RT&A, No 4 (89)
Volume 20, December 2025

SOME REFINEMENTS OF INEQUALITIES FOR POLYNOMIALS

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Abstract

The study of inequalities for polynomials plays a central role in mathematical analysis, with numerous results exploring the relationships between a polynomial and its derivative. Over time, several refinements and generalizations have been established, strengthening classical inequalities and extending them to broader settings. In this paper, we present new refinements that further enhance existing results on polynomial inequalities. Our findings not only sharpen earlier theorems, but also provide generalized forms that encompass a wider class of polynomials. The refinements introduced here not only strengthen known inequalities, but also provide a framework that connects different strands of results in this area. This contributes to a deeper understanding of the behavior of polynomials in the complex domain and establishes new directions for further investigation.

Keywords: Complex polynomials, Complex zeros, Inequalities, Complex domain

1. Introduction

Inequalities for polynomials and their derivatives have long been a central topic in mathematical analysis, particularly within approximation theory and complex function theory. Classical results, beginning with the work of Zygmund and Bernstein, established foundational inequalities that describe the relationship between a polynomial and its derivative in the unit circle. These results have been refined and extended by many researchers, often imposing conditions on the location of zeros of the polynomial, leading to sharper bounds and broader applicability.

DOI: https://doi.org/10.24412/1932-2321-2025-489-772-780

Despite this progress, there remains scope for further refinements that unify different strands of existing results and provide stronger generalizations. The purpose of this work is to build upon earlier contributions by developing new inequalities for complex polynomials that not only sharpen known bounds, but also extend them to more general classes of polynomials with specific zero restrictions.

Suppose that P_η be the set of all polynomials of degree at most η, then for $P \in P_\eta$, Zygmund proved that

$$\|P\|_{\mathtt{q}} \geq \frac{1}{\eta}\|P'\|_{\mathtt{q}} \tag{1}$$

where $\mathtt{q}$ is positive.

In literature, so many generalizations and refinements of this result (1) exists. Recently, a generalization of the result (1) was proved. In this paper, we prove a refinement of the result(1) which in turn provides a generalization of several other results.

Let P_η be the class of all polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree η.
Define,

$$\|P\|_{\mathtt{q}} := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\Theta})|^{\mathtt{q}} d\Theta \right\}^{\frac{1}{\mathtt{q}}}, \quad 0 < \mathtt{q} < \infty \|P\|_\infty := \max_{|z|=1} |P(z)|$$

Also,

$$\|P\|_0 := exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} log|P(e^{i\Theta}|d\Theta \right\} \quad for \;\; P \in P_\eta$$

If $P \in P_\eta$,

$$\|P'\|_{\mathtt{q}} \leq \eta \|P\|_{\mathtt{q}}, \quad \mathtt{q} > 0. \tag{2}$$

inequality (2) was found out by Zygmund [15]. Letting $\mathtt{q} \to \infty$ in (2) , we get

$$\|P'\|_\infty \leq \eta \|P\|_\infty \tag{3}$$

which is a well-known Bernstein's inequality.

Arestov[1] proved that (2) is true for $\mathtt{q} \in [0, 1)$ as well.
If $P \in P_\eta$ such that P(z) doesnot vanish in $|z| < 1$, then (3) and (2) can be respectively replaced by

$$\|P'\|_\infty \leq \frac{\eta}{2}\|P\|_\infty \tag{4}$$

and

$$\|P'\|_{\mathtt{q}} \leq \frac{\eta}{\|1+z\|_{\mathtt{q}}}\|P\|_{\mathtt{q}}, \quad \mathtt{q} > 0. \tag{5}$$

The inequality (4) was conjectured by Erdös and later verified by Lax [10], inequality (5) is due to Bruijn [3] for $\mathtt{q} \geq 1$. Rahman and Schmeisser[12] verified that (5) is valid for $\mathtt{q} \in [0, 1)$ as well. Turan [14] proved that, if $P \in P_\eta$ vanishes in $|z| \leq 1$,then

$$\|F'\|_\infty \geq \frac{\eta}{2}\|P\|_\infty. \tag{6}$$

equality holds in both the inequalities (4) and (6) for $P(z) = l + \tau z^\eta$, where $|l| = |\tau|$.
Also, Govil and Rahman [6] proved that, if $P \in P_\eta$ does not vanishes in $|z| < h, h \geq 1$, then

$$\|P'\|_{\mathtt{q}} \leq \frac{\eta}{\|h+z\|_{\mathtt{q}}}\|P\|_{\mathtt{q}}, \;\; \mathtt{q} \geq 1 \tag{7}$$

Gardener and Weems [5] and Rather [13] independently verified the validity of (7) for $0 < \mathtt{q} < 1$ as well.

Also, Aziz and Rather [2] proved that if $P \in P_\eta$ does not vanishes in $|z| < h, h \geq 1$, then for every $\mathtt{q} > 0$

$$\|P'\|_{\mathtt{q}} \leq \frac{\eta}{\|\Delta_{h,1} + z\|_{\mathtt{q}}} \|P\|_{\mathtt{q}}, \quad \mathtt{q} \geq 1 \tag{8}$$

where $\Delta_{h,1} = \dfrac{\eta|a_0|h^2 + h^2|a_1|}{\eta|a_0| + h^2|a_1|}$

Malik [11] proved that if $P \in P_\eta$ vanishes in $|z| \leq 1$, then for every $\mathtt{q}$*positive*.

$$\|P'\|_{\mathtt{q}} \geq \frac{\eta}{\|1 + z\|_{\mathtt{q}}} \|P\|_{\mathtt{q}}. \tag{9}$$

Aziz and Rather [2] generalised the inequality (9) as if $P \in P_\eta$ vanishes in $|z| \leq h, h \leq 1$, then for every $\mathtt{q} > 0$

$$\|P'\|_{\mathtt{q}} \geq \frac{\eta}{\|1 + t_{h,1}z\|_{\mathtt{q}}} \|P\|_{\mathtt{q}}. \tag{10}$$

and

$$\|P'\|_{\infty} \geq \frac{\eta}{\|1 + t_{h,1}z\|_{\mathtt{q}}} \|P\|_{\mathtt{q}}. \tag{11}$$

where $t_{h,1} = \dfrac{\eta|a_\eta|h^2 + |a_{\eta-1}|}{\eta|a_\eta| + |a_{\eta-1}|}$

Govil et al. [7] demonstrated the following two theorems, in which they generalised (7) and (9), and also the inequality (12) by involving some coefficients of P(z).

Theorem 1. If $P \in P_\eta$ does not vanishes in $|z| < h, h \geq 1$, then

$$\|P'\|_{\infty} \leq \frac{\eta}{1+h} \frac{(1 - |\alpha|)(1 + h^2|\alpha|) + h(\eta - 1)|\mu - \alpha^2|}{(1 - |\alpha|)(1 - h + h^2 + h|\alpha|) + h(\eta - 1)|\mu - \alpha^2|} \|P\|_{\infty}, \tag{12}$$

where

$$\alpha = \frac{ha_1}{\eta a_0} \text{ and } \mu = \frac{2h^2 a_2}{\eta(\eta - 1)a_0}$$

Theorem 2. If $P \in P_\eta$ vanishes in $|z| \leq h, h \leq 1$, then

$$\|P'\|_{\infty} \geq \frac{\eta}{1+h} \frac{(1 - |\beta|)(1 + h^2|\beta|) + h(\eta - 1)|\gamma - \beta^2|}{(1 - |\beta|)(1 - h + h^2 + h|\beta|) + h(\eta - 1)|\gamma - \beta^2|} \|P\|_{\infty}, \tag{13}$$

where

$$\beta = \frac{\overline{a}_{\eta-1}}{\eta h \overline{a}_\eta} \text{ and } \gamma = \frac{2\overline{a}_{\eta-2}}{\eta(\eta - 1)h^2 \overline{a}_\eta}$$

Recently, Krishnadas and Chanam [9] demonstrated the following two theorems in which they extended inequalities (14) and (16) to $L_{\mathtt{q}}$ norms. ,

Theorem 3. If $P \in P_\eta$ does not vanishes in $|z| < h, h \geq 1$, then for every $\mathrm{q} > 0$

$$\|P'\|_{\mathrm{q}} \leq \frac{\eta}{\|\Gamma + z\|_{\mathrm{q}}} \|P\|_{\mathrm{q}}. \tag{14}$$

where

$$\Gamma = h \frac{(1-|\alpha|)(|\alpha| + h^2) + h(\eta - 1)|\mu - \alpha^2|}{(1-|\alpha|)(|\alpha|h^2 + 1) + h(\eta - 1)|\mu - \alpha^2|} \tag{15}$$

$$\alpha = \frac{ha_1}{\eta a_0} \text{ and } \mu = \frac{2h^2 a_2}{\eta(\eta - 1)a_0}$$

Theorem 4. If $P \in P_\eta$ vanishes in $|z| \leq h, h \leq 1$, then for every $\mathrm{q} > 0$

$$\|P'\|_{\mathrm{q}} \geq \frac{\eta}{\|1 + Sz\|_{\mathrm{q}}} \|P\|_{\mathrm{q}}. \tag{16}$$

where

$$S = h \frac{(1-|\beta|)(|\beta| + h^2) + h(\eta - 1)|\gamma - \beta^2|}{(1-|\beta|)(|\beta|h^2 + 1) + h(\eta - 1)|\gamma - \beta^2|} \tag{17}$$

$$\beta = \frac{\overline{a}_{\eta-1}}{\eta h \overline{a}_\eta} \text{ and } \gamma = \frac{2\overline{a}_{\eta-2}}{\eta(\eta - 1)h^2 \overline{a}_\eta}$$

2. Main Results

In this paper,first we obtain the following result which includes not only a refinement of Theorem 1.3 but also provides some generalization of other results.

Theorem 5. If $P \in P_\eta$ does not vanishes in $|z| < h, h \geq 1$, then for every $\mathrm{q} > 0$

$$\|P' + \tau\eta m z^{\eta-1}\|_{\mathrm{q}} \leq \frac{\eta}{\|\Gamma + z\|_{\mathrm{q}}} \|P\|_{\mathrm{q}}. \tag{18}$$

where Γ is defined by (15) and $m = \min\limits_{|z|=h} |P(z)|$.

Corollary 1. If we put $m = 0$ and let $\mathrm{q} \to \infty$ in (18), the Theorem (5) reduces to Theorem (5) by using the inequality (18).

Corollary 2. For $m = 0$, inequality (18) reduces to inequality (16).

Next, we prove the theorem as a refinement of Theorem (4). In fact we prove

Theorem 6. If $P \in P_\eta$ vanishes in $|z| \leq h, h \leq 1$, then for every $\mathrm{q} > 0$

$$\|P' + \tau\eta m z^{\eta-1}\|_{\mathrm{q}} \geq \frac{\eta}{\|1 + Sz\|_{\mathrm{q}}} \|P\|_{\mathrm{q}}. \tag{19}$$

where S is defined by (17).

Corollary 3. For $m = 0$ Theorem (6) reduces to Theorem (4).

Instead of proving Theorem (6), we prove a more general result, from which Theorem (6),follows as a special case.

Theorem 7. If $P \in P_\eta$ vanishes in $|z| \le h, h \le 1$, then for every $\mathsf{q} > 0, r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$.

$$\|P' + \tau\eta m z^{\eta-1}\|_{s\mathsf{q}} \ge \frac{\eta}{\|1 + Sz\|_{r\mathsf{q}}}\|P\|_{\mathsf{q}}. \tag{20}$$

where S is defined by (17).

Corollary 4. Put $r = 0$ or $s = 0$, we obtain Theorem (6).

3. Lemmas

In this section we present some lemmas which will help us to prove our results.

Lemma 1. If $P \in P_\eta$ does not vanishes in $|z| < h, h \ge 1$, then

$$\Gamma|P'(z)| \le |\mathcal{Q}'(z)| \tag{21}$$

where Γ is defined by (15) and $\mathcal{Q}(z) = z^\eta \overline{P(\frac{1}{\overline{z}})}$

Above Lemma (1) is due to Govil et al. [7]. By applying Lemma (1) to the polynomial $P(z) = P(z) + m\tau z^\eta$, we immediately get the following result.

Lemma 2. If $P \in P_\eta$ does not vanishes in $|z| < h, h \ge 1$, then for any complex number τ with $|\tau| \le 1$,

$$\Gamma|P'(z) + \tau\eta m z^{n-1}| \le |\mathcal{Q}'(z)| \tag{22}$$

where Γ is defined by (15) and $m = \underset{|z|=h}{min}|P(z)|$.

Lemma 3. If $P \in P_\eta$ vanishes in $|z| \le h, h \le 1$, then on $|z| = 1$

$$|\mathcal{Q}'(z)| \le S|P'(z) + \eta m\tau z^{\eta-1}| \tag{23}$$

where S is defined by (17).

Proof of Lemma 3: Since $P(z)$ vanishes in $|z| \le h, h \le 1$, then the polynomial $\mathcal{Q}(z) = z^\eta \overline{P(\frac{1}{\overline{z}})}$ does not vanishes in $|z| < \frac{1}{h}, \frac{1}{h} \ge 1$. Thus applying Lemma (2) to the polynomial $\mathcal{Q}(z)$, we have

$$|\mathcal{Q}'(z)| \le h\frac{(1-|\beta|)(|\frac{1}{h^2}\beta| + 1) + \frac{1}{h}(\eta-1)|\gamma - \beta^2|}{(1-|\beta|)(|\beta| + h^2) + \frac{1}{h}(\eta-1)|\gamma-\beta^2|}\left|P'(z) + \eta m\tau z^{\eta-1}\right|$$

Where

$$\beta = \frac{1/h}{\eta}\frac{\overline{a}_{\eta-1}}{\overline{a}_\eta} = \frac{\overline{a}_{\eta-1}}{\eta h\overline{a}_\eta} \text{ and } \gamma = \frac{2/h^2}{\eta(\eta-1)h^2}\frac{\overline{a}_{\eta-2}}{\overline{a}_\eta} = \frac{2\overline{a}_{\eta-2}}{\eta(\eta-1)h^2\overline{a}_\eta}.$$

Then,

$$|\mathcal{Q}'(z)| \le h\frac{(1-|\beta|)(|\beta| + h^2) + h(\eta-1)|\gamma-\beta^2|}{(1-|\beta|)(|\beta|h^2 + 1) + h(\eta-1)|\gamma-\beta^2|}\left|P'(z) + \eta m\tau z^{\eta-1}\right|$$

which proves Lemma (3).

Lemma 4. If $P \in P_\eta$, then for every l, $0 \le l < 2\pi$ and $\mathsf{q} > 0$

$$\int_0^{2\pi}\int_0^{2\pi}|\mathcal{Q}'(e^{i\Theta}) + e^{il}P'(e^{i\Theta})|^{\mathsf{q}}d\Theta dl \le 2\pi\eta^{\mathsf{q}}\int_0^{2\pi}|P(e^{i\Theta})|^{\mathsf{q}}d\Theta \tag{24}$$

The above Lemma (4) is due to Aziz [2].

Lemma 5. If $P \in P_\eta$, then for every l, $0 \le l < 2\pi$,$\mathrm{q} > 0$ and for any complex number τ with $|\tau| \le 1$,

$$\int_0^{2\pi}\int_0^{2\pi} |\mathcal{Q}'(e^{i\Theta}) + e^{il}\left\{P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right\}|^{\mathrm{q}} d\Theta dl \le 2\pi\eta^{\mathrm{q}} \int_0^{2\pi} |P(e^{i\Theta})|^{\mathrm{q}} d\Theta \tag{25}$$

Proof of Lemma 5: By applying Lemma (4) to the polynomial $P(z) = P(z) + m\tau z^n$, we can easily get the proof of Lemma (5).

Lemma 6. Let z be any complex and independent of l , where l is any real, then for $\mathrm{q} > 0$

$$\int_0^{2\pi} |1 + ze^{il}|^{\mathrm{q}} dl = \int_0^{2\pi} |e^{il} + |z||^{\mathrm{q}} dl \tag{26}$$

Lemma (6) is due to Gardner and Govil [4].

4. Proofs of Theorems

Proof of Theorem 5: As $P(z)$ does not vanishes in $|z| < h, h \ge 1$ hence, by Lemma (2), we have

$$\Gamma|P'(z) + \tau\eta m z^{\eta-1}| \le |\mathcal{Q}'(z)| \tag{27}$$

where Γ is defined by (15) and $m = \min\limits_{|z|=h}|P(z)|$.
For each Real l and $G \ge r \ge 1$, we have

$$|G + e^{il}| \ge |r + e^{il}|$$

Then, for every $\mathrm{q} > 0$, we have

$$\int_0^{2\pi} |G + e^{il}|^{\mathrm{q}} dl \ge \int_0^{2\pi} |r + e^{il}|^{\mathrm{q}} dl \tag{28}$$

For points $e^{i\Theta}$, $0 \le \Theta < 2\pi$, for which $P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}$ does not vanishes, we denote $G = \left|\dfrac{\mathcal{Q}'(e^{i\Theta})}{P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}}\right|$ and $r = \Gamma$, then by (27), we for every $\mathrm{q} > 0$

$$\begin{aligned}
&\int_0^{2\pi} \left|\mathcal{Q}'(e^{i\Theta}) + e^{il}\left\{P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right\}\right|^{\mathrm{q}} dl \\
&= \left|P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathrm{q}} \int_0^{2\pi} \left|\frac{\mathcal{Q}'(e^{i\Theta})}{P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}} + e^{il}\right|^{\mathrm{q}} dl \\
&= \left|P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathrm{q}} \int_0^{2\pi} \left|\left|\frac{\mathcal{Q}'(e^{i\Theta})}{P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}}\right| + e^{il}\right|^{\mathrm{q}} dl \quad by \quad \text{Lemma}(6) \\
&= \left|P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathrm{q}} \int_0^{2\pi} \left|G + e^{il}\right|^{\mathrm{q}} dl \\
&\ge \left|P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathrm{q}} \int_0^{2\pi} \left|r + e^{il}\right|^{\mathrm{q}} dl
\end{aligned}$$

Using (28),hence,

$$\int_0^{2\pi} \left|\mathcal{Q}'(e^{i\Theta}) + e^{il}\left\{P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right\}\right|^{\mathrm{q}} dl \ge \left|P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}\right|^{\mathrm{q}} \int_0^{2\pi} \left|\Gamma + e^{il}\right|^{\mathrm{q}} dl \tag{29}$$

for $e^{i\Theta}$, $0 \leq \Theta < 2\pi$, for which $P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta}$ does not vanishes.
For points $e^{i\Theta}$, $0 \leq \Theta < 2\pi$, for which $P'(e^{i\Theta}) + nm\tau e^{i(n-1)\Theta}$ vanishes, (29) trivially holds. Hence, using (29) in Lemma (5), we get for each $\mathfrak{q} > 0$,

$$\int_0^{2\pi} \left| P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta} \right|^{\mathfrak{q}} d\Theta \int_0^{2\pi} \left| \Gamma + e^{il} \right|^{\mathfrak{q}} \leq 2\pi\eta^{\mathfrak{q}} \int_0^{2\pi} |P(e^{i\Theta})|^{\mathfrak{q}} d\Theta$$

which is equivalent to

$$\left\{ \int_0^{2\pi} \left| P'(e^{i\Theta}) + \eta m\tau e^{i(\eta-1)\Theta} \right|^{\mathfrak{q}} d\Theta \right\}^{\frac{1}{\mathfrak{q}}} \leq \eta \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \Gamma + e^{il} \right|^{\mathfrak{q}} dl \right\}^{-\frac{1}{\mathfrak{q}}} \left\{ \int_0^{2\pi} \left| P(e^{i\Theta}) \right|^{\mathfrak{q}} d\Theta \right\}^{\frac{1}{\mathfrak{q}}} \tag{30}$$

which proves Theorem (5).

Proof of Theorem 7: Since $P(z)$ vanishes in $|z| \leq h, h \leq 1$, $P^{'}(z)$ also vanishes in $|z| \leq h, h \leq 1$. Hence, by Gauss-Lucas Theorem,

$$z^{\eta-1}\overline{P(\frac{1}{\overline{z}})} = \eta \mathcal{Q}(z) - z\mathcal{Q}'(z) \tag{31}$$

vanishes in $|z| \geq \frac{1}{h}, \frac{1}{h} \geq 1$. Further, since $P(z)$ vanishes in $|z| \leq h, h \leq 1$, we have by Lemma (3),

$$\begin{aligned} |\mathcal{Q}'(z)| &\leq S|P'(z) + \eta m\tau z^{\eta-1}| \\ &= S\left\{|P'(z)| + t\eta m\right\} \quad for \;\; |z| = 1. \end{aligned} \tag{32}$$

where S is defined by (17) and $|\tau| = t$.
For $|z| = 1$, we also have

$$|P'(z) = |\eta \mathcal{Q}(z) - z\mathcal{Q}'(z)|. \tag{33}$$

Using (33) in (32) , we have on $|z| = 1$

$$|\mathcal{Q}'(z)| \leq S\left\{|\eta \mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m\right\} \tag{34}$$

Thus, by (31) and (34),

$$\psi(z) = \frac{z\mathcal{Q}'(z)}{S\left\{|\eta \mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m\right\}}$$

is analytic in $|z| \leq 1, |\psi(z)| \leq 1$ on $|z| = 1$ and $\psi(0) = 0$. Therefore, $1 + S\psi(z)$ is subordinate to the function $1 + Sz$ for $|z| \leq 1$. Hence, by a well known property of subordination [8], we have for every $\mathfrak{q} > 0$

$$\int_0^{2\pi} |1 + S\psi(e^{i\Theta})|^{\mathfrak{q}} d\Theta \geq \int_0^{2\pi} |1 + Se^{i\Theta}|^{\mathfrak{q}} d\Theta \tag{35}$$

Now,

$$1 + S\psi(z) = 1 + \frac{z\mathcal{Q}'(z)}{|\eta \mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m} = \frac{\eta \mathcal{Q}(z)}{|\eta \mathcal{Q}(z) - z\mathcal{Q}'(z)| + t\eta m}$$

which implies for $|z| = 1$,

$$\begin{aligned} |\eta \mathcal{Q}(z)| =& |1 + S\psi(z)| \left| \eta \mathcal{Q}(z) - z\mathcal{Q}'(z) \right| + t\eta m \\ &= |1 + S\psi(z)| \left| P'(z) \right| + t\eta m \quad by \quad (33) \end{aligned}$$

As $|P(z)| = |\mathcal{Q}(z)|$ on $|z| = 1$, by preceding inequality we have

$$\eta|P(z)| = |1 + S\psi(z)| \left| P'(z) \right| + t\eta m \quad on \quad |z| = 1 \tag{36}$$

Then for every $q > 0$ and $0 \leq \Theta < 2\pi$, we have

$$\eta^{q} \int_0^{2\pi} |P(e^{i\Theta})|^{q} d\Theta = \int_0^{2\pi} |1 + S\psi(e^{i\Theta})|^{q} \left\{ \left| P'(e^{i\Theta}) \right| + t\eta m \right\}^{q} d\Theta$$

Applying Holder's inequality to the above inequality, we have for $r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$ and for every $q > 0$

$$\eta^{q} \int_0^{2\pi} |P(e^{i\Theta})|^{q} d\Theta \leq \left\{ \int_0^{2\pi} \left| 1 + S\psi(e^{i\Theta}) \right|^{rq} d\Theta \right\}^{\frac{1}{r}} \left\{ \int_0^{2\pi} \left\{ \left| P'(e^{i\Theta}) \right| + t\eta m \right\}^{sq} d\Theta \right\}^{\frac{1}{s}}$$

which implies

$$\eta \left\{ \int_0^{2\pi} |P(e^{i\Theta})|^{q} d\Theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} \left| 1 + S\psi(e^{i\Theta}) \right|^{rq} d\Theta \right\}^{\frac{1}{qr}} \left\{ \int_0^{2\pi} \left\{ \left| P'(e^{i\Theta}) \right| + t\eta m \right\}^{sq} d\Theta \right\}^{\frac{1}{sq}}$$

using (35) in the above inequality , we have

$$\eta \left\{ \int_0^{2\pi} |P(e^{i\Theta})|^{q} d\Theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} \left| 1 + Se^{i\Theta} \right|^{rq} d\Theta \right\}^{\frac{1}{qr}} \left\{ \int_0^{2\pi} \left\{ \left| P'(e^{i\Theta}) \right| + t\eta m \right\}^{sq} d\Theta \right\}^{\frac{1}{sq}}$$

by choosing argument of τ as in the proof of Theorem (6) , we get the above inequality as

$$\eta \left\{ \int_0^{2\pi} |P(e^{i\Theta})|^{q} d\Theta \right\}^{\frac{1}{q}} \leq \left\{ \int_0^{2\pi} \left| 1 + Se^{i\Theta} \right|^{rq} d\Theta \right\}^{\frac{1}{qr}} \left\{ \int_0^{2\pi} \left| P'(e^{i\Theta}) + \tau\eta m e^{i(n-1)\Theta} \right|^{sq} d\Theta \right\}^{\frac{1}{sq}}$$

which proves the theorem.

5. CONCLUSION

One of the interesting results on the L_q inequalities of complex polynomials is Aziz and Rathers™s extension of Malik™s inequality for the class of polynomials having all zeros in the disk $|z| \leq k, k \leq 1$. Krishnadas and Chanam [9] demonstrated the generalized form of that inequality. They succeeded in obtaining the L_q analogue of an inequality concerning a polynomial $P(z) = \sum_{v=0}^{n} a_v z^v$ of degree n having no zero in $|z| \leq k, k \geq 1$ instead of having all zeros in $|z| \leq k, k \leq 1$ .In this paper of our discussion, We proved a refinement of the inequality due to Krishnadas and Chanam [9] to the class of polynomials $P(z) = \sum_{v=0}^{n} a_v z^v$ having no zeros in $|z| \leq K, K \geq 1$ with the help of which several more results can be generalised.

References

[1] Arestov, V. V. (1982). On inequalities for trigonometric polynomials and their derivative. *Mathematics of the USSR-Izvestiya*, 18:1-17.

[2] Aziz, A. and Rather, N. A. (2004). Some Zygmund type Lq inequalities for polynomials. *Journal of Mathematical Analysis and Applications*, 289:14-29.

[3] Bruijn, N. G.(1947). Inequalities concerning polynomials in the complex domain. *Nederal. Akad. Wetensch. Proceeding*, 50:1265-1272.

[4] Gardner, R. B. and Govil, N. K. (1995). An Lq inequality for a polynomial and its derivative. *Journal of Mathematical Analysis and Applications*, 194:720-726.

[5] Gardner, R. and Weems, A. (1998). A Bernstein-type Lp inequality for a certain class of polynomials. *Journal of Mathematical Analysis and Applications*, 219:472-478.

[6] Govil, N. K. and Rahman, Q. I. (1969). Functions of exponential type not vanishing in a half-plane and related polynomials. *Transactions of the American Mathematical Society*, 137:501-517.

[7] Govil, N.K. , Rahman, Q. I. and Schmeisser, G. (1979). On the derivative of a polynomial. *Illinois Journal of Mathematics*, 23:319-329.

[8] Hille, E. Analytic Function Theory, Vol. II, Ginn. and Company, New York, Toronto (1962).

[9] Krishnadas, K. and Chanam, B. (2021). L^{γ} Inequalities Concerning Polynomials. *International Journal of Applied Mathematics*, 34:979-994.

[10] Lax, P. D. (1944). Proof of a Conjecture of P. Erdos on the derivative of a polynomial. *Bulletin of the American Mathematical Society*, 50:509-513.

[11] Malik, M. A. (1969). On the derivative of a polynomial. *Journal of the London Mathematical Society*, 57-60.

[12] Rahman, Q. I. and Schmeisser, G. (1988). Lp inequalities for polynomials. *Journal of Approximation Theory*, 53:26-32.

[13] Rather, N. A. Extremal Properties and Location of the Zeros of Polynomi- als. PhD. Thesis, University of Kashmir, (1998).

[14] Turan, P. Uber die ableitung von polynomen. *Compositio Mathematica*, 7:89-95(1939).

[15] Zygmund, A. (1932). A remark on conjugate series. *Proceedings of London Mathematical Society*, 34:392-400.