METRIC DIMENSION OF GRAPHS WITH SEQUENTIAL PENDANT EDGE EXTENSION

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Abstract

The metric dimension of a graph measures the smallest number of vertices (landmarks) needed so that every vertex in the graph can be uniquely identified by its distances to these landmarks. In this study, we investigate a specific structural modification called Sequential Pendant Edge Extension (SPEE) in which pendant edges are attached one at a time to an existing graph. We analyse whether such extensions affect the metric dimension of graph families. These results provide both theoretical insights and practical assurance in designing reliable and scalable network models.

Keywords: Sequential Pendant Edge Extension (SPEE), Metric Dimension, Pendant Edge, Graph Resolvability, Edge Extension.

I. Introduction

Metric dimension is a fundamental concept in the domain of graph theory, which indicates the smallest possible size of a set of vertices required to uniquely distinguish every vertex of the graph. This minimum set, referred to as a set of landmarks or resolving set, ensures that each vertex can be individually identified based on its distances to the vertices in that set. Let graph $G = (V, E)$, where V collection of nodes (vertices) and E collection of links (edges) which connects the vertices. A subset $B \subseteq V$ referred to as a resolving set of G if for every tuple of vertices $(p, q) \in V$, there exist a vertex $m \in B$ such that the distance between m and p is different from the distance between m and q.

The concept of metric dimension in graph theory was first introduced by Slater [1] in the context of "leaves of trees". Harary and Melter [2] independently introduce the idea of metric dimension. The study of metric dimension is not only of theoretical interest but also has wide-ranging applications across science, engineering, and technology in which structure of a chemical compound can be represented by a labelled graph with vertex and edge specifying the atoms and bond types, respectively. Other uses for resolving sets can be found in a number of contexts, such as robot navigation, the coin-weighing problem, drug discovery, connected joins in graphs, and mastermind game tactics [3], network discovery and verification [4]. Metric dimension theory has been studied for various graph classes such as paths, cycles, complete graphs and wheel graphs and for operations like Cartesian products, joins and vertex addition. The metric dimension $\beta(G) = 1$ for any graph G exists only when G is a path mentioned in [5].

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In [6] mentioned that the graphs with metric dimension k cannot have a subgraph isomorphic to $K_{2^k+1} - (2^{k-1} - 1)K_2$. By joining k cycles at a common vertex in each of the k cycles, estimate the metric dimension for some unique types of graphs in [7]. The metric dimension of wheel for larger value of n and also disprove the observation made by [8]. In [9] investigated the dimension of a connected graph by the addition of a single vertex and formula is developed for the dimension of a wheel. Also, the existence of a k-dimensional graph for every integer $k \geq 2$ is shown. In [10] shown metric dimension of comb product of graphs. In [11] the metric dimension of Jahangir graph 2.

Recent studies have continued to expand the applications of metric dimension in dynamic and evolving networks. For instance, [12] investigated the metric dimension in hybrid network models with pendant and chordal extensions, showing that structural modifications may preserve or alter resolvability depending on attachment rules. Similarly,[13] studied the stability of resolving sets under iterative graph operations, highlighting that pendant edge additions offer a unique case of invariance. Despite several studies on metric dimension across classical graphs and graph operations, the impact of sequential pendant edge extension (SPEE) has remained largely unexplored.

Existing literature often focuses on single pendant additions or generalized extensions but does not address whether repeated, stepwise pendant attachments preserve or alter the metric dimension across different graph families. Understanding the stability of metric dimension under SPEE is crucial for analyzing evolving and dynamic networks. Motivated by [12], [13], [14], [15]; we have obtained some results.

II. Preliminaries

I. Metric Dimension

Let graph $G = (V, E)$, where V collection of nodes (vertices) and E collection of links (edges) which is connecting the vertices. A subset $B \subseteq V$ referred to as a resolving set of G if for every tuple of vertices $(p, q) \in V$, there exist a vertex $m \in B$ such that the distance between m and p is different from the distance between m and q. The metric dimension of a graph denotes as $\beta(G)$.

II. Graph Resolvability

Graph resolvability refers to the ability to uniquely determine the positions of each vertex with in a graph by measuring distances to a selected subset of reference vertices, known as a resolving set (or landmark set). A graph is said to be resolvable if there exists at least one resolving set whose pairwise distance vectors called metric representations are distinct for every vertex in the graph. The size of the smallest such set is the graph's metric dimension.

III. Sequential Pendant Edge Extension (SPEE)

Sequential pendant edge extension describes the process of adding one-degree leaf vertices to a graph one at a time, in a specified order. At each step, a new vertex is attached by a single edge (a pendant edge) to an existing vertex, extending the graph's structure without altering the distances among the original vertices.

IV. Wheel Graph

A wheel graph is a kind of graph that has a centre vertex (the hub) attached to a set of peripherals vertices (the rim). The rim vertices also link together in a cyclic pattern, making a cycle. The wheel graph is denoted by W_{n},, where n is the number of vertices on the rim. The metric dimension of the wheel graph is presented below:

$$\beta(W_n)=\begin{cases} 3, & for\ n = 3\ and\ 6 \\ 2, & for\ n = 4\ and\ 5 \\ \frac{2n+2}{5}, & for\ n \geq 7 \end{cases}. \tag{1}$$

V. Cycle Graph

A cycle graph is a basic graph formed by a single cycle, which is an enclosed loop of vertices linked by an edge. It is indicated as C_n, where n is the number of nodes in the cycle. Metric dimension of cycle graph is presented below:

$$\beta\ (C_n) = 2, for\ n \geq 3. \tag{2}$$

VI. Complete Graph

A complete graph is a basic undirected graph where every pair of distinct vertices are joined by a single edge. A complete graph with n vertices, abbreviated as K_n is a graph with n vertices that are contiguous to each other. Metric dimension of complete graph is presented below:

$$\beta(K_n) = n - 1\ , for\ \ n \geq 2. \tag{3}$$

III. Metric Dimension of Graphs with SPEE

In graph theory, a pendant edge is simply an edge connected to a vertex with degree 1. This means it links to a pendant vertex with a single neighbouring edge. The presence of this edge has the potential to impact the metric dimension of graph. Let's look at the details.

When a pendant edge links to a pendant vertex, it can serve as a landmark to uniquely identify both the edge and the vertex. As a result, the presence of this pendant edge has no effect on the metric dimension of graph because the existing pendant vertex serves as a land mark. If the pendant edge connects with a non-pendant vertex the metric dimension of a graph with a pendant edge might remain constant or increase. It is conceivable for the metric dimension to remain constant in some instances where the addition of the pendant edge does not modify the structure or connectivity of the graph in such a manner that it changes the minimum set of vertices required to uniquely identify all others.

For example, consider a graph with non-pendant vertex v, and its adjacent vertices u and w. If there is an edge between u and w and a pendant edge is added to vertex v, then the metric dimension of the graph would remain the same. The reason for this is that even though the pendant edge introduces a new connection to vertex v, it does not alter the existing relationships between the vertices u and w. As a result, the minimum set of vertices needed to uniquely identify all other vertices in the graph remains same. We consider different classes of graphs including wheel graph, cycle graph and specific graph families with known metric dimension properties. By analysing these families of graphs, we establish conditions under which the metric dimension is preserved when pendant edge is attached sequentially.

I. Metric dimension of graph obtained by wheel graph with SPEE

Let be a graph W_n' obtained by wheel graph with SPEE and vertex set of obtained graph $V({W_n}') =$

$\{v, v_1, v_2, v_3, \dots, v_n\}$ and w be the pendant vertex linked to vertex $v_q \in V(W_n)$; $1 \le q \le n$. Several distinct cases arise while evaluating the metric dimension of the derived graph, as presented below:

Case I: For $n = 4$ and 5, Consider the set B=$\{v_2, v_3\} \subseteq V(W_n')$, then the metric representations of derived graph are presented below:

Metric tuples of the apex vertices

$$r(v_q|B) = \begin{cases} (1,2), q = 1 \\ (0,1), q = 2 \\ (1,0), q = 3 \\ (2,1), q = 4 \\ (2,2), q = 5 \end{cases}$$

Metric representation of the pendant vertex w

$$r(w|B) = \begin{cases} (2,3), if\ w\ corresponding\ to\ v_1 \\ (2,0), if\ w\ corresponding\ to\ v_3 \\ (3,2), if\ w\ corresponding\ to\ v_4 \\ (3,3), if\ w\ corresponding\ to\ v_5 \end{cases}$$

Since all metric representations are unique, metric dimension of obtained graph (W_4') and (W_5') is 2.

Case II: For $n = 3$, Consider the set $B = \{v_1, v_2, v_3\} \subseteq V(W_n')$ then the metric representations of derived graph are presented below:

Metric tuples of the apex vertices

$$r\ (v_q|B) = \begin{cases} (0,1,1), q = 1 \\ (1,0,1), q = 2 \\ (1,1,0), q = 3 \end{cases}$$

Metric representation of the pendant vertex w

$$r(w|B) = \begin{cases} (1,2,2), if\ w\ corresponding\ to\ v_1 \\ (2,1,2), if\ w\ corresponding\ to\ v_2 \\ (1,2,2), if\ w\ corresponding\ to\ v_3 \end{cases}$$

The subsequent cases illustrate that assuming $|B| = 2$, the set B fails to function as a resolving set.

- Consider the vertex set $B = \{v, v_q: 1 \le i \le 3\} \subseteq V(W_n')$. In this configuration, it is observed that the metric representation of the vertices $r(v_{((q+1)mod3+1)}|B) = (1,1) = r(v_{q(mod3)+1}|B)$; with respect to B are identical. This contradicts the requirement for a resolving set, where in each vertex must have a unique metric representation. Hence, B cannot be considering a resolving set under the assumption $|B| = 2$.
- Consider a vertex set $B = \{v_i, w: 1 \le i \le 3\} \subseteq V(W_n')$ and w be the pendant vertex corresponding to v_j $(1 \le j \le 3)$. In this setup, the metric representation of the vertices $r(v_k|B) = (1,2) = r(v|B)$ with respect to B is identical. This defines the criteria for a resolving set, which states that each vertex must have a unique metric representation. As a result, B cannot be considered a resolving set under the premise that $|B| = 2$.
- Consider the vertex set $B = \{v_i, v_j: 1 \le i \le 3, 1 \le j \le 3\}$. It is noticed that the metric representation of vertex $r(v_k|B) = (1,1) = r(v|B)$ with respect to B is identical. This goes against the criteria for a resolving set, which states that each vertex must have a distinct metric representation. Assuming that $|B|$=2, B cannot be regarded a resolving set.
- Consider a vertex set $B = \{v, w\} \subseteq V(W_3')$ and w corresponding to v_i $(1 \le i \le 3)$. Here we find that $r(v_j|B) = (1,2) = r(v_k|B)$ which contradict our assumption. Consider the vertex set $B = \{v, w\} \subseteq V(W_3')$ where w corresponds to a rim vertex v_i for some $(1 \le i \le 3)$.In this configuration,

it is observed that the metric representations of vertices $r(v_j|B) = (1,2) = r(v_k|B)$ with respect to B are identical. This contradicts the requirement for a resolving set, which stipulates that each vertex must have a unique metric representation. Therefore, under the assumption $|B|$=2, B cannot be considered a resolving set.

Above consider the vertex set $B = \{v_1, v_2, v_3\} \subseteq V(W_n')$. In this configuration, each vertex in the graph has a unique metric representation with respect to B. Consequently, the metric dimension of graph W_n' is 3.

Case III: For n=6, Consider the set B=$\{v_1, v_3, v_5\} \subseteq V(W_n')$ then the metric representations of derived graph are presented below:
Metric tuples of the apex vertices

$$r(v_q|B) = \begin{cases} (0,2,2), q = 1 \\ (1,1,2), q = 2 \\ (2,0,2), q = 3 \\ (2,1,1), q = 4 \\ (2,2,0), q = 5 \\ (1,2,1), q = 6 \end{cases}$$

Metric representation of the pendant vertex w

$$r(w|B) = \begin{cases} (1,3,3), if\ w\ corresponding\ to\ v_1 \\ (2,2,3), if\ w\ corresponding\ to\ v_2 \\ (3,1,3), if\ w\ corresponding\ to\ v_3 \\ (3,2,2), if\ w\ corresponding\ to\ v_4 \\ (3,3,1), if\ w\ corresponding\ to\ v_5 \\ (2,3,2), if\ w\ corresponding\ to\ v_6 \end{cases}$$

The subsequent cases illustrate that assuming $|B|$ = 2, the set B fails to function as a resolving set.

- Consider the vertex set $B = \{v, v_q: 2 \leq q \leq 5\} \subseteq V(W_6')$. In this configuration, $r(v_{q-1}|B) = (1,1) = r(v_{q+1}|B)$ metric representation is identical. This indicates that B fails to distinguish between these vertices. Furthermore, for $i = 6$, $B = \{v_6, v\}$ obtain $r(v_1|B) = (1,1) = r(v_5|B)$ and for $i = 1$, $B = \{v, v_1\}$ and $r(v_2|B) = (1,1) = r(v_6|B)$. These instances demonstrate that B does not serve as a resolving set under the assumption $|B|$=2, as it fails to uniquely identify all vertices in $W_6{}'$.
- Consider the vertex set $B = \{v, w\} \subseteq V(W_6')$, where w is the pendant vertex corresponding to a rim vertex $v_i, \{2 \leq i \leq 5\}$. In this configuration, the metric representations of vertices $r(v_{i-1}|B) = (1,2) = r(v_{i+1}|B)$ with respect to B are identical. Similarly, for i = 6 notice that $r(v_1|B) = (1,2) = r(v_5|B)$, and for i = 1 and $r(v_6|B) = (1,2) = r(v_2|B)$.These cases demonstrate that B fails to act as a resolving set for $|B|$ = 2.
- Consider the vertex set $B = \{v_I, w\} \subseteq V(W_6')$, w be the pendant vertex corresponding to $\{v_i: 2 \leq i \leq 5\}$ Here, we find that $r(v_{i-1}|B) = (1,2) = r(v_{i+1}|B)$; for $i = 6$, we obtain $r(v_1|B) = (1,2) = r(v_5|B)$ and for $i = 1$, we obtain $r(v_6|B) = (1,2) = r(v_2|B)$ which contradicts our assumption.

Every vertex in the set B=$\{v_1, v_3, v_5\} \subseteq V(W_n')$ possesses a distinct metric representation. Therefore, the metric dimension of the graph obtained by wheel graph W_n' is 3.

Case IV(a): For $n \geq 7$, When $\beta(W_n')$ = odd and w be pendant vertex linked to v_q; $(1 \leq q \leq n)$.

- For $n = 5k + 2$,where $k \geq 1$ and contains $2k + 1$ vertices, since $B = \{v_1, v_4, v_6, v_{10} \dots, v_{k-1}\} \cup \{v_{5k+1}\}$, it serves as a resolving set across all configuration.
- For $n = 5k + 3$, where $k \geq 1$ and contains $2k + 1$ vertices, since $\{v_1, v_4, v_6, v_{10}, \dots, v_{k-2}\} \cup \{v_{5k-4}, v_{5k}, v_{5k+2}\}$ it serves as a resolving set across all configuration.

Case IV(b): For $n \geq 7$, When $\beta(W_n')$ = even and w be the pendant vertex corresponding to v_I; $(1 \leq i \leq n)$.

• For $n = 5k$, where $k \geq 2$, and contains $2k$ vertices and $B = \{v_1, v_4, v_6, v_{10}, \dots, v_{k-1}\}$, it serves as a resolving set across all configuration.

• For $n = 5k + 1$, where $k \geq 2$, and contains $2k$ vertices and $B = \{v_1, v_4, v_6, v_{10}, \dots, v_{k-2}\} \cup \{v_{5k-4}, v_{5k}\}$,it serves as a resolving set across all configuration.

• For $n = 5k + 4$, where $k \geq 1$, and contains $2k + 2$ vertices and $B = \{v_1, v_4, v_6, v_{10}, \dots, v_k\}$, it serves as a resolving set across all configuration.

Hence, $\beta(W_n') \leq \left\lceil \frac{2n+2}{5} \right\rceil$.

Next, show that the metric dimension of derived graph $\beta\ (W_n{}') \geq \left\lceil \frac{2n+2}{5} \right\rceil$.

Let B is a resolving set of W_n' with $|B| \leq \left\lceil \frac{2n+2}{5} \right\rceil$.

Now if $B = \{v_1, v_2, v_3, \dots, v_r\}$; $1 \leq r \leq n$, then

$$r(p|B) = (2,2,2,2,..,2,2) = r(q|B);\ p, q\ \in V(W_n') \text{ if } B = \{v, v_1, v_2, v_3, \dots v_r), \text{ then}$$
$$r(p|B) = (1,2,2,2, \dots ,2) = r(q|B); p, q\ \in V(W_n')\ , \text{ if } B = \{w, v_1, v_2, v_3, \dots, v_r\}, \text{ and}$$

w be the pendant vertex of $v_q (1 \leq q \leq n)$, then

$$r(p|B) = (3,2,2,2, \dots ,2) = r(q|B).$$

It is observed that two vertices share identical metric representations, when $|B| \leq \left\lceil \frac{2n+2}{5} \right\rceil$, So B does not function as a resolving set for the graph W_n'. Therefore,

$$\beta\ (W_n{}') \geq \left\lceil \frac{2n+2}{5} \right\rceil.$$

Hence, we conclude that $\beta\ (W_n{}') = \left\lceil \frac{2n+2}{5} \right\rceil$.

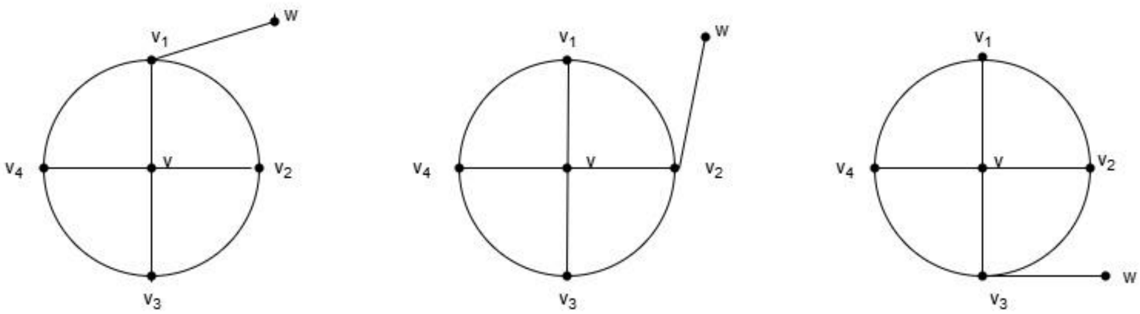

Figure 1: *Wheel graph (W_4) with SPEE*

II. Metric dimension of graph obtained by cycle graph with SPEE

Let be a graph $C_n{}'$ obtained by cycle graph with SPEE and vertex set of obtained graph $V(C_n{}') = \{ v_1, v_2, v_3 \dots \dots v_{n-1}, v_n, v_1\}$ and w be the pendant vertex linked to the vertex v_q. Several distinct cases arise while evaluating the metric dimension of the derived graph, as presented below.

Case I: For $n = 3$ and $B = \{v_1, v_2\} \subseteq V(C_n{}')$ then the metric representations of derived graph are presented below:

Metric representation of the apex vertices

$$r(v_q|B) = \begin{cases} (0,1), q = 1 \\ (1,0), q = 2 \\ (1,1), q = 3 \end{cases}$$

Metric representation of the pendant vertex w

$$r(w|B) = \begin{cases} (1,2), if\ w\ corresponding\ to\ v_1 \\ (2,1), if\ w\ corresponding\ to\ v_2 \\ (2,2), if\ w\ corresponding\ to\ v_3 \end{cases}$$

Since all metric representations of derived graph are unique, then metric dimension of derived graph is 2.

Case II: For $n = 5$ and $B = \{v_3, v_4\} \subseteq V(C_n{}')$ then the metric representations of derived graph are presented below:
Metric representation of the apex vertices

$$r(v_q|B) = \begin{cases} (2,2), if\ q = 1 \\ (1,2), if\ q = 2 \\ (0,1), if\ q = 3 \\ (1,0), if\ q = 4 \\ (2,1), if\ q = 5 \end{cases}$$

Metric *representation* of the pendant vertex w

$$r(w|B) = \begin{cases} (3,3), if\ w\ corresponding\ to\ v_1 \\ (2,3), if\ w\ corresponding\ to\ v_2 \\ (1,2), if\ w\ corresponding\ to\ v_3 \\ (2,1), if\ w\ corresponding\ to\ v_4 \\ (3,2), if\ w\ corresponding\ to\ v_5 \end{cases}$$

Above all metric representations are unique hence the metric dimension of the derived graph is 2. Similarly, w be pendant vertex v_q, then for any value of *n*, *B* be the basis of $C_n{}'$ and contain only two vertex such that $B=\{v_{q+1}, w|1 \leq q \leq n-1\}$ and if $q = n$ then $B = \{v_1, w\}$. Since each vertex has a distinct metric representation, the set B qualifies as a resolving set across all configurations. Hence the metric dimension of graph $C_n{}'$ obtained by cycle graph with sequential pendant edge extension is 2.

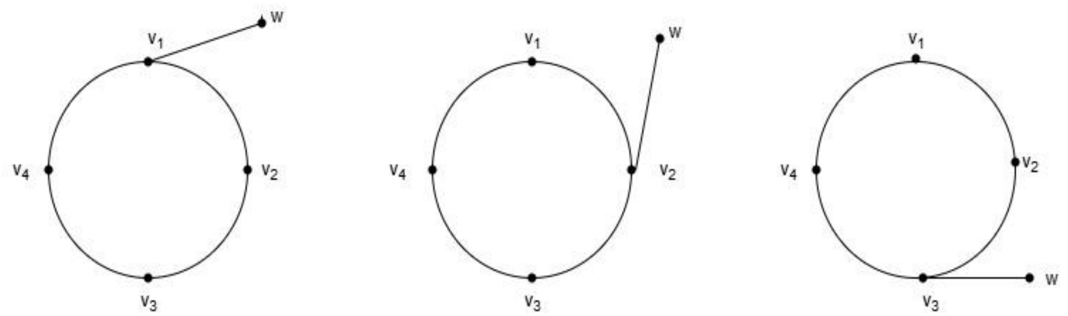

Figure 2: *Cycle Graph* (C_4)*with SPEE*

III. Metric dimension of graph obtained by complete graph with SPEE

Let be a graph $K_n{}'$ obtained by complete graph with SPEE and vertex set of obtained graphs $V(K_n{}') = \{v_1, v_2, v_3, \dots, v_n\}$ and w be the pendant vertex linked to vertex $v_q; 1 \leq q \leq n$. Several distinct cases arise while evaluating the metric dimension of the derived graph, as presented below.

Case I: For *n*=3, $B = \{v_1, v_2\}$ and then the metric representations of derived graph are presented below:

Metric tuple of the vertices relative to reference set B

$$r(v_q|B) = \begin{cases} (0,1), if\ q = 1 \\ (1,0), if\ q = 2 \\ (1,1), if\ q = 3 \end{cases}$$

Metric representation of the pendant vertex w with respect to B

$$r(w|B) = \begin{cases} (1,2,2,2), if\ w\ corresponding\ to\ v_1 \\ (2,1,2,2), if\ w\ corresponding\ to\ v_2 \\ (2,2,1,2), if\ w\ corresponding\ to\ v_3 \\ (2,2,2,1), if\ w\ corresponding\ to\ v_4 \\ (2,2,2,2), if\ w\ corresponding\ to\ v_5 \end{cases}$$

The following cases demonstrate that B is not a resolving set in this situation if we assume $|B| = 2$.

- Consider a vertex set $B = \{v_q, w: 1 \leq i \leq 5\} \subseteq V(K'_n)$ and w be the pendant vertex corresponding to v_q. Here, we find that $r(v_k|B) = (1,2) = r(v_j|B)$ which contradicts our assumptions.
- Consider the vertex set $B = \{v_i, v_j: 1 \leq i \leq 5, 1 \leq j \leq 5)$. Here we find that $r(v_k|B) = (1,1) = r(v_i|B)$ which contradicts our assumptions.
- Consider a vertex set $B = \{v_i, w\} \subseteq V(K'_n)$ and w corresponding to v_j $(1 \leq j \leq 5)$. Here we find that $r(v_k|B) = (1,1) = r(v_i|B)$ which contradicts our assumptions.

The following cases demonstrate that B is not a resolving set in this situation if we assume |B| = 3

- Consider the vertex set $B = \{v_i, v_j, v_k: 1 \leq i \leq 5, 1 \leq j \leq 5, 1 \leq k \leq 5)$. Here we find that r $(v_m|B) = (1,1) = r(v_l|B)$ which contradicts our assumption.
- Consider a vertex set $B = \{v_i, v_j, w: 1 \leq i \leq 5, 1 \leq j \leq 5\} \subseteq V(K'_n)$ and w be the pendant vertex corresponding to v_i. Here, we find that $r(v_k|B) = (1,1,2) = r(v_l|B)$ which contradicts our assumption.
- Consider a vertex set $B = \{v_i, v_j, w: 1 \leq i \leq 5, 1 \leq j \leq 5\} \subseteq V(K'_n)$ and w be the pendant vertex corresponding to v_j. Here, we find that $r(v_k|B) = (1,1,2) = r(v_l|B)$ which contradicts our assumption.
- Consider a vertex set $B = \{v_i, v_j, w: 1 \leq i \leq 5, 1 \leq j \leq 5\} \subseteq V(K'_n)$ and w be the pendant vertex corresponding to $v_k (1 \leq k \leq 5)$. Here, we find that $r(v_m|B) = (1,1,2) = r(v_l|B)$ which contradicts our assumption.

Hence the metric dimension of $V(K'_5) = 4$.

Case III: For $n \geq 6$, $V(K'_n) = \{v_1, v_2, v_3, v_4, \ldots, v_l, \ldots, v_n\}$ and $B = \{v_1, v_2, v_3, v_4, v_5 \ldots, v_{n-1}\}$ then metric representation of $V(K'_n)$ with respect to B.

Metric representation of the vertices

$$r(v_q|B) = \begin{cases} (0,1,1,1 \ldots,1), if\ q = 1 \\ (1,1,0,1 \ldots,1), if\ q = 3 \\ \vdots \\ (1,1,1,1 \ldots,0), if\ q = n-1 \\ (1,1,1,1, \ldots,1), if\ q = n \end{cases}$$

Metric representation of the pendant vertex w

$$r(w|B) = \begin{cases} (1,2,2,2,\dots,2), if\ w\ corresponding\ to\ v_1 \\ (2,1,2,2,\dots,2), if\ w\ corresponding\ to\ v_2 \\ (2,2,1,2,\dots,2), if\ w\ corresponding\ to\ v_3 \\ \vdots \\ (2,2,2,2,\dots,1), if\ w\ corresponding\ to\ v_{n-1} \\ (2,2,2,2,\dots,2), if\ w\ corresponding\ to\ v_n \end{cases}$$

Above all metric representations are unique. Hence the metric dimension of graph K_n' obtained by complete graph with sequential pendant edge extension is $n-1$.

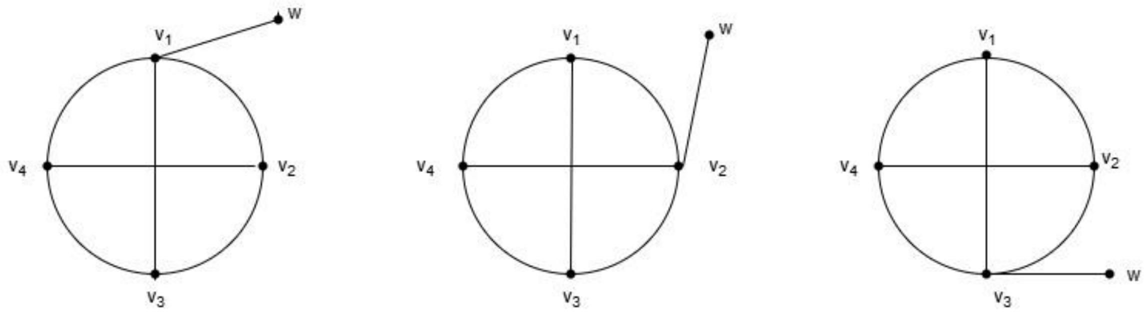

Figure 3: *Complete Graph (K_4) with SPEE*

IV. Results

Table 1: *Metric dimension of graph under SPEE*

Graph type	Number of vertices	Metric dimension $\beta(G)$
Wheel graph	$n \geq 7$	$\left\lceil \frac{2n+2}{5} \right\rceil$
Cycle graph	$n \geq 2$	2
Complete graph	$n \geq 2$	n

V. Discussion

When pendant edges are attached externally and sequentially to a graph, the existing shortest-path distances between its original vertices remain unchanged. In the case of a wheel graph, the central hub continues to serve as the unique reference point for distance measurements: no matter how many pendant edges are appended to the rim, the hub's distances to all other vertices are unaffected. Consequently, the metric dimension of the wheel graph remains constant.

Similarly, a complete graph inherently has every pair of vertices at distance 1. Adding pendant edges to any vertex introduces new vertices at distance 1 or 2 from the core, but these additions do not disturb the uniform distance structure among the original vertices. Hence, the minimal resolving set for the complete graph continues to suffice, and its metric dimension is preserved.

For a cycle graph, two diametrically opposite vertices form a resolving set because their distance profiles uniquely label every vertex. Sequentially attaching pendant edges does not alter the distances between these two key vertices—or between either key vertex and any other original cycle vertex. As a result, the cycle's metric dimension remains invariant under these modifications.

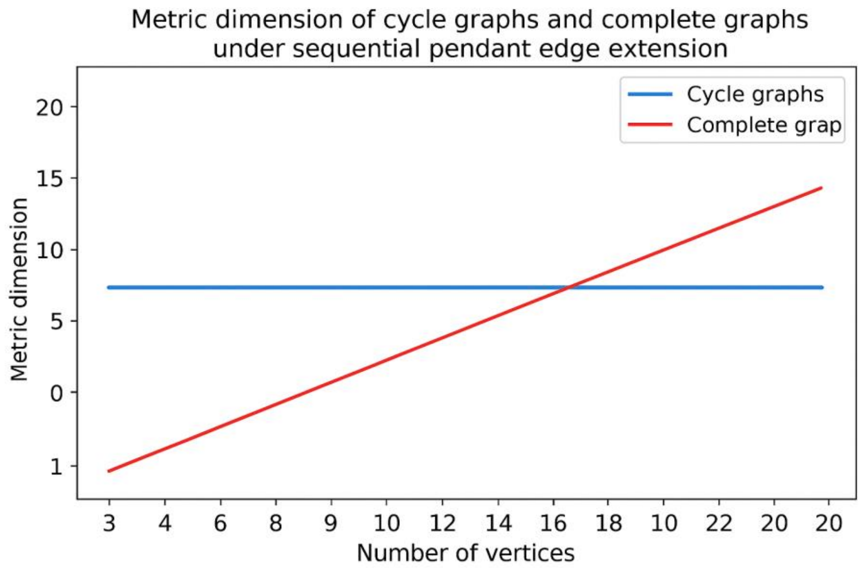

Figure 4: *Metric dimension of cycle and complete graph with SPEE*

VI. Conclusion

Across wheel graphs, complete graphs, and cycle graphs, the sequential external attachment of pendant edges does not affect the distances among the original vertices or the effectiveness of their minimal resolving sets. Therefore, the metric dimension of each of these graph families remains unchanged despite pendant-edge extensions. This stability underscores the robustness of metric dimension as a graph invariant, ensuring consistent vertex identification even under systematic structural augmentations.

I. Future scope

There is considerable potential to extend this study beyond the classical families of wheel, complete, and cycle graphs. One avenue is to investigate more complex or irregular topologies—such as bipartite graphs, trees with high branching factors, cactus graphs, and scale-free or small-world networks—to determine whether metric-dimension invariance under sequential pendant-edge addition persists. Analogous analyses could explore the effect of attaching multiple pendant edges at once or of performing other systematic modifications (for example, subdividing edges or adding chords) to see how these operations compare in their impact on resolving sets.

Another promising direction is to incorporate richer edge-weight or directionality models, reflecting real-world networks where distances are asymmetric or cost-driven. Developing efficient, incremental algorithms that maintain minimal resolving sets in large, evolving networks—possibly under random or adversarial growth processes—would have practical value for dynamic localization, network monitoring, and sensor placement. Finally, forging connections between metric dimension stability and other graph invariants (such as domination numbers, zero-forcing parameters, or rigidity measures) could yield a unified theoretical framework for predicting how structural augmentations influence multiple properties of a network simultaneously

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