PARAMETRIC ESTIMATION OF THE PROCESS CAPABILITY INDEX $\mathbb{S}''_{pk}$ AND ITS APPLICATION TO ELECTRONIC INDUSTRIES

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Abstract

The proposed index is the process capability index used in the electronics industries to measure the capability of the process. This article focuses on process capability index, specifically applicable to normal random variables. The article has three main objectives: Firstly, we explore various classical estimation methods for the proposed index from frequentist approaches for normal distributions and compare their performance based on mean squared errors. Second, we calculate the classical confidence interval for the proposed index, which includes the asymptotic confidence interval. Third, we examine both Bayes point and interval estimation under symmetric and asymmetric loss functions for the proposed index. A Monte Carlo and Markov Chain Monte Carlo simulation study is conducted to compare the performance of the classical and the Bayes estimates of the proposed index for some set of parameters. Finally, to demonstrate the applicability of this index, two real data sets from the electronics industry are re-analyzed.

Keywords: Asymptotic confidence interval, Bayesian estimation, classical methods of estimation, point and interval estimation.

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Abbreviations and notations

$\mathbb{ACI}$	:	Asymptotic confidence interval.
$\mathbb{AW}$	:	Average width.
$\mathbb{BE}$	:	Bayes estimate.
$\mathbb{CI}$	:	Confidence interval.
$\mathbb{CP}$	:	Coverage probability.
$\mathbb{CME}$	:	Cramér-von-Mises estimator.
$\mathbb{CDF}$	:	Cumulative distribution function.
$\mathbb{HPD}$	:	Highest posterior density.
$\mathbb{IG}$	:	Inverse gamma.
$\mathbb{KLF}$	:	K-loss function.
$\mathbb{LSE}$	:	Least squares estimator.
$\mathbb{LINEX}$	:	Linear exponential.
$\mathbb{LSLF}$	:	Logarithmic squared error loss function.
$\mathbb{MCMC}$	:	Markov Chain Monte Carlo.
$\mathbb{MLE}$	:	Maximum likelihood estimator.
$\mathbb{MPSE}$	:	Maximum product of spacing estimator.
$\mathbb{MSE}$	:	Mean squared error.
$\mathbb{MSELF}$	:	Modified squared error loss function.
$\mathbb{MC}$	:	Monte Carlo.
$\mathbb{NLM}$	:	Non-linear minimization.
$\mathbb{PE}$	:	Percentile estimator.
$\mathbb{PR}$	:	Posterior risk.
$\mathbb{PDF}$	:	Probability density function.
$\mathbb{PCI}$	:	Process capability index.
$\mathbb{SELF}$	:	Squared error loss function.
$\mathbb{WLSE}$	:	Weighted least square estimator.
$\mathbb{WSELF}$	:	Weighted squared error loss function.

1. Introduction

Across all manufacturing sectors, processes are routinely evaluated to determine their capacity using various instruments and methods to verify whether a process under observation meets a given set of requirements or standards. $\mathbb{PCI}$ is the tool of choice for manufacturers because it has been proven to help with decision-making and improve process performance. $\mathbb{PCI}$s have garnered significant attention, particularly in the field of quality assurance. When studying traditional capability indices, it is often assumed that with a process mean κ and process standard deviation ζ a normal probability model is used. This assumption has contributed to the widespread interest in $\mathbb{PCI}$s, particularly in quality assurance. For a deeper understanding of various capability indices, readers may refer to: $\mathbb{C}_p$ Juran, Gryna and Bingham [11], $\mathbb{C}_{pk}$ Kane [12], $\mathbb{C}_{pm}$ Chang, Cheng and Spiring [4]; Hsiang and Taguchi [10], $\mathbb{C}_{pmk}$ Pearn, Kotz and Jhonson [15]; Wu and Pearn [27], $\mathbb{C}_p(u,v)$ Vannman [25], $\mathbb{S}_{pk}$ Boyles [1]; Chen and Pearn [2]; Saha [17] and many others. The Boyles [1] $\mathbb{PCI}$, $\mathbb{S}_{pk}$ was defined as

$$\mathbb{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\kappa}{\zeta}\right) + \frac{1}{2}\Phi\left(\frac{\kappa-\mathbb{L}}{\zeta}\right)\right\} \tag{1}$$

here, $\mathbb{U}$ and $\mathbb{L}$ represent the upper and lower specification limits, respectively, and $\Phi(\cdot)$ is the $\mathbb{CDF}$ of $N(0,1)$ and defined as:

$$\mathbb{F}\left(y|\kappa,\zeta\right) = \Phi\left(\frac{y-\kappa}{\zeta}\right) \quad ; \quad y,\kappa \in \mathcal{R}, \zeta > 0 \tag{2}$$

with the $\mathbb{PDF}$ of normal distribution,

$$f(y|\kappa,\zeta) = \frac{1}{\zeta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\kappa}{\zeta}\right)^2} \;\; ; \;\; y,\kappa \in \mathcal{R}, \zeta > 0. \tag{3}$$

One of the drawbacks of $\mathbb{S}_{pk}$ is that it ignores the target value of the process. Its inability to represent a situation in which the process is not centred on the target value is another disadvantage. To address this problem, a new index was created that considers deviations from the desired value when determining a process's capabilities. In the article by Saha [17], using $\mathbb{SELF}$ a capability index, a measure designed to assess the performance of a process relative to its specification limits. This index was subsequently cited and discussed by Saha, Devi and Pareek [18], while its inferential properties were rigorously investigated by Saha et al. [19]. Building upon this prior foundation, the present study develops a novel asymmetric loss-based process capability index, denoted as $\mathbb{S}''_{pk}$, specifically for processes that can be modeled by a normal distribution.

The proposed index is formulated by incorporating the $\mathbb{LINEX}$ loss function by Varia [26], as a tool for decision-making under asymmetric loss conditions. This loss function has proven useful in various statistical inference settings where the cost of overestimation and underestimation differ. In process capability analysis, Saha and Dey [21] have previously applied the $\mathbb{LINEX}$ loss to address such asymmetries, making it a natural choice for the development of $\mathbb{S}''_{pk}$. Mathematically, the $\mathbb{LINEX}$ loss function is expressed as: $\mathbb{L}_2(\kappa - \mathbb{T}) = 2\frac{e^{\varpi(\kappa-\mathbb{T})} - \varpi(\kappa-\mathbb{T}) - 1}{\varpi^2}$, where ϖ is a fixed parameter that controls the degree of asymmetry of $\mathbb{L}_2$. For small values of ϖ (e.g., ϖ = 0.01), this loss function becomes nearly symmetric and resembles the quadratic loss function, $\mathbb{L}(\kappa - \mathbb{T}) = (\kappa - \mathbb{T})^2$ Erfanian and Gildeh [9].

The index $\mathbb{S}''_{pk}$, as developed by Dey, Saha and Park [7] and further refined by Saha and Dey [20] is obtained by incorporating the $\mathbb{LINEX}$ loss function into the denominator of the conventional $\mathbb{S}_{pk}$ $\mathbb{PCI}$. This modification results in a capability measure that directly accounts for asymmetric loss considerations in evaluating process performance, and is given by

$$\mathbb{S}''_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\kappa}{\sqrt{\zeta^2 + 2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}}}\right) + \frac{1}{2}\Phi\left(\frac{\kappa-\mathbb{L}}{\sqrt{\zeta^2 + 2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}}}\right)\right\} \tag{4}$$

The term $2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}$ captures two types of variation: (i) is within-process variation, which reflects the natural variability of individual observation around the process mean κ, (ii) between-process deviation, which represents the departure of the process mean κ from the desired target value $\mathbb{T}$. Together, both the inherent dispersion of the process and its alignment with the target specifications.

The primary objective of this study is to examine the performance characteristics of six well-known frequentist estimation methods previously employed by Dey et al. [8]. These estimators include the $\mathbb{MLE}$, $\mathbb{LSE}$,$\mathbb{WLSE}$, $\mathbb{PE}$, $\mathbb{CME}$ and $\mathbb{MPSE}$. We focuses on their application to the $\mathbb{S}''_{pk}$ under various combinations of sample sizes and parameter settings. The evaluation criterion for these estimators is their $\mathbb{MSE}$s, which balances both bias and variability in estimation. Since the theoretical comparison of these estimators is analytically intractable owing to the complex form of $\mathbb{S}''_{pk}$ and the absence of closed form variance expression, a comprehensive simulation study is undertaken. The primary focus is on the $\mathbb{MLE}$, given its widespread use and desirable asymptotic properties. The second objective is the study is to derive an $\mathbb{ACI}$ for $\mathbb{S}''_{pk}$.

To assess the performance of the interval, the $\mathbb{AW}$ and $\mathbb{CP}$ are calculated. The third objective is to obtain the Bayesian point and interval estimates for $\mathbb{S}''_{pk}$. The novelty of this study lies in the fact that, to date, there have been no reports on the estimation of $\mathbb{S}''_{pk}$ or the construction of $\mathbb{ACI}$ using the $\mathbb{MLE}$ method. Thus, this study aims to provide applied statisticians and quality control engineers with guidance on selecting the optimal estimation method among the six distinct frequentist approaches. To assess interval quality, both $\mathbb{AW}$ and $\mathbb{CP}$ are computed. The third objective is to develop Bayesian point and interval estimates for $\mathbb{S}''_{pk}$, offering a probabilistic alternative to the frequentist framework. The novelty of this work lies in the fact that, to the best of our knowledge, no prior research has addressed the estimation of $\mathbb{S}''_{pk}$ or the construction of $\mathbb{ACI}$ for it using $\mathbb{MLE}$ approach.

Rest of the article is structured as follows. In Section 2, the estimation methods for $\mathbb{S}''_{pk}$ are introduced, and the corresponding estimators are derived. This section also includes the construction of $\mathbb{ACI}$ for $\mathbb{S}''_{pk}$ based on the $\mathbb{MLE}$, as well as Bayesian estimation using five different loss functions. Section 3 presents a comprehensive $\mathbb{MC}$ and $\mathbb{MCMC}$ simulation study to compare the performances of $\mathbb{S}''_{pk}$ based on the various estimation methods. In Section 4, two real datasets from the electronics industry are analyzed for illustration. Finally, Section 5 provides the concluding remarks.

2. Estimation of the index $\mathbb{S}''_{pk}$

This section investigates the point and interval estimation of the index $\mathbb{S}''_{pk}$ from classical and Bayesian point of view, respectively.

2.1. Classical estimation

A random variable $\mathbb{Y}$ has $N(\kappa, \zeta)$, with its $\mathbb{CDF}$, given in equation (2) and $\mathbb{PDF}$, given in equation (3) where κ, ζ are the parameters of the distribution and $\Phi(\cdot)$ is the $\mathbb{CDF}$ of the standard normal variate. In the context of the considered model, the analytical forms of the derivatives and necessary transformations are not available in closed form for the parameter estimation procedures. Consequently, for all the considered estimation methods the parameter estimates were obtained using the $\mathbb{NLM}$ technique. A short formulation of the considered classical methods of estimation are provided in brief in Table 1, followed by the final expression for the derived $\mathbb{PCI}$ based on the estimated values of the parameters. Note that only the method of MLE and MPSE satisfied the invariance property and hence we got the MLE and MPSE of the index $\mathbb{S}''_{pk}$ by using the invariance property and for the repmaing considered methods we have used the plug-in method to the obatin the estimate of the index. The detailed description of these menthods of estimation in the context of process capability analysis the readers may the articles Saha, Devi and Pareek [18], Saha et al. [19], Saha and Dey [21] and many more.

Asymptotic confidence interval

$\mathbb{ACI}$ of $\mathbb{S}''_{pk}$ in case of normal distribution, can be derived with $100(1-\omega)\%$ $\mathbb{CI}$. $\mathbb{ACI}$ of $\mathbb{S}''_{pk}$ can be obtained as:

$$\frac{\hat{\mathbb{S}}''_{pk} - \mathbb{S}''_{pk}}{\sqrt{var(\hat{\mathbb{S}}''_{pk})}} \sim N(0,1)$$

where,

$$\begin{aligned} var(\hat{\mathbb{S}}''_{pk}) &= \left(\frac{\partial \mathbb{S}''_{pk}}{\partial \kappa}\right)^2 \times \left(-\frac{1}{E\left(\frac{\partial^2 ln\ L(\kappa,\zeta)}{\partial \kappa^2}\right)}\right) + \left(\frac{\partial \mathbb{S}''_{pk}}{\partial \zeta}\right)^2 \times \left(-\frac{1}{E\left(\frac{\partial^2 ln\ L(\kappa,\zeta)}{\partial \zeta^2}\right)}\right) \\ &\quad -2\left(\frac{\partial \mathbb{S}''_{pk}}{\partial \kappa}\right)\left(\frac{\partial \mathbb{S}''_{pk}}{\partial \zeta}\right) \times \left(-\frac{1}{E\left(\frac{\partial^2 ln\ L(\kappa,\zeta)}{\partial \kappa \partial \zeta}\right)}\right). \end{aligned}$$

The elements of Fisher information matrix

$$I(\theta) = E\left\{ \begin{matrix} \left(\frac{\partial^2 ln\ L(\kappa,\zeta)}{\partial \kappa^2}\right) & \left(\frac{\partial^2 ln\ L(\kappa,\zeta)}{\partial \kappa \partial \zeta}\right) \\ \left(\frac{\partial^2 ln\ L(\kappa,\zeta)}{\partial \zeta \partial \kappa}\right) & \left(\frac{\partial^2 ln\ L(\kappa,\zeta)}{\partial \zeta^2}\right) \end{matrix} \right\}$$

Table 1: *Classical methods of estimation and corresponding estimated PCI*

Method	Function to be optimized	Expression of estimated index S''_{pk}
$\mathbb{MLE}$	$log\left(\left(\frac{1}{2\pi\zeta^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y-\kappa}{\zeta}\right)^2}\right)$	$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\hat{\kappa}_{mle}}{\sqrt{\hat{\zeta}^2_{mle}+\frac{2(e^{\omega(\hat{\kappa}_{mle}-\mathbb{T})}-\omega(\hat{\kappa}_{mle}-\mathbb{T})-1)}{\omega^2}}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\kappa}_{mle}-\mathbb{L}}{\sqrt{\hat{\zeta}^2_{mle}+\frac{2(e^{\omega(\hat{\kappa}_{mle}-\mathbb{T})}-\omega(\hat{\kappa}_{mle}-\mathbb{T})-1)}{\omega^2}}}\right)\right\}$
$\mathbb{LSE}$	$\sum_{i=1}^{n}\left\{\mathbb{F}(y_{(i:n)}\mid\kappa,\zeta)-\frac{i}{n+1}\right\}^2$	$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\hat{\kappa}_{lse}}{\sqrt{\hat{\zeta}^2_{lse}+\frac{2(e^{\omega(\hat{\kappa}_{lse}-\mathbb{T})}-\omega(\hat{\kappa}_{lse}-\mathbb{T})-1)}{\omega^2}}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\kappa}_{lse}-\mathbb{L}}{\sqrt{\hat{\zeta}^2_{lse}+\frac{2(e^{\omega(\hat{\kappa}_{lse}-\mathbb{T})}-\omega(\hat{\kappa}_{lse}-\mathbb{T})-1)}{\omega^2}}}\right)\right\}$
$\mathbb{WLSE}$	$\sum_{i=1}^{n} w_i\left\{\mathbb{F}(y_{(i:n)}\mid\kappa,\zeta)-\frac{i}{n+1}\right\}^2$	$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\hat{\kappa}_{wlse}}{\sqrt{\zeta^2_{wlse}+\frac{2(e^{\omega(\hat{\kappa}_{wlse}-\mathbb{T})}-\omega(\hat{\kappa}_{wlse}-\mathbb{T})-1)}{\omega^2}}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\kappa}_{wlse}-\mathbb{L}}{\sqrt{\hat{\zeta}^2_{wlse}+\frac{2(e^{\omega(\hat{\kappa}_{wlse}-\mathbb{T})}-\omega(\hat{\kappa}_{wlse}-\mathbb{T})-1)}{\omega^2}}}\right)\right\}$
$\mathbb{PE}$	$\sum_{j=1}^{n}\left\{y_{(i:n)}-Q(p_i\mid\kappa,\zeta)\right\}^2$	$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\hat{\kappa}_{pe}}{\sqrt{\zeta^2_{pe}+\frac{2(e^{\omega(\hat{\kappa}_{pe}-\mathbb{T})}-\omega(\hat{\kappa}_{pe}-\mathbb{T})-1)}{\omega^2}}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\kappa}_{pe}-\mathbb{L}}{\sqrt{\hat{\zeta}^2_{pe}+\frac{2(e^{\omega(\hat{\kappa}_{pe}-\mathbb{T})}-\omega(\hat{\kappa}_{pe}-\mathbb{T})-1)}{\omega^2}}}\right)\right\}$
$\mathbb{CME}$	$\frac{1}{12n}+\sum_{i=1}^{n}\left\{\mathbb{F}(y_{(i:n)}\mid\kappa,\zeta)-\frac{2i-1}{2n}\right\}^2$	$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\hat{\kappa}_{cme}}{\sqrt{\zeta^2_{cme}+\frac{2(e^{\omega(\hat{\kappa}_{cme}-\mathbb{T})}-\omega(\hat{\kappa}_{cme}-\mathbb{T})-1)}{\omega^2}}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\kappa}_{cme}-\mathbb{L}}{\sqrt{\hat{\zeta}^2_{cme}+\frac{2(e^{\omega(\hat{\kappa}_{cme}-\mathbb{T})}-\omega(\hat{\kappa}_{cme}-\mathbb{T})-1)}{\omega^2}}}\right)\right\}$
$\mathbb{MPSE}$	$\frac{1}{(n+1)}\sum_{i=1}^{n+1}\log\left(\mathbb{F}(y_{(i:n)}\mid\kappa,\zeta)-\mathbb{F}(y_{(i-1:n)}\mid\kappa,\zeta\right)$	$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathbb{U}-\hat{\kappa}_{mpse}}{\sqrt{\zeta^2_{mpse}+\frac{2(e^{\omega(\hat{\kappa}_{mpse}-\mathbb{T})}-\omega(\hat{\kappa}_{mpse}-\mathbb{T})-1)}{\omega^2}}}\right)+\frac{1}{2}\Phi\left(\frac{\hat{\kappa}_{mpse}-\mathbb{L}}{\sqrt{\hat{\zeta}^2_{mpse}+\frac{2(e^{\omega(\hat{\kappa}_{mpse}-\mathbb{T})}-\omega(\hat{\kappa}_{mpse}-\mathbb{T})-1)}{\omega^2}}}\right)\right\}.$

can easily be obtained, where

$$E\left(\frac{\partial^2 lnL(\kappa,\zeta)}{\partial\kappa^2}\right) = -\frac{n}{\zeta^2}, E\left(\frac{\partial^2 lnL(\kappa,\zeta)}{\partial\zeta^2}\right) = -\frac{2n}{\zeta^2} \text{ and } E\left(\frac{\partial^2 lnL(\kappa,\zeta)}{\partial^2\kappa^2\zeta^2}\right) = 0$$

After, obtaining partial derivative of $\mathbb{S}''_{pk}$ with respect to κ and ζ. Using, $(\Phi^{-1})'(y) = \frac{1}{\Phi'(\Phi^{-1}(y))}$, with $\Phi'(=\phi)$ as the first derivative of Φ. We get it as:

$$\begin{aligned}\frac{\partial\mathbb{S}''_{pk}}{\partial\kappa} &= \frac{1}{6}\frac{1}{\Phi'\left(3\mathbb{S}''_{pk}\right)} \times \left\{\phi\left(\frac{\mathbb{U}-\kappa}{\sqrt{\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}}}\right)\right.\\ &\left(-\frac{1}{\sqrt{\left(\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}\right)}} - \frac{(\mathbb{U}-\kappa)(e^{\varpi(\kappa-\mathbb{T})}-1)}{\varpi\left(\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}\right)^{3/2}}\right)\\ &+ \phi\left(\frac{\kappa-\mathbb{L}}{\sqrt{\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}}}\right)\\ &\left.\left(\frac{1}{\sqrt{\left(\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}\right)}} - \frac{(\kappa-\mathbb{L})(e^{\varpi(\kappa-\mathbb{T})}-1)}{\varpi\left(\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}\right)^{3/2}}\right)\right\}\end{aligned}$$

$$\begin{aligned}\frac{\partial\mathbb{S}''_{pk}}{\partial\zeta} &= \frac{1}{6}\frac{1}{\Phi'\left(3\mathbb{S}''_{pk}\right)} \times \left\{\phi\left(\frac{\mathbb{U}-\kappa}{\sqrt{\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}}}\right)\right.\\ &\left(\frac{-(\mathbb{U}-\kappa)\zeta}{\left(\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}\right)^{3/2}}\right) + \phi\left(\frac{\kappa-\mathbb{L}}{\sqrt{\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}}}\right)\\ &\left.\left(\frac{-(\kappa-\mathbb{L})\zeta}{\left(\zeta^2+2\frac{e^{\varpi(\kappa-\mathbb{T})}-\varpi(\kappa-\mathbb{T})-1}{\varpi^2}\right)^{3/2}}\right)\right\}\end{aligned}$$

Hence,the $(1-\omega)100\%$ $\mathbb{AIC}$ of $\mathbb{S}''_{pk}$ can be obtained as:

$$\left\{\left(\hat{\mathbb{S}''}_{pk} - \mathbb{Z}_{(\frac{\omega}{2})}\sqrt{var(\hat{\mathbb{S}''}_{pk})}\right), \left(\hat{\mathbb{S}''}_{pk} + \mathbb{Z}_{(\frac{\omega}{2})}\sqrt{var(\hat{\mathbb{S}''}_{pk})}\right)\right\}$$

where, $\mathbb{Z}_{(\omega/2)}$ is upper $(\omega/2)^{th}$ quantile of the $N(0,1)$.

2.2. Bayesian estimation

This section focuses on the Bayesian estimation of the newly developed index $\mathbb{S}''_{pk}$. In Bayesian theory, the accuracy of estimation depends on the careful selection of the prior distribution and loss function. The prior reflects initial beliefs about the parameters, while the loss function quantifies estimation errors. Both choices are essential for guiding the posterior distribution towards an accurate estimate. The importance of these components is highlighted by Shiau, Chiang and Hung [22], Spiring et al. [23], and Dey and Saha [6], who demonstrate how the appropriate selection of prior and loss function can enhance the robustness and efficiency of Bayesian estimation. The $\mathbb{PDF}$ of the considered prior for the parameters κ and $\zeta^2 = \tau$ are given by

$$\pi_1(\kappa|\tau) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2\tau}(\kappa-\kappa_0)^2}, \qquad \kappa, \kappa_0 \in R, \tau > 0. \tag{5}$$

$$\pi_2(\tau) = \frac{\beta^\alpha}{\varpi(\alpha)} \tau^{-(\alpha+1)} e^{-\frac{\beta}{\tau}}, \qquad \alpha, \beta, \tau > 0. \tag{6}$$

where, the hyper-parameters α, β, and κ_0 are assumed to be known. Hence, the joint prior density function of κ and τ can be written as

$$\pi(\kappa, \tau) = \frac{1}{\sqrt{2\pi\tau}} \frac{\beta^\alpha}{\varpi(\alpha)} e^{-\frac{1}{2\tau}(\kappa-\kappa_0)^2 - \frac{\beta}{\tau}} \tau^{-(\alpha+1)} \tag{7}$$

The joint posterior distribution is obtained by combining the likelihood function and prior distribution, and it is given by;

$$p(\kappa, \tau|y) \propto (\tau)^{-\frac{(n+3)}{2} - \alpha} e^{-\frac{1}{2\tau}\left\{S_{kk} + (n+1)(\kappa-\kappa_1)^2\right\}} \tag{8}$$

where, $\kappa_1 = \frac{n\bar{y}+\kappa_0}{n+1}$ and $S_{kk} = S_y + 2\beta + \frac{n(\kappa_0-\bar{y})^2}{n+1}$. The marginal posterior distributions of the parameter τ and κ given y are in the following equations;

$$\pi(\tau|y) \propto \tau^{-\left(\frac{n}{2}+\alpha+1\right)} e^{\left(-\frac{1}{\tau}\left(\frac{S_{kk}}{2}\right)\right)} \tag{9}$$

$$\pi(\kappa|y) \propto \frac{1}{\sqrt{\pi}} \frac{\varpi\left(\frac{n+1+2\alpha}{2}\right)}{\varpi(n+2\alpha)} \frac{\sqrt{(n+1)}}{\sqrt{S_{kk}}} \left\{1 + \frac{(\kappa-\kappa_1)^2}{S_{kk}/(n+1)}\right\}^{-\left(\frac{n+1+2\alpha}{2}\right)} \tag{10}$$

respectively. The Eqn. (9) is the marginal posterior distribution of τ which represents inverse gamma distribution, i.e., $\tau \sim \mathbb{IG}(\frac{n+2\alpha}{2}, \frac{S_{kk}}{2})$. In Eqn. (10), shows the marginal posterior distribution of κ which is unknown distribution. However, $\sqrt{N}\left(\frac{\kappa-\kappa_1}{\sqrt{S_{kk}/(n+1)}}\right)$ follows a t-distribution with $N = n + 2\alpha$ degrees of freedom. Since the joint posterior distribution does not have a closed-form expression, the marginal posterior distributions will be used to generate posterior random deviates through the $\mathbb{MCMC}$ algorithm. Using the marginal posterior distributions of κ and τ, the $\mathbb{BE}$ of the index $\mathbb{S}''_{pk}$ is obtained by generating $\mathbb{MCMC}$ random deviates. This is done using five different loss functions: $\mathbb{SELF}$, $\mathbb{WSELF}$, $\mathbb{MSELF}$, $\mathbb{LSLF}$, and $\mathbb{KLF}$. The expressions of the loss functions, the associated $\mathbb{BE}$ and $\mathbb{PR}$ are given in tabular form in the following Table 2.

Table 2: *BE and PR under different loss functions.*

Name of the loss function	Form of the loss function	Bayes estimates	Posterior risk
$\mathbb{SELF}$	$(\theta - d)^2$	$E(\theta\|X)$	$V(\theta\|X)$
$\mathbb{WSELF}$	$\frac{(\theta-d)^2}{\theta}$	$\left(E\left(\theta^{-1}\|X\right)\right)^{-1}$	$E(\theta\|X) - \left(E\left(\theta^{-1}\|X\right)\right)^{-1}$
$\mathbb{MSELF}$	$\left(1 - \frac{d}{\theta}\right)^2$	$\frac{E\left(\theta^{-1}\|X\right)}{E\left(\theta^{-2}\|X\right)}$	$1 - \frac{E\left(\theta^{-1}\|X\right)^2}{E\left(\theta^{-2}\|X\right)}$
$\mathbb{LLF}$	$(\log\theta - \log d)^2$	$\exp\left(E\left(\log\theta\|X\right)\right)$	$Var\left(\log\theta\|X\right)$
$\mathbb{KLF}$	$\left(\sqrt{\frac{d}{\theta}} - \sqrt{\frac{\theta}{d}}\right)$	$\sqrt{\frac{E(\theta\|X)}{E\left(\theta^{-1}\|X\right)}}$	$E\{E(\theta\|X)\left(E\left(\theta^{-1}\|X\right)\right) - 1\}$

The following steps have been utilized in order to obtain posterior samples from the respective marginal distributions.

Step-1: Set preliminary estimate value $(\tau^{(0)}, \kappa^{(0)})$
Step-2: To start with $p = 1$.
Step-3: Generate a new sampler for τ and κ as follows: $\tau^{(p)} \sim \pi(y|\tau^{(p-1)})$, $\kappa^{(p)} \sim \pi(y|\kappa^{(p)})$
Step-4: We obtain the posterior samples of size $\mathbb{P}$ for parameters τ and κ. After repeating the

step 2 and step 3 for $r = 1, 2, 3, ..., \mathbb{P}(= 10000)$ times.

Step-5: We get a sequence ${\mathbb{S}''_{pk}}^{(p)}$ by using the step-4 in above sequence.

Step-6: After receiving the posterior samples, the corresponding 𝔹𝔼 of $\mathbb{S}''_{pk}$ can be obtained from the above 𝕄ℂ𝕄ℂ algorithm based on the above defined loss functions.

Step-7: The Bayes credible interval for the proposed index $\mathbb{S}''_{pk}$ is constructed by utilizing the sequences of random deviates ${\mathbb{S}''_{pk}}^{(p)}$, $r = 1, 2, 3, ..., \mathbb{P}$ obtained through step 5. Using, algorithm the $(1-\omega)100\%$ highest posterior density credible interval of the index is given by, $\left\{{\mathbb{S}''_{pk}}^{(p_1^*)}, {\mathbb{S}''_{pk}}^{(p_1^*+(1-\delta)\mathbb{P})}\right\}$, where, (p_1^*) satisfying,

$${\mathbb{S}''_{pk}}^{(p_1^*+(1-\delta)\mathcal{P})} - {\mathbb{S}''_{pk}}^{(p_1^*)} = min\left({\mathbb{S}''_{pk}}^{(p_1+(1-\delta)\mathcal{P})} - {\mathbb{S}''_{pk}}^{(p_1)}\right)$$

3. Simulation and discussion

This section employs 𝕄ℂ simulations to evaluate the performances of the ℙℂ𝕀 $\mathbb{S}''_{pk}$ across different estimation methods by analyzing their 𝕄𝕊𝔼s and simulated risks. Additionally, the 𝔸𝕎s and ℂℙs of the 𝔸ℂ𝕀 is evaluated to measure the performance. The simulation study examines sample sizes n = 20, 30, 50, 100 and values of ϖ = 0.01, 1, 5, 10, with several sets of parameter values: (κ, ζ) = (6.5,1.25), (6.5,1.50), (6.5,1.75), (7.5,1.25), (7.5,1.50), (7.5,1.75). The lower, target and upper specification limits are set at 0.50, 7.0, and 12.50, respectively.

In Tables 3 and 4, the values of $\mathbb{S}''_{pk}$ along with the average estimates and 𝕄𝕊𝔼s for each estimation method, are presented. The results of the 𝔸ℂ𝕀 using 𝕄𝕃𝔼 as the estimation method for $\mathbb{S}''_{pk}$ is shown in Table 5, with various values of ϖ. The finding from tables reveals that, among all the estimation methods, 𝕄𝕃𝔼 yields the best performance in terms of 𝕄𝕊𝔼s. Furthermore, the 𝕄𝕊𝔼*s* decreases, with increase in sample size, demonstrating the consistency of the estimators. Additionally, as κ increases, the true value of $\mathbb{S}''_{pk}$ also increases. The analysis incorporated 𝔸𝕎s and ℂℙs by using 𝕄𝕃𝔼, as the method of estimation for the 𝔸ℂ𝕀 of $\mathbb{S}''_{pk}$, because 𝕄𝕃𝔼 is performing better among all other estimation methods. The findings also indicate that as sample size increases 𝔸𝕎 decrease and ℂℙ increases. Next, the Bayes point and interval estimation for the index have been considered under the above mentioned priors using symmetric and asymmetric loss functions. The 𝔹𝔼 and ℙℝ along with coverage probability (ℂℙ) and average width (𝔸𝕎) are calculated for the considered setup, and presented in Tables 6 and 7 for four different values of ϖ. The values of hyper-parameters in Bayesian setup are chosen as $\alpha = 1.55$, $\beta = 0.05$ and $\kappa_0 = 5$ so that it coincides with the parameter value with smallest variation in the simulation study. The simulation study suggests that with the increase in sample size posterior risk decreases for all setups. Also as further, the ℂℙs increases and tend to the nominal value and their 𝔸𝕎s decreases with increase in sample size. Moreover, the ℙℝ is decreases for small values of ϖ and increase for large values of ϖ. As ϖ value increase from 0.01 to 5, ℙℝ decreases and for $\varpi = 10$, ℙℝ increases.

Table 3: *Estimates and MSE for $\varpi = (0.01, 1)$ using* $\mathbb{MLE}$, $\mathbb{LSE}$, $\mathbb{WLSE}$, $\mathbb{PE}$, $\mathbb{CME}$, *and* $\mathbb{MPSE}$ *with* $\mathbb{S}''_{pk}$.

$\mathbb{PCI}$			$\mathbb{MLE}$		$\mathbb{LSE}$		$\mathbb{WLSE}$		$\mathbb{PE}$		$\mathbb{CME}$		$\mathbb{MPSE}$	
Parameters	S''_{pk}	n	$\mathbb{MLE}$	$\mathbb{MSE}$	$\mathbb{LSE}$	$\mathbb{MSE}$	$\mathbb{WLSE}$	$\mathbb{MSE}$	$\mathbb{PE}$	$\mathbb{MSE}$	$\mathbb{CME}$	$\mathbb{MSE}$	$\mathbb{MPSE}$	$\mathbb{MSE}$
							$\varpi = 0.01$							
S'_{pk}=1.48556	1.48573	20	1.50973	0.05563	1.3974	0.07406	1.37641	0.08297	1.38714	0.08044	1.52055	0.07949	1.66466	0.10776
κ=6.5		30	1.51405	0.0407	1.43546	0.05086	1.40451	0.0629	1.41229	0.0617	1.54164	0.06193	1.62528	0.07026
ζ=1.25		50	1.49475	0.0227	1.44589	0.02983	1.4013	0.04624	1.40921	0.04446	1.53128	0.0375	1.5689	0.0336
		100	1.48988	0.01182	1.48081	0.01616	1.41114	0.03725	1.4181	0.03406	1.54251	0.02502	1.53389	0.01523
S'_{pk}=1.26491	1.26502	20	1.28681	0.04183	1.19082	0.0539	1.16949	0.05955	1.18102	0.05966	1.29958	0.0606	1.42174	0.08114
κ=6.5		30	1.28335	0.02826	1.21746	0.03671	1.18246	0.0496	1.1895	0.04851	1.31155	0.0448	1.38185	0.04938
ζ= 1.50		50	1.28028	0.01731	1.24824	0.02223	1.20178	0.03866	1.20289	0.03476	1.32169	0.03135	1.34511	0.02632
		100	1.26432	0.00764	1.25798	0.01245	1.1987	0.02871	1.20185	0.02716	1.31211	0.0193	1.3031	0.01003
S'_{pk}=1.09888	1.09895	20	1.1113	0.03166	1.03214	0.04193	1.00968	0.05134	1.01655	0.04934	1.12731	0.04841	1.22943	0.06043
κ=6.5		30	1.1044	0.01984	1.05015	0.02608	1.01985	0.03932	1.02446	0.0359	1.13177	0.03145	1.19013	0.03351
ζ=1.75		50	1.10313	0.01201	1.07601	0.01617	1.03592	0.03052	1.03577	0.02639	1.14055	0.02235	1.1598	0.01763
		100	1.09833	0.00623	1.09709	0.00949	1.05642	0.03173	1.04423	0.02139	1.14586	0.01583	1.13279	0.00797
S'_{pk}=1.29514	1.29501	20	1.35925	0.06525	1.2459	0.08371	1.23488	0.08935	1.23673	0.09022	1.41289	0.1125	1.48825	0.1236
κ=7.5		30	1.32959	0.03824	1.24445	0.0572	1.21999	0.06755	1.22185	0.06935	1.39272	0.08191	1.42258	0.06284
ζ=1.25		50	1.31198	0.02261	1.26614	0.03401	1.2263	0.04865	1.22565	0.0502	1.38202	0.05387	1.37209	0.03213
		100	1.30772	0.01199	1.3012	0.02075	1.239	0.03641	1.23851	0.03764	1.38477	0.03764	1.34297	0.01544
S'_{pk}=1.11905	1.11897	20	1.16814	0.04617	1.06321	0.06068	1.05601	0.07508	1.05235	0.06658	1.2105	0.08493	1.28458	0.08875
κ=7.5		30	1.15126	0.03144	1.08925	0.04362	1.05333	0.05965	1.0606	0.05556	1.21497	0.06688	1.23594	0.05361
ζ=1.50		50	1.13297	0.01695	1.09885	0.02465	1.03958	0.03946	1.05172	0.03807	1.19164	0.03921	1.18718	0.02492
		100	1.12821	0.00762	1.12154	0.01406	1.04692	0.02906	1.05964	0.0289	1.18909	0.0257	1.16006	0.00991
S'_{pk}=0.98703	0.98698	20	1.02856	0.0361	0.94299	0.04519	0.92968	0.05672	0.93065	0.05082	1.07033	0.06495	1.1319	0.07127
κ=7.5		30	1.01381	0.02207	0.96256	0.03077	0.92945	0.04231	0.93504	0.03891	1.06941	0.04744	1.08659	0.03833
ζ=1.75		50	1.00174	0.01399	0.9754	0.01855	0.92764	0.03084	0.93518	0.02915	1.05831	0.03154	1.05072	0.0204
		100	0.99934	0.00633	0.99583	0.01165	0.9368	0.02195	0.94477	0.02194	1.05638	0.02263	1.02734	0.00842
							$\varpi = 1$							
S'_{pk}=1.48556	1.50093	20	1.54139	0.06236	1.42627	0.08019	1.47147	0.09572	1.42094	0.08414	1.55618	0.08735	1.69591	0.11878
κ=6.5		30	1.52667	0.0411	1.44099	0.05413	1.45834	0.08239	1.41482	0.06862	1.55539	0.06396	1.64056	0.06983
ζ=1.25		50	1.51413	0.02351	1.4684	0.03287	1.46756	0.05831	1.42706	0.04925	1.56253	0.0432	1.59114	0.03457
		100	1.51023	0.01106	1.49936	0.01771	1.47763	0.04416	1.43954	0.03594	1.56744	0.02792	1.55541	0.01486
S'_{pk}=1.26491	1.27436	20	1.29643	0.04233	1.20125	0.05452	1.22541	0.06688	1.18827	0.06145	1.31711	0.06199	1.43435	0.08622
κ=6.5		30	1.29546	0.02956	1.22941	0.04057	1.23985	0.05689	1.20392	0.05138	1.33153	0.05125	1.39652	0.05171
ζ= 1.50		50	1.29237	0.0164	1.25428	0.02257	1.23972	0.04062	1.21166	0.03624	1.33481	0.03256	1.36072	0.02605
		100	1.281	0.00753	1.27715	0.01201	1.24399	0.02875	1.21991	0.02556	1.33618	0.02056	1.32075	0.0104
S'_{pk}=1.09888	1.10506	20	1.11731	0.03055	1.03241	0.043	1.04352	0.05404	1.0175	0.05026	1.13383	0.04727	1.23852	0.05837
κ=6.5		30	1.12228	0.02119	1.07021	0.0275	1.06828	0.04013	1.04165	0.03699	1.15774	0.03597	1.21266	0.03925
ζ=1.75		50	1.11932	0.01315	1.10169	0.018	1.0851	0.03482	1.05733	0.02832	1.17056	0.02757	1.17991	0.02078
		100	1.10738	0.00601	1.10289	0.00978	1.07307	0.02204	1.05395	0.02014	1.1568	0.01623	1.14214	0.00794
S'_{pk}=1.29514	1.27988	20	1.32608	0.07087	1.20482	0.08855	1.18578	0.10199	1.1956	0.0951	1.37427	0.11544	1.44935	0.12568
κ=7.5		30	1.30185	0.0418	1.21902	0.06	1.186	0.07443	1.19691	0.07197	1.36705	0.08177	1.3894	0.06607
ζ=1.25		50	1.29009	0.02224	1.24581	0.03845	1.19514	0.05248	1.21004	0.05255	1.36564	0.0573	1.34705	0.03075
		100	1.2913	0.01271	1.28035	0.02177	1.2012	0.03874	1.21781	0.03905	1.36199	0.03807	1.32559	0.01606
S'_{pk}=1.11905	1.10964	20	1.13742	0.05005	1.03606	0.0659	1.02036	0.07682	1.02583	0.07232	1.18146	0.08512	1.2483	0.09114
κ=7.5		30	1.12559	0.03479	1.05665	0.04835	1.01894	0.06295	1.028	0.06048	1.17779	0.06642	1.20342	0.05291
ζ=1.50		50	1.13057	0.01937	1.0986	0.02801	1.0507	0.04068	1.05941	0.03955	1.19726	0.04675	1.18268	0.02786
		100	1.11861	0.00816	1.11822	0.01581	1.05543	0.02749	1.06537	0.02766	1.19019	0.02928	1.14879	0.0104
S'_{pk}=0.98703	0.98084	20	1.00616	0.03804	0.91884	0.0464	0.90215	0.05522	0.90585	0.05247	1.04566	0.06109	1.10816	0.07216
κ=7.5		30	0.99718	0.02348	0.94917	0.03271	0.91957	0.04654	0.92475	0.04173	1.05906	0.04917	1.06448	0.03626
ζ=1.75		50	0.99335	0.01551	0.96958	0.02291	0.91637	0.03616	0.92419	0.03519	1.05139	0.03663	1.04073	0.02146
		100	0.99114	0.00651	0.99023	0.0115	0.92799	0.02217	0.9343	0.02209	1.04856	0.02155	1.01891	0.00861

Table 4: *Estimates and MSE for $\varpi = (5, 10)$ using $\mathbb{MLE}$, $\mathbb{LSE}$, $\mathbb{WLSE}$, $\mathbb{PE}$, $\mathbb{CME}$, and $\mathbb{MPSE}$ with S''_{pk}.*

PCI			MLE		LSE		WLSE		PE		CME		MPSE	
Parameters	S''_{pk}	n	MLE	MSE	LSE	MSE	WLSE	MSE	PE	MSE	CME	MSE	MPSE	MSE
							$\varpi = 5$							
S'_{pk}=1.48556	1.53889	20	1.59324	0.06026	1.46702	0.08273	1.45189	0.09192	1.45845	0.08885	1.62362	0.09496	1.76865	0.12926
κ=6.5		30	1.58533	0.04587	1.49351	0.06484	1.3176	0.13923	1.47684	0.07548	1.63622	0.08389	1.70984	0.08222
ζ=1.25		50	1.56487	0.02289	1.51782	0.03526	1.2828	0.13729	1.47765	0.05047	1.63121	0.05149	1.64732	0.03721
		100	1.55006	0.01076	1.53751	0.01826	1.43669	0.12438	1.46597	0.04139	1.61504	0.0319	1.59962	0.01495
S'_{pk}=1.26491	1.29734	20	1.35465	0.04813	1.24484	0.0606	1.06366	0.08883	1.23409	0.06659	1.38294	0.07671	1.50611	0.10412
κ=6.5		30	1.32116	0.02887	1.24935	0.03892	1.2882	0.0558	1.2223	0.05205	1.36593	0.04977	1.42866	0.05218
ζ= 1.50		50	1.31847	0.0176	1.28468	0.02688	1.29327	0.05042	1.23507	0.04337	1.37572	0.04162	1.39303	0.02918
		100	1.30092	0.00852	1.29766	0.01439	1.28712	0.03783	1.23365	0.03112	1.36087	0.02515	1.34317	0.01141
S'_{pk}=1.09888	1.11995	20	1.15947	0.03488	1.07904	0.04338	1.16182	0.06879	1.06661	0.04903	1.19991	0.05793	1.28996	0.07506
κ=6.5		30	1.14241	0.02383	1.08558	0.02879	1.15953	0.06176	1.05501	0.04027	1.18532	0.03975	1.23683	0.04278
ζ=1.75		50	1.14095	0.01333	1.11131	0.01982	1.16947	0.05451	1.06784	0.03084	1.1906	0.03095	1.20479	0.02226
		100	1.12842	0.00643	1.12791	0.01176	1.20609	0.07041	1.07597	0.02368	1.18667	0.02182	1.1651	0.00882
S'_{pk}=1.29514	1.17186	20	1.1582	0.12618	1.03587	0.15858	0.9794	0.20195	1.02848	0.16505	1.21419	0.17051	1.2465	0.18054
κ=7.5		30	1.16385	0.08842	1.07561	0.10884	1.01181	0.14251	1.05677	0.1207	1.23377	0.12717	1.22707	0.11584
ζ=1.25		50	1.15853	0.05114	1.11895	0.07458	1.02441	0.10924	1.06961	0.09736	1.23899	0.09529	1.20199	0.06148
		100	1.17011	0.02647	1.16385	0.03821	1.04035	0.06816	1.0997	0.05751	1.24917	0.05555	1.19539	0.03005
S'_{pk}=1.11905	1.04003	20	1.00723	0.11626	0.90104	0.14595	0.8186	0.2092	0.8889	0.15349	1.06276	0.15405	1.09218	0.16447
κ=7.5		30	1.00357	0.07264	0.93174	0.09543	0.83074	0.15439	0.90196	0.11314	1.06513	0.10318	1.0618	0.09073
ζ=1.50		50	1.01948	0.04	0.99133	0.05491	0.87071	0.10286	0.94007	0.0768	1.09401	0.06923	1.06024	0.0477
		100	1.02335	0.02132	1.02147	0.02935	0.88638	0.06776	0.95856	0.04739	1.09416	0.04012	1.04693	0.02359
S'_{pk}=0.98703	0.93358	20	0.88986	0.08857	0.79397	0.11611	0.70573	0.19855	0.77518	0.12914	0.93626	0.11598	0.96694	0.11941
κ=7.5		30	0.9017	0.05798	0.83901	0.07308	0.72887	0.14564	0.80845	0.08695	0.95826	0.07643	0.95628	0.07442
ζ=1.75		50	0.90131	0.03197	0.87699	0.04407	0.76338	0.10699	0.8323	0.0614	0.96934	0.05151	0.93736	0.037
		100	0.90406	0.01714	0.90517	0.02378	0.80379	0.08258	0.84619	0.04014	0.96887	0.03	0.92539	0.01845
							$\varpi = 10$							
S'_{pk}=1.48556	1.56048	20	1.61498	0.06574	1.4898	0.09286	1.51911	0.09226	1.48008	0.10055	1.67702	0.07352	1.80093	0.14361
κ=6.5		30	1.59903	0.04237	1.49328	0.06347	1.5007	0.07451	1.46878	0.07623	1.6686	0.05144	1.7327	0.08006
ζ=1.25		50	1.58983	0.02417	1.53368	0.03692	1.51779	0.06088	1.48549	0.05795	1.68242	0.04295	1.6795	0.04089
		100	1.5682	0.01168	1.55913	0.0208	1.53165	0.04601	1.49642	0.04176	1.68063	0.03338	1.61911	0.016
S'_{pk}=1.26491	1.31021	20	1.36396	0.05491	1.24816	0.07341	1.29292	0.0895	1.2329	0.08237	1.53892	0.09034	1.5301	0.12427
κ=6.5		30	1.34239	0.03004	1.27162	0.04365	1.30462	0.06471	1.24255	0.05828	1.57186	0.08733	1.4553	0.05651
ζ= 1.50		50	1.33814	0.01802	1.29649	0.02545	1.30354	0.04873	1.24627	0.04151	1.58511	0.08536	1.41341	0.03072
		100	1.31884	0.00812	1.31415	0.01616	1.3034	0.04165	1.25233	0.03331	1.6044	0.09115	1.36168	0.01134
S'_{pk}=1.09888	1.12819	20	1.16441	0.03786	1.07051	0.0515	1.13796	0.09465	1.05227	0.06094	1.47814	0.14919	1.30302	0.08319
κ=6.5		30	1.14457	0.02196	1.08811	0.03356	1.16327	0.07696	1.05719	0.04608	1.19369	0.04522	1.24188	0.04032
ζ=1.75		50	1.14451	0.01354	1.12093	0.01925	1.18335	0.06687	1.06993	0.03426	1.20186	0.03233	1.21006	0.02226
		100	1.13768	0.00685	1.13711	0.01281	1.19075	0.06359	1.08805	0.02311	1.20214	0.02404	1.17498	0.00959
S'_{pk}=1.29514	0.86982	20	0.87835	0.26348	0.74793	0.28032	0.66193	0.32228	0.73249	0.28881	0.97904	0.36532	0.94731	0.34949
κ=7.5		30	0.87024	0.20383	0.77598	0.22345	0.66415	0.26205	0.74471	0.23789	0.95896	0.28386	0.91341	0.24731
ζ=1.25		50	0.8743	0.15817	0.82733	0.17654	0.69242	0.20741	0.78167	0.19107	0.96411	0.22629	0.8988	0.17901
		100	0.8795	0.08495	0.86205	0.0957	0.66561	0.13551	0.78944	0.11161	0.95209	0.12354	0.89228	0.09186
S'_{pk}=1.11905	0.81598	20	0.75027	0.22186	0.64376	0.24586	0.49482	0.32284	0.62671	0.2554	0.81894	0.26525	0.8101	0.28193
κ=7.5		30	0.76837	0.16898	0.70173	0.19073	0.52002	0.26983	0.66731	0.20668	0.85371	0.21299	0.80589	0.19884
ζ=1.50		50	0.78091	0.1282	0.75279	0.14587	0.54739	0.21987	0.70099	0.16708	0.87723	0.17035	0.80457	0.14313
		100	0.79504	0.07604	0.78972	0.08707	0.55708	0.16482	0.71371	0.10946	0.87899	0.10335	0.80868	0.08191
S'_{pk}=0.98703	0.76429	20	0.69021	0.18864	0.59243	0.20125	0.51141	0.24028	0.57176	0.21262	0.75118	0.2088	0.74778	0.23758
κ=7.5		30	0.7078	0.14734	0.63559	0.16988	0.52561	0.22064	0.5968	0.19218	0.77175	0.18159	0.74931	0.17035
ζ=1.75		50	0.72775	0.10531	0.69762	0.1195	0.56043	0.16941	0.6382	0.14363	0.80784	0.13387	0.75284	0.11839
		100	0.72854	0.06593	0.72386	0.07551	0.56825	0.12233	0.64628	0.10185	0.80113	0.08501	0.74224	0.07063

Table 5: *Estimates of* $\mathbb{S}''_{pk}$*,* $\mathbb{AW}$*, and* $\mathbb{CP}$ *for* $\mathbb{ACI}$ *using* $\mathbb{MLE}$ *with* $\varpi = (0.01, 1, 5, 10)$.

$\mathbb{PCI}$		$\mathbb{ACI}$							
Parameter	n	$\mathbb{AW}$	$\mathbb{CP}$	$\mathbb{AW}$	$\mathbb{CP}$	$\mathbb{AW}$	$\mathbb{CP}$	$\mathbb{AW}$	$\mathbb{CP}$
		$\varpi = 0.01$		$\varpi = 1$		$\varpi = 5$		$\varpi = 10$	
$\mathbb{S}''_{pk}$		1.48573		1.50093		1.53889		1.56048	
	20	0.93426	0.948	0.91256	0.946	0.92149	0.943	0.95738	0.943
κ=6.5	30	0.75246	0.949	0.73929	0.955	0.75045	0.944	0.77221	0.957
ζ=1.25	50	0.58055	0.957	0.56893	0.962	0.57932	0.956	0.59191	0.959
	100	0.40919	0.961	0.40182	0.964	0.40423	0.961	0.41613	0.963
$\mathbb{S}''_{pk}$		1.26502		1.27436		1.29734		1.31021	
	20	0.80269	0.949	0.78449	0.933	0.79416	0.943	0.80985	0.942
κ=6.5	30	0.65378	0.953	0.64050	0.945	0.64471	0.941	0.66070	0.945
ζ=1.50	50	0.50052	0.957	0.49191	0.951	0.49220	0.956	0.50511	0.952
	100	0.35038	0.962	0.34582	0.959	0.34814	0.959	0.35444	0.953
$\mathbb{S}''_{pk}$		1.09895		1.10501		1.11995		1.12819	
	20	0.70423	0.942	0.69684	0.941	0.70924	0.938	0.74602	0.942
κ=6.5	30	0.570053	0.945	0.55931	0.944	0.56555	0.941	0.57962	0.947
ζ=1.75	50	0.43880	0.949	0.43302	0.951	0.43177	0.948	0.43938	0.950
	100	0.30605	0.95	0.30274	0.953	0.30265	0.953	0.30798	0.955
$\mathbb{S}''_{pk}$		1.29501		1.27988		1.17186		0.86982	
	20	0.90836	0.937	0.93838	0.940	1.34251	0.917	1.87011	0.746
κ=7.5	30	0.73317	0.941	0.76631	0.943	1.11147	0.919	1.70115	0.769
ζ=1.25	50	0.56096	0.948	0.59405	0.947	0.88337	0.933	1.43707	0.805
	100	0.39808	0.951	0.41617	0.951	0.63742	0.941	1.13160	0.868
$\mathbb{S}''_{pk}$		1.11897		1.10964		1.04003		0.81598	
	20	0.77505	0.946	0.81823	0.934	1.16887	0.918	1.72788	0.729
κ=7.5	30	0.63075	0.949	0.66029	0.941	0.99399	0.926	1.54389	0.736
ζ=1.50	50	0.48786	0.953	0.51171	0.949	0.76979	0.938	1.29780	0.777
	100	0.34420	0.955	0.36017	0.952	0.55928	0.950	1.05467	0.828
$\mathbb{S}''_{pk}$		0.98698		0.98084		0.93358		0.76429	
	20	0.68964	0.929	0.71578	0.940	1.07325	0.910	1.50443	0.720
κ=7.5	30	0.55773	0.942	0.58301	0.947	0.90575	0.931	1.41512	0.741
ζ=1.75	50	0.42885	0.949	0.44916	0.949	0.70608	0.943	1.22068	0.782
	100	0.30382	0.953	0.31620	0.951	0.49292	0.950	1.00100	0.841

Table 6: *Estimates of $\mathbb{S}''_{pk}(\varpi)$, $\mathbb{S}'_{pk}$ and $\mathbb{S}_{pk}$ along with $\mathbb{ABE}$, $\mathbb{AW}$, $\mathbb{PR}$ and $\mathbb{CP}$ for $\mathbb{S}''_{pk}(\varpi)$ with, ϖ = (0.01, 1).*

	$\varpi = 0.01$						$\varpi = 1$					
	SELF	WSELF	MSELF	LLF	KLF		SELF	WSELF	MSELF	LLF	KLF	
n	ABE PR	ABE PR	ABE PR	ABE PR	ABE PR	AW CP	ABE PR	ABE PR	ABE PR	ABE PR	ABE PR	AW CP
	κ=6.5		**ζ=1.25**		**$\mathbb{S}_{pk}$=1.6000**		**$\mathbb{S}'_{pk}$=1.4855**		**$\mathbb{S}''_{pk}(0.01)$=1.4857**		**$\mathbb{S}''_{pk}(1)$=1.5009**	
20	1.4808	1.4477	1.4146	1.4643	1.4642	0.8539	1.5069	1.4762	1.4454	1.4916	1.4915	0.8319
	0.0494	0.0330	0.0230	0.0228	0.0230	0.9440	0.0467	0.0307	0.0210	0.0207	0.0209	0.9330
30	1.4791	1.4562	1.4333	1.4677	1.4676	0.7124	1.5080	1.4866	1.4651	1.4973	1.4972	0.6970
	0.0341	0.0229	0.0158	0.0517	0.0158	0.9520	0.0325	0.0214	0.0145	0.0143	0.0145	0.9430
50	1.4781	1.4640	1.4499	1.4711	1.4711	0.5599	1.5067	1.4934	1.4801	1.5001	1.5000	0.5496
	0.0208	0.0141	0.0097	0.0096	0.0097	0.9550	0.0200	0.0133	0.0089	0.0089	0.0089	0.9450
100	1.4821	1.4749	1.4677	1.4785	1.4785	0.4009	1.4988	1.4920	1.4853	1.4954	1.4954	0.3925
	0.0106	0.0072	0.0049	0.0049	0.0049	0.9620	0.0102	0.0068	0.0046	0.0045	0.0045	0.9490
			κ=6.5	**ζ=1.50**	**$\mathbb{S}_{pk}$=1.3333**	**$\mathbb{S}'_{pk}$=1.2649**	**$\mathbb{S}''_{pk}(0.01)$=1.2650**		**$\mathbb{S}''_{pk}(1)$=1.2744**			
20	1.2677	1.2386	1.2095	1.2532	1.2531	0.7416	1.3101	1.2826	1.2549	1.2964	1.2963	0.7343
	0.0373	0.0291	0.0237	0.0234	0.0236	0.9490	0.0365	0.0275	0.0217	0.0213	0.0216	0.9550
30	1.2669	1.2469	1.2269	1.2569	1.2569	0.6158	1.2978	1.2791	1.2603	1.2885	1.2884	0.6042
	0.0255	0.0200	0.0162	0.0160	0.0161	0.9490	0.0245	0.0187	0.0147	0.0146	0.0147	0.9520
50	1.2709	1.2586	1.2463	1.2648	1.2647	0.4841	1.2846	1.2730	1.2614	1.2788	1.2788	0.4743
	0.0156	0.0122	0.0098	0.0097	0.0098	0.9540	0.0149	0.0116	0.0092	0.0091	0.0091	0.9550
100	1.2627	1.2565	1.2503	1.2596	1.2596	0.3441	1.2748	1.2689	1.2630	1.2719	1.2719	0.3386
	0.0078	0.0062	0.0050	0.0049	0.0049	0.9600	0.0076	0.0059	0.0047	0.0047	0.0047	0.9560
			κ=6.5	**ζ=1.75**	**$\mathbb{S}_{pk}$=1.1428**	**$\mathbb{S}'_{pk}$=1.0988**	**$\mathbb{S}''_{pk}(0.01)$=1.0989**		**$\mathbb{S}''_{pk}(1)$=1.1051**			
20	1.1024	1.0763	1.0500	1.0894	1.0893	0.6546	1.1235	1.0987	1.0737	1.1111	1.1110	0.6452
	0.0292	0.0261	0.0245	0.0241	0.0244	0.9450	0.0282	0.0248	0.0228	0.0224	0.0226	0.9500
30	1.0994	1.0816	1.0637	1.0905	1.0905	0.5418	1.1198	1.1028	1.0858	1.1113	1.1113	0.5334
	0.0198	0.0178	0.0166	0.0164	0.0166	0.9480	0.0191	0.0169	0.0155	0.0153	0.0154	0.9520
50	1.0916	1.0808	1.0700	1.0862	1.0862	0.4214	1.1086	1.0984	1.0880	1.1035	1.1035	0.4148
	0.0118	0.0108	0.0101	0.0100	0.0100	0.9540	0.0114	0.0103	0.0094	0.0093	0.0094	0.9440
100	1.0960	1.0905	1.0851	1.0933	1.0933	0.3006	1.1067	1.1015	1.0962	1.1041	1.1041	0.2966
	0.0060	0.0054	0.0050	0.0050	0.0050	0.9540	0.0058	0.0052	0.0048	0.0047	0.0048	0.9590
			κ=7.5	**ζ=1.25**	**$\mathbb{S}_{pk}$=1.3870**	**$\mathbb{S}'_{pk}$=1.2951**	**$\mathbb{S}''_{pk}(0.01)$=1.2950**		**$\mathbb{S}''_{pk}(1)$=1.2799**			
20	1.3330	1.2984	1.2641	1.3157	1.3156	0.8286	1.3585	1.3193	1.2802	1.3390	1.3387	0.8859
	0.0469	0.0345	0.0266	0.0265	0.0268	0.9480	0.0534	0.0391	0.0303	0.0298	0.0303	0.9360
30	1.3249	1.3008	1.2769	1.3129	1.3128	0.6931	1.3258	1.2986	1.2716	1.3122	1.3121	0.7336
	0.0325	0.0241	0.0185	0.0185	0.0186	0.9510	0.0363	0.0271	0.0211	0.0210	0.0212	0.9490
50	1.3110	1.2961	1.2812	1.3036	1.3035	0.5454	1.3126	1.2958	1.2789	1.3042	1.3042	0.5778
	0.0199	0.0150	0.0115	0.0115	0.0116	0.9450	0.0223	0.0169	0.0131	0.0130	0.0131	0.9530
100	1.3073	1.2996	1.2919	1.3035	1.3035	0.3921	1.2901	1.2814	1.2728	1.2857	1.2857	0.4110
	0.0102	0.0077	0.0059	0.0059	0.0060	0.9450	0.0112	0.0086	0.0067	0.0067	0.0068	0.9590
			κ=7.5	**ζ=1.50**	**$\mathbb{S}_{pk}$=1.1735**	**$\mathbb{S}'_{pk}$=1.1190**	**$\mathbb{S}''_{pk}(0.01)$=1.1189**		**$\mathbb{S}''_{pk}(1)$=1.1096**			
20	1.1612	1.1310	1.1010	1.1461	1.1460	0.7231	1.1848	1.1507	1.1164	1.1678	1.1676	0.7716
	0.0358	0.0302	0.0268	0.0266	0.0269	0.9480	0.0405	0.0341	0.0305	0.0299	0.0303	0.9360
30	1.1573	1.1363	1.1155	1.1468	1.1468	0.6027	1.1587	1.1355	1.1121	1.1471	1.1470	0.6336
	0.0246	0.0209	0.0184	0.0184	0.0185	0.9530	0.0270	0.0233	0.0209	0.0206	0.0209	0.9410
50	1.1424	1.1295	1.1166	1.1359	1.1359	0.4729	1.1302	1.1157	1.1012	1.1230	1.1229	0.4961
	0.0149	0.0129	0.0115	0.0114	0.0115	0.9590	0.0164	0.0145	0.0131	0.0130	0.0131	0.9410
100	1.1302	1.1235	1.1169	1.1269	1.1269	0.3389	1.1256	1.1182	1.1108	1.1219	1.1219	0.3553
	0.0076	0.0067	0.0059	0.0059	0.0060	0.9620	0.0083	0.0074	0.0067	0.0066	0.0067	0.9500
			κ=7.5	**ζ=1.75**	**$\mathbb{S}_{pk}$= 1.0220**	**$\mathbb{S}'_{pk}$=0.9870**	**$\mathbb{S}''_{pk}(0.01)$=0.9869**		**$\mathbb{S}''_{pk}(1)$=0.9808**			
20	1.0415	1.0151	0.9887	1.0283	1.0282	0.6405	1.0280	0.9985	0.9686	1.0134	1.0132	0.6673
	0.0280	0.0265	0.0262	0.0260	0.0264	0.9370	0.0302	0.0295	0.0307	0.0299	0.0303	0.9400
30	1.0215	1.0032	0.9849	1.0123	1.0123	0.5294	1.0281	1.0076	0.9869	1.0179	1.0178	0.5594
	0.0190	0.0183	0.0183	0.0182	0.0184	0.9390	0.0210	0.0205	0.0209	0.0205	0.0207	0.9490
50	1.0115	1.0001	0.9887	1.0058	1.0058	0.4171	0.9981	0.9855	0.9727	0.9918	0.9918	0.4356
	0.0116	0.0114	0.0114	0.0114	0.0114	0.9430	0.0126	0.0127	0.0130	0.0129	0.0130	0.9500
100	0.9997	0.9939	0.9880	0.9968	0.9968	0.2983	0.9910	0.9845	0.9780	0.9877	0.9877	0.3111
	0.0059	0.0059	0.0059	0.0059	0.0059	0.9480	0.0064	0.0065	0.0066	0.0066	0.0066	0.9520

Table 7: *Estimates of $\mathbb{S}''_{pk}(\varpi)$, $\mathbb{S}'_{pk}$ and $\mathbb{S}_{pk}$ along with $\mathbb{ABE}$, $\mathbb{AW}$, $\mathbb{PR}$ and $\mathbb{CP}$ for $\mathbb{S}''_{pk}(\varpi)$ with, ϖ = (5, 10).*

	$\varpi = 5$						$\varpi = 10$					
	SELF	WSELF	MSELF	LLF	KLF		SELF	WSELF	MSELF	LLF	KLF	
n	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{AW}$ $\mathbb{CP}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{ABE}$ $\mathbb{PR}$	$\mathbb{AW}$ $\mathbb{CP}$
			κ=6.5	**ζ=1.25**	**$\mathbb{S}_{pk}$=1.6000**	**$\mathbb{S}'_{pk}$=1.4856**	**$\mathbb{S}''_{pk}(5)$=1.5389**		**$\mathbb{S}''_{pk}(10)$=1.5605**			
20	1.5827	1.5518	1.5195	1.5675	1.5672	0.8475	1.6106	1.5714	1.4916	1.5930	1.5908	0.8892
	0.0481	0.0309	0.0212	0.0199	0.0201	0.9370	0.0535	0.0392	0.0583	0.0243	0.0268	0.9280
30	1.5710	1.5499	1.5284	1.5605	1.5604	0.7062	1.5974	1.5743	1.5475	1.5861	1.5858	0.7322
	0.0331	0.0212	0.0139	0.0136	0.0137	0.9460	0.0358	0.0231	0.0176	0.0146	0.0148	0.9480
50	1.5570	1.5440	1.5308	1.5505	1.5505	0.5536	1.5789	1.5652	1.5514	1.5721	1.5720	0.5695
	0.0203	0.0130	0.0085	0.0084	0.0085	0.9460	0.0215	0.0136	0.0088	0.0087	0.0087	0.9480
100	1.5462	1.5395	1.5328	1.5428	1.5428	0.3957	1.5683	1.5614	1.5544	1.5648	1.5648	0.4067
	0.0103	0.0067	0.0044	0.0043	0.0043	0.9500	0.0109	0.0070	0.0045	0.0044	0.0045	0.9530
			κ=6.5	**ζ=1.50**	**$\mathbb{S}_{pk}$=1.3333**	**$\mathbb{S}'_{pk}$=1.2649**	**$\mathbb{S}''_{pk}(5)$=1.2973**		**$\mathbb{S}''_{pk}(10)$=1.3102**			
20	1.3340	1.3049	1.2720	1.3198	1.3194	0.7446	1.3757	1.3273	1.1824	1.3572	1.3507	0.8034
	0.0372	0.0290	0.0265	0.0223	0.0228	0.9420	0.0443	0.0484	0.1313	0.0339	0.0467	0.9460
30	1.3311	1.3121	1.2925	1.3217	1.3216	0.6139	1.3488	1.3256	1.2856	1.3381	1.3371	0.6397
	0.0251	0.0190	0.0151	0.0144	0.0146	0.9450	0.0276	0.0232	0.0334	0.0172	0.0181	0.9510
50	1.3112	1.2997	1.2882	1.3055	1.3054	0.4753	1.3267	1.3147	1.3022	1.3208	1.3207	0.4875
	0.0150	0.0114	0.0089	0.0088	0.0088	0.9490	0.0158	0.0120	0.0096	0.0091	0.0092	0.9530
100	1.3126	1.3068	1.3009	1.3097	1.3097	0.3416	1.3227	1.3167	1.3106	1.3197	1.3197	0.3477
	0.0077	0.0059	0.0045	0.0045	0.0045	0.9500	0.0080	0.0060	0.0046	0.0046	0.0046	0.9520
			κ=6.5	**ζ=1.75**	**$\mathbb{S}_{pk}$=1.1429**	**$\mathbb{S}'_{pk}$=1.0989**	**$\mathbb{S}''_{pk}(5)$=1.1199**		**$\mathbb{S}''_{pk}(10)$=1.1282**			
20	1.1733	1.1448	1.1075	1.1597	1.1589	0.6743	1.1797	1.1005	0.8478	1.1572	1.1364	0.7412
	0.0307	0.0284	0.0355	0.0289	0.0258	0.9470	0.0389	0.0792	0.2905	0.0611	0.1773	0.9350
30	1.1574	1.1401	1.1217	1.1489	1.1487	0.5434	1.1661	1.1351	1.0474	1.1541	1.1501	0.5840
	0.0197	0.0174	0.0163	0.0152	0.0153	0.9510	0.0237	0.0310	0.0943	0.0263	0.0348	0.9410
50	1.1375	1.1273	1.1170	1.1324	1.1324	0.4181	1.1512	1.1399	1.1265	1.1457	1.1455	0.4306
	0.0116	0.0102	0.0092	0.0090	0.0091	0.9510	0.0124	0.0113	0.0121	0.0099	0.0100	0.9420
100	1.1326	1.1275	1.1223	1.1301	1.1300	0.2980	1.1371	1.1318	1.1264	1.1345	1.1345	0.3021
	0.0059	0.0052	0.0046	0.0046	0.0046	0.9530	0.0060	0.0053	0.0047	0.0047	0.0047	0.9450
			κ=7.5	**ζ=1.25**	**$\mathbb{S}_{pk}$=1.3870**	**$\mathbb{S}'_{pk}$=1.2951**	**$\mathbb{S}''_{pk}(5)$=1.1719**		**$\mathbb{S}''_{pk}(10)$=0.8698**			
20	1.2169	1.1157	0.9773	1.1712	1.1646	1.1543	1.0234	0.6668	0.2546	0.8868	0.8118	1.4698
	0.0903	0.1011	0.1526	0.1021	0.1138	0.9330	0.1750	0.3566	0.7657	0.5990	1.3627	0.8920
30	1.2036	1.1335	1.0474	1.1709	1.1678	1.0003	0.9780	0.7168	0.3920	0.8693	0.8305	1.3380
	0.0672	0.0701	0.0877	0.0678	0.0720	0.9430	0.1430	0.2612	0.5781	0.4236	0.6598	0.8990
50	1.1787	1.1341	1.0836	1.1574	1.1561	0.8173	0.9391	0.7628	0.5421	0.8614	0.8440	1.1788
	0.0447	0.0445	0.0491	0.0420	0.0434	0.9440	0.1076	0.1763	0.3605	0.2685	0.3406	0.9150
100	1.1761	1.1534	1.1291	1.1650	1.1647	0.6039	0.8995	0.7974	0.6780	0.8521	0.8464	0.9773
	0.0243	0.0227	0.0224	0.0206	0.0209	0.9500	0.0704	0.1022	0.1750	0.1421	0.1575	0.9280
			κ=7.5	**ζ=1.50**	**$\mathbb{S}_{pk}$=1.1735**	**$\mathbb{S}'_{pk}$=1.1191**	**$\mathbb{S}''_{pk}(5)$=1.0400**		**$\mathbb{S}''_{pk}(10)$= 0.8160**			
20	1.0664	0.9592	0.7996	1.0195	1.0104	1.0623	0.8622	0.5007	0.1600	0.7235	0.6356	1.2980
	0.0777	0.1072	0.2121	0.1291	0.1508	0.9190	0.1444	0.3615	0.8417	0.8542	2.8453	0.8990
30	1.0408	0.9660	0.8677	1.0067	1.0023	0.9161	0.8465	0.5734	0.2502	0.7366	0.6869	1.1938
	0.0573	0.0748	0.1237	0.0877	0.0958	0.9350	0.1186	0.2730	0.6997	0.5653	1.0768	0.9070
50	1.0398	0.9970	0.9458	1.0197	1.0181	0.7256	0.8231	0.6359	0.3989	0.7429	0.7196	1.0627
	0.0357	0.0428	0.0587	0.0472	0.0493	0.9400	0.0908	0.1872	0.4701	0.3488	0.4870	0.9200
100	1.0296	1.0082	0.9847	1.0192	1.0188	0.5334	0.8234	0.7174	0.5834	0.7759	0.7677	0.8787
	0.0192	0.0214	0.0252	0.0225	0.0229	0.9440	0.0592	0.1060	0.2262	0.1695	0.1960	0.9310
			κ=7.5	**ζ=1.75**	**$\mathbb{S}_{pk}$=1.0220**	**$\mathbb{S}'_{pk}$=0.9870**	**$\mathbb{S}''_{pk}(5)$=0.9336**		**$\mathbb{S}''_{pk}(10)$= 0.7643**			
20	0.9361	0.8223	0.6372	0.8884	0.8758	0.9646	0.7851	0.3963	0.1050	0.6467	0.5265	1.1588
	0.0651	0.1138	0.2958	0.1644	0.2076	0.9210	0.1188	0.3888	0.8942	1.0364	9.5793	0.8990
30	0.9319	0.8580	0.7520	0.8993	0.8936	0.8175	0.7538	0.4661	0.1662	0.6429	0.5775	1.0583
	0.0464	0.0738	0.1575	0.1013	0.1151	0.9320	0.0976	0.2876	0.7962	0.7318	1.9367	0.9080
50	0.9174	0.8739	0.8180	0.8975	0.8953	0.6587	0.7382	0.5429	0.2950	0.6580	0.6272	0.9460
	0.0298	0.0435	0.0753	0.0556	0.0590	0.9460	0.0752	0.1953	0.5815	0.4408	0.7137	0.9050
100	0.9138	0.8935	0.8702	0.9041	0.9035	0.4750	0.7466	0.6357	0.4900	0.6982	0.6874	0.7962
	0.0155	0.0203	0.0290	0.0248	0.0253	0.9500	0.0507	0.1109	0.2904	0.2091	0.2563	0.9180

4. Applications

This section deals with the real life application of the considered study. Considering two real data sets one is for aluminium foil and other is for liquid crystal display, before including there data sets in our study we use "R programming" for goodness of fit test, in particular use *fitdistrplus* Delignette-Muller and Dutang, [5] package to get the K-S test value and p-value for given data sets.

Data Set I

The dataset, consisting of 50 observation, is sourced from Saha [17]; Saha and Dey [21] and records the voltage of aluminium foil used by an electronics manufacturer. The production specifications of the voltage are set as ($\mathbb{L}$, $\mathbb{T}$, $\mathbb{U}$) =(510, 520, 530). This voltage range is crucial to prevent the aluminium foil from breaking, which would lead to its rejection.

Using the $fitdistrplus$ package, we firstly check for normality test and then obtained the estimated values of κ and ζ as $519.7560, 1.7658$, respectively. Additionally, the one sample Kolmogorov-Smirnov statistic and its p-values are $(0.0734, 0.9505)$, respectively, indicated a good fit.

Data Set II

The data, sourced from Peng [16]; Wu and Pearn [27] and also used by Saha and Dey [21], represents the glass substrate thickness (in mm) of the $\mathbb{STN}$-$\mathbb{LCD}$, (liquid crystal display). If the product's characteristic falls outside this range, $\mathbb{L} = 0.63$ and $\mathbb{U} = 0.77$, with $\mathbb{T} = 0.70$ the lifetime or reliability of the $\mathbb{STN}$-$\mathbb{LCD}$ will be deemed unacceptable and the product will be rejected. Before moving forward, we first examine whether the normal distribution is appropriate for analyzing the given dataset. For dataset II, the $\mathbb{MLE}$s of κ and ζ are found to be $\hat{\kappa}_{mle} = 0.7096$, $\hat{\zeta}_{mle} = 0.0193$. The one sample (K-S statistic, p-values) are (0.0707, 0.8250), respectively.
For given specifications of two data sets the classical and Bayes estimates for S''_{pk} with four different ϖ values are calculated. In classical setup, $\mathbb{MLE}$ and $\mathbb{MPSE}$ are considered for providing the point and interval estimates ($\mathbb{AIC}$) and are reported in Table 8. The Bayes estimates are obtained for five different loss functions along with $\mathbb{HPD}$ credible intervals are also obtained and shown in Table 8. Further, the Bayes point and interval estimates of the considered $\mathbb{PCI}$ are computed under the assumption of non-informative priors using different loss functions for various values of $\varpi = 0.01$ to 10. the values of hyper-parameters are chosen as $\alpha = (1.05, 1.55)$, $\beta = (0.01, 0.001)$ and $\kappa_0 = (520, 0.70)$, respectively. The obtained result is reported in Table 8. The mixing of the chain based on generated sequence of posterior samples under $\mathbb{MCMC}$ approach for both the data sets is investigate through the corresponding trace plot, which is given in Figure 1. This suggests that the generated samples are effectively mixed and converge to the true parameter values.

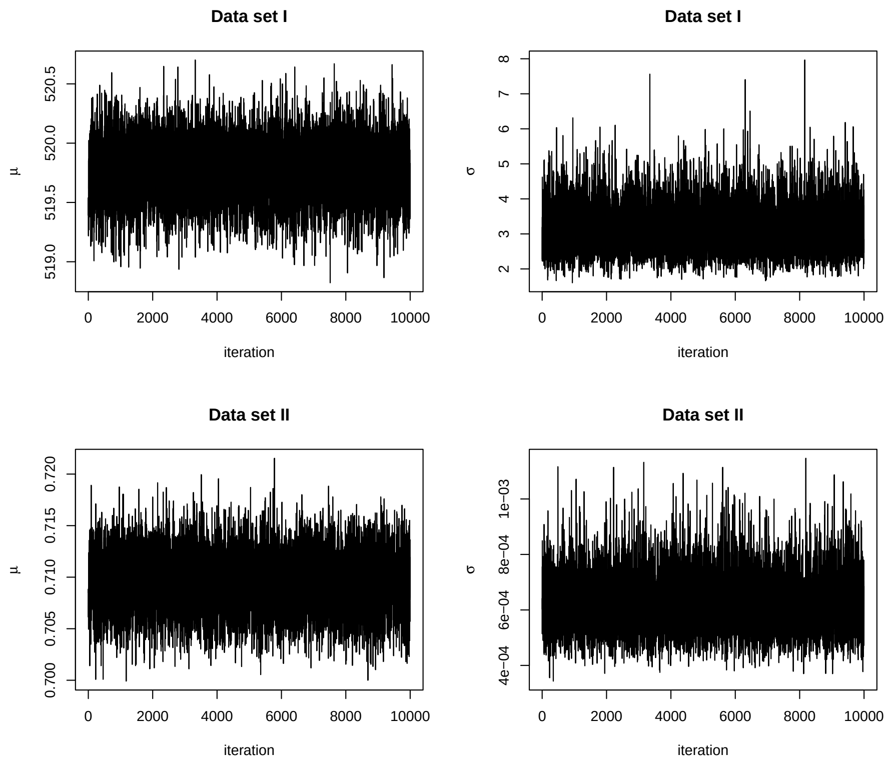

Figure 1: *Trace plot for the considered real data sets I and II.*

Table 8: *Classical and Bayesian estimates of $\mathbb{S}''_{pk}$ using $\mathbb{MLE}$ and $\mathbb{MPSE}$ for real data sets with $\varpi = (0.01, 1, 5, 10)$.*

Point and Interval Estimates							
Data set	n	Method of Estimation	Classical Estimate	Width			
				$\mathbb{ACI}$	$\mathbb{GCI}$	$\mathbb{BCI}$	
						$\mathbb{P}$- boot	$\mathbb{S}$- boot
$\varpi = 0.01$							
I	50	$\mathbb{MLE}$	1.8539	0.7475	0.7295	0.7399	0.7330
		$\mathbb{MPSE}$	1.8359	0.7400	0.7350	0.8223	0.8154
II	79	$\mathbb{MLE}$	1.0012	0.3532	0.3446	0.3656	0.3607
		$\mathbb{MPSE}$	0.9966	0.3517	0.3453	0.3742	0.3745
$\varpi = 1$							
I	50	$\mathbb{MLE}$	1.8553	0.7440	0.7207	0.7682	0.7725
		$\mathbb{MPSE}$	1.8372	0.7366	0.7464	0.8137	0.8021
II	79	$\mathbb{MLE}$	1.0009	0.3535	0.3400	0.3618	0.3556
		$\mathbb{MPSE}$	0.9963	0.3519	0.3441	0.3824	0.3858
$\varpi = 5$							
I	50	$\mathbb{MLE}$	1.8593	0.7361	0.7254	0.7578	0.7462
		$\mathbb{MPSE}$	1.8411	0.7289	0.7174	0.8085	0.8032
II	79	$\mathbb{MLE}$	0.9998	0.3547	0.3448	0.3785	0.3676
		$\mathbb{MPSE}$	0.9952	0.3532	0.3498	0.4060	0.3954
$\varpi = 10$							
I	50	$\mathbb{MLE}$	1.8624	0.7327	0.7364	0.7889	0.7839
		$\mathbb{MPSE}$	1.8441	0.7255	0.7497	0.7669	0.7858
II	79	$\mathbb{MLE}$	0.9983	0.3563	0.3443	0.3452	0.3568
		$\mathbb{MPSE}$	0.9937	0.3547	0.3516	0.4090	0.3950
Bayes Point Estimates and Credible Interval Estimates							
		$\mathbb{BE}$					Width
		$\mathbb{SELF}$	$\mathbb{WSELF}$	$\mathbb{MSELF}$	$\mathbb{LLF}$	$\mathbb{KLF}$	$\mathbb{CI}$
$\varpi = 0.01$							
I	50	1.8588	1.8398	1.8207	1.8493	1.8493	0.7309
II	79	0.9889	0.9811	0.9733	0.9850	0.9850	0.3403
$\varpi = 1$							
I	50	1.8633	1.8445	1.8254	1.8539	1.8538	0.7291
II	79	0.9882	0.9805	0.9729	0.9843	0.9844	0.3426
$\varpi = 5$							
I	50	1.8719	1.8541	1.8362	1.8631	1.8630	0.7061
II	79	0.9867	0.9789	0.9713	0.9828	0.9828	0.3382
$\varpi = 10$							
I	50	1.8730	1.8532	1.8308	1.8634	1.8631	0.7149
II	79	0.9854	0.9775	0.9696	0.9814	0.9814	0.3436

5. Conclusions

In this article, the point and interval estimation have been considered under classical and Bayesian paradigm for newly proposed index $\mathbb{S}''_{pk}$, when the process is normally distributed. Under classical estimation for the index, six methods were studied. One approach was explored to obtain the interval estimation: $\mathbb{ACI}$ based on $\mathbb{MLE}$. In order to evaluate the effectiveness of each of these point and interval estimations, we conducted Monte Carlo simulation studies with various layouts, obtaining $\mathbb{MSE}s$ for point estimation and $\mathbb{AW}s$, $\mathbb{CP}s$ for interval estimation, respectively. According to the simulation studies, observed that $\mathbb{MLE}$ performed better in terms of their $\mathbb{MSE}$ for point estimation, whereas $\mathbb{ACI}$ shows appropriate pattern in terms of $\mathbb{AW}s$ for interval estimation. The Bayes (point and interval) estimations are carried out for the index $\mathbb{S}''_{pk}$ with five different loss functions. The same setups were used for the Bayesian estimation. The comparison was considered with the help of average $\mathbb{BE}$ and corresponding $\mathbb{PR}$ obtained by simulation study. According to this, $\mathbb{LLF}$ has the lowest $\mathbb{PR}$ and $\mathbb{SELF}$ has the highest $\mathbb{BE}$ with decreasing $\mathbb{AW}$ and increase in $\mathbb{CP}$ as sample size rise. Two real data sets were re-analyzed, and the results were consistent with the findings of the considered simulation study. Real life experiment revels that $\mathbb{SELF}$ has the largest $\mathbb{BE}$ for all four sets of ϖ values. For further study the present study may be extended for the censored scheme. We hope that the results obtained in this study will help the practitioners in the manufacturing industries in their decision making.

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