

A COMPREHENSIVE EXPLORATION OF QUASI RAMA DISTRIBUTION WITH SIGNIFICANT STATISTICAL PROPERTIES AND APPLICATIONS

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Abstract

In this study, we propose a novel extension of the quasi Rama distribution referred to as the length-biased quasi Rama (LBQR) distribution, developed by applying the length-biased technique to the classical quasi Rama distribution. The proposed distribution is thoroughly investigated and several of its important statistical properties are derived including moments, harmonic mean, order statistics, reliability function, hazard rate function, reverse hazard rate function, moment generating function, characteristic function, Renyi entropy as well as Bonferroni and Lorenz curves. Furthermore, the parameters of the new distribution are estimated using the method of maximum likelihood estimation. To evaluate the practical relevance and predictive performance of the length biased quasi Rama distribution, it is applied to two real lifetime data sets, demonstrating its applicability and effectiveness in modeling real-world lifetime phenomena.

Keywords: Length biased distribution, quasi Rama distribution, reliability analysis, entropy

I. Introduction

In distribution theory, the adaptation of diversified models to classical distributions has gained remarkable momentum largely because classical distributions often fail to provide an adequate fit for complex real-world data. By incorporating an additional parameters or modifying existing structures, these enhanced models offer a better representation of data variability and complexity. In such situation, a compelling case arises to introduce an additional parameter to existing baseline distribution and this additional parameter improves its performance and brings flexibility in their behavior. This additional parameter creates a new understanding of existing classical distribution

and this additional parameter is approached through reputed weighted technique. Fisher [4] studied this concept to examine how the ascertainment biases can affect the form of distribution of recorded observations which reformulated by Rao [14] in a unifying theory were the observations fall in non-experimental, non-replicated and non-random manner. Weighted distributions are suitable in situation of unequal probability sampling like actuarial sciences, biomedicine, biostatistics, ecology and survival data analysis. This idea has been applied as a device for collection of suitable model for observed data and these distributions are applicable only if the observations are recorded without any repetition, experiment and random process. When weight function depends on length of units of interest, resulting distribution is termed as length biased distribution. The idea of length biased distribution was introduced by Cox [3] in renewal theory and it has been found useful in biometry, wildlife studies, family history, intermediate events and analysis of lifetime data. When observations are selected with probability proportional to their length resulting distribution is termed as length biased. Length biased distribution is a particular case of more general form known as weighted distribution and therefore length biased refers to sample data where probability of recording an observation depends upon magnitude of observations. Length biased sampling situation occurs because sometimes it is impossible to work with truly random sample from population of interest. Thus length biased distribution has also been applied in development of proper models for lifetime data.

Many authors have contributed significantly to develop length biased probability distributions along with their applications. Nanuwong and Bodhisuwan [12] constructed the length biased Beta-Pareto distribution with properties and its application. Klinjan and Aryuyuen [7] proposed the length-biased power Garima distribution with application. Ganaie and Rajagopalan [6] studied the length biased power quasi Lindley distribution with properties and applications. Pushkarna and Mustafa [13] derived length biased weighted Ishita distribution with real life data sets. Saghir and Khadim [16] obtained mathematical properties of length biased weighted Maxwell distribution. Al-omari et al. [1] derived the power length biased Suja distribution with properties and application. Mudasir and Ahmad [11] discussed structural properties of length biased Nakagami distribution. Mustafa and Khan [10] presented length biased powered inverse Rayleigh distribution. Beghriche and Zeghdoudi [2] obtained size biased gamma Lindley distribution. Sharma et al. [19] described length and area biased Maxwell distributions. Ganaie and Rajagopalan [5] derived the length biased two parameter Pranav distribution with characterizations and applications. Salama et al. [17] discussed length biased weighted exponentiated inverted exponential distribution with properties and estimation.

A quasi Rama distribution is a recently executed new distribution introduced by Shanker et al. [18] which is a particular case of Rama distribution. The proposed quasi Rama distribution is a two parametric distribution and some of its statistical properties based upon moments, mean residual life function, hazard rate function, stochastic ordering and mean deviation have been described. Further its parameters are estimated by method of moments and maximum likelihood estimation.

II. Length Biased Quasi Rama (LBQR) Distribution

The quasi Rama distribution with its probability density function given by

$$f(x; \theta, \alpha) = \frac{\theta^3}{(\alpha\theta^2 + 6)} (\alpha + \theta x^3) e^{-\theta x} ; x > 0, \theta > 0, \alpha > 0 \quad (1)$$

and quasi Rama distribution with its cumulative distribution function given by

$$F(x; \theta, \alpha) = 1 - \left(1 + \frac{\left(\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x \right)}{\left(\alpha\theta^2 + 6 \right)} \right) e^{-\theta x}; x > 0, \theta > 0, \alpha > 0 \quad (2)$$

The random variable X under non-negative circumstances consists the probability density function $f(x)$ and its weight function be $w(x)$ that is non-negative, then probability density function of weighted random variable X_w will be defined as

$$f_w(x) = \frac{w(x) f(x)}{E(w(x))}, x > 0$$

Here in this article, we have to acquire the length biased model of quasi Rama distribution, since in weights if $w(x) = x$ the performed distribution is referred to as length biased distribution and its probability density function is defined as

$$f_1(x; \theta, \alpha) = \frac{x f(x; \theta, \alpha)}{E(x)} \quad (3)$$

Now $E(x) = \int_0^{\infty} x f(x; \theta, \alpha) dx$

$$E(x) = \frac{\left(\alpha\theta^2 + 24 \right)}{\theta \left(\alpha\theta^2 + 6 \right)} \quad (4)$$

After substituting the equations (1) and (4) in equation (3), we will get probability density function of length biased quasi Rama distribution as

$$f_1(x; \theta, \alpha) = \frac{x\theta^4}{\left(\alpha\theta^2 + 24 \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} \quad (5)$$

Now the cumulative distribution function of length biased quasi Rama distribution will be obtained as

$$\begin{aligned} F_1(x; \theta, \alpha) &= \int_0^x f_1(x; \theta, \alpha) dx \\ &= \int_0^x \frac{x\theta^4}{\left(\alpha\theta^2 + 24 \right)} \left(\alpha + \theta x^3 \right) e^{-\theta x} dx \\ &= \frac{1}{\left(\alpha\theta^2 + 24 \right)} \left(\alpha\theta^4 \int_0^x x e^{-\theta x} dx + \theta^5 \int_0^x x^4 e^{-\theta x} dx \right) \end{aligned} \quad (6)$$

After simplification of expression (6), we determine the cumulative distribution function of length biased quasi Rama distribution as

$$F_1(x; \theta, \alpha) = \frac{1}{\left(\alpha\theta^2 + 24 \right)} \left(\alpha\theta^2 \gamma(2, \theta x) + \gamma(5, \theta x) \right) \quad (7)$$

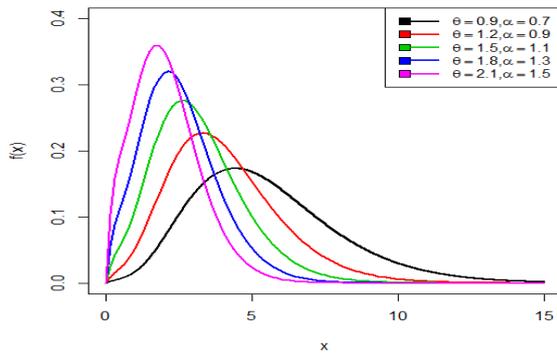


Figure 1: Pdf plot of LBQR distribution

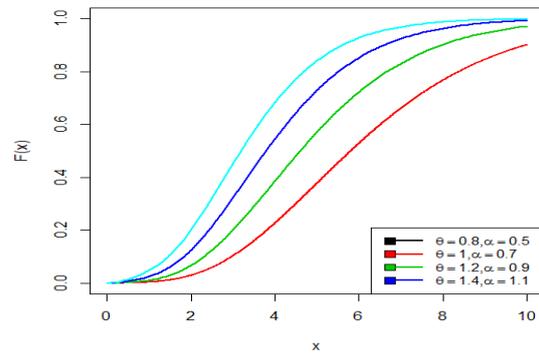


Figure 2: Cdf plot of LBQR distribution

III. Reliability Analysis

In this portion, some fundamental aspects via reliability function, hazard function, reverse hazard function and mills ratio of length biased quasi Rama distribution has been derived. The reliability function of length biased quasi Rama distribution is given by

$$R(x) = 1 - F_1(x; \theta, \alpha)$$

$$R(x) = 1 - \frac{1}{(\alpha \theta^2 + 24)} \left(\alpha \theta^2 \gamma(2, \theta x) + \gamma(5, \theta x) \right)$$

The hazard function of performed length biased quasi Rama distribution is given by

$$h(x) = \frac{f_1(x; \theta, \alpha)}{1 - F_1(x; \theta, \alpha)}$$

$$h(x) = \frac{x \theta^4 (\alpha + \theta x^3) e^{-\theta x}}{\left(\alpha \theta^2 + 24 \right) - \left(\alpha \theta^2 \gamma(2, \theta x) + \gamma(5, \theta x) \right)}$$

The reverse hazard function of length biased quasi Rama distribution is given by

$$h_r(x) = \frac{f_1(x; \theta, \alpha)}{F_1(x; \theta, \alpha)}$$

$$h_r(x) = \frac{x \theta^4 (\alpha + \theta x^3) e^{-\theta x}}{\left(\alpha \theta^2 \gamma(2, \theta x) + \gamma(5, \theta x) \right)}$$

The Mills Ratio of length biased quasi Rama distribution is given by

$$M.R = \frac{1}{h_r(x)} = \frac{\left(\alpha \theta^2 \gamma(2, \theta x) + \gamma(5, \theta x) \right)}{x \theta^4 (\alpha + \theta x^3) e^{-\theta x}}$$

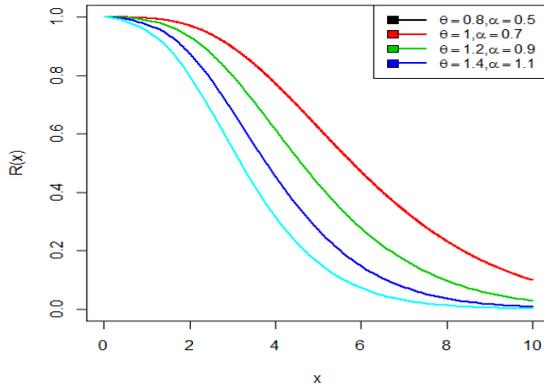


Figure 3: Reliability plot of LBQR distribution

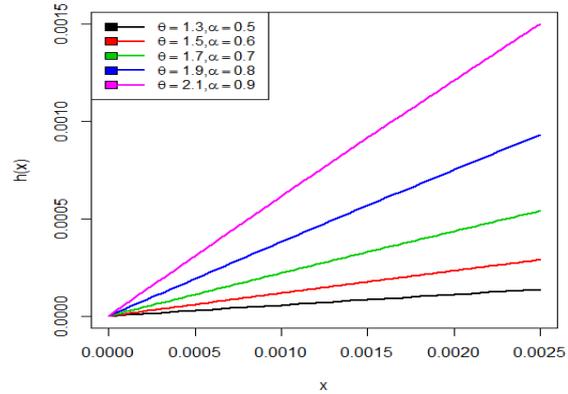


Figure 4: Hazard plot of LBQR distribution

IV. Order Statistics

Order statistics is a comprehensive tool in inference and non-parametric statistics which plays a crucial role in arranging the independent and identically distributed random variables. Suppose the order statistics $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ of a random sample X_1, X_2, \dots, X_n from a continuous distribution with its probability density function $f_x(x)$ and cumulative distribution function $F_x(x)$, then probability density function of r th order statistics $X_{(r)}$ is defined as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_x(x) (F_x(x))^{r-1} (1 - F_x(x))^{n-r} \quad (8)$$

Now substituting the equations (5) and (7) in equation (8), we will get probability density function of r th order statistics $X_{(r)}$ of length biased quasi Rama distribution as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{x\theta^4}{(\alpha\theta^2 + 24)} (\alpha + \theta x^3) e^{-\theta x} \right) \left(\frac{1}{(\alpha\theta^2 + 24)} (\alpha\theta^2 \gamma(2, \theta x) + \gamma(5, \theta x)) \right)^{r-1} \times \left(1 - \frac{1}{(\alpha\theta^2 + 24)} (\alpha\theta^2 \gamma(2, \theta x) + \gamma(5, \theta x)) \right)^{n-r}$$

Hence for $r = n$, we have acquired the probability density function of higher order statistic $X_{(n)}$ of length biased quasi Rama distribution as

$$f_{x(n)}(x) = \frac{n x \theta^4}{(\alpha\theta^2 + 24)} (\alpha + \theta x^3) e^{-\theta x} \left(\frac{1}{(\alpha\theta^2 + 24)} (\alpha\theta^2 \gamma(2, \theta x) + \gamma(5, \theta x)) \right)^{n-1}$$

and for $r = 1$, we have acquired the probability density function of first order statistic $X_{(1)}$ of length biased quasi Rama distribution as

$$f_{x(1)}(x) = \frac{n x \theta^4}{(\alpha \theta^2 + 24)} (\alpha + \theta x^3) e^{-\theta x} \left(1 - \frac{1}{(\alpha \theta^2 + 24)} (\alpha \theta^2 \gamma(2, \theta x) + \gamma(5, \theta x)) \right)^{n-1}$$

V. Statistical Measures

In this portion, we will have derived some fundamental properties like moments, harmonic mean, moment generating and characteristic function of length biased quasi Rama distribution those are

I. Moments

Let X be the random variable consists length biased quasi Rama distribution, then r^{th} order moment $E(X^r)$ of performed new distribution will be derived by using the expression as

$$\begin{aligned} \mu_r' &= E(X^r) = \int_0^{\infty} x^r f_1(x; \theta, \alpha) dx \\ &= \int_0^{\infty} x^r \frac{x \theta^4}{(\alpha \theta^2 + 24)} (\alpha + \theta x^3) e^{-\theta x} dx \\ &= \frac{\theta^4}{(\alpha \theta^2 + 24)} \left(\alpha \int_0^{\infty} x^{(r+2)-1} e^{-\theta x} dx + \theta \int_0^{\infty} x^{(r+5)-1} e^{-\theta x} dx \right) \end{aligned} \quad (9)$$

After the simplification of expression (9), we get

$$\mu_r' = E(X^r) = \frac{(\alpha \theta^2 \Gamma(r+2) + \Gamma(r+5))}{\theta^r (\alpha \theta^2 + 24)} \quad (10)$$

Hence by applying $r = 1$ in equation (10), we have obtained the mean of length biased quasi Rama distribution which is

$$\mu_1' = \frac{(2\alpha \theta^2 + 120)}{\theta (\alpha \theta^2 + 24)}$$

Now by applying $r = 2, 3$ and 4 in equation (10), we have obtained other three moments of length biased quasi Rama distribution as

$$\begin{aligned} \mu_2' &= \frac{(6\alpha \theta^2 + 720)}{\theta^2 (\alpha \theta^2 + 24)} \\ \mu_3' &= \frac{(24\alpha \theta^2 + 5040)}{\theta^3 (\alpha \theta^2 + 24)} \end{aligned}$$

$$\begin{aligned} \mu_4' &= \frac{\left(120\alpha\theta^2 + 40320\right)}{\theta^4\left(\alpha\theta^2 + 24\right)} \\ \text{Variance} &= \frac{\left(2\alpha^2\theta^4 + 384\alpha\theta^2 + 2880\right)}{\theta^2\left(\alpha\theta^2 + 24\right)^2} \\ \text{S.D}(\sigma) &= \frac{\sqrt{\left(2\alpha^2\theta^4 + 384\alpha\theta^2 + 2880\right)}}{\theta\left(\alpha\theta^2 + 24\right)} \\ \text{C.V} &= \frac{\sqrt{\left(2\alpha^2\theta^4 + 384\alpha\theta^2 + 2880\right)}}{\left(2\alpha\theta^2 + 120\right)} \end{aligned}$$

II. Harmonic mean

The harmonic mean of explored distribution is obtained by applying the expression as

$$\begin{aligned} \text{H.M} &= E\left(\frac{1}{x}\right) = \int_0^{\infty} \frac{1}{x} f_1(x; \theta, \alpha) dx \\ &= \int_0^{\infty} \frac{\theta^4}{\left(\alpha\theta^2 + 24\right)} \left(\alpha + \theta x^3\right) e^{-\theta x} dx \\ &= \frac{\theta^4}{\left(\alpha\theta^2 + 24\right)} \int_0^{\infty} \left(\alpha + \theta x^3\right) e^{-\theta x} dx \end{aligned} \tag{11}$$

After the simplification of expression (11), we get

$$\text{H.M} = \frac{\theta(\alpha\theta + 6)}{\left(\alpha\theta^2 + 24\right)}$$

III. Moment generating function and characteristic function

Suppose X be the random variable which constitutes length biased quasi Rama distribution, then the MGF of performed distribution is obtained by applying the expression as

$$\begin{aligned} M_X(t) &= E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f_1(x; \theta, \alpha) dx \\ &= \int_0^{\infty} e^{tx} \frac{x\theta^4}{\left(\alpha\theta^2 + 24\right)} \left(\alpha + \theta x^3\right) e^{-\theta x} dx \\ &= \frac{\theta^4}{\left(\alpha\theta^2 + 24\right)} \int_0^{\infty} x \left(\alpha + \theta x^3\right) e^{-(\theta-t)x} dx \end{aligned} \tag{12}$$

After the simplification of expression (12), we get

$$M_X(t) = \frac{\theta^4}{(\theta - t)^5 (\alpha \theta^2 + 24)} \left(\alpha(\theta - t)^3 + 24\theta \right)$$

Clearly, the characteristic function of proposed distribution is given by

$$M_X(it) = \frac{\theta^4}{(\theta - it)^5 (\alpha \theta^2 + 24)} \left(\alpha(\theta - it)^3 + 24\theta \right)$$

VI. Entropy

The entropy is a fundamental term which is used in diverse fields especially to measure the amount of uncertainty or randomness present in the dataset.

I. Renyi Entropy

The Renyi entropy was originally developed in information theory which generalizes several forms of entropy and therefore Renyi entropy forms the basis of the concept of generalized dimensions in fractal dimension estimation. The Renyi entropy should be expressed as

$$\begin{aligned} T_R(\beta) &= \frac{1}{1-\beta} \log \left(\int_1^\beta f_1^\beta(x; \theta, \alpha) dx \right) \\ &= \frac{1}{1-\beta} \log \left(\left(\frac{\theta^4}{(\alpha \theta^2 + 24)} \right)^\beta \int_0^\infty x^\beta (\alpha + \theta x^3)^\beta e^{-\theta \beta x} dx \right) \end{aligned} \quad (13)$$

After the simplification of expression (13), we get

$$T_R(\beta) = \frac{1}{1-\beta} \log \left(\left(\frac{\theta^{3\beta - 3k - 1}}{(\alpha \theta^2 + 24)^\beta} \right)_{k=0}^\infty \sum_{k=0}^{\infty} \binom{\beta}{k} \alpha^{\beta-k} \theta^k \frac{\Gamma(\beta + 3k + 1)}{\beta^{\beta + 3k + 1}} \right)$$

Particularly, the Tsallis entropy of proposed LBQR distribution will be obtained by applying the expression

$$\begin{aligned} T_s(\zeta) &= \frac{1}{\zeta - 1} \left(1 - \int_0^1 f_1^\zeta(x; \theta, \alpha) dx \right) \\ &= \frac{1}{\zeta - 1} \left(1 - \left(\frac{\theta^4}{(\alpha \theta^2 + 24)} \right)^\zeta \int_0^\infty x^\zeta (\alpha + \theta x^3)^\zeta e^{-\theta \zeta x} dx \right) \end{aligned} \quad (14)$$

After the simplification of expression (14), we get

$$T_s(\zeta) = \frac{1}{\zeta - 1} \left(1 - \left(\frac{\theta^{3\zeta - 3j - 1}}{(\alpha\theta^2 + 24)^\zeta} \right) \sum_{j=0}^{\infty} \binom{\zeta}{j} \alpha^{\zeta - j} \theta^j \frac{\Gamma(\zeta + 3j + 1)}{\zeta^{\zeta + 3j + 1}} \right)$$

VII. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are one of comprehensive tool which has diverse applications to depict graphically the distribution of income or wealth. The bonferroni and Lorenz curves should be defined as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x f_1(x; \theta, \alpha) dx$$

$$\text{and } L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x f_1(x; \theta, \alpha) dx$$

$$\text{Here } \mu_1' = \frac{(2\alpha\theta^2 + 120)}{\theta(\alpha\theta^2 + 24)}$$

$$\begin{aligned} B(p) &= \frac{\theta(\alpha\theta^2 + 24)}{p(2\alpha\theta^2 + 120)} \int_0^q x \frac{x\theta^4}{(\alpha\theta^2 + 24)} (\alpha + \theta x^3) e^{-\theta x} dx \\ &= \frac{\theta(\alpha\theta^2 + 24)}{p(2\alpha\theta^2 + 120)} \int_0^q \frac{x^2\theta^4}{(\alpha\theta^2 + 24)} (\alpha + \theta x^3) e^{-\theta x} dx \end{aligned} \tag{15}$$

After the simplification of expression (15), we get

$$B(p) = \frac{1}{p(2\alpha\theta^2 + 120)} (\alpha\theta^2\gamma(3, \theta q) + \gamma(6, \theta q))$$

$$L(p) = \frac{1}{(2\alpha\theta^2 + 120)} (\alpha\theta^2\gamma(3, \theta q) + \gamma(6, \theta q))$$

VIII. Maximum Likelihood Estimation and Fisher's Information Matrix

In this portion, we have applied the technique of maximum likelihood estimation to estimate the parameters of length biased quasi Rama distribution. Suppose X_1, X_2, \dots, X_n be random sample of size n from the length biased quasi Rama distribution, then likelihood function should be derived as

$$L(x) = \prod_{i=1}^n f_1(x; \theta, \alpha)$$

$$\begin{aligned}
 &= \prod_{i=1}^n \left(\frac{x_i \theta^4}{(\alpha \theta^2 + 24)} \left(\alpha + \theta x_i^3 \right) e^{-\theta x_i} \right) \\
 &= \frac{\theta^{4n}}{(\alpha \theta^2 + 24)^n} \prod_{i=1}^n \left(x_i \left(\alpha + \theta x_i^3 \right) e^{-\theta x_i} \right)
 \end{aligned}$$

The log likelihood function should be expressed as

$$\log L = 4n \log \theta - n \log(\alpha \theta^2 + 24) + \sum_{i=1}^n \log x_i + \sum_{i=1}^n \log(\alpha + \theta x_i^3) - \theta \sum_{i=1}^n x_i \quad (16)$$

Hence differentiating the log likelihood expression (16) with respect to parameters θ and α . The given below normal equations should be satisfied

$$\begin{aligned}
 \frac{\partial \log L}{\partial \theta} &= \frac{4n}{\theta} - n \left(\frac{2\alpha\theta}{(\alpha \theta^2 + 24)} \right) + \sum_{i=1}^n \left(\frac{x_i^3}{(\alpha + \theta x_i^3)} \right) - \sum_{i=1}^n x_i = 0 \\
 \frac{\partial \log L}{\partial \alpha} &= -n \left(\frac{\theta^2}{(\alpha \theta^2 + 24)} \right) + \sum_{i=1}^n \left(\frac{1}{(\alpha + \theta x_i^3)} \right) = 0
 \end{aligned}$$

Therefore the above system of non-linear equations are intricated to resolve algebraically. So it is better to apply numerical technique like Newton-Raphson method for estimating the required parameters.

To get the confidence interval hence we apply the asymptotic normality results. It should be noticed if $\hat{\omega} = (\hat{\theta}, \hat{\alpha})$ consists the MLE of $\omega = (\theta, \alpha)$. The conclusion should be observed as

$$\sqrt{n}(\hat{\omega} - \omega) \rightarrow N_2(0, I^{-1}(\omega))$$

Hence $I^{-1}(\omega)$ is Fisher's information matrix.

$$I(\omega) = -\frac{1}{n} \begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) \end{pmatrix}$$

Here we state that

$$\begin{aligned}
 E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) &= -\frac{4n}{\theta^2} - n \left(\frac{(\alpha \theta^2 + 24)(2\alpha) - (2\alpha\theta)^2}{(\alpha \theta^2 + 24)^2} \right) - \sum_{i=1}^n \left(\frac{(x_i^3)^2}{(\alpha + \theta x_i^3)^2} \right) \\
 E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) &= n \left(\frac{(\theta^2)^2}{(\alpha \theta^2 + 24)^2} \right) - \sum_{i=1}^n \left(\frac{1}{(\alpha + \theta x_i^3)^2} \right)
 \end{aligned}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = -n \left(\frac{\left(\alpha \theta^2 + 24 \right) (2\theta) - (2\alpha \theta) \left(\theta^2 \right)}{\left(\alpha \theta^2 + 24 \right)^2} \right) - \sum_{i=1}^n \left(\frac{\left(x_i^3 \right)}{\left(\alpha + \theta x_i^3 \right)^2} \right)$$

Therefore ω is not known, hence $I^{-1}(\omega)$ is estimated by $I^{-1}(\hat{\omega})$ and since it should be applied to attain asymptotic confidence interval for θ and α .

IX. Application

In this portion, we have scrutinized and studied the two real lifetime data sets to investigate the performance and goodness of fit of length biased quasi Rama distribution and then fit of length biased quasi Rama distribution has been compared in order to reveal that length biased quasi Rama distribution provides a better fit in comparison over quasi Rama, Rama, Pratibha, Komal, Uma and Lindley distributions. The two real lifetime data sets are given as

Data set 1: The first given data set from civil engineering reported by Kotz and van Dorp [8] which consists of 85 hailing times observations and the observations of real lifetime data set is given below as

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80

Data set 2: The second given data set was first reported by Mahmoud and Mandouh [9] which represents the uncensored strengths of glass fibres and the observations of real data set is shown under as

1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.91, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, 4.121

The model comparison criterions along with unknown parameters are estimated thoroughly by applying the technique of R software. To examine significance and capability of length biased quasi Rama distribution as compared over quasi Rama, Rama, Pratibha, Komal, Uma and Lindley distributions, the criterions like *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected), *CAIC* (Consistent Akaike Information Criterion), Shannon's entropy $H(X)$ and $-2\log L$ have been used. The distribution performs quite better even if it has lesser criterions of *AIC*, *BIC*, *AICC*, *CAIC*, $H(X)$ and $-2\log L$ in comparison over other well known distributions. The following formulas given are used for determining the criterion values

$$AIC = 2k - 2 \log L, \quad BIC = k \log n - 2 \log L, \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$CAIC = -2 \log L + \frac{2kn}{n-k-1} \quad \text{and} \quad H(X) = -\frac{2 \log L}{n}$$

Therefore k is number of parameters in the statistical model, n is the sample size and $-2\log L$ is the maximized value of log-likelihood function under considered model.

Table 1: Shows MLE and S.E of following data sets of Performed Distributions

| Data set 1 | | | Data set 2 | | |
|--------------|--|--|--------------|--|--|
| Distribution | MLE | S.E | Distribution | MLE | S.E |
| LBQR | $\hat{\alpha} = 0.001000$ $\hat{\theta} = 0.878667$ | $\hat{\alpha} = 0.000010$ $\hat{\theta} = 0.029170$ | LBQR | $\hat{\alpha} = 0.00100$ $\hat{\theta} = 3.09420$ | $\hat{\alpha} = 0.000100$ $\hat{\theta} = 0.11986$ |
| Quasi Rama | $\hat{\alpha} = 0.0010000$ $\hat{\theta} = 0.7029134$ | $\hat{\alpha} = 0.0001000$ $\hat{\theta} = 0.0259567$ | Quasi Rama | $\hat{\alpha} = 0.00100$ $\hat{\theta} = 2.47446$ | $\hat{\alpha} = 0.000010$ $\hat{\theta} = 0.106260$ |
| Rama | $\hat{\theta} = 0.677018$ | $\hat{\theta} = 0.035522$ | Rama | $\hat{\theta} = 1.666889$ | $\hat{\theta} = 0.094815$ |
| Pratibha | $\hat{\theta} = 0.475361$ | $\hat{\theta} = 0.029030$ | Pratibha | $\hat{\theta} = 1.200580$ | $\hat{\theta} = 0.081031$ |
| Komal | $\hat{\theta} = 0.298773$ | $\hat{\theta} = 0.022844$ | Komal | $\hat{\theta} = 0.842362$ | $\hat{\theta} = 0.078397$ |
| Uma | $\hat{\theta} = 0.657943$ | $\hat{\theta} = 0.034344$ | Uma | $\hat{\theta} = 1.61434$ | $\hat{\theta} = 0.09652$ |
| Lindley | $\hat{\theta} = 0.3098973$ | $\hat{\theta} = 0.0241075$ | Lindley | $\hat{\theta} = 0.938282$ | $\hat{\theta} = 0.088962$ |

Table 2: Shows Comparison, Significance and Performance of fitted Distributions

| Data sets | Distribution | -2logL | AIC | BIC | AICC | CAIC | H(X) |
|-----------|--------------|----------|----------|----------|----------|----------|--------|
| 1 | LBQR | 332.4848 | 336.4848 | 341.3701 | 336.6311 | 336.6311 | 3.9115 |
| | Quasi Rama | 348.5044 | 352.5044 | 357.3897 | 352.6507 | 352.6507 | 4.1000 |
| | Rama | 356.3218 | 358.3218 | 360.7644 | 358.3699 | 358.3699 | 4.1920 |
| | Pratibha | 385.6311 | 387.6311 | 390.0737 | 387.6792 | 387.6792 | 4.5368 |
| | Komal | 427.3503 | 429.3503 | 431.7929 | 429.3984 | 429.3984 | 5.0276 |
| | Uma | 362.0973 | 364.0973 | 366.54 | 364.1454 | 364.1454 | 4.2599 |
| | Lindley | 423.4846 | 425.4846 | 427.9272 | 425.5327 | 425.5327 | 4.9821 |
| 2 | LBQR | 93.02548 | 97.02548 | 101.3117 | 97.2254 | 97.2254 | 1.4765 |
| | Quasi Rama | 103.6676 | 107.6676 | 111.9539 | 107.8676 | 107.8676 | 1.6455 |
| | Rama | 176.7793 | 178.7793 | 180.9224 | 178.8448 | 178.8448 | 2.8060 |
| | Pratibha | 166.1255 | 168.1255 | 170.2686 | 168.1910 | 168.1910 | 2.6369 |
| | Komal | 177.4134 | 179.4134 | 181.5565 | 179.4789 | 179.4789 | 2.8160 |
| | Uma | 170.1836 | 172.1836 | 174.3268 | 172.2491 | 172.2491 | 2.7013 |
| | Lindley | 170.9519 | 172.9519 | 175.095 | 173.0174 | 173.0174 | 2.7135 |

Obviously, it is clearly observed and revealed from the result given above in table 2 that the length biased quasi Rama distribution has lesser criterion values of AIC , BIC , $AICC$, $CAIC$, $H(X)$ and $-2\log L$ as compared over quasi Rama, Rama, Pratibha, Komal, Uma and Lindley distributions which consists that the length biased quasi Rama distribution gives a best fit in comparison over quasi Rama, Rama, Pratibha, Komal, Uma and Lindley distributions.

X. Conclusion

In this manuscript, we have approached a novel version of quasi Rama distribution termed as length biased quasi Rama distribution has been formulated. The explored new distribution is introduced by applying the length biased technique to baseline quasi Rama distribution and its several structural properties such as moments, shape of pdf and cdf, order statistics, reliability function, hazard rate function, reverse hazard rate function, moment generating and characteristic function, bonferroni and Lorenz curves, Renyi entropy and Tsallis entropy has been described. Additionally, the parameters of performed distribution are estimated by using technique of maximum likelihood estimation. Furthermore, the flexibility and superiority of proposed length biased quasi Rama distribution has been analyzed by applying the two real data sets and therefore it is realized from the result that performed length biased quasi Rama distribution gives a best fit as compared over quasi Rama, Rama, Pratibha, Komal, Uma and Lindley distributions.

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