STATISTICAL INFERENCE ON THE GENERALIZED EXPONENTIAL DISTRIBUTION BASED ON GENERALIZED ORDER STATISTICS CHARACTERIZATIONS AND APPLICATIONS

Imtiyaz A. Shah[1] and Khalid Ul Islam Rather[2]

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[1]Department of Community Medicine (Bio Statistics), Sher-I-Kashmir Institute of Medical Sciences, Srinagar (J&K) India
2Department of Statistics and Computer Science, Sher- e- Kashmir University Agricultural Sciences and Technology Jammu (J&K) India
driashah03@gmail.com, khalidstat34@gmail.com

Abstract

In this paper, we study the Some distribution functions have been characterized based on m-dual generalized order statistics and consequently m-generalized order statistics. Moreover, we show that these characterization properties provide a beneficial strategy to predict future events, which are based on past or current events and on an arbitrary distribution function. Finally, an application of these results is given for bivariate generalized exponential distribution.

Keywords: Generalized exponential distribution, Dual generalized order statistics, generalized order statistics, Generalized inverted exponential distribution, weak convergence, Random sample size

I. Introduction

Generalized order statistics introduced by [21]. A concept of generalized order statistics (GOSs) as a unified approach to a variety of models of ascendingly ordered random variables (RVs) introduced by [21]. The concept of dual GOSs, denoted by DGOSs, was introduced by [9] as a parallel concept of GOSs to enable a common approach to descendingly ordered RVs. [9] has shown that (cf. Theorem 3.3) there is a direct link between DGOSs and GOSs.

The subclasses m −GOSs and m −DGOSs of GOSs and DGOSs, respectively, contain many important models of ordered RVs such as ordinary order statistics (OOSs), lower and upper record values, k−records, sequential order statistics (SOSs) and type II censored OOSs. For any 1 ≤ r ≤ n, the marginal probability density functions (PDFs) of the rth m-GOS X(r, n; m, k) and m-DGOS X^*(r, n; m, k), based on a continuous distribution function (DF) F_X(x) = P(X ≤ x) with a PDF f_X(x), are given, respectively, by (cf. [10] and [19]).

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!}\,[\bar{F}_X(x)]^{\gamma_r^{(n)}-1}[\frac{1-[\bar{F}_X(x)]^{m+1}}{m+1}]^{r-1} f_X(x) \quad , \quad m \neq -1 \tag{1}$$

and

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!}\,{F_X}^{\gamma_r^{(n)}-1}\,[\frac{1-[F_X(x)]^{m+1}}{m+1}]^{r-1} f_X(x) \quad , \quad m \neq -1 \tag{2}$$

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where $\overline{F} = 1 - F$, $\gamma_r^{(n)} = k + (n-r)(m+1)$ and $C_{r-1}^{(n)} = \prod_{i=1}^{r} \gamma_r^{(n)}$, $1 \le r \le n$.

Classical results in characterizations can be found in [12], [13], [14] and [15]. Different results of characterization and its applications in terms of GOSs and DGOSs are derived by many authors. Among these authors are [5], [7], [11], [16], [17], [18], [19] and [20].

In this work, we consider a wide subclass of GOSs and DGOSs, specifically when $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$. This subclass is known as m-GOSs and m-DGOSs [22] (see also [9]) has derived the marginal df of the $(m-r+1)^{th}$ m-GOSs, in the form $\varphi_{m-r+1:m}^{(n,k)}(x) = I_{G_{n(x)}}(N - R + 1, R)$, where $G_n(x) = 1 - (1 - F(x))^{n+1} = 1 - \overline{F}(x)^{n+1}$, $N = \frac{k}{n+1} + m - 1$, $R = \frac{k}{n+1} + r - 1$ and $I_x(a,b) = \frac{1}{\beta(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, a, b > 0$, is the incomplete beta ratio function. Similarly, by using the results of [11] the marginal df of the $(m-r+1)^{th}$m-DGOSs, is given by $\varphi_{m-r+1:m}^{d(n,k)}(x) = I_{T_{n(x)}}(R, N - R + 1)$, where $T_n(x) = F^{n+1}(x)$.

II. Model

GIED is the generalization of the one parameter IED which was proposed by [4]. The various statistical properties and reliability estimation has also been studied in detail by [4]. A three parameter GIED has the following PDF and CDF as follows: The cumulative distribution function CDF $F_X(x)$and probability density function PDF $f_X(x)$ of the Generalized inverted exponential distribution (GIED) distribution are given by:

$$\mathrm{F_X}(x) = 1 - [1 - e^{-\lambda\ /x}]^{\alpha}, 0 \le x < \infty, \alpha\ \beta,\ \lambda > 0 \tag{3}$$

and

$$f_X(x) = \frac{\alpha\lambda\beta}{x^2} e^{-\frac{\lambda\beta}{x}} \left[1 - e^{-\frac{\lambda\beta}{x}}\right]^{\alpha-1}, 0 \le x < \infty, \alpha, \beta, \lambda > 0 \tag{4}$$

where α and β are the shape parameters and λ is the scale parameter.

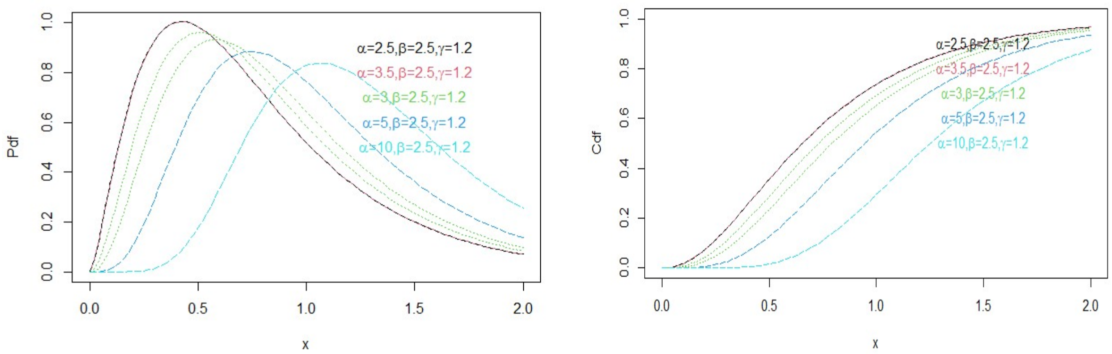

Figure 1. Possible shapes of the probability density function f(x) (left) and cumulative distribution function F(x) (right) of the Generalized inverted exponential distribution (GIED) distribution for fixed parameter values of β and λ.

The Generalized inverted exponential distribution (GIED) distribution reduces to generalization of the one parameter (IED) when α = 1.

III. Reliability Characteristics

The reliability function R(x) is an important tool for characterizing life phenomenon. R(x) is analytically expressed as $R(x) = 1 - F(x)$. Under certain predefined conditions, the reliability function R(x) gives the probability that a system will operate without failure until a specified time x. The reliability function of the Generalized Inverted Exponential Distribution (GIED) distribution is given by

$$R(x) = 1 - \left[1 - e^{-\frac{\lambda\beta}{x}}\right]^{\alpha}, 0 \leq x < \infty, \alpha, \beta, \ \lambda > \ 0 \tag{5}$$

Another important reliability characteristics is the failure rate function. The failure rate function gives the probability of failure for a system that has survived up to time x. The failure rate function h(x) is mathematically expressed $h(x) = f(x)/R(x)$. The failure rate function the Extended Generalized inverted exponential distribution (GIED) distribution is given by:

$$h(x) = \frac{\frac{\alpha\lambda\beta}{x^2} e^{-\lambda\beta/x}[1 - e^{-\lambda\beta/x}]^{\alpha-1}}{1 - [1 - e^{-\lambda\beta/x}]^{\alpha}}, 0 \leq x < \infty, \alpha, \beta, \lambda > 0$$

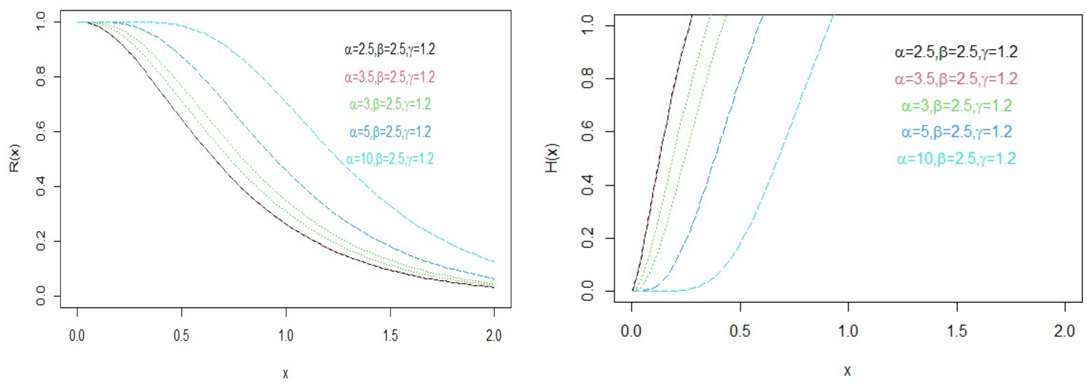

Figure 2. Possible shapes of the reliability function R(x) (left) and failure rate function h(x) (right) of the Generalized inverted exponential distribution (GIED distribution for fixed parameter values of β and λ.

Let $\{X_i\}$ and $\{Y_j\}$ be two sequences of rv's defined on the same probability space $(\varphi, \omega, *)$, where $\{X_i\}$ is i.i.d with common df $F(x)$) (the rv's $Y_i, i = 1,2, \cdots$ need not to be independent or identical). Furthermore, let $Y^*_{m-r_{m+1}:m} = Y(m - r_m + 1, m, n, k)(Y^*_{d:m-r_{m+1}:m} = Y_d(m - r_m + 1, n, m, k))$ and $X^*_{m-r_{m+1}:n} = X(m - r_m + 1, n, m, k)(X^*_{d:m-r_{m+1}:m} = X_d(m - r_m + 1, n, m, k))$ be the $(m - r_m + 1)^{th}$ m-GOSs and m-DGOSs corresponding the sequences Y_i and X_i, respectively, where $1 \leq r_m \leq m$, when the sample size itself is assumed to be a rv v_n and $(Y^*_{m-r_{m+1}:m})(Y^*_{d:m-r_{m+1}:m})$ and some regularly varying function, the limit df's of m-GOSs and m-DGOSs $(Y^*_{v_m-r_{m+1}:m})(Y^*_{d:v_m-r_{m+1}:m}), n > -1$ are derived in the following exhaustive cases:

(1) Extreme case, where $r_m = r$, is any fixed integer and $1 \leq r < m$.

(II) Central case, where $r_m \to \infty$ and $\frac{r_m}{m} \to \ \lambda \, \epsilon \ (0,1)$, as $\to \infty$.

(III) Intermediate case, where min $(r_m, m - r_m) \to \infty$ and $\frac{r_m}{m} \to \ 0$, as $m \to \infty$.

It is worth mentioning that when $n = -1, k = 1, i.e.$, the case of the record values, the above problem was studied recently by [6].

Suppose that an, $a_m, \tilde{a}_m > 0$ and $b_m, \tilde{b}_m \, \epsilon \, \Re$ are suitable normalizing constants such that

$$\Phi^{(n,k)}_{v_m-r_m+1:v_m}(a_m x+b_m)=P((X^*_{v_m-r_{m+1}:m}\le a_m x+b_m)\xrightarrow{w}\omega^{(n,k)}(x) \tag{6}$$

and

$$\Phi^{d(n,k)}_{m-r_m+1:v_m}(\tilde{a}_m x+\tilde{b}_m)=P((X^*_{d:v_m-r_{m+1}:v_m}\le \tilde{a}_m x+\tilde{b}_m)\xrightarrow{w}\omega^{d(n,k)}(x) \tag{7}$$

where $\left(\xrightarrow{w}\right)$ stands for weak convergence, as $n\to\infty, \omega^{(m,k)}(x)$ and $\omega^{d(m,k)}(x)$ are a non-degenerate df 's. Let C and C^d be the classes of all non degenerate limit df 's $\omega^{(m,k)}(x)$ and $\omega^{d(n,k)}(x)$ in (1) and (2), respectively. Recently, [6] and [9] studied the limit df 's of the bivariate extreme, central, and intermediate m-generalized order statistics (m-GOSs) and m- dual generalized order statistics (m-DGOSs) with random sample size. This study gives the corresponding results for univariate upper generalized order statistics (GOSs) and upper dual generalized order statistics (DGOSs) cases with random sample size. These results, which characterize the classes C and C^d are presented in Theorem 2.1 and Theorem 2.2, respectively, where through these theorems and all the paper the symbols $\underset{m}{\rightarrow}$ stands for convergence, as $m\to\infty$,. Moreover, the symbols $\xrightarrow{p}$ stands for convergence in probability, as $m\to\infty$. [1] has characterized exponential distribution under random dilation for adjacent GOSs. In this paper, distributional properties of the generalized order statistics (GOSs) have been used to characterize distributions for non- adjacent generalized order statistics (GOSs) under random translation, dilation and contraction. One may also refer to [5], [7], [24], [25], [17], [20] and [21] for the related results.

IV. Characteristics Results

Theorem 2.1: Let $n>-1$, For any non-degenerate distribution function (df) $\omega^{(n,k)}(x)$, $\omega^{(n,k)}(x)\in C$, if and only if one of the following conditions (I), (II) and (III) holds. Let $X(r,n,m,k)$ be the r^{th} m-generalized order statistics (GOSs) from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then,

$$\Psi^{(n,k)}_{m-r+1:m}(a_m x+b_m)=P(X^*_{m-r+1}\le a_m x+b_m)\xrightarrow{w}\psi^{(nk)}(x)=1-\exp\left(\mathcal{U}^{n+1}_{i:\beta}(x)\right) \tag{8}$$

where $X(m_1-m_2-j,n_1,m,k)$ is independent of $X(r,n_2,m,k)$ *if* and only *if* $X_1\sim\exp\left(\mathcal{U}^{n+1}_{i:\beta}(x)\right)$.

Proof. The limit df $\Psi^{(n,k)}(x)$ has the form $\Psi^{(n,k)}(x)=1-\int_0^\infty \Gamma_R\left(z\mathcal{U}^{n+1}_{i:\beta}(x)\right)dp\ (\tau\le z), i\in\{1,2,3\}$, where $\mathcal{U}_1(x)=\mathcal{U}_{1:\beta}(x)=e^{-x}, \forall\ x;\ \mathcal{U}_{2:\beta}(x)=e^{-\beta}, x>0$; and $\mathcal{U}_{3:\beta}(x)=(-x)^{\beta},\ x\le 0.\ \frac{v_m}{m}\xrightarrow{p}\tau$.

$$\Psi^{(n,k)}_{m-r_m+1:m}(a_m x+b_m)=P(X^*_{m-r_m+1}\le a_m x+b_m)\xrightarrow{w}\Psi^{(n,k)}(x)=\mathbb{N}(W^*_{i:\beta}(x)) \tag{9}$$

Where $\left(W^*_{i:\beta}(x)\right)=\frac{C^*_{\lambda(n)}}{C^*_{\lambda}}(n+1)\left(W^*_{i:\beta}(x)\right), C^*_{\lambda}=\frac{C_\lambda}{\lambda}, C_\lambda=\sqrt{\lambda(1-\lambda)}, \lambda(n)=\lambda^{1/(n+1)}$. And $\mathbb{N}(.)$ is the standard normal df, $\frac{v_n}{n}\xrightarrow{p}\tau$.

The limit df $\Psi^{(n,k)}(x)$ has the form $\Psi^{(n,k)}(x)=1-\int_0^\infty \mathbb{N}\left(\sqrt{z}W^*_{i:\beta}(x)\right)dp(\tau\le z)$ where $W_{i:\beta}(x)$, $i\in\{1,2,3,4\}$, has one and only one of the following types.

$$W_{1:\beta}(x)=\begin{cases}-\infty, & x\le 0\\ cx^{\beta}, x>0, c_2,\beta>0\end{cases} \tag{10}$$

$$W_{2:\beta}(x)=\begin{cases}-c|x|^{\beta}, & x\le 0\\ \infty, & x>0, c,\beta>0\end{cases} \tag{11}$$

$$W_{3:\beta}(x)=\begin{cases}-c_1\,|x|^{\beta}, x\le 0, c_1>0\\ c_2x^{\beta}, x>0, c_2,\beta>0\end{cases}\tag{12}$$

$$W_{4:\beta}(x)=\begin{cases}-\infty\,, x\le -1\\ 0\,, -1<x\le 1\\ \infty\,, x>1\end{cases}\tag{13}$$

$$\sqrt{r_{m+\sigma(m)}}-\sqrt{r_n}=\frac{\alpha\gamma l}{2},$$

For any sequence of integer values $\{\sigma(m)\}$ for which $\frac{\sigma(m)}{m^{1-\alpha/2}}\underset{m}{\to}\gamma$.

Where $0<\alpha<1$, $l>0$ and γ is any real number (intermediate case).

$$\Psi^{(n,k)}_{m\psi-r_m+1:m}(a_mx+b_m)=P(X^*_{m-r_m+1}\le a_mx+b_m)\overset{w}{\to}\psi^{(n,k)}(x)=\mathbb{N}((n+1)V^*_{i:\beta}(x))\tag{14}$$

$$\frac{v_m-m}{m^{1-\alpha/2}}\overset{p}{\to}\tau$$

The limit df $\Psi^{(n,k)}(x)$ has the form $\Psi^{(n,k)}(x)=\int_{-\infty}^{\infty}\mathbb{N}((n+1)V_{i:\beta}(x)-lz(1-\alpha)dp(\tau\le z)$ where $V_{i:\beta}(x)$, has one and only one of the following types.

$$V_1(x)=V_{i:\beta}(x)=x,\forall x\tag{15}$$

$$V_{i:\beta}(x)=\begin{cases}-\beta ln|x|\,, x\le 0\\ \infty\,, x>0\end{cases}\tag{16}$$

$$V_{3:\beta}(x)=\begin{cases}-\infty\,; x\le 0\\ \beta ln|x|\,; x>0\end{cases}\tag{17}$$

where β is some positive constant, which depends only on α , l and the type of F(x). Finally, if r_n does not satisfy any of conditions (i − I), (i − II) and (i − III), then $\Psi^{(n,k)}(x)$ can only have a degenerate type or does not exist.

Theorem 2.2: Let $n>-1$, For df $\omega^{(n,k)}(x)$, $\omega^{d(n,k)}(x)\,\epsilon\, C^d$, if and only if one of the following conditions (I), (II) and (III) holds. Let $X^*(r,n,\tilde{m},k)$ be the r^{th} GOSs from a sample of size n drawn from a continuous population with the $pdf\ f$ and the $df\ F$, then,

$$\Psi^{d(n,k)}_{m-r+1:m}(\tilde{a}_mx+\tilde{b}_m)=P(X^*_{d:m-r+1}\le\tilde{a}_mx+\tilde{b}_m)\overset{w}{\to}\psi^{d(nk)}(x)=1-\exp(\mathcal{U}^{n+1}_{i:\beta}(x))\tag{18}$$

Proof. The limit df $\Psi^{d(n,k)}(x)$ has the form $\Psi^{d(n,k)}(x)=1-\int_0^{\infty}\Gamma_R\left(z\mathcal{U}^{n+1}_{i:\beta}(x)\right)dp\ (\tau\le z), i\in\{1,2,3\}$,
where $\mathcal{U}^*_1(x)=\mathcal{U}^*_{1:\alpha}(x)=e^{-x}$, $\forall\, x$; $\mathcal{U}^*_2(x)=\mathcal{U}^*_{2:\alpha}(x)=e^{-x}$, $x>0$; and $\mathcal{U}^*_3(x)=\mathcal{U}^*_{3:\alpha}(x)=(-x)^{\beta}$, $x\le 0$. $\frac{v_m}{m}\overset{p}{\to}\tau$.

$$\Psi^{d(n,k)}_{m-r_m+1:m}(\tilde{a}_mx+\tilde{b}_m)=P(X^*_{m-r_m+1}\le\tilde{a}_mx+\tilde{b}_m)\overset{w}{\to}\Psi^{d(n,k)}(x)=\mathbb{N}(W^*_{i:\beta}(x))\tag{19}$$

Where $\varphi^{d(m,k)}(x)$ is a non-degenerate distribution function (df). Furthermore, let v_n be a sequence of non-negative integer valued rv's which satisfy

$$\frac{v_m-m}{m^{1-\alpha/2}}\ \overset{p}{\to}\ \tau$$

The limit distribution function (df) $\Psi^{d(n,k)}(x)$ has the form $\Psi^{d(n,k)}(x)=1-\int_0^{\infty}\mathbb{N}\,((m+1)V_{i:\beta}(x)+lz(1-\alpha)dp(\tau\le z), i=\{1,2,3,\cdots\}$. Finally, if r_m does not does not satisfy any of conditions (1 − I), (i − II) and (i − III), then $\Psi^{d(n,k)}(x)$ can only have a degenerate type or does not exist has one and only one of the following types.

Remark 1. Under conditions (I), (II), and (III), when $m=0, k=1$, Theorem 2.1 gives the upper order statistics case and Theorem 1.2 gives the lower order statistics case.

Theorem 2.3: Suppose that $X^*_{m-r_m+1:m}\in L(Y^*:\varphi|\tau)$. Let $n>-1, a_m>0$ and $b_m\in\mathcal{B}, n\ge 1$, be suitable normalizing constants for which

$$P(X^*_{m-r_m+1} \leq a_m x + b_m) \overset{w}{\rightarrow} \varphi^{(n,k)}(x) \tag{20}$$

Where $\varphi^{d(n,k)}(x)$ is a non-degenerate df. Furthermore, let ν_m be a sequence of non negative integer valued rv's which satisfy

$$\frac{\nu_m - m}{m^{1-\alpha/2}}$$

where the df $A(z) = P(\tau \leq z)$ is continuous at zero, i.e., $A(0-) = A(0) = A(0+) = 0$ (the condition $A(0-) = A(0)$ is not considered in the original Theorem 2.1 of Xie Shengrong (1997), for the maximum oos, but as we shall see in the proof of Theorem 2.1, it seems to be indispensable). Then

$$P(X^*_{m-r_m+1} \leq a_m x + b_m) \overset{w}{\rightarrow} \varphi^{(m,k)}(x) \overset{w}{\rightarrow} \begin{cases} 1 - \int_0^\infty \Gamma\left(z^0 \Psi^{m+1}_{i:\beta}(x) dA\,(z), i \epsilon \{1,2,3\}\right) \\ \text{where } r = r_n = \text{constsnt (extreme case)} \\ \int_0^\infty N(z^{\frac{\rho}{2}} W^*_{i:\beta}(x) dA(z), i \in \{1,2,3\}, \text{(central case)} \end{cases} \tag{21}$$

Furthermore, let the intermediate rank sequence $\{r_m\}$ satisfy Chibisov condition and let condition (21) be replaced by

$$\frac{\nu_m - m}{m\varphi^{-\frac{\alpha}{2}}(m)} \overset{p}{\rightarrow} \tau$$

Where $\varphi(m) \in \text{RV } 0 < \rho < 1\ \alpha, \varphi_m^{\frac{\alpha}{2}}(\frac{\varphi(mx)}{\varphi(m)} - x^0) \underset{m}{\rightarrow} 0$ and the df $A(Z)$ is symmetric and continuous at zero. Then

$$P(X^*_{m-r_m+1} \leq a_m x + b_m) \overset{w}{\rightarrow} \int_{-\infty}^{\infty} N((m+1)V_{i:\beta}(x) - l\rho z(1-\alpha) dA(z), i \in \{1,2,3\} \tag{22}$$

$$P\big(m(\tau - \epsilon) \leq \nu_m \leq n(\tau + \epsilon)\big) \underset{m}{\rightarrow} 1 \tag{23}$$

Proof: Consider, first the extreme case, where $r_m = r = \text{constant}$. Let $d > 0$ be a positive real number. Clearly, for any $\varepsilon\, \epsilon (0, d)$, (12) implies

$$P\big(m(\tau - \epsilon) \leq \nu_m \leq m(\tau + \epsilon)\big) \underset{m}{\rightarrow} 1 \tag{24}$$

On the other hand, we can write

$$P\big(X^*_{m-r_m+1} \leq a_m x + b_m\big) = P\big(X^*_{m-r_m+1} \leq a_{[\varphi(m)]} x + b_{[\varphi(m)]}\big) + P\big(m(\tau - \epsilon) > \nu_m \text{ or } \nu_m > n(\tau + \epsilon)\big)$$
$$\big(X^*_{m-r_m+1} \leq a_{[\varphi(m)]} x + b_{[\varphi(m)]}\big) = \wp_m^{(1)}(\epsilon) + \wp_m^{(2)}(\epsilon)$$

According to (24), $\wp_m^{(2)}(\epsilon) \underset{n}{\rightarrow} 0$, so calculating $\wp_m^{(1)}(\epsilon)$ is only necessary. However

$$\wp_m^{(1)}(\epsilon) = P\big(m(\tau - \epsilon) \leq \nu_m \leq n(\tau + \epsilon)\big)\big(X^*_{m-r_m+1} \leq a_{[\varphi(m)]} x + b_{[\varphi(m)]}\big)$$
$$= \int_0^\infty P\big(m(\tau - \epsilon) \leq \nu_m \leq n(\tau + \epsilon)\big) X^*_{m-r_m+1} \leq a_{[\varphi(m)]} x + b_{[\varphi(m)]} | \tau = z dA(z)$$
$$\leq P(\tau \leq d) + \int_0^\infty P\big(m(\tau - \epsilon) \leq \nu_m \leq n(\tau + \epsilon)\big) X^*_{m-r_m+1} \leq a_{[\varphi(m)]} x + b_{[\varphi(m)]} | \tau = z dA(z) z \tag{25}$$

By using the well-known inequality

$$X^*_{m'-r_m+1:m'} \leq X^*_{m''-r_m+1:m''}\ , m' \leq m''$$

we obtain

$$\wp_m^{(1)}(\epsilon) \leq P(\tau \leq d) = \int_0^\infty P\big(X^*_{[m(\tau-\epsilon)-r+1:[n(\tau-\epsilon)]}\big) \leq a_{[\varphi(m)]} x + b_{[\varphi(m)]} | \tau = z dA(z)$$

$$= P(\tau \le d) + \int_0^\infty P\big(X^*_{[m(\tau-\in)-r+1:[n(\tau-\in)]}\big) \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} | \tau = z dA(z) \tag{26}$$

Since $X^*_{m-r_m+1:m} \in L(\Delta^*: \Phi(\tau)$, we obtain

$$\wp_m^{(1)}(\epsilon) \le P(\tau \le d) + \int_0^\infty P\big(X^*_{[m(\tau-\in)-r+1:[n(\tau-\in)]}\big) \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} | \tau = z dA(z) + 0(1) \tag{27}$$

Now, let us introduce

$$\tau_d = \begin{cases} \tau \ , & \tau > d \\ d \ , & \tau \le d \end{cases}$$

Clearly,

$$P(\tau_d \le z) = \begin{cases} 0\,, & z < d \\ P(\tau \le z), & z \ge d \end{cases} \tag{28}$$

Therefore, (18) yields

$$\wp_m^{(1)}(\epsilon) \le P(\tau \le d) = P\big(X^*_{[\varphi([m(\tau_d-\in)]-r+1:[\varphi[n(\tau_d-\in)]}\big) \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} | \tau + 0(1)$$

On the other hand, since $\tau_d - \in > 0$, we obtain

$$\frac{\varphi([m(\tau_d-\in)])}{\varphi(m)} = \frac{\varphi\left(n\frac{[m(\tau_d-\in)]}{m}\right)}{\varphi(m)} \underset{m}{\rightarrow} (\tau_d-\in)^p \tag{29}$$

Note that $\frac{\varphi(mx)}{\varphi(m)} \underset{m}{\rightarrow} x^\rho$ holds locally uniformly on $(0,\infty)$, consequently, by using Theorem 2.1 (the extreme case) and we obtain

$$\lim_{n\to\infty} \wp_m^{(1)}(\epsilon) \le P(\tau \le d) + \int_0^\infty 1 - \Gamma_R((z-\in)^p \mho_{i:\beta}^{m+1}(x) dA(z)\,, i \in \{1,2,3\} \tag{30}$$

Thus, by letting $\epsilon \to 0$, then $d \to 0$, and in view of the fact that $0 = \Theta(0^-) = \Theta(0) = \Theta(0^+)$, we obtain

$$\lim_{n\to\infty} P(X^*_{\nu_m-r+1:\nu_m} \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} \le 1 - \int_0^\infty 1 - \Gamma_R((z-\in)^p \mho_{i:\beta}^{m+1}(x) dA(z)\,, i \in \{1,2,3\} \tag{31}$$

By applying a similar argument (with only the obvious modification) we can prove that $\lim_{n\to\infty} P(X^*_{\nu_m-r+1:\nu_m} \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} \ge 1 - \int_0^\infty 1 - \Gamma_R((z-\in)^p \mho_{i:\beta}^{m+1}(x) dA(z)\,, i \in \{1,2,3$ (32)

Thus, the claimed result for extreme case follows by combining (31) and (32). For proving the theorem in central case, the preceding argument can be repeated by changing the roles of Theorem 2.1, Part (I) (the extreme case), and Theorem 2.1, Part (1), to Theorem 2.1, Part (II) (the central case), and Theorem 2.1, Part (2), respectively. The details are therefore omitted. We now turn to the proof of the second part of the theorem (the intermediate case). The proof follows closely as the proof of the theorem in the extreme and central cases, with only some obvious changes. For example, in view of (22), (24) takes the form

$$P(m + m\varphi^{-\frac{\alpha}{2}}(m)(\tau - \epsilon) \le \nu_m \le (m + m\varphi^{-\frac{\alpha}{2}}(m)(\tau + \epsilon)) \underset{m}{\rightarrow} 1 \tag{32}$$

Therefore, by applying the same argument of the proof of the theorem in the extreme and central cases, and in view of the symmetry of the df $B(z)$, we obtain

$$P(X^*_{\nu_m - r_{\nu_m}+1:} \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} \le P(-d < \tau \le \int_{-\infty}^\infty N(m+1)V_{i:\beta}(x) - l\rho z(1-\alpha)\, dA(z) \tag{33}$$

where $m_n(s) = m + \left[ms\varphi^{-\frac{\alpha}{2}}(n)\right] = n\epsilon_n(s) \underset{n}{\rightarrow} \infty$ (note that for any $s > 0$, we have$\in_m(s)\ d, \in$ and τ_d are defined as the same as in the first part of the proof of the theorem. On the other hand, in view of the conditions $\varphi^{\frac{\alpha}{2}}(n)\,(\frac{\varphi(mx)}{\varphi(m)} - x^{p}) \underset{n}{\rightarrow} 0$ and$\rho < \frac{2}{\alpha}$, we obtain

$$\varphi\big(n_m(s)\big) = \varphi\left(ns\epsilon_n^{\rho}(s)\right) = \varphi(n)\epsilon_n^{\rho}(s) + 0\left(\varphi^{1-\frac{\alpha}{2}}(n)\right) = \varphi(n) + \sigma(\varphi(n)) \tag{35}$$

Where $\sigma\big(n_m(s)\big) = \Phi(m \in_n (s) = \varphi(n)\epsilon_n^{\rho}(s) + 0\left(\varphi^{1-\frac{\alpha}{2}}(n)\right).0 < \theta < 1$. Thus,

$$\frac{\sigma(\varphi(m))}{\varphi^{1-\frac{\alpha}{2}}(m)} = s\rho + \rho\theta\frac{\varphi^{\frac{\alpha}{2}}(m)}{m} + 0(1) \underset{m}{\rightarrow} s\rho \tag{36}$$

Therefore, by combining (35) and (36), we obtain

$$\frac{\varphi\big(n_m(\tau_d - \epsilon)\big) - \varphi(n)}{\varphi^{1-\frac{\alpha}{2}}(m)} = \frac{\sigma(\varphi(m))}{\varphi^{1-\frac{\alpha}{2}}(m)} \underset{n}{\rightarrow} \rho((\tau_d - \epsilon)) \tag{37}$$

which in turn, in view of Theorem 2.1, Part (III) (intermediate case) and Theorem 2.1, Part (3), after letting $d \rightarrow 0$, we obtain

$$\lim_{m\to\infty} P(X^*_{\nu_m-r+1:\nu_m} \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} \le \int_{-\infty}^{\infty} N(m+1)V_{i:\beta}(x) - l\rho z(1-\alpha)\,dA(z) \tag{38}$$

By applying a similar argument (with only the obvious modification) we can prove that

$$\lim_{m\to\infty} P(X^*_{\nu_m-r+1:\nu_m} \le a_{[\varphi(m)]}x + b_{[\varphi(m)]} \ge \int_{-\infty}^{\infty} N(m+1)V_{i:\beta}(x) - l\rho z(1-\alpha)\,dA(z) \tag{39}$$

Thus, the claimed result for intermediate case follows by combining the last two relations. This completes the proof of the theorem.

Theorem 2.4: Suppose that $X^*_{d:m-r_m+1:m} \epsilon L(X^*_d : \varphi|\tau)$, let $m > -1$, $\tilde{a}_n > 0$ and $\tilde{b}_n \in \Re, m \ge 1,$be suitable normalizing constants for which

$$P(X^*_{d:m-r_m+1:m} \le \tilde{a}_m x + \tilde{b}_m) \overset{w}{\rightarrow} \varphi^{d(n,k)}(x) \tag{40}$$

Where $\varphi^{d(n,k)}(x)$ is a non-degenerate df. Furthermore, let ν_m be a sequence of non-negative integer valued rv's which satisfy

$$\frac{\nu_m - m}{m^{1-\alpha/2}} \overset{p}{\rightarrow} \tau \tag{41}$$

where the df $A(z) = P(\tau \le z)$ is continuous at zero, then

$$P(X^*_{d:m-r_m+1:m} \le \bar{a}_m x + \bar{b}_m) \overset{w}{\rightarrow} \varphi^{(m,k)}(x) \overset{w}{\rightarrow} \begin{cases} 1 - \int_0^{\infty} \Gamma\big(z^0 \Psi_{i:\beta}^{m+1}(x) dA\,(z), i\epsilon\{1,2,3\}\big) \\ \text{where } r = r_n = \text{constsnt (extreme case)} \\ \int_0^{\infty} N(z^{\frac{\rho}{2}} W^*_{i:\beta}(x) dA(z), i \in \{1,2,3\}, \text{(central case)} \end{cases}$$

Furthermore, let the intermediate rank sequence $\{r_n\}$ satisfy Chibisov condition and let condition (41) be replaced by

$$\frac{\nu_m - m}{m\varphi^{-\frac{\alpha}{2}}(m)} \overset{p}{\rightarrow} \tau \tag{42}$$

Where $\varphi(m)\epsilon RV_{\rho}, 0 < \rho < \frac{2}{\alpha}, \varphi^{\frac{\alpha}{2}}(m), (\frac{\varphi(mx)}{\varphi(m)} - x^{\rho}) \underset{n}{\rightarrow} 0$ and the $df\ A(z)$ is symmetric and continuous at zero, then

$$P(X^*_{\nu_m-r+1:\nu_m} \le \bar{a}_{[\varphi(m)]}x + \bar{b}_{[\varphi(m)]} \underset{m}{\rightarrow} \int_{-\infty}^{\infty} N(m+1)V_{i:\beta}(x) - l\rho z(1-\alpha)\,dA(z);\ i = \{1,2,3\}$$

(43)

Without significant modifications, the method of the proof of Theorem 2.4 is the same as that Theorem 2.3, except only the obvious changes. Hence, for brevity the details of the proof are omitted.

V. Applications

Many authors have considered prediction problems based on samples of random sizes, The importance of the order statistics in the reliability theory is attributed to the fact that the r^{th} order statistics $(\mathrm{n}-\mathrm{r}+1)$ out-of-n system made up of n identical components with independent life lengths. On the other hand, in dealing with censored samples, where the life-test is terminated after observing the r^{th} failure (Type II censoring), or the termination of the test occurs after a given time lapse (Type I censoring), the complete survival times cannot usually be observed (due to time or cost). In many biological and agriculture problems, we often come across a situation where the sample size is not deterministic because either some observations get lost for various reasons, or the size of the target population and its representative sample cannot be determined well. For example, assume that the inhabitants of a populous town are exposed to a dose of radiation resulting from an atomic accident, or exposed to an infection of an unknown epidemic. Furthermore, assume that our interest focuses on the time at which r persons would die among a big random sample of size n that is drawn from the residents of this town. Since the number of infected people in this town is unknown and changes randomly with time, the drawn sample contains a random number of infected and non-infected people. Accordingly, the sample size of the sub-sample of the infected people will be a non-negative integer valued RV, e.g.N, and it will be described by a sequence of independent and identically distributed RVs $X_1, X_2, \cdots, X_N$. Therefore, the r^{th} smallest order statistic will be denoted by $X_{r:N}$, which represents the time at which r persons will die.

VI. Conclusions

In this paper, we consider two sequences $\{X^*_{m-r_m+:m}\}$ and $\{Y^*_{m-r_m+:m}\}$, $1 \le r_m \le m$, of gos. The first one is based on independent and identically distributed (i.i.d) random variables (rv's) $X_i, i = 1,2,3, \cdots$ (with common df 's F), while the second sequence is based on a general sequence of rv's $Y_i, i = 1,2,3, \cdots$ (these rv's need not to be independent or identical). Theorem 2.3 extends the result concerning the asymptotic behavior of the sequence $X^*_{\nu_m - r_{\nu_m}+1:\nu_m}$ of gos with random index νn (given by Theorem 2.1) to the second sequence$Y^*_{\nu_m - r_{\nu_m}+1:\nu_m}$, provided that $Y^*_{m-r_m+1:m} \in L(Y^*;\ \varphi|\tau)$ and $Y^*_{d:\,m-r_m+1:m} \in L(Y^*_d;\ \varphi|\tau)$ i.e., the sequences $Y^*_{m-r_m+1:m}$ and $Y^*_{d:\,m-r_m+1:m}$ dependent on τ (where τ is the limit rv of the suitably normalized sequence ν_m in probability) in connecting $X^*_{m-r_m+1:m}$ and $X^*_{d:\,m-r_m+1:m}$ respectively, and $\Phi(n) \in RV_\rho$. Moreover, Theorem 2.4. gives a similar extension for the dual generalized order statistics (DGOSs).

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