BAYESIAN STUDY OF HYBRID WEIBULL-EXPONENTIAL POWER MODEL BY MEANS OF HAMILTONIAN MONTE CARLO ALGORITHM

M. A. Alhassan[1*], A. Yahaya[2], O. O. Ishaq[1,3], B. Abba[4], A. B. Muhammad[1,5]

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[1]Department of Statistics, Aliko Dangote University of Science and Technology, Wudil, Kano 700001, Nigeria
[2]Department of Statistics, Ahmadu Bello University, Zaria, Kaduna 800001, Nigeria
[4]Department of Mathematics, Yusuf Maitama Sule University, Kano 700001, Nigeria
[4]School of Mathematics and Statistics, Central South University, Changsha, 410083, China
[1*]ma23038@gmail.com, [2]Abubakaryahaya@abu.edu.ng, [3]babington4u@gmail.com,
[4]babba@yumsuk.edu.ng, [5]mrwudil@gmail.com

Abstract

In this paper, we derived two important properties, viz: moment and order statistics, for the Hybrid Weibull-Exponential Power actuarial reliability model. We considered Bayesian procedures for estimating the parameters of the model. We tested the performance of the estimation approach by means of simple Monte Carlo experiment, hence, found the estimator consistent for the model parameters. Subsequently, we utilized the Bayesian inferential technique to further explore the overall adequacy the model in real—life through two sets of failure time reliability data. However, at the complexity point of the Bayesian method in dealing with the complex posterior of the model, we ushered-in the robust Hamiltonian Monte Carlo simulation algorithm, and thus, swiftly enhanced the posterior inference. The Add-on MLE examination on the model parameters was briefly conducted to additionally check the model's performance in real life. On the premise of the evaluation criteria and goodness-of-fit statistics results obtained from the MLE computation, we, once again, found clearly that the model is superior among the similar models compared. The HWEP model is, thus, apt for reliability analyses of dual failures as well as dual-component failed systems.

Keywords: Bayesian, parameter estimation, statistical reliability, Hamiltonian Monte Carlo, simulation algorithm

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I. Introduction

Recently, Alhassan et al. [1], established four-parameter hybrid Weibull exponential power (HWEP) life casting distribution based on actuarial reliability assumption. The model, thus, describes the actual failure of dual-component systems when either or both of the components cease to function. It also applies in the analysis of dual failures in a system. Assuming that the behavior of either of the failures (or failed components), respectively follow the Weibull or exponential power distribution. The HWEP model was formulated based on the additive failure rate methodology [17]. [1] showed that, to a higher significance than several models, the distribution fits much adequately in modeling times-to-failure data such as the benchmark dataset in [9], and generally, life data. [1] also observed that the hazard rate (HR) of the grand HWEP distribution assumes increasing trend at $\alpha = 1.0$ and $\tau = 1.5$, and also, reveals a decreasing pattern at $\alpha = 0.8$ and $\tau = 2.5$. Additionally, its HR follows the rule of the legend bathtub trend at $\alpha = 14$ and $\tau = 0.15, 0.35, 0.55$ and 0.75 respectively, among other values tuning. Thus, validate its appropriateness in monitoring and providing unbiased assessment of the entire system lifecycle. Other notable reliability attributes such as the survival, mean residual life and mean time-to-failure, have been obtained and elaborated in [1]. And so, point to the model's adequateness in system reliability, especially the series systems. Furthermore, the give and take relationship between HWEP's HR and mean residual life functions in describing the complete life cycle of engineered systems, was shown. The ML estimators for the model parameters were also proven consistent.

In the current paper, we aim to explore the suitability of the Bayesian estimators for estimating the HWEP model parameters under the quadratic error loss function, succored by the robust Hamiltonian Monte Carlo (HMC) algorithm for posterior simulation. The HMC technique promotes swift and successful Bayesian computations especially in situations where the posterior density of the target distribution appears analytically complex. Its power in prevailing over such abnormalities in real practices has been shown by Betancourt and Girolami [8], by delving into its role in hierarchical modeling. In practice, many authors leveraged the power of the HMC algorithm to enhance their inferential computations. For instance, with the help of HMC technique for enhanced precision, Abba et al. [4], performed Bayesian computations of the Flexible Dhillon-Weibull Competing Risks model, hence, accurately led to a swift inference of the model's complex posterior. Al-Essa et al. [6], used the HMC algorithm to sample from the posterior, and verified the Bayes estimates of the Modified Exponential-Weibull model, eventually, the estimates were found accurate. [16], provided Bayesian analysis of the Additive Chen-Weibull model via the HMC method and found the algorithm effective for parameter convergence. [15], successfully delivered an extensive Bayesian study for Improved New Modified Weibull distribution by means of HMC method. See also [5], and the Refs within. Thus, the notable outcomes catalyzed by the HMC technique in the above literature motivated our proposal to use the HMC-guided Bayesian paradigm for the inferential analysis of our HWEP model. So, for the purpose of achieving our goal, we (I) explore further distributional and reliability properties of the HWEP model (II) employ the HMC algorithm in order to enhance the Bayesian posterior computations, (III) simulate and assess the performance of the estimator by means of simple Monte Carlo technique, (IV) apply the HWEP model on another benchmark dataset to further prove the model's firm significance in reliability studies.

Following the foregoing preamble and objectives, the paper is, moreover, categorized this way: supplementary properties of the HWEP model in both probability and reliability contexts are contained in section II. HMC fortified Bayesian computations are given in Section III. Simulation experiment for evaluating the Bayes estimates is given in section IV. Demonstration to further iterate the relevance of the HWEP model in system reliability is provided in section V using failure times dataset. Section VI wraps up the paper.

II. Methods

I. Supplementary Properties of the HWEP Model

In the previous publication [1], we established the HWEP reliability and survival model, and discussed some of its properties relating to both probability and reliability perspectives. Such as the PDF, CDF and survival function respectively given below:

$$\mathbb{g}(t;\alpha,\gamma,\nu,\tau)=\left[\alpha\gamma^{\alpha}\,t^{\alpha-1}+\tau\nu(\nu t)^{\tau-1}e^{(\nu\, t)^{\tau}}\right]e^{-(\gamma t)^{\alpha}-e^{(\nu t)^{\tau}}+1},t>0. \tag{1}$$

$$R(t;\alpha,\gamma,\nu,\tau)=e^{-(\gamma t)^{\alpha}-e^{(\nu t)^{\tau}}+1},t>0. \tag{2}$$

$$\mathbb{G}(t;\alpha,\gamma,\nu,\tau)=1-e^{-(\gamma t)^{\alpha}-e^{(\nu t)^{\tau}}+1},t>0. \tag{3}$$

The parameter vector is $\boldsymbol{\xi}=(\alpha,\gamma,\nu,\tau)'$; where, $\alpha,\tau>0$ and $\gamma,\nu\geq 0$ dually represent the corresponding shape and scale parameters. Here, we present additional properties of the novel HWEP model in respect of the said perspectives in our subsequent discussions.

a) Moment

The role of moments is pivotal in the establishment of probability distributions. They can be utilized to generate many distributional properties such as the measures of central tendency, skewness and kurtosis. Suppose that the random variable $T\sim HWEP(\boldsymbol{\xi})$, then the rth raw moment of T is given as:

$$\mathbb{E}[t^{r}]=\mu_{r}'=\int_{0}^{\infty}t^{r}d\mathbb{G}(t). \tag{4}$$

Then, using (3), we have

$$=\int_{0}^{\infty}t^{r}d1-e^{-(\gamma t)^{\alpha}-e^{(\nu t)^{\tau}}+1}=-\int_{0}^{\infty}t^{r}de^{-(\gamma t)^{\alpha}-e^{(\nu t)^{\tau}}+1},$$

$$=\int_{0}^{\infty}rt^{r-1}e^{-(\gamma t)^{\alpha}-e^{(\nu t)^{\tau}}+1}dt=\sum_{\mathscr{k},\ell=0}^{\infty}\frac{(-1)^{\mathscr{k}}\mathscr{k}^{\ell}\nu^{\ell\tau}}{\mathscr{k}!\,\ell!}\int_{0}^{\infty}rt^{\ell\tau+r-1}e^{-(\gamma t)^{\alpha}}dt.$$

Let $y=(\gamma t)^{\alpha}\Longrightarrow\frac{dy}{dt}=\alpha\gamma(\gamma t)^{\alpha-1}\Longrightarrow\frac{dy}{\alpha}=\gamma(\gamma t)^{\alpha-1}dt$. Thus,

$$\mu_{r}'=\frac{r}{\alpha}\sum_{\mathscr{k},\ell=0}^{\infty}\frac{(-1)^{\mathscr{k}}\mathscr{k}^{\ell}\nu^{\ell\tau}}{\mathscr{k}!\,\ell!}\gamma^{-(\ell\tau+1)}\int_{0}^{+\infty}y^{\frac{\ell\tau+r}{\alpha}-1}e^{-y}dy. \tag{5}$$

Where, $t^{\ell\tau+r}=\gamma^{-(\ell\tau+r)}((\gamma t)^{\alpha})^{\frac{\ell\tau+r}{\alpha}}$ and $\int_{0}^{+\infty}y^{\frac{\ell\tau+r}{\alpha}-1}e^{-y}dx\sim\Gamma\left(\frac{\ell\tau+r}{\alpha}\right)$. Thus,

$$\mu_{r}'=\frac{r}{\alpha}\sum_{\mathscr{k},\ell=0}^{\infty}\frac{(-1)^{\mathscr{k}}\mathscr{k}^{\ell}\nu^{\ell\tau}}{\mathscr{k}!\,\ell!}\Gamma\left(\frac{\ell+r}{\alpha}\right),r=1,2,\ldots \tag{6}$$

b) Order statistics

It is important that we derive the pdf of the rth order statistic $T_{(r)}$ of the HWEP model's random sample $T_{i,\{1\le i\le n\}}$ with parameter vector $(\boldsymbol{\xi})$. Thus, reference to [7], the pdf of $T_{(r)}$ is given by:

$$\mathbb{g}_{r:n}(t) = \frac{1}{B(r,n-r+1)}[\mathbb{G}(t)]^{r-1}[1-\mathbb{G}(t)]^{n-r}\mathbb{g}(t), \tag{7}$$

where, $B(.,.)$, is the beta function. While $\mathbb{G}(t) = 1 - e^{-H(t)}$ with $e^{-H(t)} = R(t)$ and $\mathbb{g}(t) = \hbar(t)R(t)$.

As for HWEP model, the hazard and cumulative hazard functions are respectively [1]: $\hbar(t) = \alpha\gamma^{\alpha}\, t^{\alpha-1} + \tau\nu^{\tau}t^{\tau-1}e^{(\nu t)^{\tau}}$ and $H(t) = (\gamma t)^{\alpha} + e^{(\nu t)^{\tau}} - 1$. Meanwhile, we have

$$[\mathbb{G}(t)]^{r-1} = \left[1-e^{-H(t)}\right]^{r-1} = \sum_{\Bbbk=0}^{r-1}\binom{r-1}{\Bbbk}(-1)^{\Bbbk}e^{-\Bbbk H(t)}, \tag{8}$$

and

$$[1-\mathbb{G}(t)]^{n-r} = e^{-(n-r)H(t)}. \tag{9}$$

Thus, putting (8) and (9) into (7), we have

$$\mathbb{g}_{r:n}(t) = \frac{1}{B(r,n-r+1)}\sum_{\Bbbk=0}^{r-1}\binom{r-1}{\Bbbk}(-1)^{\Bbbk}\hbar(t)e^{-(n+\Bbbk+1-r)H(t)},$$

$$= n\binom{n-r}{r-1}\sum_{\Bbbk=0}^{r-1}\binom{r-1}{\Bbbk}(-1)^{\Bbbk}\left(\alpha\gamma^{\alpha}\, t^{\alpha-1} + \tau\nu^{\tau}t^{\tau-1}e^{(\nu t)^{\tau}}\right)e^{-(n+\Bbbk+1-r)\left((\gamma t)^{\alpha}+e^{(\nu t)^{\tau}}-1\right)},$$

$$\mathbb{g}_{r:n}(t) = n\binom{n-r}{r-1}\sum_{\Bbbk=0}^{r-1}\binom{r-1}{\Bbbk}\frac{(-1)^{\Bbbk}}{(n+\Bbbk+1-r)}\mathbb{g}(t;\alpha,\gamma,\nu,\tau). \tag{10}$$

Where, $\mathbb{g}(t;\alpha,\gamma,\nu,\tau)$ is the pdf of HWEP model with parameters $\alpha, \gamma^* = (n+\Bbbk+1-r)\gamma, \nu_e = \nu + ln(n+\Bbbk+1-r)$ and τ.

II. Inferential Procedures for the HWEP Model

Here, we present the HWEP's parameter estimation by Bayesian procedures and give succinct explanation on the HMC technique.

In reliability context and otherwise, Bayesian methods hold expedient position over the frequentist approaches for providing simply conceptual approach in handling complex situations. This is for their incorporation of the prior information concerning the parameters of the population beforehand. Thus, appropriate prior selection significantly decides the outlook of Bayesian tests. Further, of most advantageous part of the Bayesian modelling to reliabilists is their applicability in dealing with the two common problematic features of reliability data, viz censoring and small sample sizes. Besides, also provide strong probabilistic interpretations. Refer to [11]. Perhaps, in complicated form of models involving high-dimensional integrals, inferences based on posterior samples may be rationally hard to work out. Albeit the situations may be low-dimensional. We will touch this aspect in our subsequent discussions. Now, we propose to estimate the HWEP's parameters by Bayes method. Assume the times of failure, $\mathfrak{D}$: $t_1, t_2, \ldots, t_n$, resulting from life test of n units, are the data observed and let the distribution in Eq. (1) be of the times of failure. Then, we have the equivalent likelihood, $\mathbb{L}(\mathfrak{D}|\boldsymbol{\xi})$, as:

$$\ell(T|\alpha,\gamma,\nu,\tau) = \prod_{i=1}^{n}\left[\alpha\gamma^{\alpha} t_i^{\alpha-1} + \tau\nu^{\tau}t_i^{\tau-1}e^{(\nu t_i)^{\tau}}\right]\exp\left\{-\sum_{i=1}^{n}\left[(\gamma t_i)^{\alpha} + e^{(\nu t_i)^{\tau}} - 1\right]\right\}. \quad (11)$$

Next, we assume an independent vague gamma prior, $P(\boldsymbol{\xi})$, for $\boldsymbol{\xi} = (\alpha,\gamma,\nu,\tau)'$, out of no prior information on the values of the parameters, thus, the inference is data-driven. Subsequent to the effective outcomes of gamma priors chosen by [10], [16], [5], etc. Thus, we have

$$P(\boldsymbol{\xi}) = \frac{\delta_k^{\beta_k}}{\Gamma(\beta_k)}\boldsymbol{\xi}^{\beta_k-1}e^{-\delta_k\xi_k}, k = 1,2,3,4,$$

where, $\delta_k, \beta_k > 0$ are the hyperparameters, usually considered known beforehand in Bayesian analysis [10, 16]. While the constant, $\frac{\delta_k^{\beta_k}}{\Gamma(\beta_k)}$, which ensures the density's integration to unity, will be overlooked. Thus, the parameters hierarchically and independently follow a gamma prior of the form

$$P(\boldsymbol{\xi}) = \boldsymbol{\xi}^{\beta_k-1}e^{-\delta_k\xi}, \delta_k, \beta_k > 0, k = 1,2,3,4,$$

where, $\boldsymbol{\xi} = (\alpha,\gamma,\nu,\tau)'$. Thus, $\alpha \sim P(\alpha|\delta_1,\beta_1)$, $\gamma \sim P(\gamma|\delta_2,\beta_2)$, $\nu \sim P(\nu|\delta_3,\beta_3)$ and $\tau \sim P(\tau|\delta_4,\beta_4)$, depending on their hyper-parameters. Consequently, the joint prior is given as:

$$P(\boldsymbol{\xi}) = P(\alpha|\delta_1,\beta_1)P(\gamma|\delta_2,\beta_2)P(\nu|\delta_3,\beta_3)P(\tau|\delta_4,\beta_4).$$

Hence, up to proportionality to the product of the prior and the likelihood, we can obtain the joint posterior, through the Bayes theorem, as:

$$P(\boldsymbol{\xi}|\mathfrak{D}) \propto \alpha^{\beta_1-1}\gamma^{\beta_2-1}\nu^{\beta_3-1}\tau^{\beta_4-1}\exp\left\{-\sum_{i=1}^{n}\left[(\gamma t)^{\alpha} + e^{-(\nu t_i)^{\tau}} - 1\right]\right\}$$
$$\times \exp\{-\delta_{\mathbf{1}}\alpha - \delta_{\mathbf{2}}\gamma - \delta_{\mathbf{3}}\nu - \delta_4\tau\}\prod_{i=1}^{n}\left[\alpha\gamma^{\alpha} t_i^{\alpha-1} + \tau\nu(\nu t_i)^{\tau-1}e^{-(\nu t_i)^{\tau}}\right]. \quad (12)$$

Yielding the posterior marginal densities of α,γ,ν and τ respectively as:

$$P(\alpha|\mathfrak{D}) \propto \alpha^{\beta_1-1}\prod_{i=1}^{n}\left[\alpha\gamma^{\alpha} t_i^{\alpha-1} + \tau\nu(\nu t_i)^{\tau-1}e^{-(\nu t_i)^{\tau}}\right] exp\left\{\sum_{i=1}^{n}\left[-(\gamma t)^{\alpha} - \delta_{\mathbf{1}}\alpha + 1\right]\right\},$$
$$P(\gamma|\mathfrak{D}) \propto \gamma^{\beta_2-1}\prod_{i=1}^{n}\left[\alpha\gamma^{\alpha} t_i^{\alpha-1} + \tau\nu(\nu t_i)^{\tau-1}e^{-(\nu t_i)^{\tau}}\right] exp\left\{\sum_{i=1}^{n}\left[-(\gamma t)^{\alpha} - \delta_{\mathbf{2}}\gamma + 1\right]\right\},$$
$$P(\nu|\mathfrak{D}) \propto \nu^{\beta_3-1}\prod_{i=1}^{n}\left[\alpha\gamma^{\alpha} t_i^{\alpha-1} + \tau\nu(\nu t_i)^{\tau-1}e^{-(\nu t_i)^{\tau}}\right] exp\left\{\sum_{i=1}^{n}\left[-e^{-(\nu t_i)^{\tau}} - \delta_{\mathbf{3}}\nu + 1\right]\right\},$$
$$P(\tau|\mathfrak{D}) \propto \tau^{\beta_4-1}\prod_{i=1}^{n}\left[\alpha\gamma^{\alpha} t_i^{\alpha-1} + \tau\nu(\nu t_i)^{\tau-1}e^{-(\nu t_i)^{\tau}}\right] exp\left\{\sum_{i=1}^{n}\left[-e^{-(\nu t_i)^{\tau}} - \delta_4\tau + 1\right]\right\},$$

Then, we determine the Bayes estimators of the unknown parameters under square error loss function. Denote $\boldsymbol{\xi}_i, i = 1, \ldots, 4$ to represent α,γ,ν and τ respectively, thus, we have

$$\hat{\boldsymbol{\xi}}_{i_{BE}} = \mathbb{E}(\boldsymbol{\xi}_i|\mathfrak{D}) = \int \boldsymbol{\xi}_i P(\boldsymbol{\xi}|\mathfrak{D})d\boldsymbol{\xi},$$
$$\hat{\hbar}_{BE}(t) = \mathbb{E}(\hbar(t;\boldsymbol{\xi})|\mathfrak{D}) = \int \hbar(t;\boldsymbol{\xi})P(\boldsymbol{\xi}|\mathfrak{D})d\boldsymbol{\xi},$$
$$\hat{R}_{BE}(t) = \mathbb{E}(R(t;\boldsymbol{\xi})|\mathfrak{D}) = \int R(t;\boldsymbol{\xi})P(\boldsymbol{\xi}|\mathfrak{D})d\boldsymbol{\xi}.$$

Conversely, the above equations may not easily be handled to attain the promising Bayesian Estimates (BEs) due to the intricate form of the posterior. Elsewise, alternate sample-based approaches like the MCMC techniques offer simply handy solutions, may such settings emerge, more so, the Metropolis-Hastings (MH) and Gibbs samplers.

However, their limitations such as the random walk behaviors render them ineffective toward dealing with the objective model. Particularly when the parameters are highly correlated or in high dimensional situations. Refer to Gupta *et al*. [10], Thach and Bris [15, 16], Abba *et al.* [2] among others. On the other hand, we opt for the HMC technique.

- Hamiltonian Monte Carlo Simulation Algorithm

Hamiltonian Monte Carlo algorithm, an instance of a unique MCMC algorithm, generalizes the MH method, was first used in statistics by Neal [13] on neural network models for Bayesian learning. By default, it works on all continuous positive objective distributions [16]. By borrowing a physics concept known as Hamiltonian dynamics (HDs) which describe objects' timely change rates upon a friction-free surface, based on their collective potential, $\mathcal{U}(\mathscr{v})$ and kinetic, $\mathcal{K}(\mathscr{k})$ energies, i.e Hamiltonian $[\mathcal{U}(\mathscr{v}) + \mathcal{K}(\mathscr{k})]$. Instead of using a probability model, the HMC suggests future states for the Markov chain through the HDs which uses the partial derivatives of the log of the posterior $[lnP(\boldsymbol{\xi}|\mathfrak{D})]$ with respect to the parameter vector, $\boldsymbol{\xi}$. By this notion, it restrains the MH sampler's actions of the random walk and its sensitiveness towards correlated settings among parameters. Thus, enables significant fusion and exploring through the target model more swiftly. See [2] for an excellent elaboration on the HMC methods. In this respect, this study adopts the HMC algorithm for posterior samples simulation using No-U-Turn Sampler (NUTS), a version of the HMC, in RStudio 4.3.0. Thus, easily draw inferences based on the samples. Now, for a drawn sample $\{\boldsymbol{\xi}_i, i = 1,2, \dots, N\}$ out of posterior density $\mathcal{P}(\boldsymbol{\xi}|\mathfrak{D})$. For significant enough i, insofar as larger than n_0, say, so that we have the sample $\{\boldsymbol{\xi}_i, i = n_0 + 1, \dots, N\}$ out of the actual posterior. Next, the BEs of the parameters α, γ, ν and τ with the corresponding $h(t)$ and $R(t)$ of the HWEP model can approximately be computed, each from their means as:

$$\hat{\boldsymbol{\xi}}_{BE} \approx \frac{1}{N - n_0} \sum_{i=n_0+1}^{N} \boldsymbol{\xi}_i,$$

$$\hat{\mathscr{h}}_{BE}(t) \approx \frac{1}{N - n_0} \sum_{i=n_0+1}^{N} \mathscr{h}(t; \boldsymbol{\xi}_i),$$

$$\hat{R}_{BE}(t) \approx \frac{1}{N - n_0} \sum_{i=n_0+1}^{N} R(t; \boldsymbol{\xi}_i),$$

where, $\hat{\boldsymbol{\xi}} = (\hat{\alpha}, \hat{\gamma}, \hat{\nu}, \hat{\tau})'$ is the one-to-one estimate of $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$. The warm-up stage, n_0, denotes the total iterations prior to determining the stationary samples. Further, for convergence assessment of sampler, $\mathscr{k}$ parallel chains are usually computed (say, $\mathscr{k} = 3, 4$ or 5) in preference to a single chain. Thus,

$$\hat{\boldsymbol{\xi}}_{BE} \approx \frac{1}{\mathscr{k}(N - n_0)} \sum_{\substack{0 \le j \le \mathscr{k} \\ n_0+1 \le i \le N}} \boldsymbol{\xi}_{i,j},$$

$$\hat{\mathscr{h}}_{BE}(t) \approx \frac{1}{\mathscr{k}(N - n_0)} \sum_{\substack{0 \le j \le \mathscr{k} \\ n_0+1 \le i \le N}} \mathscr{h}\left(t; \boldsymbol{\xi}_{i,j}\right),$$

$$\hat{R}_{BE}(t) \approx \frac{1}{\mathscr{k}(N - n_0)} \sum_{\substack{0 \le j \le \mathscr{k} \\ n_0+1 \le i \le N}} R\left(t; \boldsymbol{\xi}_{i,j}\right),$$

where, $\hat{\boldsymbol{\xi}} = (\hat{\alpha}, \hat{\gamma}, \hat{\nu}, \hat{\tau})'$ is the one-to-one estimate of $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$.

III. Simulation Experiment

The MLEs for the HWEP parameters were proven consistent in our prior publication [1]. Now, we justify the efficiency of the Bayes estimators. The proposed Bayes estimators of HWEP parameters were evaluated through the simulation experiment by adequately utilizing the R function, inverseCDF(p, CDF, ...) from HDInterval package. Where p is an n-size vector of probabilities q_j, j = 1, 2, …, n and the dots (…) denote the parameters to be passed to the CDF. For instance, for n = 1000,$\boldsymbol{\xi} = (\alpha = 14, \gamma = 0.00026, \nu = 0.00021, \tau = 0.75)'$. The steps are: (i) some starting values for $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$ and K sample sizes were picked; (ii) since the exact solution for HWEP's quantile function in [1] does not exist, then, the nonclosed form equation of the quantile function given in [1] was utilized to sample from the HWEP distribution by applying the R function mentioned above; (iii) n = 1000 samples of size K=20, 30,.., 300 were drawn and employed to obtain the posterior means, biases and mean squared errors (MSEs) for the HWEP parameters, $\boldsymbol{\xi} = (\alpha, \gamma, \nu, \tau)'$, for 1000 iterations, however, discard 50% as warm-up samples. The results are presented in Table 4.

IV. Further Applicability of the HWEP Model in Reliability Context

Further showcase of the HWEP model's reliability potential is done through the Bayesian inferential procedure using two bathtub FT datasets presented in Tables 1 and 2 respectively. Four-chain set of samples drawn from HMC procedure were adopted for the posterior computations and plotting using RStan package in RStudio. Additional MLE computation is briefly presented using Nelder Mead and BFGS optimizers in R. Again, the models for the comparative performance study are the Improved New Modified Weibull (INMW) [15], Additive Chen-Weibull (ACW) [16] and Flexible Additive Chen-Gompertz (FACG) [3]. The comparative benchmark criteria include the log-likelihood value (-ℓ), Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected AIC (AICc), Kolmogorov-Smirnov (KS), Anderson-Darling (AD), Cramér-von Mises (CVM) as well as the empirical FR plots. The case studies are as follows.

- Time to First Failures (TTFFs) of 500 Megawatt (MW) Generators

The thirty-six TTFFs of 500MW generators first used by [9], is a bathtub FR dataset. It was applied by Alhassan et al. [1] and Xu et al. [18] for evaluating the MLEs of HWEP and q-Weibull distributions respectively. It is given below.

Table 1: *TTFFs of 500MW generators (in 1000's of hours)*

0.058	0.070	0.090	0.105	0.113	0.121	0.153	0.159	0.224
0.421	0.570	0.596	0.618	0.834	1.019	1.104	1.497	2.027
2.234	2.372	2.433	2.505	2.690	2.877	2.879	3.166	3.455
3.551	4.378	4.872	5.085	5.272	5.341	8.952	9.188	11.399

- Failure and Running Times (FRTs) of Devices

The dataset in Table 2, originally used by [11], is of failure and running times of thirty devices, and was later adopted enormously in the reliability literature, for instance, see Refs. [12, 4, 14, 16].

Table 2: *FRTs of 30 devices*

2	10	13	23	23	28	30	65	80	88
106	143	147	173	181	212	245	247	261	266
275	293	300	300	300	300	300	300	300	300

Two FMs viz: electrical surge which predominated the product's early-life stage and normal product wear during the unit's wear-out phase, were observed for this dataset [11]. These coincide with a vital assumption for the proposed HWEP model [1].

III. Results

I. Simulation experiment

The results obtained from the Monte Carlo experiment for evaluating the BEs of HWEP parameters are presented in Table 3. These include the mean estimates, biases and MSEs for three distinct preset parameter values. The mean parameter values, biases and MSEs were used as the basis for evaluating the consistency of the Bayes estimators for the HWEP model parameters. Five different sample sizes were used.

Table 3: *Simulation results for evaluating BEs*

	(α = 0.6, γ = 0.2, ν = 1.0, τ = 5.0)				(α = 0.7, γ = 0.4, ν = 0.2, τ = 4.4)				(α = 0.9, γ = 0.6, ν = 0.7, τ = 0.8)			
						Means						
K	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\nu}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\nu}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\nu}$	$\hat{\tau}$
20	0.6240	0.2151	0.9981	5.0208	0.7639	0.4194	0.1989	4.4748	0.9258	0.6711	0.7658	0.8876
30	0.6004	0.2000	1.0000	5.0023	0.7715	0.4059	0.1994	4.4032	0.9575	0.6609	0.7481	0.8363
50	0.6000	0.2000	1.0000	5.0000	0.7203	0.4020	0.1997	4.3463	0.9341	0.6526	0.7257	0.8287
100	0.6000	0.2000	1.0000	5.0000	0.7046	0.4014	0.1998	4.3650	0.9046	0.6396	0.7128	0.8248
300	0.6000	0.2000	1.0000	5.0000	0.7023	0.4000	2.0000	4.3895	0.9142	0.6461	0.7273	0.8231
						Biases						
20	-0.3849	0.4240	-0.0019	0.0208	-0.2806	0.3639	-0.0011	0.0748	0.1658	-0.3742	-0.0289	0.0876
30	-0.4010	0.4004	0.0000	-0.0004	-0.2941	0.3715	-0.0006	0.0032	0.1481	-0.3425	-0.0391	0.0363
50	-0.4000	0.4000	0.0000	0.0000	-0.2980	0.3203	-0.0003	-0.0537	0.1257	-0.3659	-0.0474	0.0287
100	-0.4000	0.4000	0.0000	0.0000	-0.2986	0.3046	-0.0002	-0.0350	0.1128	-0.3954	-0.0604	0.0248
300	-0.4000	0.4000	0.0000	0.0000	-0.3000	0.3023	0.0000	-0.0105	0.1273	-0.3858	-0.0539	0.0231
						MSEs						
20	0.1527	0.2192	2e-04	0.0514	0.0885	0.2080	4e-04	0.1542	0.0563	0.2143	0.0254	0.0570
30	0.1600	0.1603	0.0000	0.0005	0.0880	0.2073	0.0000	0.1098	0.0434	0.1752	0.0196	0.0218
50	0.1600	0.1600	0.0000	0.0000	0.0900	0.1555	0.0000	0.1044	0.0238	0.1978	0.0150	0.0095
100	0.1600	0.1600	0.0000	0.0000	0.0893	0.1105	0.0000	0.0734	0.0168	0.1957	0.0129	0.0090
300	0.1600	0.1600	0.0000	0.0000	0.0900	0.0931	0.0000	0.0130	0.0244	0.1898	0.0121	0.0086

II. Bayesian inference

In each case study, appropriate values were assigned to the hyper-parameters defined in section II, and then, respectively, 4-parallel chains of posterior samples were drawn out of the posterior marginal densities. $n_0 = 1000$ were counted out as warm-up samples and the subsequent $N - n_0$ fraction was utilized to get the respective BEs ($\hat{\alpha}_{BE}$, $\hat{\gamma}_{BE}$, $\hat{\nu}_{BE}$ and $\hat{\tau}_{BE}$) of the underlying parameters. Relative to the datasets used, Table 4 and Table 5, respectively, contain the BEs and MTTFs of the HWEP components, with their corresponding Highest Posterior Density Intervals (HPDIs).

a) TTFFs of 500MW Generators

Table 4: *BEs with 95% HPDIs for the HWEP parameters and MTTF for the TTFFs data.*

Bayes			
Parameters	Estimates	SDs	95% HPDIs
$\hat{\alpha}$	1178.1	11.987	[1155.0, 1200.6]
$\hat{\gamma}$	0.0877	7.0e-5	[0.0876, 0.0878]
$\hat{\nu}$	0.2000	0.0254	[0.1507, 0.2478]
$\hat{\tau}$	0.5823	0.0596	[0.4723, 0.7055]
MTTF	0.3475	0.0425	[0.2658, 0.4280]

b) FRTs of Devices

Table 5: *BEs with 95% HPDIs for the HWEP parameters and MTTF for the FRTs data.*

Bayes			
Parameters	Estimates	SDs	95% HPDIs
$\hat{\alpha}$	0.7272	0.1343	[0.4744, 0.9783]
$\hat{\gamma}$	0.0027	0.0010	[0.0010, 0.0046]
$\hat{\nu}$	0.0027	0.0003	[0.0021, 0.0032]
$\hat{\tau}$	5.6009	3.8613	[0.6930, 13.484]
MTTF	22.873	2.5246	[18.250, 27.250]

III. Maximum Likelihood Inference for the FRTs of Devices

As a complement to the HWEP's performance demonstration in reliability modelling of FTs data via the MLE procedure as in [1], we again use the dataset in Table 2 to explore further. We present the results in Table 6, consisting of the selected comparative models based on six evaluation criteria.

Table 6: *MLEs and the comparison metrics (p-values in parentheses) for FRTs dataset*

Model	Optimized parameters	$-\ell$	AIC	BIC	AICc	KS	AD	CVM
HWEP	$\hat{\alpha}$ = 0.7365, $\hat{\gamma}$ = 0.0028, $\hat{\nu}$ = 0.0028, $\hat{\tau}$ = 5.6226	140.99	289.972	295.576	291.572	0.2675 (0.0273)	1.1498 (0.0044)	0.1760 (0.0100)
ACW	$\hat{\alpha}$ = 0.0016, $\hat{\beta}$ = 285.37, $\hat{\gamma}$ = 0.2670, $\hat{\lambda}$ = 0.0130	142.52	293.045	298.650	294.645	0.2818 (0.0171)	1.6310 (0.0003)	0.2660 (0.0007)
FACG	$\hat{\gamma}$ = 0.2672, $\hat{\alpha}$ = 0.0130, $\hat{\theta}$ = 289.86, $\hat{\lambda}$ = 0.7097	142.52	293.045	298.650	294.645	0.2818 (0.0171)	1.6306 (0.0003)	0.2659 (0.0007)
INMW	$\hat{\alpha}$ = 0.0026, $\hat{\beta}$ = 0.0148, $\hat{\gamma}$ = 0.6533, $\hat{\theta}$ = 154.30, $\hat{\lambda}$ = 0.0025	141.87	293.738	300.744	296.238	0.2783 (0.0192)	1.5153 (0.0005)	0.2459 (0.0013)

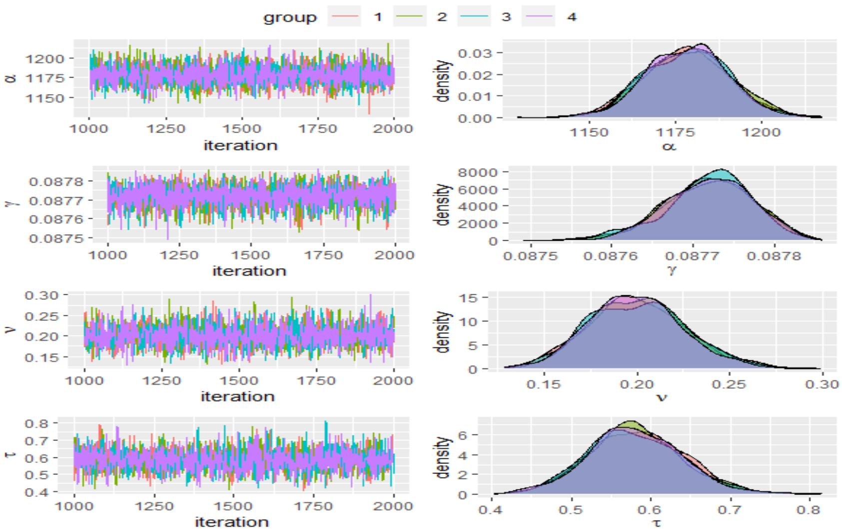
IV. Discussion

I. Simulation experiment

Using nlminb package in R, we carried out simple Monte Carlo experiment to evaluate the consistency of our Bayes estimators. Negligible biases can be observed, and with the increase in sample sizes, the biases and MSEs tend to zero. In addition, the mean estimator values returned simulate the initial assigned values. Thus, showing the consistency of the estimators.

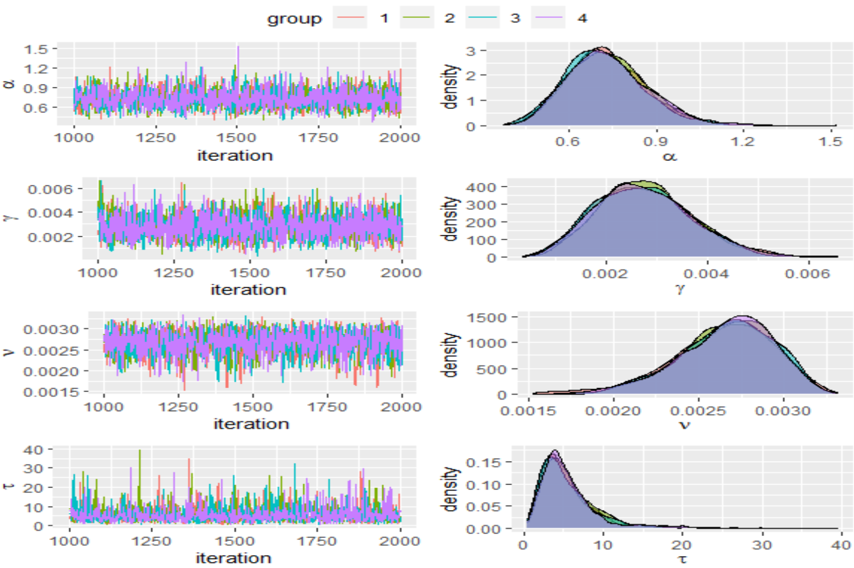
II. Bayesian inference

We can see from Table 4 and Table 5, respectively, that our BEs have been well contained by their narrow corresponding HPDIs with minimal SDs, thus, indicating consistent BEs under quadratic error loss function. Also, comparing both categories with the results of the MLEs obtained in [1], one can observe that the estimates and the intervals are respectively similar. Hence, we can reasonably infer that both estimates are satisfactory. Moreover, the posterior plots in column (a) of Figure 1 show the trace plots for the 4-parallel chains HMC simulation for TTFFs dataset. The traces vividly reveal that all the chains for a specific parameter unite rapidly with the same aiming model. While in (b) column, the estimated posterior densities for the parameters can be seen approximately distributed symmetric over the center points. Both results empirically justify the existence of consistent Bayes estimates under the quadratic loss function. Although, the posterior density curve for the scale parameter, γ, insignificantly skewed rightward. With regards to the trace plots in Figure 2(a) of the chains obtained from the HMC posterior simulation for the FRTs dataset, the same outcome can be observed as in Figure 1(a). While, the densities estimated for the parameters in Figure 2(b) can be seen insignificantly skewed leftward. Albeit, the leftward shift appears significant for the shape parameter, τ. Save the scale parameter, ν, which in contrast slightly skewed rightward. For the skewness observed in the posterior curves for the above parameters, posterior mode can be appropriately considered as the best estimate. The nature of the HMC traces for the chains in both case studies prove the precision of the HMC technique in handling the HWEP's complex posterior.



(a) (b)

Figure 1: *(a) Trace and (b) Density curves for the BEs using HMC algorithm for the TTFFs of 500MW data.*



(a) (b)

Figure 2: (a) *Trace and* (b) *Density curves for the BEs using HMC algorithm for the FRTs data*

III. Maximum Likelihood Inference for the FRTs of Devices

Moreover, we can observe from Table 6 that the HWEP model has the least likelihood ($-\ell$) value and also the minimal value of all the performance statistics among the compared models. Thus, the novel HWEP model performs better than the rest of the distributions, for describing the FRTs dataset. Figure 5 depicts the performance ranking of the models.

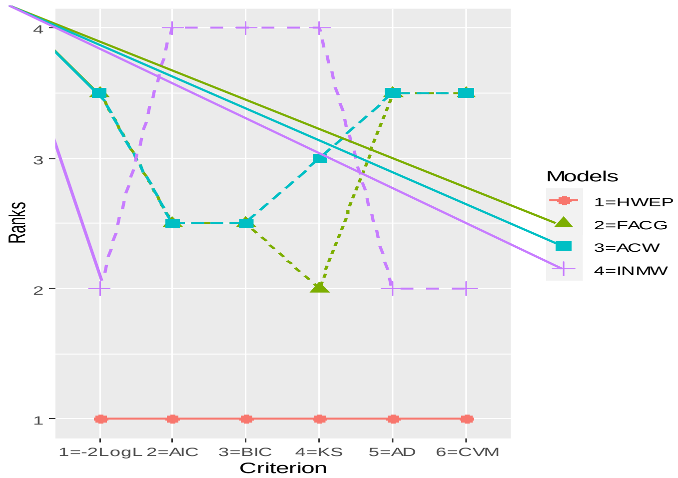


Figure 5: *Line plots showing the rankings of the trained models for FRTs data.*

Furthermore, the resulting FR curves shown in Figure 3 and 4 describing the TTFFs and FRTs dataset respectively, reveal that the HWEP model more accurately describe the FR of both datasets better than the rest of the models. Because the FR of the HWEP model follow approximately the path of the histogram of the data better than the rest of the distributions in both cases.

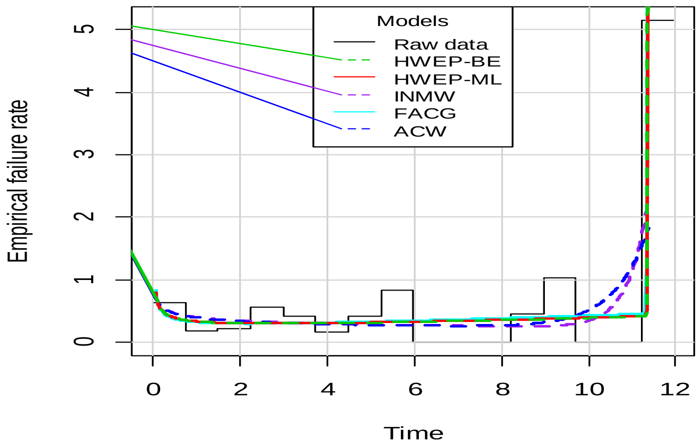


Figure 3: *Fitted FR curves for the compared models on TTFFs data*

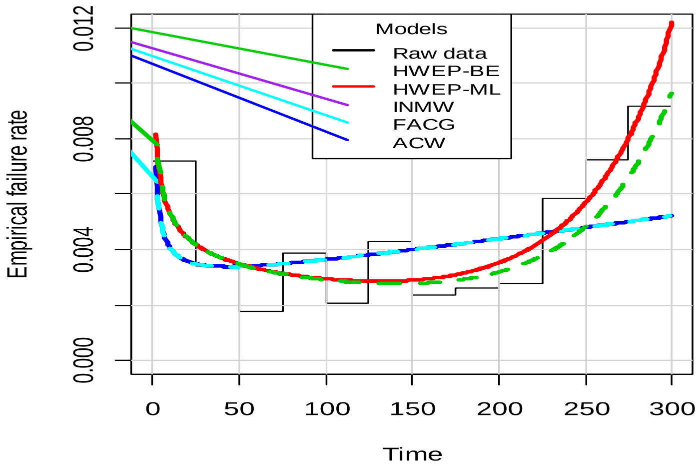


Figure 4: *Fitted FR curves for the compared models on FRTs data.*

IV Summery

In this paper, we further defined important properties of, and have given an adequate Bayesian study for the HWEP model introduced in our previous paper [1]. The HMC phenomenon successfully addressed the complexity found in the HWEP's posterior density, thus, simplified the entire Bayesian computation. We found the estimator consistent thru simulation study. We proved the fitness of the HWEP model in reliability testing on two reference datasets. Supplementary to [1], we gave an MLE computation and once again, the HWEP model out-performed all the compared models for the FTs analyzed based on the evaluation criteria. In conclusion, either the Bayesian or MLE can be used for estimating the HWEP model parameters since the estimators yield similar values. The HWEP model can better be used for describing the two datasets and other similar FTs. The HWEP model's role in Bayesian forecasting and competing risks testing can further be explored.

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Declaration of Conflict of Interest

The authors declare that there is conflict of interest

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