INVESTIGATING THE PERFORMANCE OF MAP/PH/1 QUEUEING SYSTEMS WITH CATASTROPHIC FAILURES, NEGATIVE ARRIVALS, SERVER BREAKDOWNS, REPAIRS AND LACK OF CUSTOMER TOLERANCE FOR DELAYS

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Abstract

This paper presents a comprehensive analytical study of a single-server queueing system incorporating by a Markovian Arrival Process (MAP), Phase-Type (PH) service times, and multiple complex disruptions, including catastrophic failures, negative arrivals, server breakdowns, and phase-type repairs. Utilizing matrix-analytic methods, we derive steady-state probability distributions, and system performance measures. The consequence of failure frequencies, and repair parameters on key performance metrics such as system size, and server availability is thoroughly examined. The behavior of the system is demonstrated using numerical results. The results offer valuable insights for the purposes of design and optimization of resilient and efficient service systems operating in uncertain and failure-prone environments.

Keywords: PH-Distribution, Catastrophic Failure, MAP, Breakdown, Negative Arrival, Matrix Analytic Method.

1. Introduction

Queueing systems are fundamental tools in modeling and analyzing the performance of various real-world service and communication systems, especially under uncertainty and dynamic behavior. Among these, the $MAP/PH/1$ queueing system–characterized by a MAP, a PH service time distribution, and a single server–offers a highly flexible and mathematically tractable framework for modeling complex systems. This study focuses on an advanced $MAP/PH/1$ model that incorporates several real-world complexities: catastrophic failures, negative arrivals, server breakdowns, and phase-type repair mechanisms. Catastrophic failures represent sudden events that remove all customers from the system, simulating large-scale disruptions. Negative arrivals, a less traditional but increasingly relevant concept, model scenarios where incoming entities remove one or more customers already in the queue–useful in contexts such as reverse logistics or erroneous inputs. Additionally, the model accounts for server breakdowns, reflecting interruptions in service capacity, and PH-type repairs, allowing a generalized and realistic representation of the server recovery process.

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The integration of these features aims to more accurately capture the performance and reliability characteristics of modern service systems, such as cloud computing servers, production lines, and network routers. Through analytical modeling and numerical investigation, this work provides insights into the system's behavior under varying operational parameters and disruption scenarios, ultimately aiding in the design of more robust and efficient queue-based infrastructures.

2. Literature Review

In queueing theory, a negative arrival refers to a conceptual mechanism where certain arriving entities (called negative customers) remove existing customers from the system instead of joining the queue. Gelenbe et al. [13] explored a queueing system with a one server incorporating the concept of negative arrivals. They described two distinct mechanisms for customer removal triggered by the arrival of negative customers: the first involves removing the customer currently in service, while the second eliminates the customer positioned at the queue's end. Artalejo and Gomez-Corral [2] conducted an analysis of a queueing system with a single server incorporating both repeated requests and the concept of negative arrivals. Senthil Vadivu et al. [32] analyzed a discrete-time queueing system incorporating arrival processes with correlation, the presence of negative customers, and interruptions in service. They further computed the stationary distribution corresponding to the model′s sojourn time. Lisovskaya et al. [23] studied a multi-server retrial queueing system with two orbits for positive customers. They incorporated the concept of negative customers, where a negative customer′s arrival results in the removal of every customer in the queue currently in service upon arrival before exiting the system. Nesrine and Natalia [26] examined an "$M/M/C/K$" retrial system in which customers may abandon service after an exponentially distributed waiting time and Poisson arrivals of both positive and negative customers, highlighting its relevance to call centers and e-mail support systems. Li and Zhao [22] analyzed a "$MAP/G/1$" queueing system in the presence of negative customers. They derived the stationary probability distributions for queue length and virtual sojourn time, as well as the Laplace transform corresponding to the busy period. Dimitriou [11] investigated a retrial queueing system with mixed priority customers, incorporating negative arrivals, server unreliability, and multiple vacations. Chin et al. [7] investigated a continuous-time queueing system with negative arrivals and derived its stationary distribution of the queue length.

A catastrophic event in queueing theory is a sudden disruption that causes all customers to be instantly removed from the system, including both the queue and service area. After a catastrophe, the system may undergo a recovery or repair period before resuming normal operations. Chakravarthy et al. [5] incorporated a retrial queueing system that includes MAP arrivals, customer impatience, and catastrophic failures followed by repairs. Krishna Kumar et al. [18] studied queueing systems with catastrophes and customer impatience, focusing on both transient and steady-state behavior. They also discussed some special cases of their model. Dharmaraja and Kumar [10] investigated a multi-server Markovian queueing model featuring heterogeneous-server environments subject to catastrophes, deriving its transient solution and exploring various special cases. Reni Sagayaraj et al. [29] analyzed a stochastic multi-phase "$M/M/1$" queueing system exposed to random environmental fluctuations and catastrophic failures, focusing on its steady-state behavior. Vinodhini and Vidhya [33] conducted a computational study of catastrophic queueing models in a multiphase random environment. They demonstrated the model′s relevance to dynamic traffic systems, where incidents such as collisions, weather hazards, or spilled cargo necessitate the redirection of all vehicles. Muthukumaran et al. [25] analyzed a finite-capacity single-server Markovian queueing model incorporating catastrophic effects and a Bernoulli feedback mechanism. Balasubramanian et al. [4] studied a queueing model with two servers, considering catastrophes, failures, and subsequent repairs. Ammar [1] presented a model of a single-server fluid queue subject to catastrophic failures within a randomly transitioning multi-phase environment. Chowdhury and Rani [8] derived the steady-state solution for a feedback queueing system experiencing catastrophes and balking, employing the matrix geometric approach, and carried out a cost analysis for model optimization. Janos Sztrik and Adam Toth [15] studied a finite-source single-server retrial queueing system incorporating two-way communication and catastrophic breakdowns, and performed a sensitivity analysis through simulation. Ozkar et al. [27] analyzed queueing-inventory systems with catastrophes, considering different replenishment policies. They highlighted that the application of queueing-inventory models involving warehouse catastrophes under realistic assumptions has been largely unexplored in existing literature. Additionally, they developed a cost model to optimize the performance of their proposed system. Savita and Kumar [30] analyzed a repairable "$M/M/1/K$" queueing model incorporating a threshold-based recovery mechanism to mitigate server breakdowns and catastrophes, while accounting for customer reneging and balking. They further developed a cost evaluation framework for the system. Shanmugasundaram and Baby [31] studied a three-node queueing network model with retrial and feedback features, incorporating catastrophic events under steady-state conditions. Krishna Kumar et al. [17] analyzed a queue with a single-server subject to catastrophes, failures, and repair activities from a transient perspective.

Kulkarni and Choi [20] investigated retrial queues incorporating server breakdowns and subsequent repairs. Chakravarthy et al. [6] studied a queueing model incorporating server breakdowns, vacations, repairs, and the use of a backup server. They also discussed several special-case models derived from the main system, including models characterized by an R-matrix structure. Additionally, they described the application of their model to cellular network systems. Karpagam and Somasundaram [16] developed a bulk queuing model that incorporates starting failures and a single vacation concept, with specific emphasis on its applicability to production and manufacturing operations. Kumar et al. [19] investigated a finite-capacity queueing system with server failures, customer balking, and a threshold-driven recovery policy. They discussed the model′s application to physician supervision and the management of emergency departments in urban hospitals, and also developed a cost model tailored to this context. Demircioglu et al. [9] investigated a discrete-time queueing system incorporating disasters and provided an analysis of the sojourn time. Mitrani and Puhalskii [24] analyzed an "$M/M/N$" queueing model and derived heavy traffic limiting results for multiprocessor systems experiencing breakdowns and undergoing repairs. Janani and Vijayashree [14] studied a queueing model with an unreliable server experiencing soft failures and working vacations. They also discussed the applicability of their model in the context of smart healthcare systems. Gao et al. [12] examined a non-Markovian retrial queueing system with a single server, incorporating two distinct types of breakdowns and delayed repair mechanisms. Ayyappan et al. [3] conducted a cost optimization analysis of an "$MMAP/PH(1), PH(2)/1$" preemptive priority retrial queueing system characterized by a standby server, server breakdowns, orbital search, phase-type repair distributions, and customer impatience behavior.

2.1. Real Life Applications: Cellular Base Station in a Mobile Network

Mobile users initiate calls or data sessions at varying rates, influenced by factors like time of day, location, and user behavior. These arrival patterns can be modeled using a MAP, which captures the burstiness and correlation in user activities. The service time for each user session (e.g., call setup, data transmission) can be modeled using a phase-type distribution, which represents the service process as a series of exponential phases. This approach allows for capturing complex service patterns, such as varying call durations or data transfer rates. Catastrophic failures, such as power outages or hardware malfunctions, can lead to the sudden unavailability of the base station. Negative arrivals represent the loss of users who attempt to connect during these failures but leave due to the unavailability of service. These events can be modeled using negative arrival processes, where users are lost from the system without being served. This variation highlights additional cases that illustrate the point.

2.1.1 The Romeoville Petroleum Refinery Explosion Incident (1984)

The Union Oil Company refinery in Romeoville, Illinois, USA, experienced a catastrophic explosion on July 23, 1984. The explosion resulted in the deaths of 17 workers and caused extensive damage to the facility. The explosion was triggered by a small crack in a circular weld on a pressure vessel. Attempts to close the main inlet valve to stop gas from leaking from the crack led to the crack growing larger, releasing flammable gas that ignited within moments (Catastrophic failure). The explosion and subsequent damage halted production at the refinery, leading to a loss of output and revenue. The negative arrival refers to the cessation of the refinery's intended function, effectively removing its service capacity (Negative arrival). The explosion caused significant structural damage to the refinery, including the destruction of equipment and infrastructure. The vessel, in service since 1970, had undergone numerous repairs and modifications before the incident (Breakdown). The repair process involved multiple stages, including assessment of damage, procurement of replacement parts, and reconstruction of damaged sections. The extensive repairs required careful planning and execution to restore the refinery to operational status (Repair). This incident underscores the critical importance of adhering to proper maintenance procedures and quality assurance protocols. The failure to address the small crack in the weld led to a catastrophic event, highlighting the need for thorough inspections and timely repairs to prevent such disasters. This example demonstrates how a combination of factors, including minor defects, inadequate maintenance, and delayed repairs, can culminate in a catastrophic failure with far-reaching consequences.

2.1.2 Morbi Bridge Collapse (2022)

A real-life example illustrating the scenario of catastrophic failure, negative arrival, breakdown, and phase-type repairs is the tragic collapse of the Morbi Bridge in Gujarat, India, on October 30, 2022. This 137-year-old suspension bridge over the Machchhu River, collapsed, resulting in at least 141 fatalities and over 180 injuries. The bridge had recently reopened after repairs, but the restoration was mishandled. The bridge's sudden collapse during use, leading to significant loss of life (Catastrophic Failure). The failure led to the loss of public trust and the cessation of the bridge's intended function, effectively removing its service capacity (Negative Arrival). The structural failure of the bridge, compounded by inadequate maintenance and improper repair materials, such as replacing wooden planks with aluminum (Breakdown). The restoration process was flawed, with only superficial repairs undertaken without addressing the underlying structural issues, leading to the eventual collapse (Repair). This incident underscores the critical importance of proper maintenance, adherence to safety standards, and thorough inspections to prevent such catastrophic events. In this example, the Morbi Bridge collapse serves as a poignant reminder of the consequences of neglecting comprehensive maintenance and the need for rigorous safety protocols in infrastructure management.

2.1.3 Teton Pass Road Collapse – Wyoming, USA (2024)

In June 2024, a significant portion of Teton Pass Road in Wyoming experienced a catastrophic collapse, resulting in the closure of a vital transportation route. The collapse of the road was a sudden and unexpected event, leading to immediate disruption of traffic and posing risks to public safety (Catastrophic Failure). The failure led to the cessation of the road's intended function, effectively removing its service capacity and causing delays and detours for travelers (Negative Arrival). The collapse was due to structural weaknesses and environmental factors, highlighting the vulnerability of infrastructure to unforeseen stresses (Breakdown). The repair process involved multiple stages, including assessment of damage, engineering evaluations, and phased reconstruction to restore the road to operational status (Repair). This incident underscores the importance of regular infrastructure assessments, proactive maintenance, and the need for resilient design to withstand environmental challenges. The Teton Pass Road collapse serves as a poignant reminder of the complexities involved in infrastructure management and the critical need for comprehensive planning and timely interventions to prevent such catastrophic events.

2.1.4 Ahmedabad Air India Flight crash (2025)

On June 12, 2025, There is no credible record of Air India Flight AI-171 – or any Boeing 787-8 Dreamliner from Air India – crashing after takeoff from Ahmedabad bound for London-Gatwick. The aircraft lost altitude moments after departure and struck the B.J. Medical College hostel nearby, resulting in a devastating explosion and fire. Tragically, 241 of the 242 people onboard were killed, with only one passenger surviving. Additionally, at least 39 people on the ground lost their lives, and over 60 were injured. The flight crew had issued a mayday call reporting power or thrust failure before the crash. Both black boxes have been recovered, and an international investigation–led by India′s AAIB with support from U.S., UK, Boeing, and GE teams–is underway. The incident prompted Air India and Tata Group to establish emergency aid and compensation for victims′ families, while broader aviation safety measures are being reassessed.

2.2. Overview of the Manuscript

The remainder part of the manuscript is structured as follows: Section 3 outlines the mathematical formulation of the model. Section 4 introduces the notations used in the matrix formulation and describes the matrix generation process using the Quasi-Birth"Death (QBD) approach. Section 5 presents the stability condition, derives the steady-state probability vector, and analyzes the rate matrix R. Section 6 focuses on the evaluation of system performance measures. Section 7 illustrates numerical examples with tabulated results. Finally, Section 8 summarizes the key conclusions of the study.

3. Detailed Description of the Mathematical Model

We consider a customer arrival process modeled by a MAP, represented by the pair (D_0, D_1), where D_0 corresponds to transitions without customer arrivals, and D_1 corresponds to transitions that include customer arrivals into the system. Service times are modeled using a phase-type (PH) distribution, characterized by the representation (α, T), where $T^0 + Te = 0$ such that $T^0 = -Te$. During service, if a negative customer arrival occurs, it affects the customer currently receiving service by removing both the customer and the server from the system. Additionally, catastrophic events can happen at any time–The operational status of the server (idle or busy)–and these events cause the server to fail (break down) and simultaneously remove all customers from the system. When the server experiences a breakdown due to either negative arrivals or catastrophes, the server enters a repair process. Upon completion of the repair, the server returns to service: if there are customers waiting, the server resumes servicing; otherwise, the server remains idle. While the server is under repair, customers waiting in the queue may renege (i.e., leave the system due to lack of customer tolerance for delays) based on an exponentially distributed with parameter σ. The exponentially distributed rates associated with negative arrivals, catastrophes, server breakdowns, and repairs are denoted by λ_2, ζ, τ, and γ, respectively.

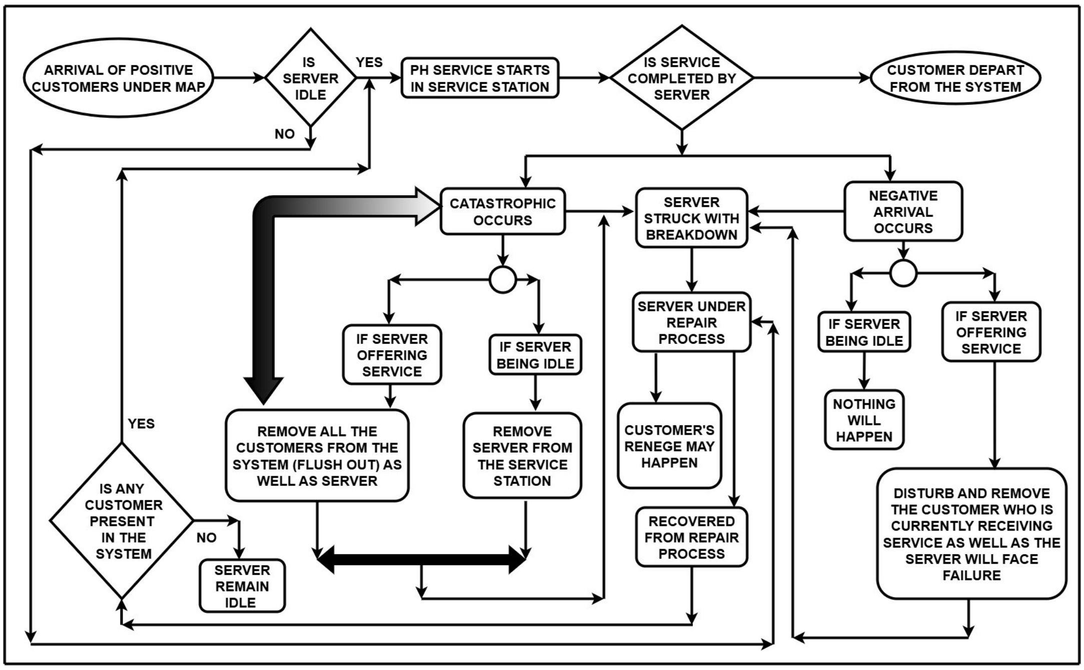

Figure 1: *Figure caption*

4. Matrix Formulation

A comprehensive description of the model′s notation is provided in this section to facilitate the generation of the QBD process.

4.1. Matrix Generation Notations

- $\otimes$ - The Kronecker multiplication operation between two matrices.
- $\oplus$ - The Kronecker sum operation between two matrices
- I_m - It signifies an identity matrix of order $(m \times m)$.
- **0** - Represents the zero matrix of suitable dimensions.
- e - A column vector of the required dimension consisting entirely of ones.
- To indicate λ_1 be the fundamental arrival rate and it is defined as $\lambda_1 = \pi_1 D_1 e_m$, where π_1 is the generator matrix $D = D_0 + D_1$, describes the transition behavior of the MAP. To conveniently determine the π_1 such that $\pi_1 D = 0$, $\pi_1 e = 1$.
- The standard service rate for arriving customers the service provided by the server during normal mode is denoted by δ, where $\delta = [\alpha(-T)^{-1}e]^{-1}$.
- $C(t)$ denotes the number of positive customers present in the system,
- $S(t)$ denotes the server status at time t, where

$$S(t) = \begin{cases} 0, & \text{if the server is being idle,} \\ 1, & \text{if the Catastrophic occurs while the server is on idle,} \\ 2, & \text{if the flush out of customers and also remove server from the system,} \\ 3, & \text{if the server is actively serving customers,} \\ 4, & \text{if the Catastrophic occurs while the server's service period,} \\ 5, & \text{if the server is being repaired,} \\ 6, & \text{if the Catastrophic occurs while the server's repair period,} \end{cases}$$

- $P(t)$ denotes the service phase under PH.
- $A(t)$ denotes the positive arrival phase under MAP.
- Let $\{(C(t)\ ,\ S(t)\ ,\ P(t)\ ,\ A(t)) : t \geq 0\ \}$ is the Continuous Time Markov Chain with the QBD process such that state space is given by

$$\mathbf{\Omega} = l(0) \bigcup_{p=1}^{\infty} l(p)$$

where
$l(0) = \{(0,q,s) : q = 0,1,2,5;\ 1 \leq s \leq m\}$

for $p \geq 1$
$l(p) = \{(p,q,r,s) : q = 3,4;\ 1 \leq r \leq n;\ 1 \leq s \leq m\} \cup \{(p,q,s) : q = 5,6;\ 1 \leq s \leq m\}$

The infinitesimal generator for the QBD process is represented by:

$$Q = \begin{bmatrix} B_{00} & B_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\ B_{10} & A_1 & A_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\ B' & A_2 & A_1 & A_0 & \mathbf{0} & \mathbf{0} & \cdots & \cdots \\ B' & \mathbf{0} & A_2 & A_1 & A_0 & \mathbf{0} & \cdots & \cdots \\ B' & \mathbf{0} & \mathbf{0} & A_2 & A_1 & A_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \end{bmatrix}$$

The entries of the block matrices of Q are described as follows:

$$B_{00} = \begin{bmatrix} D_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\zeta+\tau)I_m & \zeta I_m & \tau I_m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \gamma I_m & \mathbf{0} & \mathbf{0} & D_0 - \gamma I_m \end{bmatrix}, \quad B_{01} = \begin{bmatrix} \alpha \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_1 & \mathbf{0} \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} T^0 \otimes I_m & \mathbf{0} & \mathbf{0} & e_n \otimes \lambda_2 I_m \\ \mathbf{0} & \mathbf{0} & e_n \otimes \zeta I_m & e_n \otimes \tau I_m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \zeta I_m & \mathbf{0} \end{bmatrix}, \quad A_0 = \begin{bmatrix} I_n \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} T \oplus (D_0 - \lambda_2 I_m) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -(\zeta+\tau)I_m & \mathbf{0} & \mathbf{0} \\ \gamma\alpha \otimes I_m & \mathbf{0} & D_0 - (\gamma+\sigma)I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\zeta I_m \end{bmatrix},$$

$$A_2 = \begin{bmatrix} T^0\alpha \otimes I_m & \mathbf{0} & e_n \otimes \lambda_2 I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad B' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & e_n \otimes \zeta I_m & e_n \otimes \tau I_m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \zeta I_m & \mathbf{0} \end{bmatrix}.$$

5. Condition for Stableness

To evaluate the stability characteristics of the system, we examine the model under particular circumstances.

5.1. Assessment of Stability

Let us define the variable A as $A = A_0 + A_1 + A_2$. This indicates that the matrix A is square, with dimension $(2nm + 2m)$, and represents a matrix of irreducible infinitesimal generators.

Let $\mathbf{\Psi}$ represent the steady-state probability vector corresponding to matrix A. This vector must satisfy the conditions:

$$\mathbf{\Psi} A = 0 \text{ and } \mathbf{\Psi} e = 1.$$

where e represents a column vector whose entries are all ones.
The vector $\mathbf{\Psi}$ can be partitioned as:

$$\mathbf{\Psi} = (\Psi_0, \Psi_1, \Psi_2, \ldots),$$

where Ψ_0 has dimension $4m$, and each of Ψ_1, Ψ_2,...has dimension $2nm + 2m$.

This structure reveals that the underlying Markov process possesses a Quasi-Birth-and-Death (QBD) form. The stability of the model is ensured if the following condition is satisfied:

$$\mathbf{\Psi} A_0 e < \mathbf{\Psi} A_2 e$$

which is both a essential and complete condition for the stability of a QBD process.
Therefore, the vector Ψ is obtained by solving the system defined by the above conditions.

$$\Psi_0[(T + T^0\alpha) \oplus (D - \lambda_2 I_m)] + \Psi_2[\gamma\alpha \otimes I_m] = 0.$$
$$\Psi_1[-(\zeta + \tau) I_m] = 0.$$
$$\Psi_0[e_n \otimes \lambda_2 I_m] + \Psi_2[D - \gamma I_m] = 0.$$
$$\Psi_3[-\zeta I_m] = 0.$$

subject to normalizing condition

$$(\Psi_0 + \Psi_1) e_{nm} + (\Psi_2 + \Psi_3) e_m = 1.$$

Following algebraic derivation, the model's stability condition, expressed as $\mathbf{\Psi} A_0$e $<$ $\mathbf{\Psi} A_2$e , can be reformulated as:

$$\{\Psi_0[e_n \otimes D_1 e_m] + \Psi_1[D_1 e_m]\} < \{\zeta_0[(T^0 \otimes e_m) + (e_n \otimes \lambda_2 e_m)] + \Psi_2[\sigma e_m]\}$$

5.2. Investigation of the Steady-State Vector

Let the steady-state probability vector of the QBD matrix Q be indicated by $\mathbf{x}$, which is partitioned as $\mathbf{x} = \mathbf{x}_i$, $i \geqslant 0$. Specifically, $\mathbf{x}_0$ is a vector of dimension $4m$, while $\mathbf{x}_i$ for $i \geqslant 1$ are of dimension $(2nm + 2m)$. The vector $\mathbf{x}$ satisfies the steady-state conditions $\mathbf{x}Q = 0$ and $\mathbf{x}e = 1$, where e is a column vector of ones.

Despite this, the system remains stable, and the sub-vectors of $\mathbf{x}$, excluding $\mathbf{x}_0$, $\mathbf{x}_1$ and $\mathbf{x}_2$, which represent distinct level states, are determined by the following equations.

$$\mathbf{x}_j = \mathbf{x}_2 \, R^{j-1}, \;\; j \geqslant 3.$$

Accordingly, the rate matrix R represents the smallest solution that is non-negative to the quadratic matrix equation $R^2A_2 + RA_1 + A_0 = 0$, confirming the stability of the queuing system. The matrix R, with dimension $(2nm + 2m)$, is derived from this equation and satisfies the additional condition $RA_2e = A_0e$.

By evaluating the following expressions, the sub-vectors corresponding to $\mathbf{x}_0$, $\mathbf{x}_1$ and $\mathbf{x}_2$ are determined.

$$\mathbf{x}_0B_{00} + \mathbf{x}_1B_{10} + \mathbf{x}_2((I - R)^{(-1)}B' = 0,$$

$$\mathbf{x}_0B_{01} + \mathbf{x}_1A_1 + \mathbf{x}_2A_2 = 0.$$

$$\mathbf{x}_1A_0 + \mathbf{x}_2(A_1 + RA_2) = 0.$$

The system is subject to the following normalizing condition:

$$\mathbf{x}_0e + \mathbf{x}_1e + \mathbf{x}_2(I - R)^{-1}e = 1.$$

Therefore, the R matrix is obtained by following the mathematical procedure defined by the Logarithmic Reduction Algorithm.

5.3. Logarithmic Reduction Algorithm

The following summarizes the core steps of the Logarithmic Reduction Algorithm according to Latouche and Ramasamy [21]:

Step 0:

" $H \leftarrow (-A_1)^{-1}A_0$, $L \leftarrow (-A_1)^{-1}A_2$, $G = L$, and $T = H$ "

Step 1:

" $U = HL + LH$ "

" $M = H^2$ "

" $H \leftarrow (I - U)^{-1}M$ "

" $M \leftarrow L^2$ "

" $L \leftarrow (I - U)^{-1}M$ "

" $G \leftarrow G + TL$ "

" $T \leftarrow TH$ "

" Continue Step 1 Until $||e - Ge||_{\infty} < \epsilon$ "

Step 2:

" $R = -A_0(A_1 + A_0G)^{-1}$ "

6. Quantitative Performance Measures

In this section, we investigate the steady-state behavior of the model by evaluating key performance measures and presenting their corresponding analytical expressions.

- Probability that the server is being idle

$$P_I = \sum_{s=1}^{m} \mathbf{x}_{00s}$$

- Probability that the catastrophic occurs during server idle period

$$P_{CI} = \sum_{s=1}^{m} \mathbf{x}_{00s}$$

- Probability that the flush out of customers and remove server from the system

$$P_{FO} = \sum_{s=1}^{m} \mathbf{x}_{00s} + \sum_{p=1}^{\infty} \sum_{r=1}^{n} \sum_{s=1}^{m} \mathbf{x}_{p3rs} + \sum_{p=1}^{\infty} \sum_{s=1}^{m} \mathbf{x}_{p5s}$$

- Probability that the server providing service to customers

$$P_B = \sum_{p=1}^{\infty} \sum_{r=1}^{n} \sum_{s=1}^{m} \mathbf{x}_{p3rs}$$

- Probability that the server is under repair process for rejuvenation

$$P_R = \sum_{p=1}^{\infty} \sum_{s=1}^{m} \mathbf{x}_{p5s}$$

- The expected number of customers available in the system

$$E_{system} = \sum_{p=1}^{\infty} p\mathbf{x}_p = \mathbf{x}_1 e + \mathbf{x}_2[2(I-R)^{-1} + R(I-R)^{-2}]e$$

7. Numerical analysis

This section presents a numerical analysis of the model′s behavior, emphasizing the differences in variance and correlation structures across five different MAP representations. Notably, the first three arrival processes–ERL-A, EXP-A, and HYP-EXP-A–are renewal processes and therefore exhibit zero correlation between consecutive inter-arrival times. In contrast, MAP-NC-A and MAP-PC-A represent correlated arrival processes, with inter-arrival time correlation coefficients of -0.4804 and 0.4804, respectively. The coefficients of variation for the inter-arrival times corresponding to these five processes are 0.2500, 1.0000, 14.6365, 1.0408, and 1.0408.

Arrival in Erlang (ERL-A) :

$$D_0 = \begin{bmatrix} -4 & 4 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

Arrival in Exponential (EXP-A) :

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 1 \end{bmatrix}$$

Arrival in Hyper-exponential (HYP-EXP-A) :

$$D_0 = \begin{bmatrix} -8.2 & 0 & 0 \\ 0 & -0.82 & 0 \\ 0 & 0 & -0.082 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 5.74 & 2.05 & 0.41 \\ 0.574 & 0.205 & 0.041 \\ 0.0574 & 0.0205 & 0.0041 \end{bmatrix}$$

Arrival in MAP-Negative Correlation (MAP-NC-A) :

$$D_0 = \begin{bmatrix} -1.75 & 1.75 & 0 & 0 \\ 0 & -1.75 & 1.75 & 0 \\ 0 & 0 & -1.75 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0175 & 0 & 0 & 1.7325 \\ 3.465 & 0 & 0 & 0.035 \end{bmatrix}$$

Arrival in MAP-Positive Correlation (MAP-PC-A) :

$$D_0 = \begin{bmatrix} -1.75 & 1.75 & 0 & 0 \\ 0 & -1.75 & 1.75 & 0 \\ 0 & 0 & -1.75 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.7325 & 0 & 0 & 0.0175 \\ 0.035 & 0 & 0 & 3.465 \end{bmatrix}$$

The service process is modeled using three phase-type (PH) distributions, each normalized to obtain a consistent service time representation. The Erlang, exponential, and hyper-exponential cases are denoted by ERL-S, EXP-S, and HYP-EXP-S, respectively.

Service in Erlang (ERL-S) :

$$\alpha = \big(1, 0, 0\big), \qquad T = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

Service in Exponential (EXP-S) :

$$\alpha = \big(1\big), \qquad T = \big[-1\big]$$

Service in Hyper-exponential (HYP-EXP-S) :

$$\alpha = \big(0.8, 0.2\big), \qquad T = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}$$

7.1. Illustration:

In the Table 1, we investigate the result of the renege rate of customers (σ) on the expected count of customers available in the system (E_{System}). To fix the values of $\lambda_1 = 1,\ \lambda_2 = 0.1,\ \delta = 5,\ \zeta = 2,\ \tau = 1,\ \gamma = 3,$ such that the system remains stable.

When analyzing the arrival times combined with various service time distributions, HYP-EXPA shows a significant decrease, whereas ERL-A exhibits a slower decline compared to the other arrival time patterns. Conversely, when examining different arrival time combinations with fixed service times, ERL-S decreases gradually, EXP-S shows a moderate decline, and HYP-EXP-S decreases substantially.

In scenarios involving repair times–triggered by catastrophic events or negative arrivals–the server becomes unavailable. During these periods, customers waiting in the queue may lose patience and leave the system (renege). An increase in the reneging rate results in a corresponding decrease in the mean number of customers in the system.

Table 1: *Renege rate of customers vs. E_{system}*

SERVICES	E_{system}					
	ARRIVALS					
	σ	ERL-A	EXP-A	HYP-EXP-A	MAP-NC-A	MAP-PC-A
ERL-S	2	0.117881	0.140791	0.254385	0.133434	0.195316
	4	0.117649	0.140024	0.249009	0.132854	0.190435
	6	0.117524	0.139591	0.245143	0.132535	0.187296
	8	0.117447	0.139319	0.242307	0.132335	0.185209
	10	0.117394	0.139133	0.240177	0.132197	0.183759
	12	0.117356	0.138998	0.238540	0.132097	0.182708
	14	0.117327	0.138897	0.237251	0.132021	0.181917
	16	0.117304	0.138817	0.236217	0.131962	0.181303
	18	0.117285	0.138754	0.235372	0.131914	0.180813
	20	0.117270	0.138701	0.234669	0.131874	0.180415
	22	0.117257	0.138658	0.234078	0.131841	0.180085
	24	0.117247	0.138621	0.233573	0.131813	0.179807
	26	0.117237	0.138589	0.233138	0.131789	0.179570
	28	0.117229	0.138562	0.232760	0.131768	0.179365
EXP-S	2	0.119268	0.147839	0.277096	0.138700	0.227797
	4	0.118966	0.146940	0.271330	0.138034	0.221615
	6	0.118804	0.146427	0.267117	0.137665	0.217398
	8	0.118704	0.146101	0.263979	0.137432	0.214448
	10	0.118635	0.145876	0.261590	0.137273	0.212312
	12	0.118585	0.145713	0.259729	0.137156	0.210712
	14	0.118548	0.145590	0.258249	0.137068	0.209476
	16	0.118518	0.145493	0.257049	0.136999	0.208496
	18	0.118494	0.145415	0.256060	0.136943	0.207701
	20	0.118475	0.145351	0.255231	0.136897	0.207045
	22	0.118458	0.145298	0.254529	0.136859	0.206495
	24	0.118444	0.145253	0.253927	0.136826	0.206027
	26	0.118432	0.145214	0.253405	0.136798	0.205624
	28	0.118422	0.145180	0.252948	0.136774	0.205273
HYP-EXP-S	2	0.151057	0.185763	0.348842	0.169761	0.361502
	4	0.150195	0.184153	0.341741	0.168517	0.350852
	6	0.149697	0.183159	0.336304	0.167775	0.342590
	8	0.149375	0.182493	0.332072	0.167287	0.336104
	10	0.149150	0.182018	0.328719	0.166942	0.330927
	12	0.148983	0.181662	0.326012	0.166686	0.326724
	14	0.148856	0.181387	0.323791	0.166489	0.323257
	16	0.148755	0.181168	0.321940	0.166333	0.320354
	18	0.148673	0.180989	0.320377	0.166205	0.317893
	20	0.148605	0.180840	0.319040	0.166100	0.315782
	22	0.148548	0.180715	0.317884	0.166011	0.313954
	24	0.148499	0.180607	0.316876	0.165935	0.312355
	26	0.148457	0.180515	0.315990	0.165869	0.310946
	28	0.148421	0.180434	0.315204	0.165812	0.309695

8. Conclusion

This study examines a queueing system where customer arrival follows a MAP and service times are governed by a PH distribution. The model integrates several realistic features, including negative arrivals, catastrophic events, customer reneging, server breakdowns, and subsequent repairs. Numerical experiments, presented in tabular form, highlight the impact of varying arrival and service time parameters on the performance measures. The basis for proposing this model arises from practical applications, particularly in cellular base stations within mobile networks and real-world catastrophic scenarios. To analyze the system, we apply matrix-analytic methods to the "MAP/PH/1" queue, accounting for correlated arrivals, service variability, and system disruptions.

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