{43} & A_1^{44} \end{bmatrix}, \quad A_1^{11} = \begin{bmatrix} I_{N+1} \otimes E_{014} \end{bmatrix},$$

$$E_{014} = \begin{bmatrix} V_{17} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{18} \\ \mathbf{0} & V_{17} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{18} \\ \mathbf{0} & \mathbf{0} & V_{17} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & G18 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_{17} & \mathbf{0} & \dots & \mathbf{0} & V_{18} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{19} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{19} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{19} \end{bmatrix},$$

where, $V_{17} = D_0 - (\eta + \beta)I_m, \quad V_{18} = \beta I_m, \quad V_{19} = D_0 - \eta I_m,$

$A_1^{12} = \begin{bmatrix} I_{N+1} \otimes I_{S+1} \otimes \eta I_m \end{bmatrix}, \quad A_1^{22} = \begin{bmatrix} E_{015} & 0 \\ 0 & I_N \otimes E_{016} \end{bmatrix},$

$$E_{015} = \begin{bmatrix} V_{20} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{21} \\ \mathbf{0} & V_{20} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{21} \\ \mathbf{0} & \mathbf{0} & V_{20} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{21} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_{20} & \mathbf{0} & \dots & \mathbf{0} & V_{21} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{22} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{22} \end{bmatrix},$$

where, $V_{20} = D_0 - \beta I_m, \quad V_{21} = \beta I_m, \quad V_{22} = D_0,$

$$E_{016} = \begin{bmatrix} V_{23} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{24} \\ \mathbf{0} & V_{25} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{24} \\ \mathbf{0} & \mathbf{0} & V_{25} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{24} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_{25} & \mathbf{0} & \dots & \mathbf{0} & V_{24} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{26} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{26} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{26} \end{bmatrix},$$

where, $V_{23} = D_0 - \beta I_m, \quad V_{24} = \beta I_m, \quad V_{25} = D_0 - (\sigma_1 + \sigma_2 + \beta)I_m,$
$V_{26} = D_0 - (\sigma_1 + \sigma_2)I_m, \quad A_1^{23} = B_0^{23}, \quad A_1^{32} = \begin{bmatrix} I_{N+1} \otimes E_{017} \end{bmatrix},$
$E_{017} = \begin{bmatrix} I_S \otimes U^0 I_m & 0 \end{bmatrix}, \quad A_1^{33} = \begin{bmatrix} I_{N+1} \otimes E_{018} \end{bmatrix},$

$$E_{018} = \begin{bmatrix} V_{27} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{28} \\ \mathbf{0} & V_{27} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{28} \\ \mathbf{0} & \mathbf{0} & V_{27} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{28} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_{27} & \mathbf{0} & \dots & \mathbf{0} & V_{28} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{29} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{29} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{29} \end{bmatrix},$$

where, $V_{27} = (U \oplus D_0 + bD_1) - (\psi + \beta)I_{nm}, \quad V_{28} = \beta I_{nm}, \quad V_{29} = (U \oplus D_0 + bD_1) - \psi I_{nm}$

$A_1^{34} = \begin{bmatrix} I_N \otimes E_{019} & 0 \\ 0 & E_{019} \end{bmatrix}, \quad E_{019} = \begin{bmatrix} 0 & I_S \otimes I_l \otimes \psi I_m \end{bmatrix}, \quad A_1^{42} = \begin{bmatrix} 0 & I_S \otimes I_N \otimes E_{020} \end{bmatrix},$

$E_{020} = \begin{bmatrix} T^0 \otimes Im & 0 \\ 0 & I_S \otimes qT^0 \otimes I_m \end{bmatrix}, \quad A_1^{43} = \begin{bmatrix} I_N \otimes E_{021} & 0 \end{bmatrix}, \quad E_{021} = \begin{bmatrix} 0 \\ I_S \otimes pT^0 \gamma \otimes I_m \end{bmatrix},$

$$A_1^{44} = \left[I_N \otimes E_{022} \right], \quad E_{022} = \begin{bmatrix} V_{30} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{31} \\ \mathbf{0} & V_{30} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{31} \\ \mathbf{0} & \mathbf{0} & V_{30} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{31} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & V_{30} & \mathbf{0} & \dots & \mathbf{0} & V_{31} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{32} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & V_{32} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & V_{32} \end{bmatrix},$$

where, $V_{30} = (T \oplus D_0) - \beta I_{lm}, \qquad V_{31} = \beta I_{lm}, \qquad V_{32} = T \oplus D_0.$

4. Analysis of Stability Condition

A generator matrix is indicated by the equation $A = A_0 + A_1 + A_2$. In the steady state, let ζ be the probability vector of A that satisfies $\zeta e = 1$ and $\zeta A = 0$. The vector ζ is divided by

$$\zeta = (\zeta_0, \zeta_1, \zeta_2, \zeta_3) = (\zeta_{000}, \zeta_{001}, \dots, \pi_{00S}, \pi_{100}, \zeta_{101}, \dots, \zeta_{10S}, \zeta_{N00}, \zeta_{N01}, \dots, \zeta_{N0S}, \zeta_{010}, \zeta_{012}, \\ \dots, \zeta_{01S}, \zeta_{110}, \zeta_{111}, \dots, \zeta_{11S}, \zeta_{N10}, \zeta_{N11}, \dots, \zeta_{N1S}, \zeta_{020}, \zeta_{021}, \dots, \zeta_{02S}, \zeta_{120}, \zeta_{121}, \dots, \zeta_{12S}, \dots, \\ \zeta_{N21}, \zeta_{N22}, \dots, \zeta_{N2S}, \zeta_{030}, \zeta_{031}, \dots, \zeta_{03S}, \zeta_{130}, \zeta_{131}, \dots, \zeta_{13S}, \zeta_{N30}, \zeta_{N31}, \zeta_{N32}, \dots, \zeta_{N3S},)$$

where ζ_0 has a dimension of (N+1)(S+1)m, ζ_1 has a dimension of(N+1)(S+1)m , ζ_2 has a dimension of (N+1)Snm and ζ_3 has a dimension of (N+1)(S+1)lm.
Subject to normalizing condition

$$\sum_{c_3=0}^{N} \sum_{c_4=0}^{S} \zeta_0 e_m + \sum_{c_3=0}^{N} \sum_{c_4=0}^{S} \zeta_1 e_m + \sum_{c_3=0}^{N} \sum_{c_4=1}^{S} \zeta_2 e_{nm} + \sum_{c_3=1}^{N} \sum_{c_4=0}^{S} \zeta_3 e_{lm} = 1.$$

Within the framework of QBD, our model's stability should meet the necessary and sufficient conditions of $\zeta A_0 e < \zeta A_2 e$, which is found to be of the below form,

$$\sum_{c_3=0}^{N} \sum_{c_4=0}^{S} \zeta_{0c_3c_4} [I_{N+1} \otimes I_{S+1} \otimes D_1 e_m] + \sum_{c_3=0}^{N} \sum_{c_4=0}^{S} \zeta_{1c_3c_4} [I_{N+1} \otimes D_1 e_m] \\ + \sum_{c_3=0}^{N} \sum_{c_4=1}^{S} \zeta_{2c_3c_4} [I_{N+1} \otimes I_S \otimes I_n \otimes (1-b) D_1 e_{nm}] + \sum_{c_3=1}^{N} \sum_{c_4=1}^{S} \zeta_{3c_3c_4} [I_N \otimes I_{S+1} \otimes I_l \otimes D_1 e_{lm}] \\ < \sum_{c_3=0}^{N} \sum_{c_4=0}^{S} \zeta_{1c_3c_4} [I_N \otimes I_S \otimes \gamma \otimes \sigma_1 I_m].$$

4.1. Invariant Probability Vector Analysis

Let ν symbolize the infinitesimal generator Q's invariant probability vector and this is divided into $\nu = (\nu_0, \nu_1, \nu_2, ...)$. Mention that ν_0, ν_1, ν_2,... have a dimension of (2m[(N+1)(S+1)]+Snm(N+1)+Nlm(S+1)) and the vector ν satisfies $\nu Q = 0$ and $\nu e = 1$. Additionally, after the stability needed of the model is met, the stationary probability vector ν can be obtained by applying the following equation:
$\nu_i = \nu_0 R^i, \quad i \geq 1,$
Second-degree matrix equation
$R^2 A_2 + R A_1 + A_0 = 0,$
is met by the Neuts (1984) least non-negative solution R. The rate matrix is obtained from the matrix quadratic equation. The requirement is satisfied by the rate matrix R, whose order is provided by (2m[(N+1)(S+1)]+Snm(N+1)+Nlm(S+1)).

$RA_2e = A_0e$,
the sub vector ν_0 can be found by solving the subsequent equation
$\nu_0(B_{00} + RA_2) = 0$,
the normalizing condition is subject to
$\nu_0 e_{2m[(N+1)(S+1)]+Snm(N+1)+Nlm(S+1)} = 1$.
Before solving the set of equations mentioned above, we first need to calculate the rate matrix R. This can be done using the Logarithmic Reduction Approach [14], which simplifies the process of finding R.

5. Analysis of Active period

- The time frame that begins when users enter the empty system and concludes when the system size first falls to zero is known as the busy period. Therefore, the first phase of the shift from level 1 to level 0 is the analogue of the busy period. When a state at any other level is visited at least once, the busy cycle is the time it takes to return to level zero.
- Let's use the concept of the basic time to examine the busy time. The QBD process is nothing more than the initial transition period between the levels k_1 to $k_1 - 1$, where $k_1 \geq 1$.
- The boundary states, or the circumstances when $k_1 = 0$, require a distinct discussion. For any level k_1 where $k_1 \geq 1$, it is evident that there are 2m[(N+1)(S+1)]+Snm(N+1)+Nml(S+1) states. For this reason, the h^{th} state of level k_1 may be denoted as (k_1, h) when each state are placed in lexicographic order.
- Let $H_{hh'}(k_1, x)$ represent the probability with condition that, subject to the restriction that it begin in the state (k_1, h) at time $t = 0$, the QBD process visits level k_1-1 to modify k_1 transitions to the left and also enters the state (k_1, h'). Let us present the idea of the joint transform.
$\bar{H}_{hh'}(z,s) = \sum_{k_1=1}^{\infty} z_1^k \int_0^{\infty} e^{-sx}\, dH_{hh'}(k_1, x); |z| \leq 1, Re(s) \geq 0$, and the matrix is denoted as $\bar{H}(z,s) = \bar{H}hh'(z,s)$.
The following equations can be easily satisfied by the matrix $\bar{H}(z,s)$:
$\bar{H}(z,s) = z[sI - A_1]^{-1}A_2 + [sI - A_1]^{-1}A_0\bar{H}^2(z,s)$.
- Without taking into account the boundary states, the initial travel time should be computed with the matrix $H = H_{hh'} = \bar{H}(1,0)$ after evaluating the rate matrix R, we can quickly get the matrix H by applying the result
$H = -[A_1 + RA_2]^{-1}A2$.
- In the absence of such, it is possible to calculate the H matrix values by applying the idea of a logarithmic reduction process. $\bar{H}^{0,0}(1,0)$ provides the following equation, which is related to boundary level zero.
$\bar{H}^{0,0}(z,s) = [sI - H_{00}]^{-1}A_0\bar{H}(z,s)$.
- The next moments are simple to evaluate because the three matrices namely, H, $\bar{H}^{(1,0)}$ and $\bar{H}^{(0,0)}(1,0)$ are stochastic. At $s = 0, z = 1$,
$\overrightarrow{J_1} = -\frac{\partial}{\partial s}\bar{H}(z,s) = -[A_1 + A_0(H+I)]^{-1}e$,
$\overrightarrow{J_2} = \frac{\partial}{\partial s}\bar{H}(z,s) = -[A_1 + A_0(H+I)]^{-1}A_2e$,
$\overrightarrow{J_1}^{(0,0)} = -\frac{\partial}{\partial z}\bar{H}^{(0,0)}(z,s) = -H_{00}^{-1}[e + A_0\overrightarrow{J_1}]$,
$\overrightarrow{J_2}^{(0,0)} = \frac{\partial}{\partial z}\bar{H}^{(0,0)}(z,s) = -H_{00}^{-1}[A_0\overrightarrow{J_2}]$.

6. Performance Measures

We examine our model's qualitative behavior under steady conditions. Let x(c_1,c_2, c_3, c_4, c_5, c_6, c_7) stand for the probability of the stable state, where the quantity of clients in the orbit I is c_1, quantity of interrupted consumers in the orbit II is c_3, server status is c_2 and c_4, c_5, c_6, c_7 represent the number of items, service phases, repair phases and arrival phases, respectively.

- chance of the server being idle: $P_{idle} = \sum_{c_1=0}^{\infty} \sum_{c_3=0}^{N} \sum_{c_4=0}^{S} \sum_{c_7=1}^{m} x_{c_1 1 c_3 c_4 c_7}$
- Chance of the server is on vacation: $P_{vac} = \sum_{c_1=0}^{\infty} \sum_{c_3=0}^{N} \sum_{c_4=0}^{S} \sum_{c_7=1}^{m} x_{c_1 0 c_3 c_4 c_7}$
- Chance of the server offering service: $P_{busy} = \sum_{c_1=0}^{\infty} \sum_{c_3=0}^{N} \sum_{c_4=1}^{S} \sum_{c_5=1}^{n} \sum_{c_7=1}^{m} x_{c_1 2 c_3 c_4 c_5 c_7}$
- chance of the server is in repair: $P_{repair} = \sum_{c_1=0}^{\infty} \sum_{c_3=1}^{N} \sum_{c_4=0}^{S} \sum_{c_6=1}^{l} \sum_{c_7=1}^{m} x_{c_1 3 c_3 c_4 c_6 c_7}$
- The chance that an interrupted clients is stopped from entering into orbit II: $P_b = \sum_{c_1=0}^{\infty} x_{c_1 3 N e}$
- Expected quantity of clients in orbit I: $E[N_1] = \sum_{c_1=0}^{\infty} c_1 x_{c_1 e}$
- Expected quantity of clients in orbit II: $E[N_2] = \sum_{c_3=0}^{N} c_3 \sum_{c_1=0}^{\infty} \sum_{c_2=0}^{3} x_{c_1 c_2 c_3 e}$
- Expected number of clients in the system: $E_{system} = E[N_1] + E[N_2] + P_{busy}$
- Expected retrial rate: $E_{RR} = \sigma_1 \sum_{c_1=1}^{\infty} \sum_{c_3=0}^{N} \sum_{c_4=1}^{S} \sum_{c_7=1}^{m} x_{c_1 1 c_3 c_4 c_7} + \sigma_2 \sum_{c_1=0}^{\infty} \sum_{c_3=1}^{N} \sum_{c_4=1}^{S} \sum_{c_7=1}^{m} x_{c_1 1 c_3 c_4 c_7}$
- Expected inventory level: $E_{IL} = \sum_{c_1=0}^{\infty} \sum_{c_3=0}^{N} \sum_{c_4=1}^{S} \sum_{c_7=1}^{m} c_4 x_{c_1 0 c_3 c_4 c_7} + \sum_{c_1=0}^{\infty} \sum_{c_3=0}^{N} \sum_{c_4=1}^{S} \sum_{c_7=1}^{m} c_4 x_{c_1 1 c_3 c_4 c_7}$ $+ \sum_{c_1=0}^{\infty} \sum_{c_3=0}^{N} \sum_{c_4=1}^{S} \sum_{c_5=1}^{n} \sum_{c_7=1}^{m} c_4 x_{c_1 2 c_3 c_4 c_5 c_7} + \sum_{c_1=0}^{\infty} \sum_{c_3=1}^{N} \sum_{c_4=1}^{S} \sum_{c_6=1}^{l} \sum_{c_7=1}^{m} c_4 x_{c_1 3 c_3 c_4 c_6 c_7}$
- Expected Reorder rate: $E_{ROR} = \sum_{c_1=0}^{\infty} \sum_{c_3=0}^{N} \sum_{c_5=1}^{n} \sum_{c_7=1}^{m} x_{c_1 2 (s+1) c_3 c_5 c_7 (U^0 \otimes I_m)}$

7. Analysis of Costs

The approximate overall expense of our model is given below, with the cost elements (per unit time) related to various system measures.

- ETC: The expected sum of cost per unit time.
- C_H: The cost of carrying inventory per unit of time.
- C_W: cost per unit of time for the customer to wait in orbit.
- C_S: Setup cost per first item order.

$ETC = C_H E_{IL} + C_S E_{ROR} + C_W E_{system}$

8. Numerical Implementation

In this section, we examine the output of our system using graphical and numerical representations. With a mean value of 1, these arrival processes $ERL-A$, $EXP-A$ and $HEX-A$ have zero correlation and it correspond to the renewal process.

Erlang Arrival(ERL-A): $D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

Exponential Arrival(EXP-A): $D_0 = [-1]$, $D_1 = [1]$

Hyper exponential Arrival(HEX-A): $D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -1.90 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$

Consider the following PH distributions for the service and repair Progression:

Erlang Service(ERL-S): $\gamma = [1, 0]$; $U = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$

Erlang Repair(ERL-R): $\alpha = [1, 0]$; $T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$

Exponential Service(EXP-S): $\gamma = 1$; $U = [-1]$

Exponential Repair(EXP-R): $\alpha = 1$; $T = [-1]$

Hyper Exponential Service(HEX-S): $\gamma = [0.8, 0.2]$; $U = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}$

Hyper Exponential Repair(HEX-R): $\alpha = [0.8, 0.2]$; $T = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}$

8.1. Illustration 1

We investigate retrial rate (σ_1) affects the (E_{Orbit}) in the tables 1 to 3. Fix $\lambda = 1$, $\mu = 8$, $\sigma_2 = 50$, $\tau = 2$, $\psi = 1$, $\beta = 2$, $N = 5$, $p = 0.5$, $b = 0.8$, $S = 10$ and $s = 5$. The following analysis is based on the observations from table 1 to table 3:
The E_{Orbit} falls as we enhance the retrial rate, taking into account both arrival and service periods. From the viewpoint of arrival periods, E_{Orbit} falls significantly for HEX-A and gradually for ERL-A when the retrial rate rises. In comparison to EXP-A, E_{Orbit} falls significantly in ERL-S and slowly in HEX-S when considering retrial times.

Table 1: *Retrial rate (σ_1) vs E_{OrbitI} - ERL-S*

σ_1	ERL-A	EXP-A	HEX-A
61	1.60633595	2.20408030	2.80572168
62	1.60374522	2.19685734	2.78896850
63	1.60122499	2.18988720	2.77295257
64	1.59877238	2.18315631	2.75762501
65	1.59638463	2.17665207	2.74294121
66	1.59405917	2.17036279	2.72886039
67	1.59179356	2.16427756	2.71534517
68	1.58958546	2.15838625	2.70236127
69	1.58743268	2.15267939	2.68987711
70	1.58533314	2.14714812	2.67786365

Table 2: *Retrial rate (σ_1) vs E_{OrbitI} - EXP-S*

σ_1	ERL-A	EXP-A	HEX-A
61	1.65963406	2.32077465	3.11207557
62	1.65645172	2.31137417	3.08666802
63	1.65336072	2.30232643	3.06251226
64	1.65035709	2.29361123	3.03951620
65	1.64743708	2.28520995	3.01759664
66	1.64459718	2.27710535	2.99667822
67	1.64183404	2.26928144	2.97669252
68	1.63914454	2.26172341	2.95757725
69	1.63652569	2.25441747	2.93927561
70	1.63397471	2.24735081	2.92173568

Table 3: *Retrial rate(σ_1) vs E_{OrbitI} - HEX-S*

σ_1	$ERL-A$	$EXP-A$	$HEX-A$
61	2.23983642	3.49702951	8.76338751
62	2.22645590	3.45412770	8.55390871
63	2.21362288	3.41364195	8.35774148
64	2.20130368	3.37537115	8.17365341
65	2.18946736	3.33913601	8.00055975
66	2.17808547	3.30477620	7.83750189
67	2.16713176	3.27214787	7.68362958
68	2.15658199	3.24112158	7.53818599
69	2.14641376	3.21158057	7.40049528
70	2.13660629	3.18341922	7.26995199

8.2. Illustration 2

The effects of the vacation rate (η) in relation to E_{OrbitI} have been examined in figures 2 to 4. Fix $\lambda = 1$, $\mu = 8$, $\sigma_2 = 50$, $\tau = 2$, $\psi = 1$, $\beta = 2$, $N = 5$, $p = 0.5$, $b = 0.8$, $S = 10$, $\sigma_1 = 60$ and $s = 5$. We note that from 2 to 4, as follows:

E_{OrbitI} increases with the rise of vacation rate for different possible arrival and service time groupings. As the values for different arrival times are correlated, E_{system} decreases more fast for $HEX-A$ and more slowly for $ERL-A$. Likewise, E_{system} decreases quickly for $ERL-S$ and gradually for $HEX-S$.

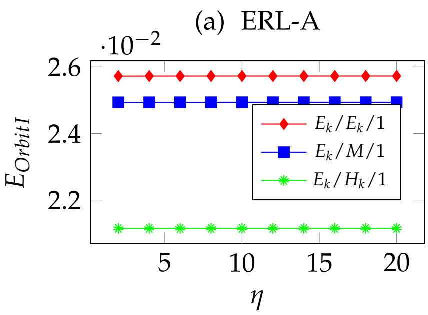

Figure 2: *Vacation rate vs E_{OrbitI}*

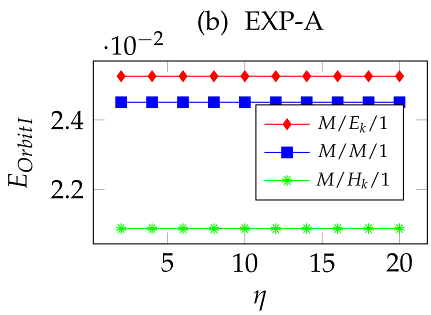

Figure 3: *Vacation rate vs E_{OrbitI}*

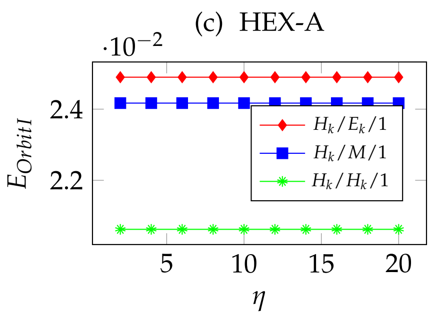

Figure 4: *Vacation rate vs E_{OrbitI}*

8.3. Illustration 3

To examine the effect on the E_{system} in the corresponding figures 5 to 13 of the vacation rate and retrial rate (Orbit II) (σ_2). To assure satisfaction of stability criteria, make adjustments to $\lambda = 1$, $\mu = 8$, $\eta = 2$, $\tau = 2$, $\psi = 1$, $\beta = 2$, $N = 5$, $p = 0.5$, $b = 0.8$, $S = 10$, $\sigma_1 = 60$ and $s = 5$.

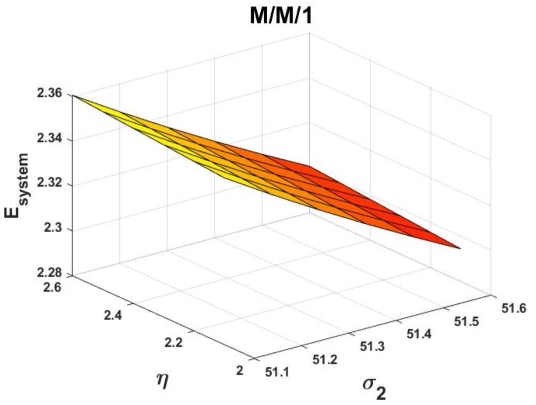

Figure 5: *Vacation and Retrial Rate(Orbit II) vs* E_{system}

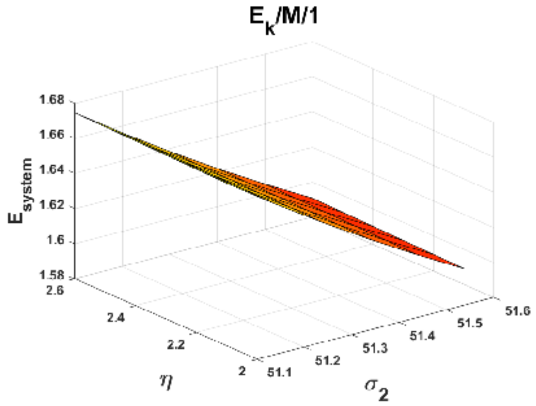

Figure 6: *Vacation and Retrial Rate(Orbit II) vs* E_{system}

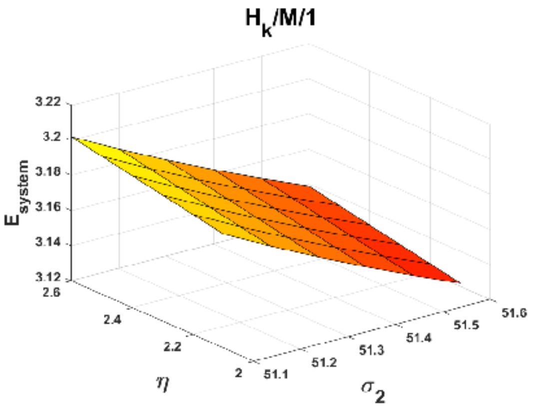

Figure 7: *Vacation and Retrial Rate(Orbit II) vs* E_{system}

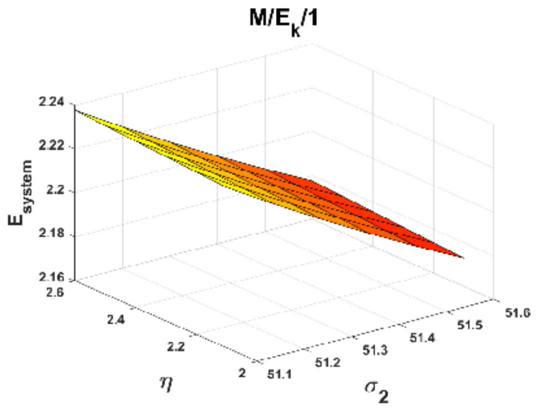

Figure 8: *Vacation and Retrial Rate(Orbit II) vs* E_{system}

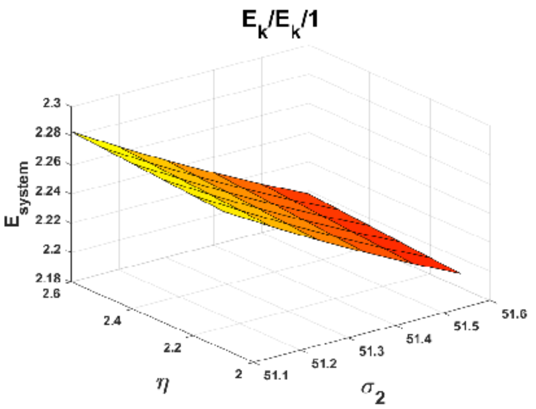

Figure 9: *Vacation and Retrial Rate(Orbit II)vs* E_{system}

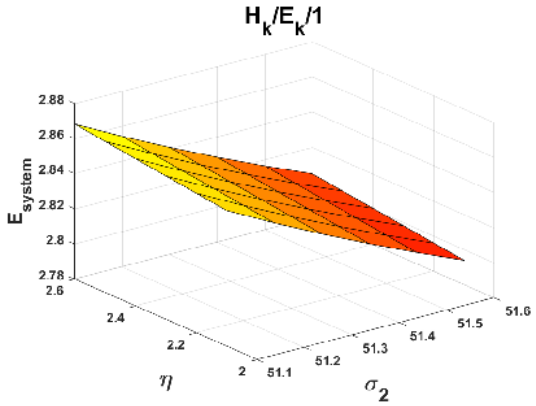

Figure 10: *Vacation and Retrial Rate(Orbit II) vs* E_{system}

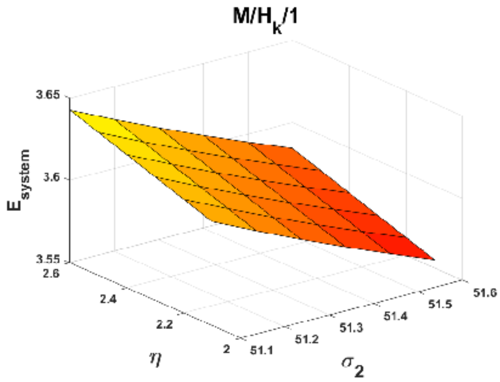

Figure 11: *Vacation and Retrial Rate(Orbit II)vs* E_{system}

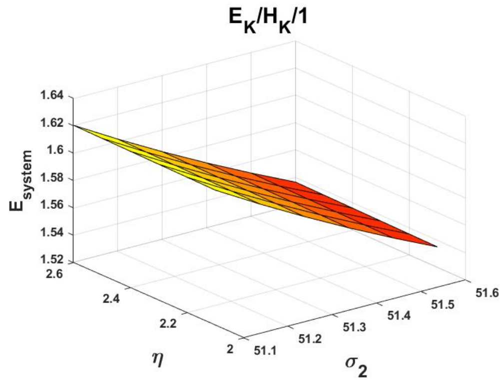

Figure 12: *Vacation and Retrial Rate(Orbit II)vs* E_{system}

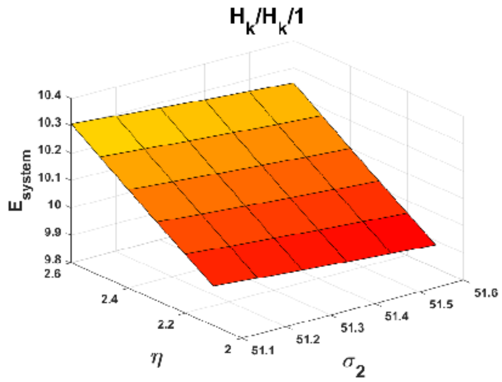

Figure 13: *Vacation and Retrial Rate(Orbit II) vs* E_{system}

From the figure 5 to 13, we can see:
We investigate the link between E_{system}, the retrial rate (σ_2), and the vacation rate of the server over a range of potential arrival and service time combinations. When there is a fixed arrival rate, the E_{system} decreases as the server's retrial rate and vacation rate rise. Taking into account the service times, E_{system} decreases significantly for ERL-S and somewhat for EXP-S. As a result, E_{System} declines quickly for HEX-A and slowly for ERL-A.

9. Conclusion

We examined a double orbit retrial queueing inventory model for a single server that included orbital search, repair, breakdown, discouragement, single vacation and constant retrial rate. We explored server status, inventory levels and the number of customers in the system using matrix analytic techniques in steady state conditions. By using this method, we were able to evaluate the effectiveness and performance of the system, determine optimal approaches for inventory and service management and learn about the system parameters. This model can be expanded for further work to include multi-servers and two different kinds of customers.

References

[1] Abdul Reiyas, M. and Jeganathan, K.(2023). A classical retrial queueing inventory system with two component demand rate. *International Journal of Operational Research*, 47(4):508-533.

[2] Amirthakodi, M. and Sivakumar, B.(2015). An inventory system with service facility and finite orbit size for feedback customers. *OPSEARCH*, 52(2):225-255.

[3] Artalejo, J.R.(2008). New results in retrial queueing systems with breakdown of the server. *Statistica Neerlandica*, 48(1):23-36.

[4] Avrachenkov, K., Dudin, A. and Klimenok, V.(2010). Retrial queueing model MMAP/M_2/1 with two orbits. *Springer-Verlag Berlin Heidelberg*, 6235:107-118.

[5] Ayyappan, G. and Gowthami, R.(2021). Analysis of $MAP/PH/1$ retrial with constant rate, working vacations, abandonment, flush out, search of customers, breakdown and repair. *International Journal of Mathematics in Operational Research* , 42(3):310-331.

[6] Ayyappan, G. and Arulmozhi, N.(2024). Numerical investigation of retrial queueing inventory system with a constant retrial rate, working vacation, flush out, collision and impatient customers.*Reliability: Theory and Applications* , 19(1):744-763.

[7] Ayyappan, G. and Meena, S.(2023). Analysis of $MAP/PH/1$ Retrial inventory queueing system with phase type vacation and unreliable server. *International Journal of Mathematics in Operational Research*, DOI: 10.1504/IJMOR.2023.10061611.

[8] Babu, D., Krishnamoorthy, A. and Joshua, V.C.(2018). retrial queue with search of interrupted customers from the finite orbit. *Springer Nature Switzerland AG*, 912:360-371.

[9] Dhibar, S. and Jain, M.(2023). Strategic behaviour for M/M/1 double orbit retrial queue with imperfect service and vacation. *International Journal of Mathematics in Operational Research*, 25(3):369-385.

[10] Dimitriou, I.(2017). A two class retrial system with coupled orbit queues. *Probability in the Engineering and Informational sciences*, 31(2):139-179.

[11] Falin, G.I. and Templeton, J.G.C.(1997). Retrial queues.*Chapman and Hall*, London New York.

[12] Jain, M., Bhagat, A. and Shekhar, C.(2015). Double orbit finite retrial queues with priority customers and service interruptions. *Applied Mathematics and Computation*, 253:324-344.

[13] Krishnamoorthy, A., Joshua, V.C. and Mathew, A.P. (2017). A retrial queueing system with abandonment and search for priority customers. *Springer International Publishing AG*, 700:98-107.

[14] Latouche, G. and Ramaswami, V.(1999). Introduction of matrix analytic methods in stochastic modeling. *Society for Industrial and Applied Mathematics* , Philadelphia.

[15] Melikov, A., Chakravarthy, S. and Aliyeva, S.(2023). A retrieval queueing model with feedback. *Queueing Models and Service Management (QMSM)*, 6(1):63-95.

[16] Melikov, A., Aliyeva, S.,Sztrik, J.(2021). Retrial queues with unreliable servers and delayed feedback.*Mathematics*, 9(19):1-23.

[17] Nithya, N. and Anbazhagan, N., Amutha, S., Jeganathan, K., Park,G.C., Joshi,G.P. and Cho, W.(2023). Controlled arrivals on the retrial queueing–inventory system with an essential interruption and emergency vacationing server. Mathematics, 11(16):3560.

[18] Sangha, S.S. and Jain, M.(2019). FM/FM/1 double orbit retrial queue with customers' joining strategy: A parametric nonlinear programing approach. *Applied Mathematics and Computation*, 362:124542.

[19] Schwarz, M., Saucer, C.,Daduna, H., Kulik, R. and Szekli, R.(2006). M/M/1 queueing systems with inventory, Queueing systems, 54(1):55-78.

[20] Singla, N. and Kaur, H.(2021). A two-state retrial queueing model with feedback and balking. *Reliability: Theory and Applications* , 16(2):142-155.

[21] Sivakumar, B.(2009). A perishable inventory system with retrial demands and a finite population. *Journal of Computational and Applied Mathematics*, 224(1):29-38.

[22] Ushakumari, P.V. (2017). A retrial inventory system with an unreliable server. *International Journal of Mathematics in Operational Research*, 10(2):190-210.