DECISION MAKING UNDER UNCERTAINTY BASED ON GENERALIZED FERMATEAN FUZZY NUMBERS

Tanuj Kumar[1,*], Gajraj Singh[2], Birjesh Kumar[3]

•

[1] Department of Mathematics, SRM Institute of Science and Technology,
Delhi NCR Campus, Ghaziabad, Uttar Pradesh, India
[2] Discipline of Statistics, School of Sciences,
Indira Gandhi National Open University, New Delhi, India
[3] Department of Statistics, Ramanujan College,
University of Delhi, New Delhi, India,
gajrajsingh@ignou.ac.in, birjesh.kumar@ramanujan.du.ac.in
* corresponding author: tanujkhutail@gmail.com

Abstract

It has been observed that Fermatean fuzzy sets, which provide a more comprehensive framework than intuitionistic fuzzy sets, offer significant advantages in decision-making processes. This study first formulates the generalized representation of Fermatean fuzzy numbers, including their membership and non-membership functions as well as the associated arithmetic operations. Furthermore, novel defuzzification and ranking techniques are proposed to convert Fermatean fuzzy information into crisp values. To validate the approach, a statistical application is presented for determining the sample range under generalized trapezoidal Fermatean fuzzy numbers using the proposed defuzzification and ranking methods. Finally, a case study is provided to demonstrate the practical applicability of generalized Fermatean fuzzy numbers in real-world decision-making contexts.

Keywords: decision making, Fermatean fuzzy sets, Pythagorean fuzzy sets, Intuitionistic fuzzy sets, defuzzification, score function

1. Introduction

In traditional decision-making approaches, it is often assumed that all criteria and their corresponding weights are precisely defined, enabling a straightforward ranking of alternatives. However, such methods frequently fail in real-world situations where ambiguity and uncertainty are present. To address these challenges, Zadeh [1] introduced fuzzy set theory (FST) as an extension of classical set theory, employing membership functions to model uncertainty. FST has proven effective in handling uncertainty in decision-making problems, leading to the development of numerous FST-based techniques applied in decision science. Building on this foundation, Atanassov [2] extended FST to intuitionistic fuzzy set theory (IFST) by incorporating membership and non-membership functions along with a hesitation index to better capture imprecise information. Intuitionistic fuzzy sets (IFSs) are now widely studied and applied across various domains of science and technology for managing uncertainty in problem-solving. In particular, generalized and standard forms of triangular and trapezoidal intuitionistic fuzzy numbers are commonly employed in decision science.

DOI: https://doi.org/10.24412/1932-2321-2025-489-781-794

2. Literature Review

Numerous techniques for intuitionistic fuzzy decision-making have been developed in the literature by various researchers, including Atanassov et al. [3], [4], Liu and Wang [5], Li et al. [6], Shaw and Roy [7], Shaw and Roy [8], Ban and Tuse [9], Zeng et al. [10], and Che et al. [11]. These techniques have been successfully applied to various real-world decision-making scenarios. Kumar and Bajaj [12] proposed complex intuitionistic fuzzy soft sets with distance measures and entropies for use in multi-criteria decision-making problems. Garg and Rani [13] introduced a generalized weighted average aggregation operator for combining multiple complex intuitionistic fuzzy sets using t-norm operations, and also proposed a multi-criteria decision-making technique based on these operators. Jose [14] provided a model for decision-making in multiple-person problems in an intuitionistic fuzzy environment. Joshi and Kumar [15] developed an accuracy function for interval-valued IFSs and applied it to the problem of group decision making with many attributes. Fan et al. [17] proposed intuitionistic fuzzy weighted geometric and arithmetic averaging operators, which can be used to aggregate intuitionistic fuzzy information in multi-attribute decision-making problems. Additionally, numerous types of IFNs have been developed to address complex situations in real-world problems, such as the generalized parabolic IFN presented by Dutta et al. [18] for use in decision-making processes.

Yager [20] introduced a specific type of fuzzy set named Pythagorean fuzzy set, for which several aggregation operators have been established by Yager [21]. Ren et al. [24] recommended the use of Pythagorean fuzzy TODIM for MCDM problems, while Peng and Yang [22] used Pythagorean fuzzy aggregation operators to evaluate investments in Internet stocks. Reformat and Yager [23] implemented PFNs to construct an itemized list of the nominated films from the Netflix competition directory. Furthermore, Kumar et al. [16] introduced the accuracy function and distance measures of interval-valued Pythagorean fuzzy sets, which were applied to decision-making problems.

3. Research Objectives and Methodology

Quality control experts often face situations where alternative grade possibilities must be described by criteria such that the sum of favoring and disfavoring degrees exceeds one in certain real-world decision-making contexts. For instance, a decision expert may propose x_i as a potential solution satisfying criterion C_j, with a maximum satisfaction level of 0.9 and a maximum dissatisfaction level of 0.6. This violates the constraints imposed by both Pythagorean and intuitionistic fuzzy sets, rendering these approaches unsuitable for representing the expert™s assessment. To address such cases, Senapati and Yager [25] introduced Fermatean fuzzy sets, a more general framework than both intuitionistic and Pythagorean fuzzy sets, for decision-making problems. In this study, we explicitly examine generalized Fermatean fuzzy numbers along with their associated ranking and defuzzification methods, which are broadly applicable in statistics and problem-solving.

The structure of this paper is organized as follows. Section 2 introduces the fundamental definitions of generalized Fermatean fuzzy numbers (GFFNs) and intuitionistic fuzzy numbers, together with their arithmetic operations. Section 3 presents the defuzzification methods of Fermatean fuzzy numbers, while Section 4 focuses on their ranking methods. In Section 5, the practical applications of the proposed defuzzification and ranking approaches are demonstrated through decision-making problems and the determination of the sample range. Finally, Section 6 provides the concluding remarks.

In this section, we review the fundamental definitions of intuitionistic fuzzy sets and Fermatean fuzzy sets, along with their associated arithmetic operations.

4. Intuitionistic and Fermatean Fuzzy sets

Let X be a non-empty set of discourse and $A \subseteq V$ is said to be a fuzzy set (FS) of V defined by $A = \{\langle x, \mu_A(v)\rangle : v \in V\}$, where $\mu_A(v) : V \to [0, 1]$ is a membership function of A or degree of belonging of v in A. Thus, fuzzy set is a collection of objects with degree of membership [1]. A more generalize aspects of fuzzy sets was introduced by Atanassov [26] and widely used in various fields of science and technology.

Definition 2.1 Let $V = \{v_1, v_2, \ldots, v_n\}$ be a non-empty finite universal set, an Intuitionistic Fuzzy Set (IFS) $\tilde{A}$ in V is an object having the form:

$$A = \{< v, \phi_A(v), \psi_A(v) > | v \in V\}, \tag{1}$$

where the functions $\phi_A : V \to [0, 1]$ and $\psi_A : V \to [0, 1]$ defined the degree of membership and non-membership of an element $v \in V$ such that the condition $0 \leq \phi_A(v) + \psi_A(v) \leq 1$ is fulfilled. The quantity defined by $\varphi_A(v) = (1 - \phi_A(v) - \psi_A(v))$ is called the degree of indeterminacy of an element $v \in A$ [2].

Definition 2.2 [27]: An Intuitionistic fuzzy subset $\tilde{A}$ of the real line satisfies the following conditions:

1. $\tilde{A}$ is normal, i.e., there exist some $v_0 \in$ such that $\phi_{\tilde{A}}(v_0) = 1$, $\psi_{\tilde{A}}(v_0) = 0$.
2. $\tilde{A}$ is convex for the membership function $\phi_{\tilde{A}}(v)$ i.e.,
$$\phi_{\tilde{A}}(\lambda v_1 + (1-\lambda)v_2) \geq \min(\phi_{\tilde{A}}(v_1), \phi_{\tilde{A}}(v_2))$$
for all $v_1, v_2 \in, \lambda \in [0, 1]$.
3. $\tilde{A}$ is concave for the non-membership function i.e.,
$$\psi_{\tilde{A}}(v)\psi_{\tilde{A}}(\lambda v_1 + (1-\lambda)v_2) \leq \max(\psi_{\tilde{A}}(v_1), \psi_{\tilde{A}}(v_2))$$
for all $v_1, v_2 \in, \lambda \in [0, 1]$.

4.1. Fermatean Fuzzy Sets

Definition 2.3 [28] Let $V = \{v_1, v_2, \ldots, v_n\}$ be a non-empty finite universal set, an A Fermatean Fuzzy Set (FFS) $\tilde{A}$ in V is an object having the form:

$$A = \{< v, \phi_A(v), \psi_A(v) > | v \in V\}, \tag{2}$$

where the functions $\phi_A : V \to [0, 1]$ and $\psi_A : V \to [0, 1]$ defined the degree of membership and non-membership of an element $v \in V$ such that the condition $0 \leq \phi_A^3(v) + \psi_A^3(v) \leq 1$ is fulfilled. The quantity defined by $\varphi_A(v) = \sqrt[3]{(1 - \phi_A^3(v) - \psi_A^3(v))}$ is called the degree of indeterminacy of an element $v \in A$.

In addition, we provide a clear presentation of the generalized variant of Fermatean fuzzy numbers, including their membership and non-membership functions, as well as their fundamental arithmetic operations.

Definition 2.4 A generalized trapezoidal fermatean fuzzy number (GTPFFN) $\tilde{A}$ denoted by, $\tilde{A}_{Tp} = \langle [a_1, a_2, a_3, a_4; \mu], [a_1', a_2, a_3, a_4'; \nu] \rangle$ is a Fermatean fuzzy set on a real number set , whose membership function and non-membership functions are defined as:

$$\phi_{\tilde{A}}(x) = \begin{cases} \mu \left(\frac{x-a_1}{a_2-a_1}\right)^{\frac{1}{3}}, & a_1 \leq x \leq a_2 \\ \mu, & a_2 \leq x \leq a_3 \\ \mu \left(\frac{a_4-x}{a_4-a_3}\right)^{\frac{1}{3}}, & a_3 \leq x \leq a_4 \\ 0, & otherwise \end{cases} \tag{3}$$

and

$$\psi_{\tilde{A}}(x) = \begin{cases} \left(\frac{(a_2-x)+\nu^3(x-a_1')}{a_2-a_1'}\right)^{\frac{1}{3}}, & a_1' \leq x \leq a_2 \\ \nu, & a_2 \leq x \leq a_3 \\ \left(\frac{(x-a_3)+\nu^3(a_4'-x)}{a_4'-a_3}\right)^{\frac{1}{3}}, & a_3 \leq x \leq a_4' \\ 0, & otherwise \end{cases} \tag{4}$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_4'$, $0 \leq \mu, \nu \leq 1$ and $0 \leq \mu + \nu \leq 1$. Note that a GTFFN is called a triangular Fermatean fuzzy number (TRFFN), if $a_3 = a_2$, $a_3 = a_4$ & $a_3 = a_4'$ and it is denoted by $\tilde{A}_{Tr} = \langle [a_1, a_2, a_3; \mu], [a_1', a_2, a_3'; \nu] \rangle$.

Definition 2.5 Let $\alpha, \beta \in [0, 1]$ be fixed numbers such that $0 \leq \alpha \leq \mu, \nu \leq \beta \leq 1$ and $0 \leq \alpha^3 + \beta^3 \leq 1$. A (α, β)-cuts generated by FFS A is defined as

$$A_{\alpha,\beta} = \{\langle v, \phi_A(v) \geq \alpha, \psi_A(v) \leq \beta \rangle : v \in V\}. \tag{5}$$

Definition 2.6 The α-cut of an FFN $\tilde{A}$ is defined as

$$\tilde{A}^{\mu}_{\alpha} = \{\langle v, \phi_A(v) \geq \alpha \rangle : v \in V\}, 0 \leq \alpha \leq \mu. \tag{6}$$

Thus, by the definition, the α-cut of an FFN $\tilde{A}$ is

$$\tilde{A}^{\mu}_{\alpha} = [L_{\tilde{A}}(\alpha), U_{\tilde{A}}(\alpha)] = \left[a_1 + \frac{\alpha^3(a_2 - a_1)}{\mu^3}, a_4 - \frac{\alpha^3(a_4 - a_3)}{\mu^3}\right], 0 \leq \alpha \leq \mu. \tag{7}$$

Definition 2.7 The β-cut of an GTPFFN $\tilde{A}$ is defined as

$$\tilde{A}^{\mu}_{\alpha} = \{\langle v, \psi_A(v) \leq \beta \rangle : v \in V\}, \nu \leq \beta \leq 1. \tag{8}$$

Thus, by the definition, the β-cut of an GTPFFN $\tilde{A}$ is

$$\tilde{A}^{\nu}_{\beta} = [L_{\tilde{A}}(\beta), U_{\tilde{A}}(\beta)] = \left[\frac{a_2(1-\beta^3) + a_1'(\beta^3 - v^3)}{1-\nu^3}, \frac{a_3(1-\beta^3) + a_4'(\beta^3 - v^3)}{1-\nu^3}\right], \nu \leq \beta \leq 1. \tag{9}$$

Definition 2.8: Let $\alpha, \beta \in [0, 1]$ be fixed numbers such that $0 \leq \alpha \leq \mu, \nu \leq \beta \leq 1$ and $0 \leq \alpha^3 + \beta^3 \leq 1$. The α and β-cuts of a GTRFFN $\tilde{A}_T = \langle [a_1, a_2, a_3); \mu], [a_1', a_2, a_3'; \nu] \rangle$ are defined as follows:

$$\tilde{A}^{\mu}_{\alpha} = [L_{\tilde{A}}(\alpha), U_{\tilde{A}}(\alpha)] = \left[a_1 + \frac{\alpha^3(a_2 - a_1)}{\mu^3}, a_3 - \frac{\alpha^3(a_3 - a_2)}{\mu^3}\right] \tag{10}$$

and

$$\tilde{A}^{\nu}_{\beta} = [L_{\tilde{A}}(\beta), U_{\tilde{A}}(\beta)] = \left[\frac{a_2(1-\beta^3) + a_1'(\beta^3 - v^3)}{1-\nu^3}, \frac{a_2(1-\beta^3) + a_3'(\beta^3 - v^3)}{1-\nu^3}\right]. \tag{11}$$

The usual arithmetic, namely addition, subtraction, multiplication and division between $\tilde{A}_{Tp}$ and $\tilde{B}_{Tp}$ are given as follows:

Let $\tilde{A}_{Tp} = \langle [a_1, a_2, a_3, a_4; \mu_1], [a_1', a_2, a_3, a_4'; \nu_1] \rangle$ and $\tilde{B}_{Tp} = \langle [b_1, b_2, b_3, b_4; \mu_2], [b_1', b_2, b_3, b_4'; \nu_2] \rangle$ be two generalized trapezoidal Fermatean fuzzy numbers, then we have

1.
$$\tilde{A} + \tilde{B} = \left\langle \begin{matrix} [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; \mu = \min(\mu_1, \mu_2)], \\ [a_1' + b_1', a_2 + b_2, a_3 + b_3, a_4' + b_4'; \nu_2 = \max(\nu_1, \nu_2)] \end{matrix} \right\rangle$$

2.
$$\tilde{A} - \tilde{B} = \left\langle \begin{matrix} [a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1; \mu = \min(\mu_1, \mu_2)], \\ [a_1' - b_4', a_2 - b_3, a_3 - b_2, a_4' - b_1'; \nu_2 = \max(\nu_1, \nu_2)] \end{matrix} \right\rangle$$

3.
$$\tilde{A} * \tilde{B} = \left\langle \begin{matrix} [a_1 b_1,\, a_2 b_2,\, a_3 b_3,\, a_4 b_4;\, \mu = \min(\mu_1,\, \mu_2)], \\ [a_1' b_1',\, a_2 b_2,\, a_3 b_3,\, a_4' b_4';\, \nu_2 = \max(\nu_1,\, \nu_2)] \end{matrix} \right\rangle$$

4.
$$\tilde{A} / \tilde{B} = \left\langle \begin{matrix} [a_1/b_4,\, a_2/b_3,\, a_3/b_2,\, a_4/b_1;\, \mu = \min(\mu_1,\, \mu_2)], \\ [a_1'/b_4',\, a_2/b_3,\, a_3/b_2,\, a_4'/b_1';\, \nu_2 = \max(\nu_1,\, \nu_2)] \end{matrix} \right\rangle$$

5.
$$k * \tilde{A} = \left\langle [ka_1,\, ka_2,\, ka_3,\, ka_4;\, \mu_1], \quad [ka_1',\, ka_2,\, ka_3,\, ka_4';\, \nu_1] \right\rangle, \text{ if } k > 0.$$

$$k * \tilde{A} = \left\langle [ka_4,\, ka_4,\, ka_2,\, ka_1;\, \mu_1], \quad [ka_4',\, ka_3,\, ka_2,\, ka_1';\, \nu_1] \right\rangle \text{ if } k < 0.$$

In this section, we will present the defuzzification method for Fermatean fuzzy numbers, which incorporating all the information related to Fermatean fuzzy numbers into the process of defuzzification, using the α-cut technique.

5. Defuzzification of Fermatean Fuzzy Numbers

Let $\tilde{A}_{Tp} = \langle [a_1, a_2, a_3, a_4; \mu_1], [a_1', a_2, a_3, a_4'; \nu_1] \rangle$ be a generalized trapezoidal Fermatean fuzzy number with the following membership and non-membership functions:

$$\phi_{\tilde{A}}(x) = \begin{cases} \mu \left(\frac{x - a_1}{a_2 - a_1} \right)^{\frac{1}{3}}, & a_1 \le x \le a_2 \\ \mu, & a_2 \le x \le a_3 \\ \mu \left(\frac{a_4 - x}{a_4 - a_3} \right)^{\frac{1}{3}}, & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases} \tag{12}$$

and

$$\psi_{\tilde{A}}(x) = \begin{cases} \left(\frac{(a_2 - x) + \nu^3 (x - a_1')}{a_2 - a_1'} \right)^{\frac{1}{3}}, & a_1' \le x \le a_2 \\ \nu, & a_2 \le x \le a_3 \\ \left(\frac{(x - a_3) + \nu^3 (a_4' - x)}{a_4' - a_3} \right)^{\frac{1}{3}}, & a_3 \le x \le a_4' \\ 0, & otherwise \end{cases}, \tag{13}$$

respectively.

The score function is defined as the difference between membership and non-membership functions:

$$S(\tilde{A}) = \mu(\tilde{A}) - \nu(\tilde{A})$$
$$= \begin{cases} \omega(x - \kappa); & \kappa \le x \le a_2 \\ (\mu^3 - \nu^3); & a_2 \le x \le a_3 \\ \rho(\tau - x); & a_3 \le x \le \tau \end{cases}$$

where

$$\kappa = \frac{\mu^3 a_1 (a_2 - a_1') + (a_2 - a_1)(a_2 - a_1' \nu^3)}{\mu^3 (a_2 - a_1') + (1 - \nu^3)(a_2 - a_1)}, \omega = \frac{\mu^3 (a_2 - a_1') + (1 - \nu^3)(a_2 - a_1)}{(a_2 - a_1)(a_2 - a_1')}$$

$$\tau = \frac{\mu^3 a_4 (a_4' - a_3) + (a_4 - a_3)(a_3 - a_4' \nu^3)}{\mu^3 (a_4' - a_3) + (1 - \nu^3)(a_4 - a_3)}, \rho = \frac{\mu^3 (a_4' - a_3) + (1 - \nu^3)(a_4 - a_3)}{(a_4 - a_3)(a_4' - a_3)}$$

with $0 \le \mu, \nu \le 1$ and $0 \le \mu + \nu \le 1$.

The α-cut of $\tilde{S} = \langle(\kappa, a_2, a_3, \tau); \mu, \nu\rangle$ is defined as

$$\begin{aligned}\tilde{A}(\alpha) &= \left[\tilde{A}_L(\alpha), \tilde{A}_R(\alpha)\right] \\ &= \left[\kappa + \frac{\alpha}{\omega}, \tau - \frac{\alpha}{\rho}\right]. \end{aligned} \tag{14}$$

The defuzzification formula for GTPFFNs is defined as follows:

$$\begin{aligned} DTP_f &= \int_0^{\mu^3-\nu^3} \left(\lambda L^{-1}(\alpha) + (1-\lambda)R^{-1}(\alpha)\right) d\alpha \\ &= \left[(\lambda\kappa + (1-\lambda)\tau)(\mu^3-\nu^3) + \frac{(\lambda\rho - (1-\lambda)\omega)(\mu^3-\nu^3)^2}{2\rho\omega}\right] \end{aligned} \tag{15}$$

To calculate the weighted average of the defuzzified values of the membership and non-membership functions of GTPFFNs, a parameter λ with $0 < \lambda < 1$ is introduced. In this research, we set $\lambda = \frac{1}{2}$, although other values of λ could be suitable for addressing weighted decision-making problems.

Note that a GTFFN is called a generalized triangular Fermatean fuzzy number (GTRFFN), if we replace a_3 by a_2, and a_4 by a_3 & a'_4 by a_3 and it is denoted by $\tilde{A}_{Tr} = \left\langle [a_1, a_2, a_3; \mu], [a'_1, a_2, a'_3; \nu]\right\rangle$ and the defuzzification formula for GTRFFNs is defined as

$$\begin{aligned} DTR_f &= \int_0^{\mu^3-\nu^3} \left(\lambda L^{-1}(\alpha) + (1-\lambda)R^{-1}(\alpha)\right) d\alpha \\ &= \left[(\lambda\kappa + (1-\lambda)\tau)(\mu^3-\nu^3) + \frac{(\lambda\rho - (1-\lambda)\omega)(\mu^3-\nu^3)^2}{2\rho\omega}\right], \end{aligned} \tag{16}$$

where

$$\kappa = \frac{\mu^3 a_1(a_2-a'_1) + (a_2-a_1)(a_2-a'_1\nu^3)}{\mu^3(a_2-a'_1) + (1-\nu^3)(a_2-a_1)}, \ \omega = \frac{\mu^3(a_2-a'_1) + (1-\nu^3)(a_2-a_1)}{(a_2-a_1)(a_2-a'_1)}$$

$$\tau = \frac{\mu^3 a_3(a'_3-a_2) + (a_3-a_2)(a_2-a'_3\nu^3)}{\mu^3(a'_3-a_2) + (1-\nu^3)(a_3-a_2)}, \ \rho = \frac{\mu^3(a'_3-a_2) + (1-\nu^3)(a_3-a_2)}{(a_3-a_2)(a'_3-a_2)}$$

with $0 \le \mu, \nu \le 1$ and $0 \le \mu + \nu \le 1$.

6. Ranking of GTPFFNs

In this section, we establish a method for ranking GTPFFNs, which involves evaluating the mean value of the membership and non-membership functions of GTPFFNs.

Let $\tilde{A}_{Tp} = \left\langle [a_1, a_2, a_3, a_4; \mu], [a'_1, a_2, a_3, a'_4; \nu]\right\rangle$ be a GTPFFN whose (α, β)-cuts are given by

$$\tilde{A}^{\mu}_{\alpha} = \left[a_1 + \frac{\alpha^3(a_2-a_1)}{\mu^3}, a_4 - \frac{\alpha^3(a_4-a_3)}{\mu^3}\right], 0 \le \alpha \le \mu. \tag{17}$$

$$\tilde{A}^{\nu}_{\beta} = \left[\frac{a_2(1-\beta^3) + a'_1(\beta^3-v^3)}{1-\nu^3}, \frac{a_3(1-\beta^3) + a'_4(\beta^3-v^3)}{1-\nu^3}\right], \nu \le \beta \le 1. \tag{18}$$

The mean value of membership and non-membership functions of GTPFFNs is defined as

$$m = \frac{1}{2}\int_0^{\mu} \left[a_1 + \frac{\alpha^3(a_2-a_1)}{\mu^3} + a_4 - \frac{\alpha^3(a_4-a_3)}{\mu^3}\right] d\alpha. \tag{19}$$

$$m = \frac{3\mu(a_1+a_4) + \mu(a_2+a_3)}{8}. \tag{20}$$

$$n = \frac{1}{2}\int_{\nu}^{1}\left[\frac{a_2(1-\beta^3)+a_1'(\beta^3-v^3)}{1-\nu^3}+\frac{a_3(1-\beta^3)+a_4'(\beta^3-v^3)}{1-\nu^3}\right]d\alpha.$$

$$n = \frac{(a_2+a_3)\left(\nu^4-4\nu+3\right)+(a_1'+a_4')(3\nu^4-4\nu^3+1)}{8(1-\nu^3)}. \tag{21}$$

The value of membership function $V_\mu(A)$ and non-membership function $V_\nu(A)$ are defined as follows:

$$V_\mu(A) = m\int_{0}^{\mu}\left[a_4-\frac{\alpha^3(a_4-a_3)}{\mu^3}-a_1-\frac{\alpha^3(a_2-a_1)}{\mu^3}+\right]d\alpha.$$

$$V_\mu(A) = \frac{\mu^2\left[9(a_4^2-a_1^2)+6(a_3a_4-a_1a_2)+(a_3^2-a_2^2)\right]}{32}. \tag{22}$$

$$V_\nu(A) = n\int_{\nu}^{1}\left[\frac{a_3(1-\beta^3)+a_4'(\beta^3-v^3)}{1-\nu^3}-\frac{a_2(1-\beta^3)+a_1'(\beta^3-v^3)}{1-\nu^3}\right]d\beta.$$

$$V_\nu(A) = \frac{\left[\theta^2(a_4'^2-a_1'^2)+2\eta\theta(a_3a_4'-a_1'a_2)+\eta^2(a_3^2-a_2^2)\right]}{32(1-\nu^3)}, \tag{23}$$

where $\eta = \left(\nu^4-4\nu+3\right)$ and $\theta = (3\nu^4-4\nu^3+1)$.

The ranking order of $\tilde{A}_{Tp}$ is defined as

$$R(\tilde{A}_{Tr},\lambda) = \lambda V_\mu(A)+(1-\lambda)V_\nu; \quad 0 \le \lambda \le 1 \tag{24}$$

and the ranking order of $\tilde{A}_{Tr}$ is defined as

$$R(\tilde{A}_{Tr},\lambda) = \lambda V_\mu(A)+(1-\lambda)V_\nu; \quad 0 \le \lambda \le 1, \tag{25}$$

$$V_\mu(A) = \frac{\mu^2\left[9(a_3^2-a_1^2)+6(a_2a_3-a_1a_2)\right]}{32} \tag{26}$$

and

$$V_\nu(A) = \frac{\left[\theta^2(a_3'^2-a_1'^2)+2\eta\theta(a_2a_3'-a_1'a_2)\right]}{32(1-\nu^3)}, \tag{27}$$

here $\eta = \left(\nu^4-4\nu+3\right)$ and $\theta = (3\nu^4-4\nu^3+1)$.

Thus, for two GFFNs $\tilde{A}$ and $\tilde{B}$, the the ranking order depend on the values of $R(\tilde{A},\lambda)$ and $R(\tilde{B},\lambda)$ and described as follows:

- $(R(\tilde{A},\lambda)) < (R(\tilde{B},\lambda))$, if $R(\tilde{A},\lambda) < R(\tilde{B},\lambda)$.
- $(R(\tilde{A},\lambda)) \ge (R(\tilde{B},\lambda))$, if $R(\tilde{A},\lambda) \ge R(\tilde{B},\lambda)$.

7. Applications of Fermatean Fuzzy Sets

7.1. Range of Samples under Trapezoidal Fermatean Fuzzy numbers based on Ranking Order

Since order statistics depend on the sequence of sample values, statistical methods do not work well with datasets that contain uncertainty or inaccuracies from measurement tools or data collection methods. To address these challenges, we introduce a fuzzification step to the data first to reduce the uncertainty and then one can perform statistical methods or decision-making procedures on the defuzzified data to obtain insights and draw conclusions.

For instance, in statistical quality control charts, the range of the data is important for determining the limits of the chart; here we show how to calculate the range of the uncertain sample data. Take sample data from a normal distribution with mean 20 and standard deviation

1 and each sample of size $n = 5$ is fuzzified into a trapezoidal Fermatean fuzzy numbers in Table 2. Rank each fuzzy observation in each sample using the proposed ranking formula (Eq. 24) and list the ranking order (RO) of each sample observation in Table 3. Calculate the range of each sample by selecting the highest and lowest ranking orders of the Fermatean fuzzy numbers; list the ranges in Table 4. Calculate the mean range for the whole data set using the proposed defuzzification formula (Eq. 15), which is 0.8642.

Table 1: *Data drawn from normal distribution*

S.No.	X_1	X_2	X_3	X_4	X_5
S_1	20.5377	20.6715	19.8978	18.9109	21.4193
S_2	21.8339	18.7925	19.7586	20.0326	20.2916
S_3	17.7412	20.7172	20.3192	20.5525	20.1978
S_4	20.8622	21.6302	20.3129	21.1006	21.5877
S_5	20.3188	20.4889	19.1351	21.5442	19.1955
S_6	18.6923	21.0347	19.9699	20.0859	20.6966
S_7	19.5664	20.7269	19.8351	18.5084	20.8351
S_8	20.3426	19.6966	20.6277	19.2577	19.7563
S_9	23.5784	20.2939	21.0933	18.9384	20.2157
S_{10}	22.7694	19.2127	21.1093	22.3505	18.8342
S_{11}	18.6501	20.8884	19.1363	19.3844	18.8520
S_{12}	23.0349	18.8529	20.0774	20.7481	20.1049
S_{13}	20.7254	18.9311	18.7859	19.8076	20.7223
S_{14}	19.9369	19.1905	18.8865	20.8886	22.5855
S_{15}	20.7147	17.0557	19.9932	19.2352	19.3331
S_{16}	19.7950	21.4384	21.5326	18.5977	20.1873
S_{17}	19.8759	20.3252	19.2303	18.5776	19.9175
S_{18}	21.4897	19.2451	20.3714	20.4882	18.0670
S_{19}	21.4090	21.3703	19.7744	19.8226	19.5610
S_{20}	21.4172	18.2885	21.1174	19.8039	18.2053

Table 2: *Fuzzifying sample data and ranking ordering*

S.No.	D'_1	D_1	D_2	D_3	D_4	D'_4	μ	ν	*Rank*	*RO*
S_{11}	20.5337	20.5353	20.5361	20.5389	20.5397	20.5416	0.9311	0.1029	0.0702	2
S_{12}	20.6707	20.6710	20.6712	20.6717	20.6719	20.6723	0.9602	0.0763	0.0156	5
S_{13}	19.8945	19.8958	19.8964	19.8987	19.8994	19.9010	0.9689	0.1248	0.0571	3
S_{14}	18.9087	18.9096	18.9100	18.9116	18.9121	18.9132	0.9084	0.0729	0.0368	4
S_{15}	21.4147	21.4166	21.4175	21.4207	21.4216	21.4239	0.9152	0.1326	0.0803	1
S_{21}	21.8312	21.8323	21.8328	21.8347	21.8352	21.8366	0.9996	0.0578	0.0566	2
S_{22}	18.7903	18.7912	18.7916	18.7932	18.7936	18.7947	0.9107	0.1462	0.0335	3
S_{23}	19.7585	19.7585	19.7585	19.7586	19.7586	19.7586	0.9775	0.1317	0.0004	5
S_{24}	20.0282	20.0300	20.0308	20.0339	20.0347	20.0369	0.9084	0.0900	0.0739	1
S_{25}	20.2903	20.2908	20.2911	20.2920	20.2922	20.2929	0.9800	0.0931	0.0241	4
S_{31}	17.7366	17.7384	17.7393	17.7425	17.7434	17.7457	0.9182	0.0764	0.0703	1
S_{32}	20.7165	20.7168	20.7169	20.7175	20.7176	20.7180	0.9136	0.1369	0.0123	5
S_{33}	20.3163	20.3175	20.3180	20.3201	20.3207	20.3221	0.9550	0.0645	0.0538	3
S_{34}	20.5483	20.5500	20.5508	20.5538	20.5547	20.5568	0.9622	0.0851	0.0792	2
S_{35}	20.1952	20.1963	20.1968	20.1986	20.1991	20.2004	0.9402	0.0576	0.0469	4
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
S_{201}	21.4135	21.4150	21.4157	21.4183	21.4190	21.4208	0.9344	0.1084	0.0671	2
S_{202}	18.2879	18.2882	18.2883	18.2886	18.2888	18.2890	0.9906	0.1380	0.0088	5
S_{203}	21.1133	21.1149	21.1157	21.1186	21.1194	21.1214	0.9261	0.1094	0.0734	1
S_{204}	19.8038	19.8039	19.8039	19.8040	19.8040	19.8041	0.9425	0.0813	0.0020	4
S_{205}	18.2045	18.2048	18.2050	18.2056	18.2057	18.2061	0.9179	0.0923	0.0126	3

Table 3: *Ranges of samples and mean range*

S.No.	R'_1	R_1	R_2	R_3	R_4	R'_4	μ	ν
1	0.7441	0.7456	0.7463	0.7489	0.7497	0.7516	0.9152	0.1326
2	0.2697	0.2714	0.2723	0.2753	0.2762	0.2783	0.9084	0.1317
3	0.1682	0.1668	0.1661	0.1637	0.1629	0.1612	0.9136	0.1369
4	0.5526	0.5513	0.5506	0.5483	0.5477	0.5460	0.9123	0.0991
5	0.0651	0.0632	0.0623	0.0590	0.0581	0.0557	0.9043	0.0669
6	0.6077	0.6089	0.6095	0.6116	0.6122	0.6137	0.9081	0.1429
7	1.1621	1.1615	1.1611	1.1600	1.1597	1.1588	0.9306	0.1009
8	1.0815	1.0829	1.0836	1.0859	1.0866	1.0883	0.9226	0.1122
9	2.4811	2.4827	2.4835	2.4863	2.4872	2.4892	0.9258	0.0909
10	3.5138	3.5148	3.5153	3.5170	3.5175	3.5188	0.9459	0.1463
11	0.5361	0.5346	0.5339	0.5312	0.5305	0.5286	0.9099	0.1413
12	2.2897	2.2886	2.2880	2.2860	2.2854	2.2840	0.9680	0.0834
13	1.7955	1.7938	1.7929	1.7898	1.7889	1.7867	0.9490	0.1118
14	1.0458	1.0476	1.0486	1.0519	1.0528	1.0551	0.9071	0.1022
15	0.7547	0.7560	0.7567	0.7590	0.7596	0.7613	0.9083	0.1331
16	0.0973	0.0961	0.0955	0.0933	0.0927	0.0912	0.9292	0.0932
17	1.0982	1.0969	1.0962	1.0938	1.0932	1.0915	0.9418	0.1483
18	2.3008	2.3022	2.3030	2.3055	2.3062	2.3080	0.9669	0.1145
19	1.5946	1.5951	1.5954	1.5963	1.5965	1.5972	0.9121	0.1090
20	1.3094	1.3110	1.3118	1.3146	1.3154	1.3174	0.9261	0.1094
$\bar{R}$	1.1734	1.1736	1.1736	1.1739	1.1739	1.1741	0.9043	0.1483

In the following, we propose a decision-making approach to choose the best option among several alternatives using Fermatean fuzzy numbers. We illustrate the practical importance of Fermatean fuzzy decision making by using the ranking of GFFNs in the method.

7.2. Multi-Criteria Decision Making Problem

Assume that r decision makers DM_1, DM_2, DM_3, $\ldots$, DM_r want to select the best option from a set of n options, say A_1, A_2, A_3, $\ldots$, A_n, based on m criteria C_1, C_2, C_3, $\ldots$, C_m. The steps in the computation are as follows:

1. The GFFNs (Low, Medium, High, and Very High) are linguistic variables used to rank the alternatives according to each criterion, and the decision makers choose the linguistic weight variables for each criterion's importance weight.
2. The decision makers give each criterion an importance weight based on the linguistic weight variables.
3. To determine the aggregated Fermatean fuzzy weight w_i of the criterion C_j, utilize the following formula:
$$w_j = \frac{[w_j^1 + w_j^2 + \ldots + w_j^r]}{r}, \quad j = 1, 2, \ldots, m. \tag{28}$$
4. Using the rating values assigned by the decision makers to the options across the criteria, construct the Fermatean fuzzy decision matrix D.

$$D = \begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_m \\ \hline A_1 & r_{11} & r_{12} & \cdots & r_{1m} \\ A_2 & r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \ldots & \ldots & \ldots & \ldots \\ A_n & r_{n1} & r_{n2} & \cdots & r_{nm} \end{array}$$

5. We can compute the weighted average decision using the recommended arithmetic operations as follows:
$$D_i = \sum_{j=1}^{m} r_{ij} w_j, \quad i = 1, 2, \ldots, n. \tag{29}$$
6. The ranking order formula given by Equation (25) is applied to compute the index value $I(D_i, \lambda)$ for each average decision D_i by considering different values of $\lambda \in [0, 1]$.
7. Ultimately, the decision expert selects the alternative with the highest index value.

A committee consisting of three decision-makers, DM_1, DM_2, and DM_3, is formed to select the most suitable applicant for a teaching position from among three candidates, A_1, A_2, and A_3. The evaluation is carried out based on four benefit criteria, C_1, C_2, C_3, and C_4.

1. The decision-makers first determine the linguistic weight variable for each criterion, as shown in Table 4, and then use the linguistic rating variable in Table 5 to evaluate the alternatives with respect to each criterion.

Table 4: *Linguistic variable for each criterion's weight*

L	<[0.18 0.20 0.26; 0.4], [0.16 0.20 0.27; 0.2] >
M	<[0.43 0.48 0.54; 0.6], [0.41 0.48 0.56; 0.4]>
H	<[0.61 0.65 0.72; 0.7], [0.59 0.65 0.75; 0.5] >
VH	<[0.84 0.88 0.95; 0.9], [0.81 0.88 0.98; 0.6] >

Table 5: *Linguistic variable for the ratings*

P	<[0.20 0.24 0.30; 0.5], [0.18 0.24 0.33; 0.1] >
F	<[0.43 0.49 0.57; 0.6], [0.39 0.49 0.59; 0.3] >
G	<[0.65 0.78 0.83; 0.8], [0.61 0.78 0.84; 0.5] >
VG	<[0.85 0.89 0.92; 0.9], [0.83 0.89 0.95; 0.6] >

2. The importance of each criterion is evaluated in Table 6 using the linguistic weighting variables defined in Table 4.

Table 6: *Criterion's weight given by Decision Makers*

Decision Makers →	2*D_1	2*D_2	2*D_3
Criterion ↓			
C_1	M	L	M
C_2	H	VH	H
C_4	VH	H	VH
C_4	VH	VH	VH

3. Table 6 is employed to aggregate the criteria weights and equation (28) is then applied to compute the overall fuzzy weight w_j of each criterion C_j.

$$w_1 =< [0.350.390.45; 0.53], [0.330.390.46; 0.60] >$$

$$w_2 =< [0.690.730.80; 0.77], [0.660.730.83; 0.53] >$$

$$w_3 =< [0.760.800.87; 0.83], [0.740.800.90; 0.57] >$$

$$w_4 =< [0.840.880.95; 0.90], [0.810.880.98; 0.60] > .$$

4. The fuzzy decision matrix D is constructed based on the ratings provided by the decision-makers.

$$D = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ A_1 & P & F & F & G \\ A_2 & G & F & VG & G \\ A_3 & VG & VG & G & G \end{matrix}$$

As a result, the fuzzy decision matrix D is created as shown below

Table 7: *Fuzzy decision matrix D*

	C_1	C_2	C_3	C_4
A_1	<[0.20 0.24 0.30;0.5], [0.18 0.24 0.33;0.1]>	<[0.43 0.49 0.57;0.6], [0.39 0.49 0.59;0.3]>	<[0.43 0.49 0.57;0.8], [0.39 0.49 0.59;0.5]>	<[0.65 0.78 0.83;0.9], [0.61 0.78 0.84;0.6]>
A_1	<[0.65 0.78 0.83;0.5], [0.61 0.78 0.84 ;0.1]>	<[0.39 0.49 0.59 ;0.5], [0.14 0.24 0.34 ;0.1]>	<[0.85 0.89 0.92;0.5], [0.83 0.89 0.95;0.1]>	<[0.65 0.78 0.83;0.5], [0.61 0.78 0.84;0.1]>
A_1	<[0.85 0.89 0.92;0.5], [0.83 0.89 0.95;0.1]>	<[0.85 0.89 0.92;0.5], [0.83 0.89 0.95;0.1]>	<[0.65 0.78 0.83;0.5], [0.61 0.78 0.84;0.1]>	<[0.65 0.78 0.83;0.5], [0.61 0.78 0.84;0.1]>

5. Now, we evaluate average decision using the suggested arithmetic operations and index values of all $D_i's$ are calculated for different values of $\lambda \in [0, 1]$, as shown in Table 7.

$$D_i = \sum_{j=1}^{m} r_{ij} w_j, \quad i = 1, 2, \ldots, n. \tag{30}$$

Table 8: *$D_i's$ with their index values for different λ values*

2*	D_1	D_2	D_3
	<[1.24,1.53,1.87;0.69], [1.10,1.53,2.00;0.53]>	<[1.69,2.06,2.43;0.74], [1.40,1.88,2.35;0.53]>	<[1.92,2.30,2.66;0.77], [1.77,2.30,2.81;0.56]>
λ	Index value of D_1	Index value of D_2	Index value of D_3
0.0	0.106	0.108	0.154
0.1	0.201	0.287	0.363
0.2	0.295	0.466	0.571
0.3	0.390	0.645	0.780
0.4	0.484	0.824	0.989
0.5	0.579	1.003	1.198
0.6	0.673	1.182	1.406
0.7	0.767	1.361	1.615
0.8	0.862	1.540	1.824
0.9	0.956	1.719	2.032
1.0	1.051	1.898	2.241

6. It is evident from the above tables that, for any value of $\lambda \in [0, 1]$, the ranking order of D_1, D_2, and D_3 remains the same, i.e., $D_3 > D_2 > D_1$. Consequently, A_3 is identified as the best alternative, with the overall ranking given by $A_3 > A_2 > A_1$.

8. Conclusions

The present study has demonstrated the explicit representation of generalized Fermatean fuzzy numbers together with their fundamental arithmetic operations. In addition, novel defuzzification and ranking methods have been proposed to transform generalized Fermatean fuzzy observations into crisp values. A statistical application has been included to illustrate the use of the proposed defuzzification and ranking techniques in determining the sample range under trapezoidal generalized Fermatean fuzzy numbers. Furthermore, a decision-making problem has been examined to highlight the practicality and applicability of generalized Fermatean fuzzy numbers in real-world scenarios.

Disclosure Statement

The authors declare that they do not have any known financial or other competing interests which could appear to have affected the work presented in this document.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8:338–356. https://doi.org/10.1016/S0019-9958(65)90241-X.

[2] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[3] Atanassov, K., Pasi, G., and Yager, R. (2002). Intuitionistic fuzzy interpretations of multi-person multi-criteria decision making. In *First International IEEE Symposium on Intelligent Systems*, pages 859–868. https://doi.org/10.1080/00207720500382365.

[4] Li, D. F. (2005). Multiattribute decision making models and methods using intuitionistic fuzzy sets. *Journal of Computer and System Sciences*, 70(1):73–85.

[5] Liu, H. W. and Wang, G. (2007). Multi-criteria decision-making methods based on intuitionistic fuzzy sets. *European Journal of Operational Research*, 179:220–233.

[6] Li, D. F., Nan, J. X., and Zhang, M. J. (2010). A ranking method of triangular intuitionistic fuzzy numbers and application to decision making. *International Journal of Computational Intelligence Systems*, 3(5):522–530.

[7] Shaw, A. K. and Roy, T. K. (2012). Some arithmetic operations on triangular intuitionistic fuzzy number and its application on reliability evaluation. *International Journal of Fuzzy Mathematics and Systems*, 2(4):363–382.

[8] Shaw, A. K. and Roy, T. K. (2013). Trapezoidal intuitionistic fuzzy number with some arithmetic operations and its application on reliability evaluation. *International Journal of Mathematics in Operational Research*, 5(1):55–73.

[9] Ban, A. I. and Tuse, D. A. (2014). Trapezoidal/triangular intuitionistic fuzzy numbers versus interval-valued trapezoidal/triangular fuzzy numbers and applications to multicriteria decision making methods. *Notes on Intuitionistic Fuzzy Sets*, 20(2):43–51.

[10] Zeng, X. T., Li, D. F., and Yu, G. F. (2014). A value and ambiguity-based ranking method of trapezoidal intuitionistic fuzzy numbers and application to decision making. *The Scientific World Journal*, 2014:8. https://doi.org/10.1155/2014/560582.

[11] Chen, S., Mu, Z., and Zeng, S. (2015). Atanassov™s intuitionistic fuzzy decision making with probabilistic information and distance measure. *Journal of Intelligent & Fuzzy Systems*, 28(1):317–325.

[12] Kumar, T. and Bajaj, R. K. (2014). On complex intuitionistic fuzzy soft sets with distance measures and entropies. *Journal of Mathematics*, 2014.

[13] Garg, H. and Rani, D. (2018). Some generalized complex intuitionistic fuzzy aggregation operators and their application to multicriteria decision-making process. *Arabian Journal for Science and Engineering*, 44(3):1–20.

[14] Jose, S. (2018). Intuitionistic fuzzy approach to multi person decision making. *Advances in Fuzzy Mathematics*, 13(1):9–14.

[15] Joshi, D. and Kumar, S. (2018). Improved accuracy function for interval-valued intuitionistic fuzzy sets and its application to multi-attributes group decision making. *Cybernetics and Systems*, 49(1):64–76. https://doi.org/10.1080/01969722.2017.1412890.

[16] Kumar, T., Bajaj, R., and Ansari, M. D. (2020). On accuracy function and distance measures of interval-valued Pythagorean fuzzy sets with application to decision making. *Scientia Iranica*, 27(4):2127–2139.

[17] Fan, C. L., Song, Y., Fu, Q., Lei, L., and Wang, X. (2018). New operators for aggregating intuitionistic fuzzy information with their application in decision making. *IEEE Access*, 6:27214–27238.

[18] Dutta, P., Saikia, B., and Doley, D. (2020). Decision making under uncertainty via generalized parabolic intuitionistic fuzzy numbers. In Castillo, O., Jana, D., Giri, D., and Ahmed, A. (eds), *Recent Advances in Intelligent Information Systems and Applied Mathematics. ICITAM 2019. Studies in Computational Intelligence*, vol. 863, pp. 243–256. Springer, Cham. https://doi.org/10.1007/978-3-030-34152-7_18.

[19] Dutta, P., Boruah, H., and Ali, T. (2011). Fuzzy arithmetic with and without using α-cut method: a comparative study. *International Journal of Latest Trends in Computing*, 2(1):99–107.

[20] Yager, R. R. (2013). Pythagorean fuzzy subsets. In *Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting*, Edmonton, Canada, pp. 57–61.

[21] Yager, R. R. (2014). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, 22:958–965.

[22] Peng, X. and Yang, Y. (2015). Some results for Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 30:1133–1160.

[23] Reformat, M. Z. and Yager, R. R. (2014). Suggesting recommendations using Pythagorean fuzzy sets illustrated using Netflix movie data. In Laurent, A., Strauss, O., Bouchon-Meunier, B., and Yager, R. R. (eds), *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pp. 546–556. Springer, Berlin.

[24] Ren, P., Xu, Z., and Gou, X. (2016). Pythagorean fuzzy TODIM approach to multi-criteria decision making. *Applied Soft Computing*, 42:246–259.

[25] Senapati, T. and Yager, R. R. (2020). Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, 11(2):663–674.

[26] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96.

[27] Burillo, P., Bustince, H., and Mohedano, V. (1994). Some definition of intuitionistic fuzzy number. In *Proceedings of Fuzzy Based Expert Systems, Fuzzy Bulgarian Enthusiasts*, Sofia, Bulgaria, September 28–30. https://doi.org/10.1007/s13369-020-05176-0.

[28] Senapati, T. and Yager, R. R. (2019). Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, 11:663–674. https://doi.org/10.1007/s12652-019-01377-0.