STRESS-STRENGTH MODELLING: A COMPARATIVE STUDY USING SRS, RSS AND MRSS FOR INVERSE NAKAGAMI DISTRIBUTION

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Abstract

The present research endeavours to use the ranked set sampling technique in place of the simple random sampling technique in estimating the reliability model in the field of reliability engineering and manufacturing industries. The stress-strength reliability is widely used in all field of engineering and in the manufacturing industry. In this research article, we take the case when the lifetime data of any machinery followed the Inverse Nakagami distribution, and by considering the stress-strength model, we compare the standard ranked set sampling and median ranked set sampling estimate with the corresponding simple random sampling estimate through Monte Carlo simulation technique. The final stage of this study involves corroborating the findings through an analysis of a benchmark dataset on elevators and their motors utilized in farming machinery.

Keywords: Stress-strength reliability, Median ranked set sampling, Ranked set sampling, Inverse Nakagami distribution, Monte Carlo simulation

I. Introduction

In the literature, the conceptual framework of the stress-strength model was first introduced by Birnbaum [2], with further advancements made in his joint work with McCarty [16]. The phrase 'stress-strength' was first coined by Church and Harris [6] in their academic article, in which they performed important investigations using parametric and non-parametric techniques. Following this, various scholars selected different probabilistic models to estimate the stress-strength relation- ships. Some of these selections were summarized by Johnson [10]. A comprehensive overview of all methodologies and discoveries related to the stress-strength paradigm across the previous four decades was compiled by Kotz et al. [12]. Different researchers work on stress-strength model by taking different cases of distribution, like Downton [8] take the case of Normal distribution, Constantine et al. [7] take the case of Gamma distribution, Awad and Gharraf [1] take Burr distribution case, Kundu and Gupta [13] take Generalised Exponential distribution case and so many others distribution are taken to estimate stress-strength reliability.

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Based on existing literature, no researchers have explored the case of the Inverse Nakagami distribution until now. The Inverse Nakagami distribution (INAD) was proposed by Louzada et al. [15] as an extension of the Nakagami distribution, which was originally introduced by Nakagami [18] himself. The versatility of the Nakagami distribution has led to its adoption in multiple fields, ranging from communications technology to medical image analysis, water resource management, and earthquake studies. (see, e.g., studies by Wang et al. [26], Tsui et al. [25], Sarkar et al. [22], Nakahara and Carcole [19]).

The Ranked Set Sampling (RSS) introduced by McIntyre [16]. Unlike conventional random sampling methods, RSS leverages the ranked order or order statistics of sampled observations to improve the quality and efficiency of estimations. The estimation of distribution functions using various RSS techniques has been explored by several researchers. Stokes and Sager [24], Yu and Lam [27], and Chen [4] have contributed to this area. Furthermore, investigations into the estimation of distribution functions under different RSS methods have been conducted by Kvam and Samaniego [14], and Chen [5]. Furthermore, the research contributions of Zhang et al. [29], Zamanzade and Vock [28], and Ozturk and Kavlak [20] have shed light on inferential methodologies and techniques that utilize data obtained through ranked set sampling approaches.

Muttak [17] introduced a modified variant of RSS called the median ranked set sampling (MRSS) aimed at estimating the population mean. A comprehensive literature review encompassing the research on RSS until 1995 can be found in the work of Kaur et al. [11]. Employing the MRSS technique, Samuh and Qtait [21] focused on estimating the parameters of the Exponentiated Exponential distribution. Hassan et al. [9] contributed to the field by exploring stress-strength reliability estimation utilizing the MRSS approach. More recently, Shahzad et al. [23] made a significant contribution to the MRSS field by demonstrating the effectiveness of combining ranked and actual auxiliary variable observations in mean estimation.

Here, we have consider the estimation of R = Pr(Y < X) with a focus on situation where the random stress Y and random strength X are two independent Inverse Nak- agami random variables with shape parameters (ξ_1, ξ_2) and scale parameters (ψ, ψ), respectively. The point estimator of R = Pr(Y < X), is obtained using the maximum likelihood method based on SRS, RSS and MRSS, and the efficiency of this method based on SRS and RSS is compared. In Section 2, we present a brief overview about the Inverse Nakagami distribution. Point estimation of the parameters is given in Section 3. Stress-strength reliability is calculated in Section 4. Section 5 and Section 6 comprises the point estimation of stress-strength model under SRS and under RSS, respectively. In Section 7 we take the case of MRSS and derived the point estimation of the parameters. A simulation study employing the Monte Carlo method is discussed in Section 8. Section 9 shows the real life application of the current findings, and lastly Section 10 provides concluding remarks for the paper.

II. Preliminary

When a random variable X follows the Inverse Nakagami distribution INAD (ξ, ψ) with shape parameter ξ, where ξ is constrained to be greater than 0, and scale parameter ψ, where ψ is greater than 0 then its PDF and CDF is given by

$$f(x, \xi, \psi) = \frac{2}{\Gamma(\xi)}\left(\frac{\xi}{\psi}\right)^{\xi} x^{-2\xi-1} \exp\left(-\frac{\xi}{\psi}\, x^{-2}\right); \quad x > 0, \psi > 0, \xi > 0 \tag{1}$$

and

$$F(x) = \frac{\Gamma\left(\xi, \frac{\xi}{\psi}\, x^{-2}\right)}{\Gamma(\xi)}; \quad x > 0, \psi > 0, \xi > 0 \tag{2}$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}\, e^{-t} dt$

The reliability function of INAD (ξ, ψ) is

$$R(t) = \frac{\gamma\left(\xi, \frac{\xi}{\psi}\, t^{-2}\right)}{\Gamma\xi}; \quad t > 0, \psi > 0, \xi > 0$$

where $\gamma(a,x) = \int_\infty^x t^{a-1}\, e^{-t} dt$

The hazard rate of INAD (ξ, ψ) is

$$h(t) = \frac{\frac{2}{\Gamma(\xi)}\left(\frac{\xi}{\psi}\right)^{\xi} t^{-2\xi-1} \exp\left(-\frac{\xi}{\psi}\, t^{-2}\right)}{\gamma\left(\xi, \frac{\xi}{\psi}\, t^{-2}\right)}; t > 0, \psi > 0, \xi > 0$$

Special Cases of INAD (ξ, ψ)

1. If ξ = 1, then INAD (ξ, ψ) becomes the Inverse Rayleigh distribution.
2. If ξ = 0.5, then INAD (ξ, ψ) becomes the Inverse half-normal distribution.
3. If ψ = 1, ξ = λ, λ = 1, 2, 3 ..., then INAD (ξ, ψ) becomes the Inverse Chi distribution.
4. There is one special case where INAD (ξ, ψ) moves to newly proposed Inverse Hayt distribution for 0 < ξ < 1.

III. Parameters estimation

If we take a random sample $x_1, x_2, \ldots, x_n$ from the INAD (ξ, ψ) of size n, then the likelihood function of the Inverse Nakagami distribution INAD (ξ, ψ) is given by

$$L(x, \xi, \psi) = \frac{\left(2\xi^{\xi}\right)^n}{(\Gamma\xi)^n(\psi^{\xi})^n} \prod_{i=1}^{n} (x_i)^{(-2\xi-1)} \exp\left(-\frac{\xi}{\psi}\sum_{i=1}^{n} x_i^{-2}\right) \tag{3}$$

We can formulate the likelihood function as follows

$$\log L = n\log 2 - n\log\Gamma\xi + n\xi\log\xi - n\xi\log\psi - (2\xi+1)\sum_{i=1}^{n}\log x_i - \frac{\xi}{\psi}\sum_{i=1}^{n} x_i^{-2} \tag{4}$$

Differentiating the log likelihood function of INAD (ξ, ψ) given in Eq. (4) with respect to ψ in the case when ξ is known and equating the resulting equation equal to zero, we get

$$\hat{\psi} = \frac{\sum_{i=1}^{n} x_i^{-2}}{n} \tag{5}$$

Similarly, differentiating the log likelihood function of INAD (ξ, ψ) given in Eq. (4) with respect to ξ in the case when ψ is known

$$\frac{\partial logL}{\partial\xi} = n(log\xi + 1) - n\left(\frac{\partial}{\partial\xi}\, log\Gamma\xi\right) - nlog\,\psi - 2\sum_{i=1}^{n} log\, x_i - \frac{1}{\psi}\sum_{i=1}^{n} x_i^{-2} \tag{6}$$

equating the Eq. (6) to zero, we get

$$\frac{\partial logL}{\partial\xi} = \log\xi - \left(\frac{\partial}{\partial\xi}\log\Gamma\xi\right) - \log\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-2}\right) + \frac{1}{n}\sum_{i=1}^{n}\log x_i^{-2} = 0 \tag{7}$$

ML estimator of ξ ($\hat{\xi}$) can be obtained from Eq. (7) by using Newton-Raphson method beacause Eq. (7) does not yield a closed-form solution.

IV. Stress - strength reliability

To formulate the reliability expression under the stress-strength model, we assume the strength characteristic is represented by the random variable X, while the stress characteristic is denoted by the random variable Y. Both X and Y are assumed to follow the Inverse Nakagami distribution, characterized by a shared scale parameter ψ, where ψ is greater than zero. However, they differ in their shape parameters, with X having a shape parameter of ξ_1 and Y having a shape parameter of ξ_2. Letting X~ INAD (ξ_1,ψ) and Y ~ INAD (ξ_2, ψ), we can proceed to derive the reliability as

$$R = \int_0^\infty P(Y < X) f(x) dx$$

$$= \int_0^\infty \left[\frac{1}{\Gamma\xi_2}\Gamma\left(\xi_2, \frac{\xi_2}{\psi\, x^2}\right)\right] \frac{2}{\Gamma\xi_1}\left(\frac{\xi_1}{\psi}\right)^{\xi_1} x^{-2\xi_1-1}\exp\left(\frac{-\xi_1}{\psi x^2}\right) dx$$

$$= 2\left(\frac{\xi_1}{\psi}\right)^{\xi_1}\frac{1}{\Gamma\xi_1}\int_0^\infty x^{-2\xi_1-1}\exp\left(\frac{-\xi_1-\xi_2}{\psi\, x^2}\right)\sum_{m=0}^{\xi_2-1}\frac{(\xi_2)^m}{\psi^m\, x^{2m}\, m!}\, dx$$

$$= \frac{(\xi_1)^{\xi_1}}{\Gamma\xi_1}\sum_{m=0}^{\xi_2-1}\frac{(\xi_2)^m}{m!}\int_0^\infty y^{-(\xi_1+m)-1}\, e^{-\frac{1}{y}(\xi_1+\xi_2)}\, dy$$

$$R = \frac{(\xi_1)^{\xi_1}}{\Gamma\xi_1}\sum_{m=0}^{\xi_2-1}\frac{(\xi_2)^m}{m!}\left[\frac{\Gamma(\xi_1+m)}{(\xi_1+\xi_2)^{\xi_1+m}}\right] \tag{8}$$

V. Evaluating stress-strength model through SRS point estimation

We consider two independent random samples X and Y, of sizes n and m respectively, drawn from INAD's. Let X follow a INAD with a shape parameter ξ_1, while Y follows a INAD with a shape parameter ξ_2. Assuming the scale parameter ψ is known, we can employ the invariance property of Maximum Likelihood (ML) estimators to derive the ML estimator for R given in Eq. (8). Under these conditions, the ML estimators of ξ_1 and ξ_2 from Eq. (7) are given by $\hat{\xi}_{1srs}$ and $\hat{\xi}_{2srs}$. Here we use the R software to calculate the ML estimators of $\xi 1$ and $\xi 2$ from Eq. (7), respectively.
In the context of simple random sampling, the expression for the maximum likelihood estimator of R takes the following form

$$\hat{R}_{srs} = \frac{\left(\hat{\xi}_{1srs}\right)^{\hat{\xi}_{1srs}}}{\Gamma\hat{\xi}_{1srs}}\sum_{m=0}^{\hat{\xi}_{1srs}-1}\frac{\left(\hat{\xi}_{1srs}\right)^m}{m!}\left[\frac{\Gamma\left(\hat{\xi}_{1srs}+m\right)}{\left(\hat{\xi}_{1srs}+\hat{\xi}_{2srs}\right)^{\hat{\xi}_{1srs}+m}}\right] \tag{9}$$

VI. Evaluating stress-strength model through RSS point estimation

RSS represents an advanced statistical method that improves parameter estimation accuracy, especially in situations where data gathering is limited by resources or expenses. The RSS process for obtaining a sample of size n involves multiple stages. Initially, m random sets are chosen through SRS, each containing m elements. These elements are then ordered within their respective sets from smallest to largest based on judgment alone, without actual measurement. The next phase involves selecting specific units for precise measurement. This is done by measuring the unit judged smallest in the first set, the second smallest in the second set, and so on, culminating with the largest judged unit in the final set. This entire sequence constitutes one cycle. The process is repeated r times, resulting in a total ranked set sample size of n = mr, where m is the number of units per set and r represents the number of completed cycles.

I. RSS-based Maximum likelihood estimation for stress-strength model

Let us consider two ranked set samples from Inverse Nakagami distributions with different parameters. The first sample, denoted as $x_{(ij)}$, has a size of $n_1 = r_1 m_1$, where i ranges from 1 to m_1, and j from 1 to r_1. This sample is drawn from an Inverse Nakagami distribution with parameters (ξ_1, ψ). Here, m_1 represents the set size, and r_1 the number of cycles. Similarly, the second sample, $y_{(kl)}$, has a size of $n_2 = r_2 m_2$, where k ranges from 1 to m_2, and l from 1 to r_2. This sample comes from an Inverse Nakagami distribution with parameters (ξ_2, ψ). In this case, m_2 is the set size, and r_2 the number of cycles. We can now express the probability density functions (PDFs) for $x_{(ij)}$ and $y_{(kl)}$ as follows:

$$f_i(x_{ij}) = \frac{m_1!}{(i-1)!\,(m_1-i)!}[F_X(x)]^{i-1}[1-F_X(x)]^{m_1-i} f(x_{ij}) \tag{10}$$

$$g_k(y_{kl}) = \frac{m_2!}{(k-1)!\,(m_2-k)!}[F_Y(y)]^{k-1}[1-F_Y(y)]^{m_2-k} g(y_{kl}) \tag{11}$$

Now the likelihood function is given as

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} f_i(x_{ij}) \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} g_k(y_{kl})$$

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} \frac{m_1!}{(i-1)!\,(m_1-i)!}[F_X(x)]^{i-1}[1-F_X(x)]^{m_1-i} f(x_{ij})$$
$$\prod_{k=1}^{r_2}\prod_{l=1}^{m_2} \frac{m_2!}{(k-1)!\,(m_2-k)!}[F_Y(y)]^{k-1}[1-F_Y(y)]^{m_2-k} g(y_{kl})$$

Let $u = \frac{m_1!}{(i-1)!(m_1-i)!}$ and $v = \frac{m_2!}{(k-1)!(m_2-k)!}$

$$L = \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} u\ [F_X(x)]^{i-1}[1-F_X(x)]^{m_1-i} f(x_{ij})$$
$$\prod_{k=1}^{r_2}\prod_{l=1}^{m_2} v\,[F_Y(y)]^{k-1}[1-F_Y(y)]^{m_2-k} g(y_{kl})$$

$$L = u\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left(\frac{1}{\Gamma\xi_1}\right)^{i-1}\left[\Gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{ij}^2}\right)\right]^{i-1}\left(\frac{1}{\Gamma\xi_1}\right)^{m_1-i}$$
$$\left[\Gamma\xi_1 - \Gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{ij}^2}\right)\right]^{m_1-i}\frac{2}{\Gamma(\xi_1)}\left(\frac{\xi_1}{\psi}\right)^{\xi_1} x^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi}\, x_{ij}^{-2}\right)$$
$$v\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left(\frac{1}{\Gamma\xi_2}\right)^{k-1}\left[\Gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{kl}^2}\right)\right]^{k-1}\left(\frac{1}{\Gamma\xi_2}\right)^{m_2-k}$$
$$\left[\Gamma\xi_2 - \Gamma\left(\xi_2,\frac{\xi_2}{\psi y_{kl}^2}\right)\right]^{m_2-k}\frac{2}{\Gamma(\xi_2)}\left(\frac{\xi_2}{\psi}\right)^{\xi_2} y_{kl}^{-2\xi_2-1}\exp\left(-\frac{\xi_2}{\psi}y_{kl}^{-2}\right)$$

$$L = u\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left(\frac{1}{\Gamma\xi_1}\right)^{m_1}\left[\Gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{ij}^2}\right)\right]^{i-1}\left[\Gamma\xi_1 - \Gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{ij}^2}\right)\right]^{m_1-i}$$

$$2\left(\frac{\xi_1}{\psi}\right)^{\xi_1} x^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi}\,x_{ij}^{-2}\right) v\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left(\frac{1}{\Gamma\xi_2}\right)^{m_2}\left[\Gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{k-1}$$
$$\left[\Gamma\xi_2-\Gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{m_2-k} 2\left(\frac{\xi_2}{\psi}\right)^{\xi_2} y_{kl}^{-2\xi_2-1}\exp\left(-\frac{\xi_2}{\psi}\,y_{kl}^{-2}\right)$$

The combination of the lower and upper incomplete gamma functions yields the complete gamma function. This relationship can be expressed mathematically as:

$$\gamma\left(\xi,\frac{\xi}{\psi\,x^2}\right)+\Gamma\left(\xi,\frac{\xi}{\psi\,x^2}\right)=\Gamma\xi$$
$$\Gamma\xi-\Gamma\left(\xi,\frac{\xi}{\psi\,x^2}\right)=\gamma\left(\xi,\frac{\xi}{\psi\,x^2}\right)$$

Thus,

$$L=u\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\frac{2}{(\Gamma\xi_1)^{m_1}}\left[\Gamma\left(\xi_1,\frac{\xi_1}{\psi\,x_{ij}^2}\right)\right]^{i-1}\left[\gamma\left(\xi_1,\frac{\xi_1}{\psi\,x_{ij}^2}\right)\right]^{m_1-i}$$
$$\left(\frac{\xi_1}{\psi}\right)^{\xi_1} x^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi}\,x_{ij}^{-2}\right)$$
$$v\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\frac{2}{(\Gamma\xi_2)^{m_2}}\left[\Gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{k-1}\left[\gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{m_2-k}$$
$$\left(\frac{\xi_2}{\psi}\right)^{\xi_2} y^{-2\xi_2-1}\exp\left(-\frac{\xi_2}{\psi}\,y_{kl}^{-2}\right)$$
$$L=u\frac{(2)^{n_1}}{(\Gamma\xi_1)^{n_1m_1}}\left(\frac{\xi_1}{\psi}\right)^{n_1\xi_1}\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left[\Gamma\left(\xi_1,\frac{\xi_1}{\psi\,x_{ij}^2}\right)\right]^{i-1}\left[\gamma\left(\xi_1,\frac{\xi_1}{\psi\,x_{ij}^2}\right)\right]^{m_1-i}$$
$$x^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi}\,x_{ij}^{-2}\right)$$
$$v\frac{(2)^{n_2}}{(\Gamma\xi_2)^{n_2m_2}}\left(\frac{\xi_2}{\psi}\right)^{n_2\xi_2}\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left[\Gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{k-1}\left[\gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{m_2-k}$$
$$y^{-2\xi_2-1}\exp\left(-\frac{\xi_2}{\psi}\,y_{kl}^{-2}\right)$$

Taking log on both sides

$$\log L=\log L_1+\log L_2$$

where,

$$L_1=u\frac{(2)^{n_1}}{(\Gamma\xi_1)^{n_1m_1}}\left(\frac{\xi_1}{\psi}\right)^{n_1\xi_1}\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left[\Gamma\left(\xi_1,\frac{\xi_1}{\psi\,x_{ij}^2}\right)\right]^{i-1}\left[\gamma\left(\xi_1,\frac{\xi_1}{\psi\,x_{ij}^2}\right)\right]^{m_1-i}$$
$$x^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi}\,x_{ij}^{-2}\right)$$
$$L_2=v\frac{(2)^{n_2}}{(\Gamma\xi_2)^{n_2m_2}}\left(\frac{\xi_2}{\psi}\right)^{n_2\xi_2}\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\left[\Gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{k-1}\left[\gamma\left(\xi_2,\frac{\xi_2}{\psi\,y_{kl}^2}\right)\right]^{m_2-k}$$
$$y^{-2\xi_2-1}\exp\left(-\frac{\xi_2}{\psi}\,y_{kl}^{-2}\right)$$

This implies

$$\begin{aligned}\log L_1 &= \log u+n_1\log 2-n_1m_1\log\Gamma\,\xi_1+n_1\xi_1(\log\xi_1-\log\psi)\\&+\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\log\left[\Gamma\left(\xi_1,\frac{\xi_1}{\psi x_{ij}^2}\right)\right]+\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i)\log\left[\gamma\left(\xi_1,\frac{\xi_1}{\psi x_{ij}^2}\right)\right]\\&-(2\xi_1+1)\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\log x_{ij}-\frac{\xi_1}{\psi}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\frac{1}{{x_{ij}}^2}\end{aligned}\tag{12}$$

By taking the partial derivatives of Eq. (12) with respect to ξ_1 and ξ_2, we obtain:

$$\frac{\partial \log(L_1)}{\partial \xi_1} = n_1(1+\log \xi_1) - n_1 m_1 \left(\frac{\partial}{\partial \xi_1}\log \Gamma \xi_1\right) - n_1 \log \psi + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\frac{\partial}{\partial \xi_1}\log\left[\Gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{ij}^2}\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i)\frac{\partial}{\partial \xi_1}\log\left[\gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{ij}^2}\right)\right] -2\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\log x_{ij} - \frac{1}{\psi}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\frac{1}{x_{ij}^2} \tag{13}$$

$$\frac{\partial \log(L_2)}{\partial \xi_2} = n_2(1+\log \xi_2) - n_2 m_2 \left(\frac{\partial}{\partial \xi_2}\log \Gamma \xi_2\right) - n_2 \log \psi + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(k-1)\frac{\partial}{\partial \xi_2}\log\left[\Gamma\left(\xi_2, \frac{\xi_2}{\psi\, y_{kl}^2}\right)\right] + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2-k)\frac{\partial}{\partial \xi_2}\log\left[\gamma\left(\xi_2, \frac{\xi_2}{\psi\, y_{kl}^2}\right)\right] -2\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\log y_{kl} - \frac{1}{\psi}\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\frac{1}{y_{kl}^2} \tag{14}$$

To determine the ML estimators for ξ_1 and ξ_2 using RSS, we employ a numerical approach. We denote these estimators as $\hat{\xi}_{1rss}$ and $\hat{\xi}_{2rss}$, which are derived from two specific equations (referred to as Eq. (13) and Eq. (14) in the original text).

Subsequently, we apply the invariance property of ML estimators. This allows us to calculate the ML estimator for the R = Pr(Y < X), based on the RSS. This estimator can be expressed as:

$$\hat{R}_{rss} = \frac{(\hat{\xi}_{1rss})^{\hat{\xi}_{1rss}}}{\Gamma\hat{\xi}_{1rss}} \sum_{m=0}^{\hat{\xi}_{1rss}-1} \frac{(\hat{\xi}_{1rss})^m}{m!}\left[\frac{\Gamma(\hat{\xi}_{1rss}+m)}{(\hat{\xi}_{1rss}+\hat{\xi}_{2rss})^{\hat{\xi}_{1rss}+m}}\right] \tag{15}$$

VII. Evaluating stress-strength model through MRSS point estimation

The MRSS method begins by selecting n random samples, each containing n units from the target population. These units are then ranked within their respective samples based on the variable of interest. For odd-sized samples, the median unit defined as the ((n+1)/2)th smallest ranked unit is chosen from each sample. In even-sized samples, the selection alternates: the (n/2)th smallest ranked unit is picked from the first half of the samples, while the ((n/2)+1)th smallest ranked unit is selected from the remaining half. This process can be repeated r times if a larger sample is required, ultimately yielding an MRSS sample of nr units. This approach ensures a representative selection of median units from the ranked samples.

When implementing MRSS, we can encounter four distinct scenarios. These scenarios are determined by the nature of the sampling process for both X and Y datasets:

1. In the first scenario, both X and Y are sampled using MRSS with odd set sizes.
2. The second scenario involves both X and Y being sampled through MRSS with even set sizes.
3. The third scenario presents a mixed approach: the strength data (X) is collected using MRSS with an odd set size, while the stress data (Y) is gathered using MRSS with an even set size.
4. The fourth scenario is the reverse of the third: the strength data (X) is obtained through MRSS with an even set size, whereas the stress data (Y) is collected using MRSS with an odd set size.

These varying scenarios provide different contexts for data analysis and interpretation in the study of strength-stress relationships.

I. First case: when set sizes are odd

Consider an MRSS sample denoted as $X_{i(g)j}$, where $i = 1,2, \dots, m_1$, $j = 1,2, \dots, r_1$; and $g = \left(\frac{m_1+1}{2}\right)$, drawn from an INAD with shape parameter ξ_1 and scale parameter ψ. This MRSS sample has a size $n_1 = m_1 r_1$, where m_1 represents the set size, and r_1 is the number of cycles. Similarly, let $Y_{k(h)l}$, where $k = 1,2, \dots, m_2$, $l = 1,2, \dots, r_2$, and $h = \left(\frac{m_2+1}{2}\right)$. This MRSS sample has a size $n_2 = m_2 r_2$, with m_2 as the set size and r_2 as the number of cycles. With these MRSS samples from Inverse Nakagami distribution for X and Y, the likelihood function can be expressed as follows:

$$L_{(1)} = \prod_{i=1}^{m_1}\prod_{j=1}^{r_1} f_g\left(x_{i(g)j}\right) \prod_{k=1}^{m_2}\prod_{l=1}^{r_2} f_h\left(y_{k(h)l}\right)$$

where,

$$f_g\left(x_{i(g)j}\right) = \frac{m_1!}{[(g-1)!]^2}\left[\frac{1}{(\Gamma\xi_1)^2}\Gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(g)j}^2}\right)\gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(g)j}^2}\right)\right]^{g-1}$$
$$\frac{2}{\Gamma\xi_1}\left(\frac{\xi_1}{\psi}\right)^{\xi_1} x_{i(g)j}^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi x_{i(g)j}^2}\right) \tag{16}$$

and

$$f_h\left(y_{k(h)l}\right) = \frac{m_2!}{[(h-1)!]^2}\left[\frac{1}{(\Gamma\xi_2)^2}\Gamma\left(\xi_2, \frac{\xi_2}{\psi\, y_{k(h)l}^2}\right)\gamma\left(\xi_2, \frac{\xi_2}{\psi\, y_{k(h)l}^2}\right)\right]^{h-1}$$
$$\frac{2}{\Gamma\xi_2}\left(\frac{\xi_2}{\psi}\right)^{\xi_2} y_{k(h)l}^{-2\xi_2-1}\exp\left(-\frac{\xi_2}{\psi y_{k(h)l}^2}\right) \tag{17}$$

Eq. (16) and (17) are represent the pdfs of $X_{i(g)j}$ and $Y_{k(h)l}$ respectively such that $X_{i(g)j} > 0$ and $Y_{k(h)l} > 0$. The log likelihood function of $L_{(1)}$ is given as

$$\log L_{(1)} = \log v + n_1 \log\alpha - \frac{\xi_1}{\psi\, x_{i(g)j}^2} - \sum_{i=1}^{m_1}\sum_{j=1}^{r_1}(2\xi_1+1)\log x_{i(g)j}$$
$$-\sum_{i=1}^{m_1}\sum_{j=1}^{r_1} 2(g-1)\log(\Gamma\xi_1) + \sum_{i=1}^{m_1}\sum_{j=1}^{r_1}(g-1)\log(Q_1 P_1)$$
$$+n_2\log\beta - \frac{\xi_2}{\psi\, y_{k(h)l}^2} - \sum_{k=1}^{m_2}\sum_{l=1}^{r_2}(2\xi_2+1)\log y_{k(h)l}$$
$$-\sum_{k=1}^{r_1}\sum_{l=1}^{r_2} 2(h-1)\log(\Gamma\xi_2) + \sum_{k=1}^{m_2}\sum_{l=1}^{r_2}(h-1)\log(Q_2 P_2) \tag{18}$$

In Eq.(17), v is the constant, $\alpha = \frac{2}{\Gamma\xi_1}\left(\frac{\xi_1}{\psi}\right)^{\xi_1}$, $\beta = \frac{2}{\Gamma\xi_2}\left(\frac{\xi_2}{\psi}\right)^{\xi_2}$, $Q_1 = \Gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(g)j}^2}\right)$, $Q_2 = \Gamma\left(\xi_2, \frac{\xi_2}{\psi\, y_{k(h)l}^2}\right)$, $P_1 = \gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(g)j}^2}\right)$, and $P_2 = \gamma\left(\xi_2, \frac{\xi_2}{\psi\, y_{k(h)l}^2}\right)$.

To determine the ML estimators for parameters ξ_1 and ξ_2, we employ a direct maximization approach on the log-likelihood function, denoted as $\log L_{(1)}$. This process involves calculating the first partial derivatives of $\log L_{(1)}$ with respect to both ξ_1 and ξ_2, independently. By setting these partial derivatives to zero and solving the resulting equations, we can derive the estimators. The expressions for these first partial derivatives, which form the basis for finding the maximum likelihood estimators, are given by:

$$\frac{\partial \log L_{(1)}}{\partial \xi_1} = n_1\left(\log \psi - 1 - \log \xi_1 - \frac{\partial}{\partial \xi_1}\log \Gamma\, \xi_1\right) - \frac{1}{\psi x_{i(g)j}^2} - \sum_{i=1}^{m_1}\sum_{j=1}^{r_1} 2\log x_{i(g)j}$$
$$- \sum_{i=1}^{m_1}\sum_{j=1}^{r_1} 2(g-1)\frac{\partial}{\partial \xi_1}\log \Gamma\, \xi_1 + \sum_{i=1}^{m_1}\sum_{j=1}^{r_1}(g-1)\frac{\partial}{\partial \xi_1}\log(Q_1 P_1) \tag{19}$$

$$\frac{\partial \log L_{(1)}}{\partial \xi_2} = n_2\left(\log \psi - 1 - \log \xi_2 - \frac{\partial}{\partial \xi_2}\log \Gamma\, \xi_2\right) - \frac{1}{\psi y_{k(h)l}^2} - \sum_{k=1}^{m_2}\sum_{l=1}^{r_2} 2\log y_{k(h)l}$$
$$- \sum_{k=1}^{m_2}\sum_{l=1}^{r_2} 2(h-1)\frac{\partial}{\partial \xi_2}\log \Gamma\, \xi_2 + \sum_{k=1}^{m_2}\sum_{l=1}^{r_2}(h-1)\frac{\partial}{\partial \xi_2}\log(Q_2 P_2) \tag{20}$$

The Eq. (19) and (20) representing the first partial derivatives do not have closed-form analytical solutions. The nature of these equations demands an iterative approach using numerical algorithms to arrive at solutions. By employing such an iterative approach, the ML estimators of the unknown parameters can be obtained based on the MRSS data when the set size is odd. Subsequently, the ML estimator of the stress-strength reliability can be derived by substituting the estimated values of the population parameters into the expression given in Eq. (8).

II. Second case: when set sizes are even

Consider a sample drawn from an Inverse Nakagami distribution with parameters (ξ_1, ψ), denoted as INAD (ξ_1, ψ), using MRSS with even set sizes. This sample can be represented as the union of two sets:$\{X_{i(q)j}, i = 1,2,\dots,q; j = 1,2,\dots,r_1\} \cup \{X_{i(q+1)j}, i = (q+1),\dots,m_1; j = 1,2,\dots,r_1\}$ where q equals $m_1/2$. Similarly, for a sample from INAD (ξ_2, ψ), using MRSS with even set sizes, we have the union of $\{Y_{k(u)l}, k = 1,2,\dots,u; l = 1,2,\dots,r_2\} \cup \{Y_{k(u+1)l}, k = (u+1),\dots,m_2; l = 1,2,\dots,r_2\}$, where u is $m_2/2$.
Given these samples, we can formulate the likelihood function $L_{(2)}$ for the observed data as follows:

$$L_{(2)} = \prod_{i=1}^{q}\prod_{j=1}^{r_1} f_q\left(x_{i(q)j}\right)\prod_{i=q+1}^{m_1}\prod_{j=1}^{r_1} f_{(q+1)}\left(x_{i(q+1)j}\right)$$
$$\prod_{k=1}^{u}\prod_{l=1}^{r_2} f_u\left(y_{k(u)l}\right)\prod_{k=u+1}^{m_2}\prod_{l=1}^{r_2} f_{(u+1)}\left(y_{k(u+1)l}\right)$$

where,

$$f_q\left(x_{i(q)j}\right) = 2\frac{m_1!}{(q-1)!\,q!}\left(\frac{1}{\Gamma\xi_1}\right)^{2q}\left[\Gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(q)j}^2}\right)\right]^{q-1}$$
$$\left[\gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(q)j}^2}\right)\right]^{q}\left(\frac{\xi_1}{\psi}\right)^{\xi_1} x_{i(q)j}^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi x_{i(q)j}^2}\right) \tag{21}$$

$$f_{(q+1)}\left(x_{i(q+1)j}\right) = 2\frac{m_1!}{(q-1)!\,q!}\left(\frac{1}{\Gamma\xi_1}\right)^{2q}\left[\Gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(q+1)j}^2}\right)\right]^{q}$$
$$\left[\gamma\left(\xi_1, \frac{\xi_1}{\psi\, x_{i(q+1)j}^2}\right)\right]^{q-1}\left(\frac{\xi_1}{\psi}\right)^{\xi_1} x_{i(q+1)j}^{-2\xi_1-1}\exp\left(-\frac{\xi_1}{\psi x_{i(q+1)j}^2}\right), \tag{22}$$

$$f_u\left(y_{k(u)l}\right) = 2\frac{m_2!}{(u-1)!\,u!}\left(\frac{1}{\Gamma\xi_2}\right)^{2u}\left[\Gamma\left(\xi_2, \frac{\xi_2}{\psi\, y_{k(u)l}^2}\right)\right]^{u-1} \tag{23}$$

$$\left[\gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{k(u)l}^2}\right)\right]^u\left(\frac{\xi_2}{\psi}\right)^{\xi_2} y_{k(u)l}^{-2\xi_2-1}\exp\left(-\frac{\xi_2}{\psi y_{k(u)l}^2}\right),$$

and

$$f_{(u+1)}\left(y_{k(u+1)l}\right)=2\frac{m_2!}{(u-1)!\,u!}\left(\frac{1}{\Gamma\xi_2}\right)^{2u}\left[\Gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{k(u+1)l}^2}\right)\right]^u$$
$$\left[\gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{k(u+1)l}^2}\right)\right]^{u-1}\left(\frac{\xi_2}{\psi}\right)^{\xi_2} y_{k(u+1)j}^{-2\xi_2-1} exp\left(-\frac{\xi_2}{\psi y_{k(u+1)l}^2}\right), \tag{24}$$

Eq. (21), (22), (23) and (24) are represent the pdfs of $X_{i(q)j}, X_{i(q+1)j}, Y_{k(u)l}$ and $Y_{k(u+1)l}$ respectively such that $x_{i(q)j} >, x_{i(q+1)j} >, y_{k(u)l}$ and $y_{k(u+1)l} > 0$. The log likelihood function of $L_{(2)}$ is given as

$$\log L_{(2)}=\log L_{(2)}{}^{*}+\log L_{(2)}{}^{**} \tag{25}$$

where,

$$\log L_{(2)}{}^{*}=\log c_1+\sum_{i=1}^{q}\sum_{j=1}^{r_1}(q-1)\log E_{1(q)}+\sum_{i=1}^{q}\sum_{j=1}^{r_1} q\log F_{1(q)}+qr_1\log\lambda_1$$
$$-\sum_{i=1}^{q}\sum_{j=1}^{r_1}(2\xi_1+1)\log x_{i(q)j}-\sum_{i=1}^{q}\sum_{j=1}^{r_1}\frac{\xi_1}{\psi x_{i(q)j}^2}$$
$$+\sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1} q\log E_{1(q+1)}+\sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}(q-1)\log F_{1(q+1)}+m_1r_1\log\lambda_1$$
$$-\sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}(2\xi_1+1)\log x_{i(q+1)j}-\sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}\frac{\xi_2}{\psi x_{i(q+1)j}^2} \tag{26}$$

and

$$\log L_{(2)}{}^{**}=\log c_2+\sum_{k=1}^{u}\sum_{l=1}^{r_2}(u-1)\log E_{2(u)}+\sum_{k=1}^{u}\sum_{l=1}^{r_2} u\log F_{2(u)}$$
$$+ur_2\log\lambda_2-\sum_{k=1}^{u}\sum_{l=1}^{r_2}(2\xi_2+1)\log y_{k(u)l}-\sum_{k=1}^{u}\sum_{l=1}^{r_2}\frac{\xi_2}{\psi y_{k(u)l}^2}$$
$$+\sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2} u\log E_{2(u+1)}+\sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}(u-1)\log F_{2(u+1)}$$
$$+m_2r_2\log\lambda_2-\sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}(2\xi_2+1)\log y_{k(u+1)l}$$
$$-\sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}\frac{\xi_2}{\psi y_{k(u+1)l}^2} \tag{27}$$

Here, c_1 and c_2 are the constants.

$\lambda_1=\frac{2}{(\Gamma\xi_1)^{2q}}\left(\frac{\xi_1}{\psi}\right)^{\xi_1}$, $\lambda_2=\frac{2}{(\Gamma\xi_2)^{2u}}\left(\frac{\xi_2}{\psi}\right)^{\xi_2}$,

$E_{1(q)}=\Gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{i(q)j}^2}\right)$, $E_{1(q+1)}=\Gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{i(q+1)j}^2}\right)$, $E_{2(u)}=\Gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{k(u)l}^2}\right)$,

$E_{2(u+1)}=\Gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{k(u+1)l}^2}\right)$, $F_{1(q)}=\gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{i(q)j}^2}\right)$, $F_{1(q+1)}=\gamma\left(\xi_1,\frac{\xi_1}{\psi\, x_{i(q+1)j}^2}\right)$,

$F_{2(u)}=\gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{k(u)l}^2}\right)$, $F_{2(u+1)}=\gamma\left(\xi_2,\frac{\xi_2}{\psi\, y_{k(u+1)l}^2}\right)$

In the process of differentiating the log-likelihood function $\log L_{(2)}$, the terms related to the $\log L_{(2)}{}^{**}$ expression will vanish when taking the derivative with respect to the parameter ξ_1. Conversely, the terms associated with the $\log L_{(2)}{}^{*}$ expression will drop out when differentiating with respect to the parameter ξ_2. The resulting first partial derivative expressions are as follows:

$$\begin{aligned}\frac{\partial \log L_2}{\partial \xi_1} = &\sum_{i=1}^{q}\sum_{j=1}^{r_1}(q-1)\frac{\partial}{\partial \xi_1}\log E_{1(q)} + \sum_{i=1}^{q}\sum_{j=1}^{r_1} q\frac{\partial}{\partial \xi_1}\log F_{1(q)} \\ &-2\sum_{i=1}^{q}\sum_{j=1}^{r_1}\log x_{i(q)j} + qr_1\left(\log \xi_1 - \log \psi - \frac{2q}{\xi_1}+1\right) \\ &-\sum_{i=1}^{q}\sum_{j=1}^{r_1}\frac{1}{\psi x_{i(q)j}} - \sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}\frac{1}{\psi x_{i(q+1)j}} \\ &+\sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1} q\frac{\partial}{\partial \xi_1}\log E_{1(q+1)} + \sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}(q-1)\frac{\partial}{\partial \xi_1}\log F_{1(q+1)} \\ &+m_1 r_1\left(\log \xi_1 - \log \psi - \frac{2q}{\xi_1}+1\right) - 2\sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}\log x_{i(q+1)j} \end{aligned} \tag{28}$$

and

$$\begin{aligned}\frac{\partial \log L_2}{\partial \xi_2} = &\sum_{k=1}^{u}\sum_{l=1}^{r_2}(u-1)\frac{\partial}{\partial \xi_2}\log E_{2(u)} + \sum_{k=1}^{u}\sum_{l=1}^{r_1} u\frac{\partial}{\partial \xi_2}\log F_{2(u)} \\ &-2\sum_{k=1}^{u}\sum_{l=1}^{r_2}\log y_{k(u)l} + ur_2\left(\log \xi_2 - \log \psi - \frac{2u}{\xi_2}+1\right) \\ &-\sum_{k=1}^{u}\sum_{l=1}^{r_2}\frac{1}{\psi y_{k(u)l}} - \sum_{k=q+1}^{m_2}\sum_{l=1}^{r_2}\frac{1}{\psi y_{k(u+1)l}} \\ &+\sum_{k=q+1}^{m_2}\sum_{l=1}^{r_2} u\frac{\partial}{\partial \xi_2}\log E_{2(u+1)} + \sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}(u-1)\frac{\partial}{\partial \xi_2}\log F_{2(u+1)} \\ &+m_2 r_2\left(\log \xi_2 - \log \psi - \frac{2u}{\xi_2}+1\right) - 2\sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}\log y_{k(u+1)l} \end{aligned} \tag{29}$$

The Eq. (28) and Eq. (29), which denote the first partial derivatives, do not have closed-form analytical solutions. As a result, an iterative numerical technique must be employed to solve these equations. By utilizing such an iterative approach, the ML estimators of the unknown parameters can be obtained when the MRSS data has even set sizes. Subsequently, the ML estimator of the $R = P_r(Y < X)$ can be derived by substituting the estimated values of the population parameters into the expression given in Eq. (8).

III. Third case: when stress has even and strength has odd set sizes

We examine the stress-strength reliability in a scenario where the samples for X and Y are drawn using different methods from Inverse Nakagami distributions (INAD). Specifically, samples of X are drawn from INAD using the Odd-set Median Ranked Set Sampling (OMRSS) technique. (ξ_1, ψ),while Y samples are drawn using Even-set Median Ranked Set Sampling technique (EMRSS) from INAD (ξ_2, ψ). For X, we denote the OMRSS observations as $X_{i(g)j}$,where i ranges from 1 to m_1, j from 1 to r_1, and g is calculated as $\frac{(m_1+1)}{2}$.

For Y, the EMRSS sample is represented as the union of two sets: $\{Y_{k(u)l}, k = 1,2,\ldots,u; l = 1,2,\ldots,r_2\} \cup \{Y_{k(u+1)l}, k = (u+1),\ldots,m_2; l = 1,2,\ldots,r_2\}$, where u equals $m_2/2$.

Given these sampling methods, we can express the likelihood function for the observed data as follows:

$$L_{(3)} = \prod_{i=1}^{m_1}\prod_{j=1}^{r_1} f_g\left(x_{i(g)j}\right) \prod_{k=1}^{u}\prod_{l=1}^{r_2} f_u\left(y_{k(u)l}\right) \prod_{k=u+1}^{m_2}\prod_{l=1}^{r_2} f_{(u+1)}\left(y_{k(u+1)l}\right)$$

$f_g(x_{i(g)j})$ and $f_u(y_{k(u)l})$ are given in Eq. (16) and (23), respectively. $f_{(u+1)}(y_{k(u+1)l})$ is given in Eq. (24). The log likelihood function of $L_{(3)}$ is given as follows:

$$\log L_{(3)} = \log L_{(3)}{}^{*} + \log L_{(3)}{}^{**} \tag{30}$$

where,

$$\begin{aligned}\log L_{(3)}{}^{*} &= \log a_1 + n_1 \log \alpha - \frac{\xi_1}{\psi\, x_{i(g)j}^2} - \sum_{i=1}^{m_1}\sum_{j=1}^{r_1} (2\xi_1 + 1)\log x_{i(g)j} \\ &- \sum_{i=1}^{m_1}\sum_{j=1}^{r_1} 2(g-1)\log(\Gamma\xi_1) + \sum_{i=1}^{m_1}\sum_{j=1}^{r_1} (g-1)\log(Q_1 P_1)\end{aligned} \tag{31}$$

and

$$\begin{aligned}\log L_{(3)}{}^{**} &= \log b_1 + \sum_{k=1}^{u}\sum_{l=1}^{r_2}(u-1)\log E_{2(u)} + \sum_{k=1}^{u}\sum_{l=1}^{r_2} u \log F_{2(u)} + u r_2 \log \lambda_2 \\ &- \sum_{k=1}^{u}\sum_{l=1}^{r_2}(2\xi_2+1)\log y_{k(u)l} - \sum_{k=1}^{u}\sum_{l=1}^{r_2}\frac{\xi_2}{\psi y_{k(u)l}^2} \\ &+ \sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2} u \log E_{2(u+1)} + \sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}(u-1)\log F_{2(u+1)} \\ &+ m_2 r_2 \log \lambda_2 - \sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}(2\xi_2+1)\log y_{k(u+1)l} \\ &- \sum_{k=u+1}^{m_2}\sum_{l=1}^{r_2}\frac{\xi_2}{\psi y_{k(u+1)l}^2}\end{aligned} \tag{32}$$

Here, a_1 and b_1 are the constants.

The ML estimators of the parameters ξ_1 and ξ_2 can be derived by maximizing the log-likelihood function $\log L_{(3)}$ directly with respect to ξ_1 and ξ_2, respectively. The resulting first partial derivative expressions with respect to ξ_1 and ξ_2 is given by Eq. (19) and (29), respectively. It appears that Eq. (19) and (29) do not have closed-form analytical solutions. As a result, a numerical technique must be utilized to obtain the solutions. Subsequently, the ML estimator of the stress-strength reliability can be obtained by substituting the estimated values of the population parameters into the expression provided in Eq. (8).

IV. Forth case: when strength has even and stress has odd set sizes

We examine the reliability estimator in a scenario where X samples are derived from the INAD distribution using EMRSS, while Y samples are obtained from the same distribution employing OMRSS. Let the $\{X_{i(q)j}, i = 1,2,\dots,q; j = 1,2,\dots,r_1\} \cup \{X_{i(q+1)j}, i = (q+1),\dots,m_1; j = 1,2,\dots,r_1\}$ be the observed EMRSS from INAD (ξ_1,ψ) distribution. Let the $Y_{k(h)l}$, where $k = 1,2,\dots,m_2, l = 1,2,\dots,r_2$ and $h = \left(\frac{m_2+1}{2}\right)$ be the observed OMRSS drawn from INAD (ξ_2,ψ) distribution. The likelihood of the observed variables will be given as follows:

$$L_{(4)} = \prod_{i=1}^{q}\prod_{j=1}^{r_1} f_q\left(x_{i(q)j}\right) \prod_{i=q+1}^{m_1}\prod_{j=1}^{r_1} f_{(q+1)}\left(x_{i(q+1)j}\right) \prod_{k=1}^{m_2}\prod_{l=1}^{r_2} f_h\left(y_{k(h)l}\right)$$

The pdfs expressions of $f_q(x_{i(q)j}), f_{(q+1)}(y_{k(q+1)l})$ and $f_h(y_{k(h)l})$ are given in Eq. (21), (22) and (17), respectively. Now, the log likelihood function of $L_{(4)}$ is given as

$$\log L_{(4)} = \log L_{(4)}{}^{*} + \log L_{(4)}{}^{**} \tag{33}$$

where,

$$\log L_{(4)}{}^{*} = \log a_2 + \sum_{i=1}^{q}\sum_{j=1}^{r_1}(q-1)\log E_{1(q)} + \sum_{i=1}^{q}\sum_{j=1}^{r_1} q\log F_{1(q)} + qr_1\log\lambda_1 - \sum_{i=1}^{q}\sum_{j=1}^{r_1}(2\xi_1+1)\log x_{i(q)j} - \sum_{i=1}^{q}\sum_{j=1}^{r_1}\frac{\xi_1}{\psi x_{i(q)j}^2} + \sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1} q\log E_{1(q+1)} + \sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}(q-1)\log F_{1(q+1)} + m_1 r_1\log\lambda_1 - \sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}(2\xi_1+1)\log x_{i(q+1)j} - \sum_{i=q+1}^{m_1}\sum_{j=1}^{r_1}\frac{\xi_2}{\psi x_{i(q+1)j}^2} \tag{34}$$

and

$$\log L_{(4)}{}^{**} = \log b_2 + n_2\log\beta - \frac{\xi_2}{\psi\, y_{k(h)l}^2} - \sum_{k=1}^{m_2}\sum_{l=1}^{r_2}(2\xi_2+1)\log y_{k(h)l} - \sum_{k=1}^{r_1}\sum_{l=1}^{r_2} 2(h-1)\log(\Gamma\xi_2) + \sum_{k=1}^{m_2}\sum_{l=1}^{r_2}(h-1)\log(Q_2P_2) \tag{35}$$

Here, a_2 and b_2 are the constants.

The ML estimators of the parameters ξ_1 and ξ_2 can be derived by directly maximizing the log-likelihood function $\log L_{(4)}$ with respect to ξ_1 and ξ_2, respectively. The resulting first partial derivative expressions with respect to ξ_1 and ξ_2 are given by Eq. (28) and (20), respectively.

It appears that Eq. (28) and (20) do not have closed-form analytical solutions. Consequently, a numerical technique must be employed to obtain the solutions. Finally, the ML estimator of the stress-strength reliability can be obtained by substituting the estimated values of the population parameters into the expression provided in Eq. (8).

VIII. Simulation study

To rigorously assess the efficacy of our proposed reliability estimation methods, we conducted a comprehensive simulation study. This computational investigation compares the performance of estimates derived from RSS and MRSS against those obtained through SRS. Our simulations span a diverse range of scenarios, providing crucial insights into the relative strengths of these sampling techniques under various conditions. The comparison is based on three key criteria: bias, mean square error (MSE), and relative efficiency (RE). The simulation setup involves various set sizes and number of cycles, specifically (m_x, m_y) = (2, 2), (3, 3), (2, 3), (3, 2), (4, 4), (5, 5), (4, 5), (5, 4),(6, 6),(7, 7),(6, 7),(7, 6),(8, 8),(9, 9),(8, 9),(9, 8) and $r_x = r_y = 5$. Consequently, the sample sizes for SRSS and MRSS are calculated as $n_1 = m_1 r_1$ and $n_2 = m_2 r_2$, respectively. For SRS samples, the sample sizes used are (n_1, n_2) = (10, 10),(15, 15),(10, 15),(15, 10),(20, 20),(25, 25),(20, 25),(25, 20),(30, 30),(35, 35),(30, 35),(35,

30),(40, 40),(45, 45),(40, 45),(45, 40). The parameter values are set to (ξ_1, ξ_2) = (2, 4), (3, 5), and (6, 8) and ψ = 3, 2 and 4 with the true system reliability values of R_{true} = 0.53909, 0.52465 and 0.51036. A total of 1000 random samples are generated from INAD (ξ_1, ψ) and INAD (ξ_2, ψ) distributions. The estimated reliability is evaluated using bias, mean square errors (MSEs) and relative efficiencies (REs).
The bias of estimate of R is defined as $Bias(\hat{R}) = E(\hat{R}) - R$. The efficiencies of RSS and MRSS with respect to SRS are defined as the ratios of their respective MSEs. Specifically, the efficiency of RSS relative to SRS is given by $\text{Effi}(R_{RSS}, R_{SRS}) = \left(\frac{\text{MSE}(R_{RSS})}{\text{MSE}(R_{SRS})}\right)$. Similarly, the efficiency of MRSS relative to SRS is $\text{Effi}(R_{MRSS}, R_{SRS}) = \left(\frac{\text{MSE}(R_{MRSS})}{\text{MSE}(R_{SRS})}\right)$.

Analysis of the simulation results, as presented in Tables 1 through 4, reveals a clear pattern of superior performance for the MRSS method. The efficiency of MRSS demonstrates a notable increase as the sample size or set size grows, highlighting its scalability and robustness. Across the various scenarios examined, MRSS consistently outperformed both RSS and SRS techniques. The efficiency gains of MRSS were substantial, with performance improvements ranging from 2 to 9 times that of SRS. This significant enhancement in efficiency underscores the potential of MRSS to dramatically improve the precision of reliability estimations, particularly in larger sample sizes.

While RSS also showed improved efficiency compared to SRS, its gains were more modest. The performance of RSS consistently fell between that of SRS and MRSS, demonstrating that while it offers improvements over simple random sampling, it does not match the marked benefits provided by MRSS. The consistent superiority of MRSS across all examined scenarios reinforces its robustness as a sampling technique. This consistency, coupled with its escalating efficiency as sample sizes increase, positions MRSS as a powerful tool for enhancing the accuracy and reliability of statistical estimations in various fields where precise sampling is crucial.

IX. Real life application

In this study, we examine two critical datasets originally explored by Louzada et al. [15] in their ground breaking research on the Inverse Nakagami distribution. These datasets offer valuable insights into the failure life of essential components in agricultural machinery: the motor and elevator systems. By repurposing this data, we aim to demonstrate the practical applications of our research in a novel context. Specifically, we utilize these datasets as stress and strength variables within our proposed reliability model, thereby bridging the gap between theoretical statistical frameworks and real-world engineering challenges. This approach not only validates the robustness of our model but also highlights its potential to enhance predictive maintenance strategies in the agricultural sector, potentially reducing downtime and improving overall operational efficiency. The datasets are shown in Table 5 and 6 respectively.

Prior to delving deeper into our analysis, it is imperative to conduct a comprehensive examination of the fundamental data characteristics. To evaluate the efficacy of our findings, we employ the Kolmogorov-Smirnov (K-S) test, a robust statistical tool, alongside its associated P-value (P-V). This methodology allows us to quantify the alignment between empirical observations and theoretical predictions. Our findings reveal encouraging results: for the initial dataset, we observe a K-S distance of 0.2222222, coupled with a P-V of 0.1267165, while the second dataset yields a K-S distance of 0.203125 and a P-V of 0.1324632. These metrics strongly suggest that our model provides an excellent fit to the observed data.To further illustrattion, we have generated visual representations of key statistical measures. Figures 1 and 2 present a suite of graphical analyses, including the probability-probability (PP) plots, and the estimated PDF and CDF for both datasets.

Table 1*: Five-Cycle Performance Assessment of R Estimators: Comparing SRS, RSS, and MRSS Using MSEs, Bias, and REs with Even X and Y Set Sizes*

Size		SRS			RSS			MRSS			Efficiency	
Sample	Set	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	RSS	MRSS
				$R_{true} = 0.53$		909, $\xi_1 = 2$, $\xi_2 = 4$, $\psi = 3$						
(10,10)	(2,2)	0.56946	0.03037	0.04324	0.49631	-0.04278	0.03769	0.58486	0.04577	0.04002	1.14717	1.08045
(20,20)	(4,4)	0.56571	0.02662	0.04445	0.58625	0.04716	0.04051	0.53355	-0.00554	0.01541	1.09729	2.88432
(30,30)	(6,6)	0.64507	0.10598	0.05312	0.52361	-0.01548	0.03893	0.54640	0.00731	0.00885	1.36447	6.00291
(40,40)	(8,8)	0.65460	0.11551	0.05303	0.55648	0.01739	0.03906	0.56059	0.02150	0.00585	1.35765	9.06352
				$R_{true} = 0.52$		465, $\xi_1 = 3$, $\xi_2 = 5$, $\psi = 2$						
(10,10)	(2,2)	0.61555	0.09090	0.06449	0.61898	0.09433	0.06371	0.61898	0.09433	0.06371	1.01227	1.01227
(20,20)	(4,4)	0.52749	0.00283	0.05350	0.48036	-0.04429	0.05148	0.51210	-0.01255	0.02167	1.03929	2.46854
(30,30)	(6,6)	0.59379	0.06914	0.06041	0.54640	0.02175	0.04822	0.52942	0.00477	0.01252	1.25297	4.82616
(40,40)	(8,8)	0.67451	0.14986	0.07686	0.59122	0.06657	0.05300	0.63883	0.11418	0.02076	1.45018	3.70180
				$R_{true} = 0.51$		036, $\xi_1 = 6$, $\xi_2 = 8$, $\psi = 4$						
(10,10)	(2,2)	0.59347	0.08311	0.09058	0.48625	-0.02411	0.08072	0.52468	0.01432	0.06753	1.12218	1.34132
(20,20)	(4,4)	0.54195	0.03159	0.07847	0.55111	0.04075	0.07787	0.48936	-0.02100	0.03016	1.00773	2.60200
(30,30)	(6,6)	0.52694	0.01658	0.08195	0.52342	0.01306	0.07941	0.49577	-0.01459	0.01746	1.03199	4.69495
(40,40)	(8,8)	0.59861	0.08825	0.08702	0.61958	0.10922	0.02351	0.49834	-0.01202	0.01260	3.70193	6.90640

Table 2: *Five-Cycle Performance Assessment of R Estimators: Comparing SRS, RSS, and MRSS Using MSEs, Bias, and REs with Odd X and Y Set Sizes*

Size		SRS			RSS			MRSS			Efficiency	
Sample	Set	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	RSS	MRSS
$R_{true} = 0.53909,\ \xi_1 = 2,\ \xi_2 = 4,\ \psi = 3$												
(15,15)	(3,3)	0.5598	0.02079	0.04426	0.52586	-0.01323	0.02323	0.51437	-0.02473	0.02296	1.90535	1.92790
(25,25)	(5,5)	0.5643	0.02520	0.04720	0.55155	0.01245	0.04387	0.54278	0.00368	0.01106	1.07598	4.26853
(35,35)	(7,7)	0.5710	0.03192	0.04354	0.52330	-0.01579	0.04222	0.56347	0.02437	0.00667	1.03138	6.53001
(45,45)	(9,9)	0.5676	0.02855	0.04601	0.57503	0.03593	0.04235	0.55684	0.01775	0.00502	1.08644	9.17347
$R_{true} = 0.52465,\ \xi_1 = 3,\ \xi_2 = 5,\ \psi = 2$												
(15,15)	(3,3)	0.5618	0.03722	0.06682	0.58101	0.05635	0.06486	0.50764	-0.01702	0.03143	1.03013	2.06363
(25,25)	(5,5)	0.5142	-0.01038	0.05891	0.53832	0.01367	0.05598	0.51871	-0.00594	0.01515	1.05219	3.69606
(35,35)	(7,7)	0.5990	0.07442	0.06588	0.51427	-0.01038	0.05891	0.53092	0.00627	0.00967	1.11837	6.81557
(45,45)	(9,9)	0.5690	0.04438	0.05424	0.53171	0.00706	0.05132	0.53450	0.00984	0.00689	1.05688	7.45084
$R_{true} = 0.51036,\ \xi_1 = 6,\ \xi_2 = 8,\ \psi = 4$												
(15,15)	(3,3)	0.6262	0.11587	0.09969	0.59814	0.08778	0.07544	0.45413	-0.05623	0.05190	1.32145	1.92098
(25,25)	(5,5)	0.5287	0.01841	0.08236	0.46952	-0.04084	0.06060	0.49233	-0.01803	0.02804	1.35903	2.93733
(35,35)	(7,7)	0.6100	0.09968	0.08841	0.52815	0.01779	0.08602	0.49821	-0.01215	0.01854	1.02769	4.64018
(45,45)	(9,9)	0.5306	0.02026	0.08540	0.54245	0.03209	0.08210	0.50403	-0.00633	0.01214	1.04019	6.76507

Table 3: *Five-Cycle Performance Assessment of R Estimators: Comparing SRS, RSS, and MRSS Using MSEs, Bias, and REs with Even X and Odd Y Set Sizes*

Size		SRS			RSS			MRSS			Efficiency	
Sample	Set	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	RSS	MRSS
$R_{true} = 0.53909,\ \xi_1 = 2,\ \xi_2 = 4,\ \psi = 3$												
(10,15)	(2,3)	0.59832	0.05923	0.02952	0.50722	-0.03187	0.02543	0.51804	-0.02105	0.02346	1.16078	1.25868
(20,25)	(4,5)	0.52992	-0.00917	0.03230	0.54190	0.00280	0.01421	0.55444	0.01534	0.00909	2.27386	3.55452
(30,35)	(6,7)	0.53044	-0.00865	0.02445	0.54381	0.00472	0.00906	0.54885	0.00976	0.00692	2.69836	3.53461
(40,45)	(8,9)	0.57242	0.03332	0.03878	0.53626	-0.00283	0.00818	0.55309	0.01400	0.00473	4.73941	8.20611
$R_{true} = 0.52465,\ \xi_1 = 3,\ \xi_2 = 5,\ \psi = 2$												
(10,15)	(2,3)	0.48004	-0.04461	0.03491	0.49116	-0.03349	0.03216	0.42064	-0.10401	0.01702	1.08525	2.05069
(20,25)	(4,5)	0.59240	0.06775	0.02786	0.51607	-0.00858	0.01607	0.53660	0.01195	0.00744	1.73327	3.74423
(30,35)	(6,7)	0.50902	-0.01563	0.04476	0.52552	0.00087	0.01579	0.52950	0.00485	0.00971	2.83468	4.61123
(40,45)	(8,9)	0.61911	0.09445	0.04332	0.53918	0.01453	0.02136	0.53723	0.01257	0.00689	2.02792	6.28423
$R_{true} = 0.51036,\ \xi_1 = 6,\ \xi_2 = 8,\ \psi = 4$												
(10,15)	(2,3)	0.57764	0.06728	0.06304	0.54479	0.03443	0.05722	0.50180	-0.00856	0.01715	1.10166	3.67651
(20,25)	(4,5)	0.54116	0.03080	0.04036	0.49541	-0.01495	0.02871	0.52390	0.01354	0.01236	1.40599	3.26508
(30,35)	(6,7)	0.57769	0.06733	0.08011	0.59971	0.08935	0.02643	0.50458	-0.00578	0.01850	3.03105	4.32964
(40,45)	(8,9)	0.57170	0.06134	0.03203	0.49048	-0.01988	0.02247	0.52134	0.01098	0.01231	2.72956	4.98361

Table 4: *Five-Cycle Performance Assessment of R Estimators: Comparing SRS, RSS, and MRSS Using MSEs, Bias, and REs with Odd X and Even Y Set Sizes*

Size		SRS			RSS			MRSS			Efficiency	
Sample	Set	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	R_{mle}	Bias	MSE	RSS	MRSS
$R_{true} = 0.53909,\ \xi_1 = 2,\ \xi_2 = 4,\ \psi = 3$												
(15,10)	(3,2)	0.57140	0.03231	0.04185	0.58546	0.04636	0.03984	0.53603	-0.00307	0.01462	1.05029	2.86314
(25,20)	(5,4)	0.49643	-0.04266	0.05727	0.49878	-0.04031	0.04846	0.53879	-0.00031	0.01506	1.18172	3.21802
(35,30)	(7,6)	0.46297	-0.07612	0.05012	0.51888	-0.02021	0.01678	0.53274	-0.00636	0.01552	2.98722	3.22860
(45,40)	(9,8)	0.58673	0.04764	0.04513	0.55541	0.01632	0.01833	0.55469	0.01560	0.00581	2.46267	7.77260
$R_{true} = 0.52465,\ \xi_1 = 3,\ \xi_2 = 5,\ \psi = 2$												
(15,10)	(3,2)	0.52226	-0.00239	0.05478	0.55902	0.03437	0.05051	0.53186	0.00721	0.04714	1.08450	1.16210
(25,20)	(5,4)	0.52432	-0.00033	0.06062	0.59991	0.07525	0.02878	0.50497	-0.01968	0.02072	2.10606	2.92622
(35,30)	(7,6)	0.52680	0.00214	0.05853	0.52225	-0.00240	0.02135	0.53108	0.00643	0.01307	2.74155	4.47936
(45,40)	(9,8)	0.53478	0.01013	0.05768	0.63350	0.10884	0.02006	0.53247	0.00782	0.00795	2.87524	7.25761
$R_{true} = 0.51036,\ \xi_1 = 6,\ \xi_2 = 8,\ \psi = 4$												
(15,10)	(3,2)	0.53337	0.02301	0.07999	0.50495	-0.00541	0.07284	0.49789	-0.01247	0.07005	1.09819	1.14201
(25,20)	(5,4)	0.53619	0.02583	0.04755	0.47904	-0.03132	0.03769	0.59782	0.08746	0.02268	1.26182	2.09649
(35,30)	(7,6)	0.55857	0.04822	0.08251	0.50128	-0.00908	0.02183	0.58730	0.07694	0.02948	2.79915	3.77979
(45,40)	(9,8)	0.56147	0.05111	0.08335	0.59939	0.08903	0.02352	0.50868	-0.00168	0.01538	3.54394	5.41820

Table 5: *Dataset Related to the Agricultural Machine's Elevator*

1	1	2	7	21
1	1	3	7	23
1	1	3	9	23
1	1	3	9	24
1	1	4	11	25
1	1	4	11	31
1	2	4	11	56
1	2	5	12	61
1	2	6	17	61
1	2	7	17	122
1	2	7	17	

Table 6: *Dataset Related to the Agricultural Machine's Motor*

1	1	2	5	11	22
1	1	2	5	11	24
1	1	3	5	12	29
1	1	3	5	12	32
1	1	3	7	13	33
1	1	4	8	16	33
1	1	4	8	17	41
1	1	4	9	17	41
1	2	4	9	18	121
1	2	5	11	18	
1	2	5	11	18	

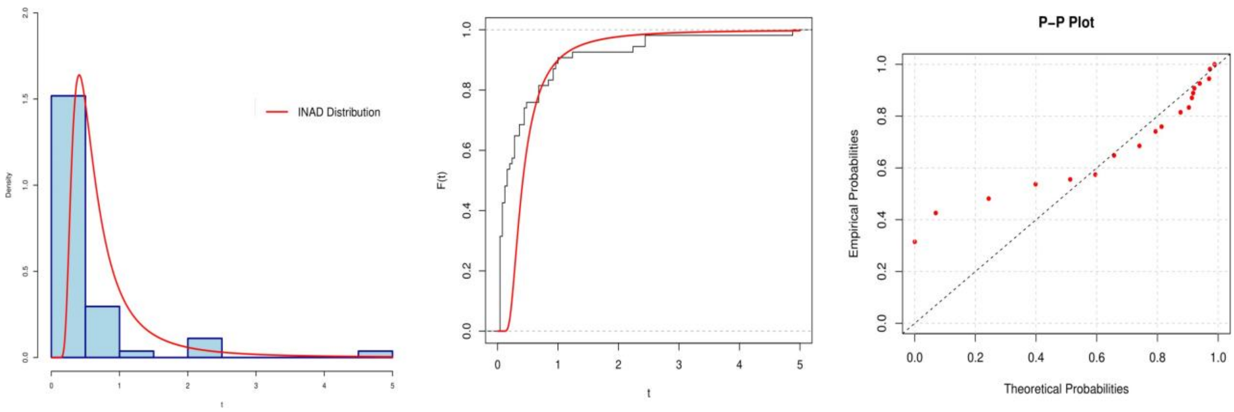

Figure 1: *The PDF, CDF and P-P Plots of the INAD distribution for Elevator dataset*

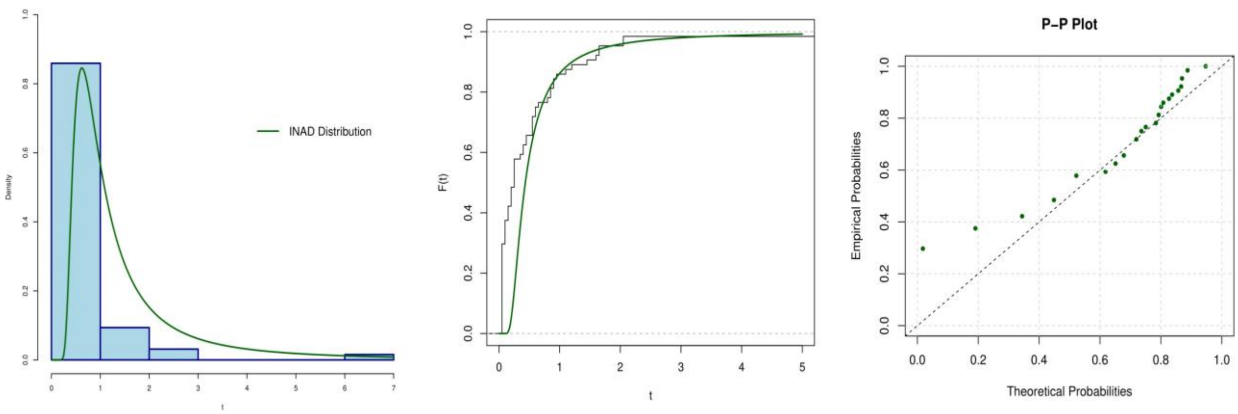

Figure 2: *The PDF, CDF and P-P Plots of the INAD distribution for Motor dataset*

Table 7: *Stress-strength R = P(X<Y) estimate of the datasets*

Sample Size	Set	SRS	RSS	MRSS
Size			Cycle = 3	
6,6	2,2	0.8577746	0.8679643	0.9364125
9,9	3,3	0.7931385	0.8590813	0.9664752
9,6	3,2	0.8244659	0.8781348	0.9754125
6,9	2,3	0.7849362	0.8546207	0.9599871
Size			Cycle = 5	
20,20	4,4	0.8250181	0.8455731	0.9236574
25,25	5,5	0.8302296	0.8632833	0.9335574
25,20	5,4	0.8085325	0.8447445	0.9678474
20,25	4,5	0.8237667	0.8503601	0.9744001

Our analysis of Data 5and 6 through three distinct sampling methodologies- MRSS, RSS, and SRS reveals a compelling pattern. As we increase the sample size, or more specifically, as the set size grows, we observe the stress-strength estimate steadily approaching unity. This trend is particularly pronounced in the case of MRSS. A close examination of Table 7 illuminates a striking feature: the stress-strength (R) estimates derived from MRSS consistently outperform those obtained via RSS and SRS. Figure 3 presents a comparative analysis of stress-strength reliability using SRS, RSS, and MRSS methods, focusing on different set sizes for the third and fifth cycles. This superiority of MRSS is not merely a numerical quirk but a recurring theme throughout our analysis. The persistent edge maintained by MRSS estimates lends credence to our earlier findings and hints at the method's robustness in specific sampling scenarios. The consistent outperformance of MRSS carries significant implications. It not only corroborates our previous conclusions but also sheds light on the potential benefits of employing MRSS in certain research contexts. This finding opens up intriguing avenues for further exploration, particularly in situations where precise stress-strength estimation is crucial.

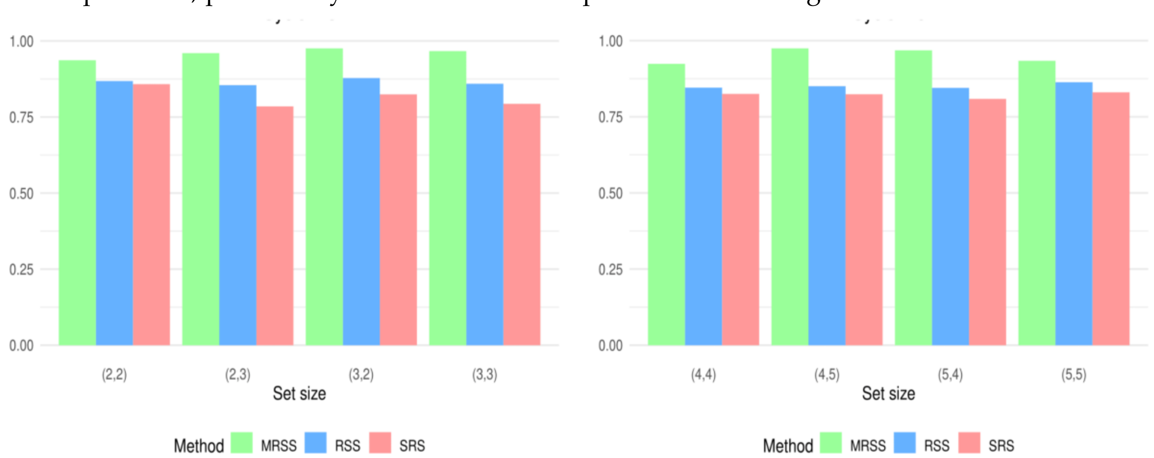

Figure 3: *Stress-strength reliability under SRS, RSS and MRSS at various set sizes for cycle 3 and 5*

X. Conclusion

This research delves into the intricate realm of stress-strength reliability estimation, focusing on scenarios where both stress and strength variables are characterized by the Inverse Nakagami distribution. Our investigation encompasses a triad of sampling methodologies: SRS, RSS, and MRSS, with the latter explored for both odd and even set sizes. Through rigorous numerical analysis, we've scrutinized the efficacy of various estimators across these diverse sampling schemes. The results unequivocally demonstrate the superiority of MRSS-based stress-strength estimates, which consistently outshine their counterparts derived from SRS and RSS data. Interestingly, RSS-based estimates typically occupy a middle ground in terms of performance, bridging the gap between SRS and MRSS. These theoretical findings find robust corroboration in real-world applications, lending credence to the practical utility of our research. The implications of this study extend beyond its immediate scope, paving the way for future investigations in this domain.

Prospective research avenues could explore the intriguing dynamics of heterogeneous sampling methods for stress and strength data. For instance, scholars might examine stress-strength estimation in scenarios where strength data is procured through RSS, while stress measurements employ MRSS with varying set sizes, or vice versa. Furthermore, the horizon of MRSS reliability inference could be expanded to encompass cases where stress and strength variables adhere to distinct probability distributions, potentially unveiling new insights in this critical field of study.

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