MODELING STRENGTH OF THE AIRCRAFT WINDOW GLASS BY TRUNCATED SHAMBHU DISTRIBUTION

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Abstract

Failure of regular distributions to appropriately model the real life scenarios demands the new distributions, which will be more useful. Truncated distributions are found to model many real datasets more precisely than corresponding regular distribution. In this paper, a novel truncated Shambhu distribution (TS) is proposed to model the data, which is naturally constrained by physical, financial, or environmental factors. Some statistical properties including moments, survival and hazard functions, are discussed. For better understanding of the proposed distribution, plots of the distribution are provided. Estimation of parameters is discussed using the maximum likelihood estimation method. Finally, an application based on real data is considered to illustrate the practical relevance of the proposed distribution.

Keywords: Lifetime distributions, survival function, hazard function, lifetime data

I. Introduction

Traditional statistical distributions often fail to capture the intricate patterns found in modern datasets, which may exhibit skewness, heavy tails, or multimodal behaviour. The proliferation of lifetime distributions arises from the fact that each is constructed under specific assumptions, and even minor modifications to these assumptions can result in the formulation of a distinct distribution. Distributions such as the exponential and Weibull have been commonly employed for modeling lifetime data across various applied fields. Researchers have attempted to introduce new distributions to model modern datasets. Gupta and Kundu [1] have generalised the exponential distribution for more flexibility. Similar attempts have been made on Weibull distribution, of which exponential distribution is considered a particular case. Sarhan and Zaindin [2] have introduced three parameter modified Weibull distribution for extra feasibility in modeling real life datasets. Use of MWD in various approaches for analysing survival datasets is increasing because of its versatility. Sutar et al.[3] analyzed the accelerated failure time based two component parallel load sharing system with component lifetimes following modified Weibull distribution. Sutar [4] developed Monte Carlo Expectation Maximization algorithm for parameter estimation of modified Weibull distribution. Shanker [5] proposed a new lifetime distribution named Shreekant distribution which gives improved fits to certain lifetime datasets than those by other existing ones. For better modeling of

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over-dispersed biological count data, Shanker et al. [6] introduced Poisson-Suja distribution as a Poisson mixture of Suja distribution. Attempts have also been made to generalise the existing distributions for enhancing practical applications. Prodhani and Shanker [7] generalised Sujatha distribution, proposed by Nwikpe and Iwoke [8]. Shanker [9] proposed Shanker distribution, which gives better fit than exponential and Lindley distribution for modeling certain real lifetime data-sets. In recent years, numerous lifetime distributions - such as the Akash (Shanker [10]), Aradhana (Shanker [11]), Sujatha (Shanker [12]), Rani (Shanker [13]) - have been proposed in the statistical literature for modeling lifetime data.

There are certain limitations for traditional distributions, which may not fit well for highly skewed, heavy tailed, multimodal datasets. The Shambhu Distribution, proposed, by Shanker [14], addresses few of these limitations by offering a versatile framework capable of accurately representing a wide range of data behaviours. This distribution provides a robust alternative for modelling lifetime and survival data where precision and adaptability are crucial.

The Shambhu distribution is a relatively lesser-known but interesting continuous probability distribution, particularly studied in the field of statistics and reliability analysis. The Shambhu distribution was introduced by Shanker [14] as a one-parameter family of distributions for modeling real life data-sets from biomedical science and engineering. The Shambhu distribution is a mixture of six distributions, viz.- exponential (θ), gamma$(2,\theta)$, gamma$(3,\theta)$, gamma$(4,\theta)$, gamma$(5,\theta)$, gamma$(6,\theta)$with mixing proportions respectively as-

$$\frac{\theta^5}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120},\quad \frac{\theta^4}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120},$$
$$\frac{2\theta^3}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120},\quad \frac{6\theta^2}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120},$$
$$\frac{24\theta}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120},\quad \frac{120}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}$$

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Shambhu distribution introduced by Shanker [14] are given by-

$$f(x,\theta) = \frac{\theta^6}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}(1+x+x^2+x^3+x^4+x^5)e^{-\theta x}; x>0, \theta>0 \qquad (1)$$

$$F(x,\theta) = 1-\left[1+\frac{\theta^5(x^5+x^4+x^3+x^2+x)+\theta^4(5x^4+4x^3+3x^2+2x)+2\theta^3(10x^3+6x^2+3x)+12\theta^2(5x^2+2x)+120\theta x}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}\right]e^{-\theta x}; x>0, \theta>0 \qquad (2)$$

Prabavathi and Elangovan [15] have recently generalised the Shambhu distribution by proposing its length biased version. Most of the times, the lifetime data is constrained between two values because of certain reasons. Using the regular lifetime distribution to model such a data may not produce favourable reasons. Therefore it is a practical need to use truncated versions of lifetime distributions in such cases. This paper attempts to define truncated Shambhu distribution, for increasing the applicability of Shambhu distribution. The rest of the paper is organised as follows. Section 2 proposes the new truncated Shambhu distribution. In section 3, the statistical properties of the truncated Shambhu distribution are studied. Application of the proposed distribution to a real life dataset is discussed in section 4. Section 5 summarises the conclusions.

II. Truncated Shambhu (TS) Distribution

The development of new truncated distributions is motivated by the need to model real-world data more accurately when observations are subject to limitations or constraints. In many practical situations, data may be constrained due to truncation—either from the left, right, or both ends—resulting in the exclusion of values that fall outside a certain range. Traditional distributions may fail to adequately capture the underlying behavior of such datasets, leading to biased parameter estimates and incorrect inferences. By defining new truncated versions of existing or novel distributions, one can better accommodate the characteristics of censored or truncated data, enhance model flexibility, and improve the accuracy of statistical inference in diverse fields such as survival analysis, reliability engineering, and economics. With this motivation, an attempt is made in this paper to propose new truncated Shambhu distribution for better modeling of truncated lifetime data. As a general scenario, truncation is considered from both the sides. Let X be the lifetime random variable (r.v.), with truncation limits as a and b, $0 < a < b < \infty$, i.e. $a < X < b$. The r.v. X is said to follow truncated Shambhu (TS) distribution truncated below a and above b, if its density is given as-

$$f_T(x;\theta,a,b) = \frac{(1+x+x^2+x^3+x^4+x^5)e^{-\theta x}}{\sum_{k=0}^{5}\frac{\gamma(k+1,a\theta)-\gamma(k+1,b\theta}{\theta^{k+1}}}; a < x < b, \theta > 0 \tag{3}$$

The distribution function (cdf) of truncated Shambhu distribution is given as follows.

$$F_T(x;\theta,a,b) = \frac{\sum_{k=0}^{5}\frac{\gamma(k+1,a\theta)-\gamma(k+1,x\theta}{\theta^{k+1}}}{\sum_{k=0}^{5}\frac{\gamma(k+1,a\theta)-\gamma(k+1,b\theta}{\theta^{k+1}}} \ ; \ a < x < b, \theta > 0 \tag{4}$$

It is denoted as $X \sim TS(\theta, a, b)$.

The plots of density for different values of a, b and θ are displayed in the Figure 1, Figure 2 and Figure 3 respectively. Also, the plots of cdf for different values of a, b and θ are displayed in the Figure 4, Figure 5 and Figure 6 respectively.

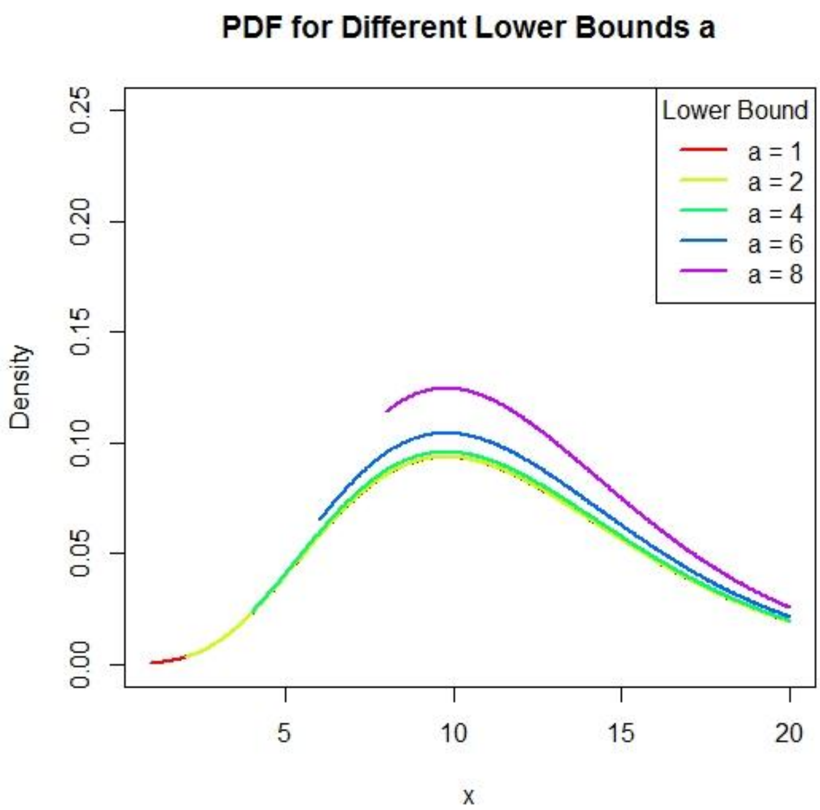

Figure 1: *Density function of TS for different values of a.*

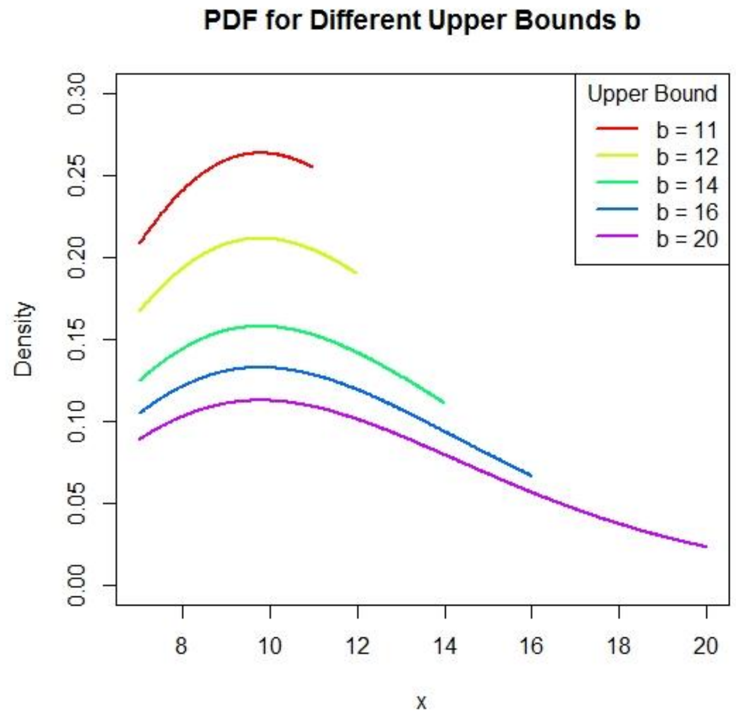

Figure 2: *Density function of TS for different values of b.*

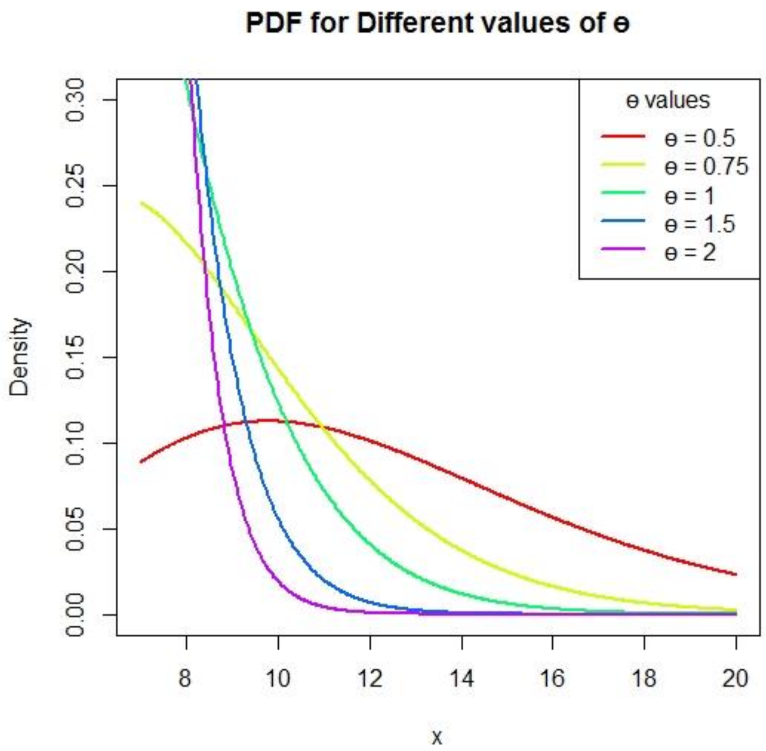

Figure 3: *Density function of TS for different values of θ.*

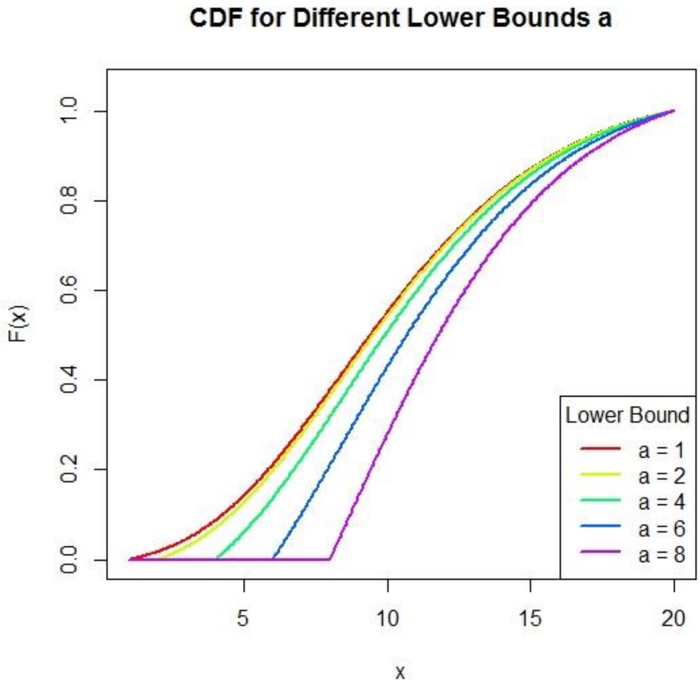

Figure 4: *Distribution function of TS for different values of a.*

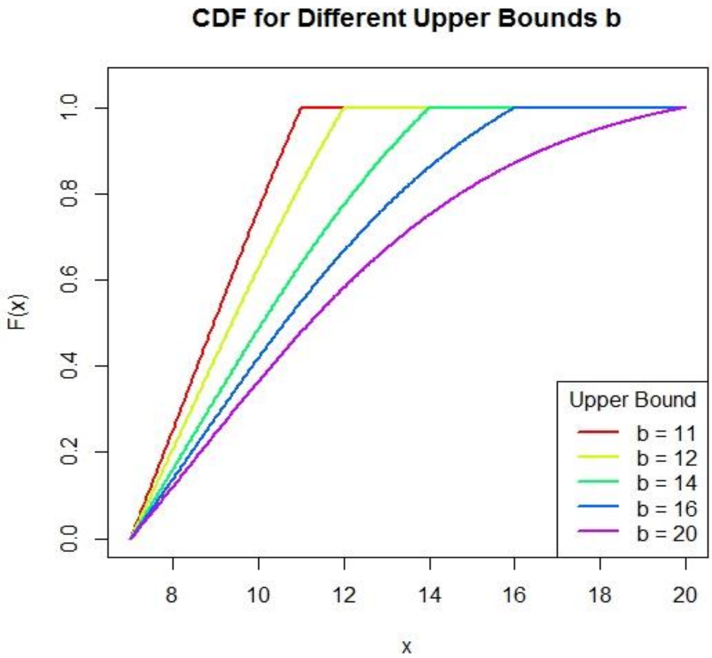

Figure 5: *Distribution function of TS for different values of b.*

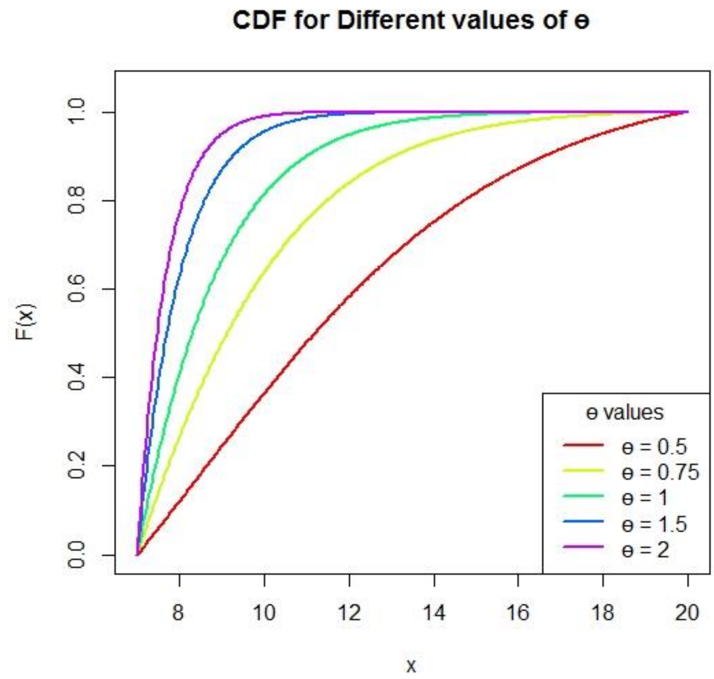

Figure 6: *Distribution function of TS for different values of θ.*

III. Statistical Properties

In this section, various fundamental properties, such as mean, variance, cdf etc. of the proposed truncated Shambhu distribution are studied.

I. **Moments**

The r^{th} moment about origin of truncated Shambhu distribution is

$$\mu_r^{'} = \frac{1}{\sum_{k=0}^{5} \frac{\gamma(k+1,a\theta)-\gamma(k+1,b\theta}{\theta^{k+1}}} \left[\sum_{i=1}^{5} \left\{\left[\frac{-1}{\theta} b^{r+i} e^{-\theta b} + \frac{1}{\theta} a^{r+i} e^{-\theta a}\right] + \frac{r+i}{\theta} \int_a^b x^{r+(i-1)} e^{-\theta x} dx\right\}\right] \quad (5)$$

Where $r = 1,2,3, \ldots$ and $\theta > 0$.

Therefore, mean and variance of this distribution are obtained as –

$$E(X) = \frac{1}{\sum_{k=0}^{5} \frac{\gamma(k+1,a\theta)-\gamma(k+1,b\theta}{\theta^{k+1}}} \left[\sum_{i=1}^{5} \left\{\left[\frac{-1}{\theta} b^{1+i} e^{-\theta b} + \frac{1}{\theta} a^{1+i} e^{-\theta a}\right] + \frac{1+i}{\theta} \int_a^b x^{i} e^{-\theta x} dx\right\}\right] \quad (6)$$

$$V(X) = \frac{1}{\sum_{k=0}^{5} \frac{\gamma(k+1,a\theta)-\gamma(k+1,b\theta}{\theta^{k+1}}} \left[\sum_{i=1}^{5} \left\{\left[\frac{-1}{\theta} b^{2+i} e^{-\theta b} + \frac{1}{\theta} a^{2+i} e^{-\theta a}\right] + \frac{2+i}{\theta} \int_a^b x^{i+1} e^{-\theta x} dx\right\}\right]$$
$$- \left\{\frac{1}{\sum_{k=0}^{5} \frac{\gamma(k+1,a\theta)-\gamma(k+1,b\theta}{\theta^{k+1}}} \left[\sum_{i=1}^{5} \left\{\left[\frac{-1}{\theta} b^{1+i} e^{-\theta b} + \frac{1}{\theta} a^{1+i} e^{-\theta a}\right] + \frac{1+i}{\theta} \int_a^b x^{i} e^{-\theta x} dx\right\}\right]\right\} \quad (7)$$

II. Survival function and hazard rate

The survival function, $S(x)$, is the probability that a subject survives longer than time x. Suppose $X$is a lifetime random variable representing the time until a specified event of interest is occurred, then the survival function of $X$is defined as $S(x) = 1 - F(x)$. From Eq. 4, the survival function of TS distribution is

$$S(x;\theta,a,b) = 1 - \frac{\sum_{k=0}^{5} \frac{\gamma(k+1,a\theta)-\gamma(k+1,x\theta)}{\theta^{k+1}}}{\sum_{k=0}^{5} \frac{\gamma(k+1,a\theta)-\gamma(k+1,b\theta)}{\theta^{k+1}}} \; ; a < x < b, \theta > 0 \tag{8}$$

Consequently, the ratio of pdf and survival function, is given by $h(x) = f(x)/S(x)$, which is called the hazard function. From the pdf in Eq.(3) and the survival function in Eq.(8), we have

$$h(x;\theta,a,b) = \frac{(1+x+x^2+x^3+x^4+x^5)e^{-\theta x}}{\sum_{k=0}^{5} \frac{\gamma(k+1,x\theta)-\gamma(k+1,b\theta)}{\theta^{k+1}}} \tag{9}$$

The graphical display of survival function as well as hazard function for different values of θ is given in Figure 7 and Figure 8 respectively.

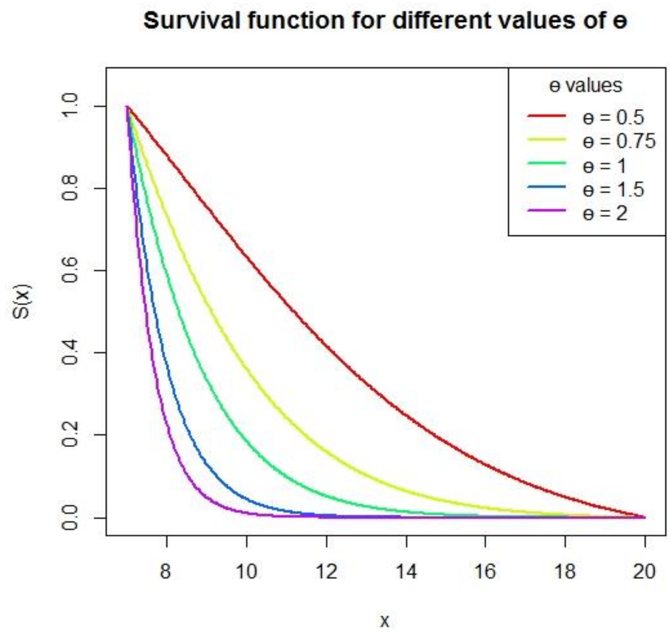

Figure 7: *Survival function of TS for different values of* θ.

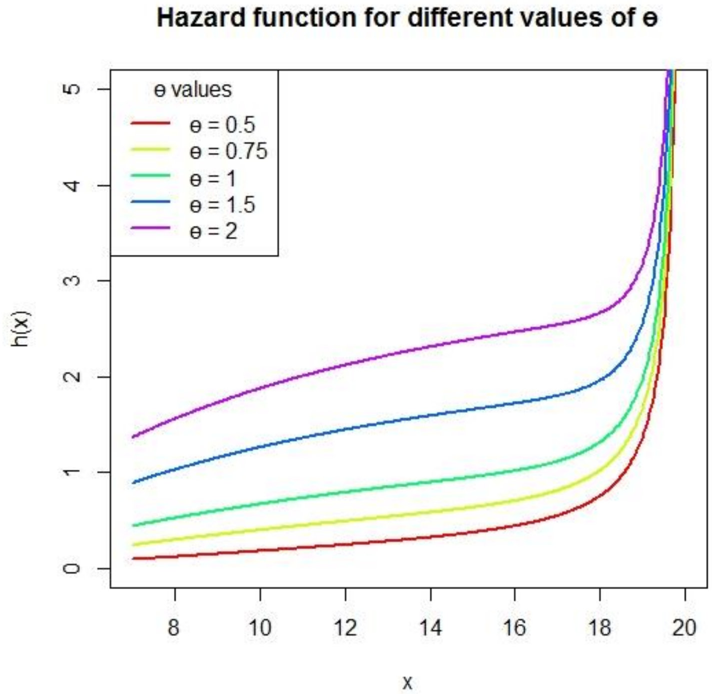

Figure 8: *Hazard function of TS for different values of* θ.

III. Parameter Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample sized n from $TS(\theta, a, b)$. Then the likelihood function L of $TS(\theta, a, b)$ is given by

$$L(\theta, a, b / x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \tag{10}$$

The log likelihood function is-

$$\log L = \sum_{i=1}^{n} \log f(x; \theta, a, b)$$

$$= \sum_{i=1}^{n} \log(1 + x_i + x_i^2 + x_i^3 + x_i^4 + x_i^5) - \theta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} log\left[\sum_{k=0}^{5} \frac{\gamma(k+1, a\theta) - \gamma(k+1, b\theta)}{\theta^{k+1}}\right]$$

$$= -\sum_{i=1}^{n} x_i - n\left\{\frac{1}{\sum_{k=0}^{5} \frac{\gamma(k+1, a\theta) - \gamma(k+1, b\theta)}{\theta^{k+1}}} \sum_{k=0}^{5} \frac{(a^{k+1}\theta^k e^{-a\theta} - b^{k+1}\theta^k e^{-b\theta})\theta^{k+1} - (\gamma(k+1, b\theta))(k+1)\theta^k}{\theta^{2(k+1)}}\right\}$$

As $a < x_i < b, i = 1, 2, \ldots, n,$ the maximum likelihood (ML) estimators for a and b are $\min\{x_1, x_2, \ldots, x_n\}$ and $\max\{x_1, x_2, \ldots, x_n\}$ respectively. Then the maximum likelihood estimate for the parameter θ is obtained by solving the equation $\frac{\partial \log L}{\partial \theta} = 0$. As this condition doesn't provide an expression in a closed form, it can be solved using any of the numerical method, such as bisection method, Newton-Raphson method etc.

IV. Application to real life dataset

To demonstrate the applicability of the proposed distribution, a real-life dataset was analyzed. The proposed truncated Shambhu distribution was fitted to this dataset, and its performance was evaluated and compared with other existing lifetime distributions using appropriate goodness-of-fit criteria. The data consists of the strengths of glass of the aircraft window, as reported Fuller et al. [16], which is given in Table 1.

The comparison for goodness of fit among various distributions is made based on different criteria, such as Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICC), Bayesian Information Criterion (BIC) and Kolmogorov-Smirnov Statistics (K-S Statistics) which are given respectively as follows:. The formulae for computing the AIC, AICC, BIC, K-S statistics are as follows-

$$AIC = -2 \log L + 2k$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$BIC = -2 \log L + k \log(n)$$

$$K - S = sup_x |F_n(x) - F_0(x)|$$

where $\log L$ denotes the log-likelihood function evaluated at the maximum likelihood estimates of the parameters, k is the parameter number of any distributions and n is the sample size, $F_n(x)$ and $F_0(x)$ are the empirical distribution function and the theoretical cumulative distribution of distributions being tested, respectively.

Table 1: *Data on strengths of aircraft window glass.*

Strength of aircraft glass window				
18.83	20.80	21.657	23.03	23.23
24.05	24.321	25.50	25.52	25.80
26.69	26.77	26.78	27.05	27.67
29.90	21.11	33.20	33.73	33.76
33.89	34.76	35.75	35.91	36.98
37.08	37.09	39.58	44.045	45.29
45.381				

Results of the parameter estimation and goodness of fit test are shown in Table 2. For testing the goodness of fit test (minimum of K-S values) in the data set, the observations about strength data of glass of the aircraft window is distributed as close to the TS distribution, with $\hat{\theta} = 0.1801312, \hat{a} = 18.83$, and $\hat{b} = 45.381$. The fit of TS distribution is more convincing than the other potential distributions such as Shambhu, truncated Ishita, Devya, Amarendra, Sujatha, Aradhana, Akash, Shanker, Lindley, Exponential, which can be seen from AIC and BIC values as given in Table 1.

Table 2: *MLE's, AIC, AICC, BIC and K-S statistics of the fitted distributions of data set*

Model	Parameter estimate	AIC	AICC	BIC	K-S statistic
TS	0.1801312	-84.1398	-84.00187	-82.70581	0.1018544
Shambhu	0.193397	225.40	225.53	226.83	0.167
Truncated Ishita	0.0883	117.9686	_	119.4025	0.1115
Devya	0.160872	229.68	229.82	231.82	0.193
Amrendra	0.128292	235.41	235.55	236.84	0.225
Sujatha	0.095610	243.50	243.64	244.94	0.270
Aradhana	0.094318	244.23	244.37	245.66	0.274
Akash	0.097062	242.68	242.82	244.11	0.266
Shanker	0.064712	254.35	254.49	255.78	0.326
Lindley	0.062988	255.99	256.13	257.42	0.333
Exponential	0.032455	276.53	276.67	277.96	0.426

IV. Conclusions

In this paper, a truncated Shambhu distribution is proposed which is more appealing than the Shambhu distribution, as the data on lifetime variables is better modeled using truncated distribution. The moments, distribution function and survival function for the proposed distribution are obtained. The procedure for parametric inference for this distribution is also discussed. To highlight the practical relevance of the proposed distributional model, a real life application is discussed. For the considered data of strengths of glass of the aircraft window, the truncated Shambhu distribution is the better model than other existing counterparts.

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