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SURVIVAL ANALYSIS OF AREA BIASED GENERALIZED SAULEH DISTRIBUTION WITH STATISTICAL PROPERTIES AND APPLICATIONS

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Abstract

The "Sauleh distribution" is a novel lifespan distribution that is suggested by this study. Among the statistical and mathematical features that have been discussed are the PDF and CDF. Numerous helpful statistical features are derived, including the Area Biased Generalized Sauleh distribution's hazard rate function, reverse hazard rate function, order statistics, moments, measure of skewness, and measure of kurtosis. It uses the well-known maximum likelihood outcome approach to estimate the parameter of the distribution. To demonstrate this distribution's adaptability and superiority over alternative distributions, a comparative analysis is conducted in the end. It establishes its significance in reliability engineering by producing a rich class of distributions that may capture various shapes and behaviors and, as a result, enable better fitting to empirical data. Real-world data sets are used to establish the use of this novel distribution. Melanoma patient survival data has demonstrated the effectiveness of the proposed distribution; it is concluded that the produced distribution provides a superior fit.

Keywords: Moments; Moment generating function; Survival function; Hazard function; Entropies; Maximum likelihood Estimation.

I. Introduction

In a variety of domains, including reliability, ecology, and healthcare, the idea of weighted distributions is used to create appropriate statistical models. For efficient statistical data modelling and prediction, weighted distributions have established a new standard when standard distributions are not suitable. Weighted distributions provide an approach to fitting the model to the unknown weight function when samples are taken from both the generated distribution and the original distribution. Weighted distributions are a concept that Fisher (1934) [6] devised to reflect the ascertainment bias.

Rao (1965) [14] later developed this concept in a unified way because the standard distributions failed to adequately capture these observations with equal probabilities in the statistical data model. By defining some of the circumstances in which the underlying model maintains its shape, Patil and Rao (1978) [11] developed the ideas of size biased sampling and weighted distributions. The weighted distribution turns into a length biased distribution when the weight function solely considers the length of the units of interest. A length-biased distribution was initially proposed by

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Cox (1962) [5] in the context of the renewal hypothesis. In broader terms, a distribution is considered size biased if the sampling procedure selects units with a probability proportional to the unit size measure. One particular instance of size biased distributions is weighted distributions, a broader variation. The weighted distributions can be used to assess both the toughness of model formulation and the difficulties in interpreting the data. Numerous academics have studied and explored the various weighted probability models, demonstrating their applicability across a range of fields. When fitting distributions of dimension at breast height (DBH) data originating from horizontal point sampling (HPS) (Grosenbaugh) inventory, Van Deusen (1986) [20] independently created the size biased distribution theory. The size biased Zeghdoudi distribution was introduced by Chouia et al. (2021) [4], who also covered its numerous statistical characteristics and uses. In 1975, Patil and Ord examined size-biased sampling and the corresponding form-invariant weighted distribution [12].

The statistical properties and application of the length biased weighted Lomax distribution were discussed by Ahmad et al. (2016) [2]. Sharma et al. (2018) investigated the area-biased Maxwell distributions and the length-biased Maxwell distributions [15]. Ganaie and Rajagopalan (2020) [7] investigated the length biased weighted quasi gamma distribution with properties and applications. VM Jaimole, D Venkatesan Perveen (2016) [12] investigated the Area biased weighted weibull distribution with applications, introducing a new area biased two parameter Pranav distribution and its use in medical sciences (2022). The estimate of area-biased Rayleigh distribution parameters was examined by Rao and Pandey (2020) [14]. The estimation and definition of the Area biased quasi-Akash distribution were recently covered by Ade et al. (2021) [1].

A new lifetime distribution that is more adaptable than the Lindley, Exponential, Shanker, Pranav, Sujatha and Sauleh distributions can therefore be found.

II. Area Biased Generalized Sauleh distribution

The probability density function of Area Biased Generalized Sauleh distribution (ABGSD) is given by

$$f(x;\alpha,\beta,\theta)=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}(\theta+\alpha x^2+\beta x^3)e^{-\theta x};x>0,\alpha>0,\beta>0,\theta>0 \tag{1}$$

And the cumulative distribution function of Area Biased Generalized Sauleh distribution (ABGSD) is given by

$$F(x;\alpha,\beta,\theta)=1-\left[1+\frac{\theta x(\beta\theta^2x^2+(\alpha\theta+3\beta)\theta x+2(\alpha\theta+3\beta))}{\theta^4+2\alpha\theta+6\beta}\right]e^{-\theta x};x>0,\theta,\alpha,\beta>0 \tag{2}$$

$$f_a(x)=\frac{x^2f(x)}{E(x^2)} \tag{3}$$

$$E(x^2)=\int_0^\infty x^2f(x)dx$$

$$E(x^2)=\int_0^\infty x^2\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}(\theta+\alpha x^2+\beta x^3)e^{-\theta x}dx$$

$$E(x^2)=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\int_0^\infty x^2\,(\theta+\alpha x^2+\beta x^3)e^{-\theta x}dx$$

$$=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\left[\theta\int_0^\infty x^2\,e^{-\theta x}dx+\alpha\int_0^\infty x^4\,e^{-\theta x}dx+\beta\int_0^\infty x^5\,e^{-\theta x}dx\right]$$

$$=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\left[\theta\int_0^\infty x^{3-1}\,e^{-\theta x}dx+\alpha\int_0^\infty x^{5-1}\,e^{-\theta x}dx+\beta\int_0^\infty x^{6-1}\,e^{-\theta x}dx\right]$$

$$=\frac{\theta^4}{\theta^4+2\alpha\theta+6\beta}\left[\frac{\theta\Gamma3}{\theta^3}+\frac{\alpha\Gamma4}{\theta^4}+\frac{\beta\Gamma5}{\theta^6}\right]$$

$$= \frac{\theta^4}{\theta^4 + 2\alpha\theta + 6\beta}\left[\frac{2\theta}{\theta^3} + \frac{24\alpha}{\theta^4} + \frac{120\beta}{\theta^6}\right]$$

$$= \frac{\theta^4}{\theta^2 + \alpha\theta + 2}\left[\frac{2\theta^{11} + 24\alpha\theta^8 + 120\beta\theta^7}{\theta^7\theta^6}\right]$$

$$= \frac{\theta^4}{\theta^4 + 2\alpha\theta + 6\beta}\left[\frac{2\theta^4 + 24\alpha\theta + 120\beta}{\theta^7\theta^6}\right]\theta^7$$

$$= \frac{1}{\theta^4 + 2\alpha\theta + 6\beta}\left[\frac{2\theta^4 + 24\alpha\theta + 120\beta}{\theta^2}\right]$$

$$E(x^2) = \frac{2\theta^4 + 24\alpha\theta + 120\beta}{\theta^2(\theta^4 + 2\alpha\theta + 6\beta)} \tag{4}$$

The probability density function of the Area Biased Generalized Sauleh distribution (ABGSD) can now be obtained by substituting equations (1) and (4) into equation (3).

$$f_a(x) = \frac{x^2\theta^4}{(\theta^4 + 2\alpha\theta + 6\beta)}(\theta + \alpha x^2 + \beta x^3)e^{-\theta x} \times \frac{\theta^2(\theta^4 + 2\alpha\theta + 2\beta)}{2\theta^4 + 24\alpha\theta + 120\beta}$$

$$f_a(x) = \frac{x^2\theta^6}{2\theta^4 + 24\alpha\theta + 120\beta}(\theta + \alpha x^2 + \beta x^3)e^{-\theta x} \tag{5}$$

And the CDF of Area Biased Generalized Sauleh distribution can be obtained a

$$F_a(x) = \int_0^x f_a(x)dx$$

$$= \int_0^x \frac{x^2\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}(\theta + \alpha x^2 + \beta x^3)e^{-\theta x}$$

$$= \frac{1}{(2\theta^4 + 24\alpha\theta + 120\beta)}\int_0^x x^2\theta^6(\theta + \alpha x^2 + \beta x^3)e^{-\theta x}dx$$

$$= \frac{1}{(2\theta^4 + 24\alpha\theta + 120\beta)}\int_0^\infty \left[\theta^7\int_0^x x^2 e^{-\theta x}dx + \alpha\theta^6\int_0^x x^4 e^{-\theta x}dx + \beta\theta^6\int_0^x x^5 e^{-\theta x}dx\right] \tag{6}$$

Put $\theta x = t \Longrightarrow \theta dx = dt \Longrightarrow dx = \frac{dt}{\theta}$ $also\ x = \frac{t}{\theta}$

$$F_a(x) = \frac{1}{(2\theta^4 + 24\alpha\theta + 120\beta)}\left[\theta^7\int_0^{\theta x}\left(\frac{t}{\theta}\right)^2 e^{-t}\frac{dt}{\theta} + \alpha\theta^6\int_0^{\theta x}\left(\frac{t}{\theta}\right)^4 e^{-t}\frac{dt}{\theta} + \beta\theta^6\int_0^{\theta x}\left(\frac{t}{\theta}\right)^5 e^{-t}\frac{dt}{\theta}\right]$$

$$F_a(x) = \frac{1}{(2\theta^4 + 24\alpha\theta + 120\beta)}[\theta^4\gamma(3,\theta x) + \alpha\theta\gamma(5,\theta x) + \beta\gamma(6,\theta x)] \tag{7}$$

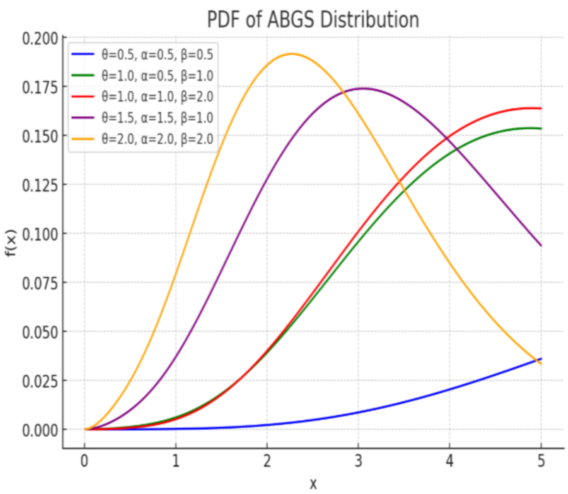

Figure 1: *PDF of the ABGSD*

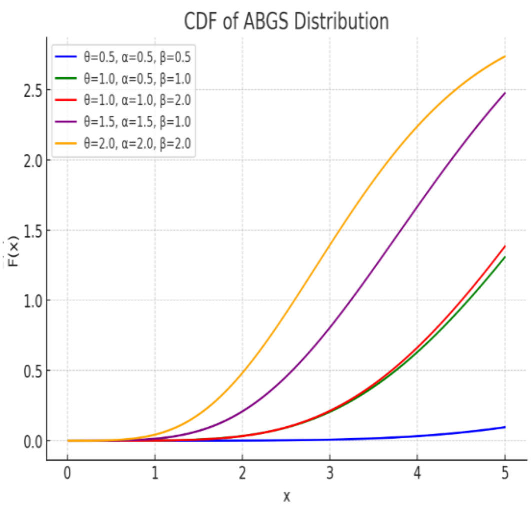

Figure 2: *CDF of the ABGSD*

III. Survival Analysis

The distribution's hazard function and reliability function are two crucial functions that we ascertain in this section. The survival function is another name for the reliability function. A system's likelihood of surviving past a given period of time is its definition. The following represents the reliability of a random variable x.

$$S(x) = 1 - F_a(x)$$

$$S(y) = 1 - \frac{1}{(2\theta^4 + 24\alpha\theta + 120\beta)}[\theta^4\gamma(3, \theta x) + \alpha\theta\gamma(5, \theta x) + \beta\gamma(6, \theta x)]$$

I. The Hazard function

The hazard function of the proposed is Area Biased Generalized Sauleh distribution obtained as

$$H(x) = \frac{f_a(x)}{S(x)} = \frac{x^2\theta^6(\theta + \alpha x^2 + \beta x^3)e^{-\theta x}}{(2\theta^4 + 24\alpha\theta + 120\beta) - \theta^4\gamma(3, \theta x) + \alpha\theta\gamma(5, \theta x) + \beta\gamma(6, \theta x)}$$

II. Reverse Hazard function

$$h_r(x) = \frac{f_a(x)}{F_a(x)} = \frac{x^2\theta^6(\theta + \alpha x^2 + \beta x^3)e^{-\theta x}}{(\theta^4\gamma(3, \theta x) + \alpha\theta\gamma(5, \theta x) + \beta\gamma(6, \theta x))}$$

III. Mills Ratio

$$M.R = \frac{1}{h_r(x)} = \frac{(\theta^4\gamma(3, \theta x) + \alpha\theta\gamma(5, \theta x) + \beta\gamma(6, \theta x))}{x^2\theta^6(\theta + \alpha x^2 + \beta x^3)e^{-\theta x}}$$

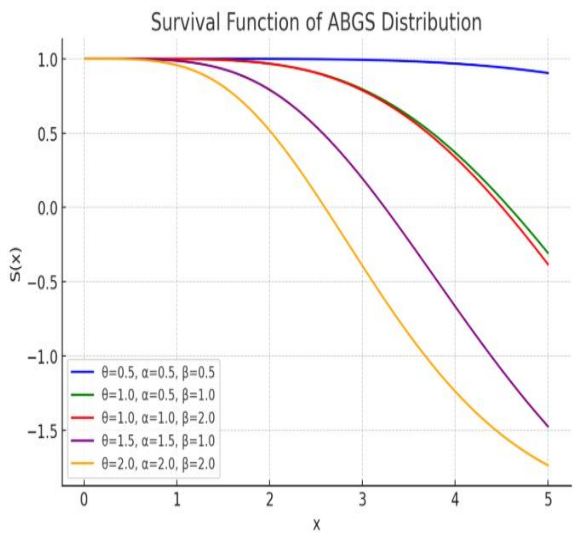

Figure 3: *Survival function ABGSD*

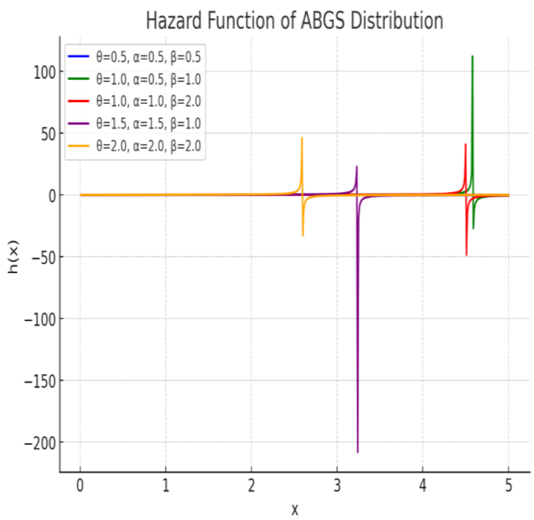

Figure 4: *Hazard function ABGSD*

IV. Statistical Properties

The statistical characteristics of the Area Biased Generalized Sauleh distribution, in particular its Order Statistics, Statistical moments, Harmonic mean, MGF, and characteristic function, are covered in this section.

I. Order Statistics

Assuming that $X_1, X_2, \ldots, X_n$ are random samples and that $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the order statistics of a random sample of size n taken from a continuous population with a probability density function $f_x(x)$ and a cumulative distribution function $F_X(X)$, the probability density function of r^th^ order statistics $X_{(r)}$ can be found using

$$f_{X(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x)[F_X(x)]^{r-1}[1-F_X(x)]^{n-r}$$

We will now obtain the probability density function of the r^{th} order statistics of the Area Biased Generalized Sauleh distribution by substituting equations (5) and (7) in equation (8).

$$f_{X(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}\frac{x^2\theta^6}{(2\theta^4+24\alpha\theta+120\beta)}(\theta+\alpha x^2+\beta x^3)e^{-\theta x}$$
$$\times\left[\frac{1}{(2\theta^4+24\alpha\theta+120\beta)}(\theta^4\gamma(3,\theta x)+\alpha\theta\gamma(5,\theta x)+\beta\gamma(6,\theta x))\right]^{r-1}$$
$$\left[\frac{1}{(2\theta^4+24\alpha\theta+120\beta)}(\theta^4\gamma(3,\theta x)+\alpha\theta\gamma(5,\theta x)+\beta\gamma(6,\theta x))\right]^{n-r}$$

Consequently, the probability density function of the Area Biased Generalized Sauleh distribution higher order statistic $X_{(n)}$ may be found as

$$f_{X(n)}(x) = \frac{nx^2\theta^6}{2\theta^2+6\alpha\theta+2\theta^5}(\theta+\alpha x^2+\beta x^3)e^{-\theta x}$$
$$\times\left[\frac{1}{(2\theta^4+24\alpha\theta+120\beta)}\left(\theta^4\gamma(3,\theta x)+\alpha\theta\gamma(5,\theta x)+\beta\gamma(6,\theta x)\right)\right]^{n-r}$$

Additionally, the probability density function of the Area Biased Generalized Sauleh distribution first order statistic $X_{(1)}$ may be found as

$$f_{X(1)}(x) = \frac{nx^2\theta^6}{(2\theta^4+24\alpha\theta+120\beta)}(\theta+\alpha x^2+\beta x^3)e^{-\theta x}$$
$$\times\left[1-\frac{1}{(2\theta^4+24\alpha\theta+120\beta)}(\theta^4\gamma(3,\theta x)+\alpha\theta\gamma(5,\theta x)+\beta\gamma(6,\theta x))\right]^{n-r}$$

II. Moments

When X represents the random variable of the Area Biased Generalized Sauleh distribution with parameters θ and α, the r^{th} order moment $E(X^r)$ of the Area Biased Generalized Sauleh distribution may be found as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_a(x)dx$$
$$= \int_0^\infty x^r\frac{x^2\theta^6}{(2\theta^4+24\alpha\theta+120\beta)}(\theta+\alpha x^2+\beta x^3)e^{-\theta x}$$
$$= \frac{\theta^6}{(2\theta^4+24\alpha\theta+120\beta)}\int_0^\infty x^{r+2}(\theta+\alpha x^2+\beta x^3)e^{-\theta x}dx$$
$$= \frac{\theta^6}{(2\theta^4+24\alpha\theta+120\beta)}\left[\theta\int_0^\infty x^{r+2}e^{-\theta x}dx+\alpha\int_0^\infty x^{r+4}e^{-\theta x}dx+\beta\int_0^\infty x^{r+5}e^{-\theta x}dx\right]$$
$$= \frac{\theta^6}{(2\theta^4+24\alpha\theta+120\beta)}\left[\theta\int_0^\infty x^{(r+2)}e^{-\theta x}dx+\alpha\int_0^\infty x^{(r+4)}e^{-\theta x}dx+\int_0^\infty x^{(r+5)}e^{-\theta x}dx\right]$$

$$\mu_r' = E(X^r) = \frac{\theta^4\Gamma r + 3 + \alpha\theta\Gamma r + 5 + \Gamma r + 6}{\theta^r(2\theta^4 + 24\alpha\theta + 120\beta)} \quad (9)$$

The mean and the last three moments of the Area Biased Generalized Sauleh distribution can now be obtained by entering r=1,2,3, and 4 in equation (9).

$$\mu_1' = E(X) = \frac{6\theta^4 + 120\alpha\theta + 720\beta}{\theta(2\theta^4 + 24\alpha\theta + 120\beta)}$$

$$\mu_2' = E(X^2) = \frac{24\theta^4 + 720\alpha\theta + 5040\beta}{\theta^2(2\theta^4 + 24\alpha\theta + 120\beta)}$$

$$\mu_3' = E(X^2) = \frac{120\theta^4 + 5040\alpha\theta + 40320\beta}{\theta^3(2\theta^4 + 24\alpha\theta + 120\beta)}$$

$$\mu_4' = E(X^4) = \frac{720\theta^4 + 40320\alpha\theta + 362000\beta}{\theta^4(2\theta^4 + 24\alpha\theta + 120\beta)}$$

$$Variance = \frac{24\theta^4 + 720\alpha\theta + 5040\beta}{\theta^2(2\theta^4 + 24\alpha\theta + 120\beta)} - \left[\frac{6\theta^2 + 24\alpha\theta + 120}{\theta(2\theta^4 + 24\alpha\theta + 120\beta)}\right]^2$$

$$S.D(\sigma) = \sqrt{\frac{24\theta^4 + 720\alpha\theta + 5040\beta}{\theta^2(2\theta^4 + 24\alpha\theta + 120\beta)} - \left[\frac{6\theta^2 + 24\alpha\theta + 120}{\theta(2\theta^4 + 24\alpha\theta + 120\beta)}\right]^2}$$

III. Hormonic mean

$$H.M = E\left[\frac{1}{x}\right] = \int_0^\infty \frac{1}{x}\frac{x^2\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}(\theta + \alpha x^2 + \beta x^3)e^{-\theta x}$$

$$= \int_0^\infty \frac{x^2\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}(\theta + \alpha x^2 + \beta x^3)e^{-\theta x}$$

$$= \frac{\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}\int_0^\infty x(1 + \alpha x + x^2)e^{-\theta x}dx$$

$$= \frac{\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}\left[\theta\int_0^\infty x\, e^{-\theta x}dx + \alpha\int_0^\infty x^3\, e^{-\theta x}dx + \beta\int_0^\infty x^4 e^{-\theta x}dx\right]$$

$$= \frac{\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}\left[\theta\int_0^\infty x^{2-1}\, e^{-\theta x}dx + \alpha\int_0^\infty x^{4-1}\, e^{-\theta x}dx + \int_0^\infty x^{5-1}e^{-\theta x}dx\right]$$

$$= \frac{\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}\left[\frac{\theta\Gamma 2}{\theta^2} + \frac{\alpha\Gamma 4}{\theta^4} + \frac{\beta\Gamma 5}{\theta^5}\right]$$

$$= \frac{\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}\left[\frac{1}{\theta^2} + \frac{6\alpha}{\theta^3} + \frac{24\beta}{\theta^4}\right]$$

$$= \frac{\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}\left[\frac{\theta^9 + 6\alpha\theta^6 + 24\beta\theta^5}{\theta^{10}}\right]$$

$$= \frac{\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}\left[\frac{\theta^4 + 6\alpha\theta + 24\beta}{\theta^{10}}\right]\theta^5$$

$$H.M = \frac{\theta(\theta^4 + 6\alpha\theta + 24\beta)}{(2\theta^4 + 24\alpha\theta + 120\beta)}$$

IV. Moment generating function & Characteristic function

$$M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx} f_a(x)dx$$

Using Taylor's series,

$$M_X(t) = \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \cdots\right] f_a(x)dx$$

$$= \int_0^\infty \sum_{k=0}^\infty \frac{(tx)^k}{k!} f_a(x)dx$$
$$= \sum_{k=0}^\infty \frac{t^k}{k!} \int_0^\infty x^k f_a(x)dx$$
$$= \sum_{j=0}^\infty \frac{t^k}{k!} \int_0^\infty \mu'_k$$
$$= \sum_{j=0}^\infty \frac{t^k}{k!} \left[\frac{\theta^4 \Gamma k + 3 + \alpha\theta\Gamma k + 5 + \Gamma k + 6}{\theta^k (2\theta^4 + 24\alpha\theta + 120\beta)} \right]$$
$$M_X(t) = \frac{1}{(2\theta^4 + 24\alpha\theta + 120\beta)} \sum_{k=0}^\infty \frac{t^k}{k!\,\theta^k} [\theta^4 \Gamma k + 3 + \alpha\theta\Gamma k + 5 + \Gamma k + 6]$$

Similarly, the characteristic function of Area Biased Generalized Sauleh distribution is given by

$$M_X(t) = \frac{1}{(2\theta^4 + 24\alpha\theta + 120\beta)} \sum_{k=0}^\infty \frac{it^k}{k!\,\theta^k} [\theta^4 \Gamma k + 3 + \alpha\theta\Gamma k + 5 + \Gamma k + 6]$$

V. Bonferroni and Lorenz curves

$$B(p) = \frac{1}{p\mu'_1} \int_0^q x f_a(x)dx \ \ and \ \ L(p) = PB(p) = \frac{1}{\mu'_1} \int_0^q x f_a(x)dx$$

$$Here, \ \ \mu'_1 = \frac{6\theta^4 + 120\alpha\theta + 720\beta}{\theta(2\theta^4 + 24\alpha\theta + 120\beta)}$$

$$B(p) = \frac{\theta(2\theta^4 + 24\alpha\theta + 120\beta)}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \int_0^q x \frac{x^2\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)} (\theta + \alpha x^2 + \beta x^3) e^{-\theta x} dx$$

$$= \frac{\theta(2\theta^4 + 24\alpha\theta + 120\beta)}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \int_0^q \frac{x^3\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)} (\theta + \alpha x^2 + \beta x^3) e^{-\theta x} dx$$
$$= \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \int_0^q x^3 (\theta + \alpha x^2 + \beta x^3) e^{-\theta x} dx$$
$$= \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \left[\theta \int_0^\infty x^3 \, e^{-\theta x} dx + \alpha \int_0^\infty x^5 \, e^{-\theta x} dx + \beta \int_0^\infty x^6 e^{-\theta x} dx \right] \quad (10)$$

Put $\theta x = t \Rightarrow \theta dx = dt \Rightarrow dx = \frac{dt}{\theta}$ $also\ x = \frac{t}{\theta}$

$$\because B(p) = \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \left[\theta \int_0^{\theta q} \left(\frac{t}{\theta}\right)^3 e^{-t} \frac{dt}{\theta} + \alpha \int_0^{\theta q} \left(\frac{t}{\theta}\right)^5 e^{-t} \frac{dt}{\theta} + \beta \int_0^{\theta q} \left(\frac{t}{\theta}\right)^6 e^{-t} \frac{dt}{\theta} \right]$$
$$= \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \left[\frac{\theta}{\theta^4} \int_0^{\theta q} t^3 e^{-t} dt + \frac{\alpha}{\theta^6} \int_0^{\theta q} t^5 e^{-t} dt + \frac{\beta}{\theta^7} \int_0^{\theta q} t^6 e^{-t} dt \right]$$
$$= \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \left[\frac{1}{\theta^3} \int_0^{\theta q} t^{4-1} e^{-t} dt + \frac{\alpha}{\theta^6} \int_0^{\theta q} t^{6-1} e^{-t} dt + \frac{\beta}{\theta^7} \int_0^{\theta q} t^{7-1} e^{-t} dt \right]$$
$$= \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \left[\frac{1}{\theta^3} \gamma(4, \theta q) + \frac{\alpha}{\theta^6} \gamma(6, \theta q) + \frac{\beta}{\theta^7} \gamma(7, \theta q) \right]$$
$$= \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \left[\frac{\theta^6 \gamma(4, \theta q) + \alpha\theta^3 \gamma(6, \theta q) + \beta\gamma(7, \theta q)}{\theta^9 \theta^6} \right] \theta^9$$
$$= \frac{\theta^7}{p(6\theta^4 + 120\alpha\theta + 720\beta)} \left[\frac{\theta^3\theta^7 \gamma(4, \theta q) + \alpha\theta^3\theta^7 \gamma(6, \theta q) + \beta\theta^3\theta^6 \gamma(7, \theta q)}{\theta^3 \theta^6 \theta^7} \right]$$
$$B(p) = \frac{1}{p(6\theta^4 + 120\alpha\theta + 720\beta)} [\theta^4 \gamma(4, \theta q) + \alpha\theta\gamma(6, \theta q) + \beta\gamma(7, \theta q)]$$

$$\because L(p) = \frac{1}{p(6\theta^4 + 120\alpha\theta + 720\beta)}\left[\theta^4\gamma(4, \theta q) + \alpha\theta\gamma(6, \theta q) + \beta\gamma(7, \theta q)\right]$$

V. Maximum likelihood estimation & fisher information matrix

$$L(x) = \prod_{i=1}^{n} f_a(x)$$

$$= \prod_{i=1}^{n}\left[\frac{x_i^2\theta^6}{(2\theta^4 + 24\alpha\theta + 120\beta)}(\theta + \alpha x^2 + \beta {x_i}^3)e^{-\theta xi}\right]$$

$$= \frac{\theta^{6n}}{(2\theta^4 + 24\alpha\theta + 120\beta)^n}\prod_{i=1}^{n}\left[x_i^2(\theta + \alpha {x_i}^2 + \beta {x_i}^3)e^{-\theta xi}\right]$$

The log likelihood function is given by

$$\log l = 6n\log\theta - n\log(2\theta^5 + 24\alpha\theta + 120\beta) + 2\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(\theta + \alpha {x_i}^2 + \beta {x_i}^3) - \theta\sum_{i=1}^{n} x_{\,i} \quad (11)$$

Now differentiating equation (11) write to α, β *and* θ we get,

$$\frac{\partial \log l}{\partial\theta} = -n\left[\frac{24\theta}{(2\theta^4 + 24\alpha\theta + 120\beta)}\right] + \sum_{i=1}^{n}\left[\frac{x_i^2}{(\theta + \alpha x_i^2 + \beta x_i^3)}\right] = 0$$

$$\frac{\partial \log l}{\partial\theta} = -n\left[\frac{120}{(2\theta^4 + 24\alpha\theta + 120\beta)}\right] + \sum_{i=1}^{n}\left[\frac{x_i^3}{(\theta + \alpha x_i^2 + \beta x_i^3)}\right] = 0$$

$$\frac{\partial \log l}{\partial\theta} = \frac{6n}{\theta} - n\left[\frac{\theta\theta^3 + 24\alpha}{(2\theta^4 + 24\alpha\theta + 120\beta)}\right] + \sum_{i=1}^{n}\left[\frac{1}{(\theta + \alpha x_i^2 + \beta x_i^3)}\right] - \sum_{i=1}^{n} x_i = 0$$

Fisher information matrix

$$E\left[\frac{\partial^2 \log l}{\partial\theta^2}\right] = n\left[\frac{(24\theta)(24\theta)}{(2\theta^4 + 24\alpha\theta + 120\beta)^2}\right] - \sum_{i=1}^{n}\left[\frac{E(x_i^2)^2}{(\theta + \alpha x_i^2 + \beta x_i^2)^2}\right]$$

$$E\left[\frac{\partial^2 \log l}{\partial\theta^2}\right] = n\left[\frac{14400}{(2\theta^4 + 24\alpha\theta + 120\beta)^2}\right] - \sum_{i=1}^{n}\left[\frac{E(x_i^2)^2}{(\theta + \alpha x_i^2 + \beta x_i^2)^2}\right]$$

$$E\left[\frac{\partial^2 \log l}{\partial\theta^2}\right] = \frac{-6n}{\theta^2} - \frac{n\,(2\theta^4 + 24\alpha\theta + 120\beta)(24\theta^2) - (\theta\theta^3 + 24\alpha)(\theta\theta^3 + 24\alpha)}{(2\theta^4 + 24\alpha\theta + 120\beta)^2} - \sum_{i=1}^{n}\left[\frac{1}{(\theta + \alpha x_i^2 + \beta x_i^3)}\right]$$

$$E\left[\frac{\partial^2 \log l}{\partial\theta^2}\right] = n\left[\frac{(24\theta)(120)}{(2\theta^4 + 24\alpha\theta + 120\beta)^2}\right] - \sum_{i=1}^{n}\left[\frac{(x_i^2)(x_i^3)}{(\theta + \alpha x_i^2 + \beta x_i^3)^2}\right]$$

$$E\left[\frac{\partial^2 \log l}{\partial\theta^2}\right] = -\frac{n\,(2\theta^4 + 24\alpha\theta + 120\beta)(24) - (24\theta)(\theta\theta^3 + 24\alpha)}{(2\theta^4 + 24\alpha\theta + 120\beta)^2} - \sum_{i=1}^{n}\left[\frac{x_i^2}{(\theta + \alpha x_i^2 + \beta x_i^3)^2}\right]$$

$$E\left[\frac{\partial^2 \log l}{\partial\theta^2}\right] = n\left[\frac{(120)(\theta\theta^3 + 24\alpha)}{(2\theta^4 + 24\alpha\theta + 120\beta)^2}\right] - \sum_{i=1}^{n}\left[\frac{x_i^2}{(\theta + \alpha x_i^2 + \beta x_i^3)^2}\right]$$

The highest probability estimate for the parameters θ is obtained from the solution to the nonlinear equation (11). R software was used to calculate the parameter values for the data sets used in this investigation.

VI. Results

For the data set under consideration, the widely utilized Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), and -2lnL were employed as the quality of fit criteria. The most adaptable and ideal distribution for the data set is thought to be the one linked to the lowest AIC, BIC, and -2InL. R-Software was used to calculate the maximum likelihood and goodness of fit criteria. In order to discuss the goodness of fit of the Area Biased Generalized Sauleh distribution, we have fitted actual data sets to it. The fit of the Area Biased Generalized Sauleh distribution has been compared to the Lindley, Pranav, exponential, and Sauleh distributions.

I. Dataset:

The survival periods (measured in months) of melanoma patients examined by Susarla and Vanryzin (1978) are shown in the data set below [18]. Table 1 below provides the data set as

Table 1: Data regarding Survival times (in months) studied by Susarla and Vanryzins

3.25	3.50	4.75	4.75	5.00	5.25	6.50	6.50	6.75	6.75	7.78	8.00	8.50
9.50	9.50	10.00	11.50	12.50	13.25	13.50	14.75	15.00	16.25	16.25	16.50	17.50
25.50	25.75	27.50	29.50	31.00	32.50	34.00	34.00	34.50	35.25	58.50		

Table: 2 Shows parameter estimations, corresponding standard errors, Criterion values and goodness of fit test.

Distribution	MLE	S.E	-2logL	AIC	BIC	P–Value
ABG Sauleh distribution	$\hat{\alpha}$=0.450988 $\hat{\theta}$=0.245345	$\hat{\alpha}$=0.6704567 $\hat{\theta}$=0.7091290	210.1678	214.9685	217.3478	0.5348
Sauleh distribution	$\hat{\theta}$=0.458904	$\hat{\theta}$=0.6120067	243.3467	244.6992	247.9832	0.6709
Exponential	$\hat{\alpha}$=4.1256789 $\hat{\theta}$=3.5690876	$\hat{\alpha}$=3.4327899 $\hat{\theta}$=0.0982347	261.3489	265.9685	268.1267	0.7649
Lindley distribution	$\hat{\alpha}$=1.4532890 $\hat{\theta}$=0.0873452	$\hat{\alpha}$=9.2112389 $\hat{\theta}$=0.0670923	256.9807	260.2436	263.3478	0.6734
Pranav distribution	$\hat{\theta}$=3.5489076	$\hat{\theta}$=0.0783242	267.3424	271.2422	274.9843	0.7890
Sujatha distribution	$\hat{\theta}$=4.2389076	$\hat{\theta}$=0.14598065	281.2367	285.0919	288.1784	0.9034

VII. Conclusion

In this study, we developed an Area Biased Generalized Sauleh distribution and identified its characteristics. Because the shape and scale parameters are flexible, the Area Biased Generalized Sauleh distribution can be used to a variety of situations. The maximum likelihood estimate is applied from the frequent estimation domain. An analysis of a patient's survival data showed how effective the distribution was. Lastly, the findings showed that the Area Biased Generalized Sauleh distribution, which is biased, fits the data better than the Lindley, Pranav, Exponential, Sujatha, and Sauleh distributions.

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