ALPHA LOGARITHMIC TRANSFORMATION OF LOMAX DISTRIBUTION WITH PROPERTIES AND APPLICATION TO SURVIVAL DATA

K.M.Sakthivel[1], Nandhini V[2], and Vidhya G[3]

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[1]Professor, Department of Statistics, Bharathiar University, Coimbatore-46
[2]Assistant Professor, Department of Mathematics, Karpagam Academy of Higher Education, Coimbatore-21
[3]Assistant Professor, Department of Mathematics, Rathinam College of Arts and Science, Coimbatore-21
Corresponding Author: nandhinivtvt@gmail.com

Abstract

Probability distribution plays a vital role in lifetime data analysis, reliability modeling, and survival studies. Classical models often exhibit limitations in flexibility when dealing with diverse real-world datasets. To address this gap, we introduce an extended model that improves the ability to capture various shapes of lifetime distributions. In this paper, we propose a new three-parameter probability distribution, termed the Alpha Logarithmic Transformation of Lomax (ALTLx) distribution, which proves to be effective for modeling lifetime data. The proposed distribution is studied in detail, and several of its statistical properties are derived, including moments, incomplete moments, quantiles, mean residual life, mean inactivity time, entropy, order statistics, and stress strength analysis. Parameter estimation is carried out using the method of maximum likelihood to ensure reliable inference. The practical applicability of the ALTLx distribution is demonstrated by fitting it to datasets representing the survival time of patients. The results indicate that it outperforms several well-known competing distributions, showing that the proposed model provides a significantly better fit. The distribution exhibits greater flexibility and effectively captures diverse patterns in lifetime data. The findings confirm that the ALTLx distribution offers superior performance in modeling lifetime and reliability data.

Keywords: Alpha logarithmic G family, Lomax distribution, Parameters, Entropy, Maximum likelihood estimation.

1. INTRODUCTION

The Lomax or Pareto Type II (shifted Pareto) distribution (1954) was proposed by K.S. Lomax. This distribution has found wide application, such as analyzing datasets from actuarial science, medical and biological sciences, and reliability modelling. Hassan and Al-Ghamdi [6] mentioned that it is used for reliability modelling and life testing [16], Harris used the Lomax distribution for income and wealth data [15], Bryson has suggested the use of Lomax distributionand it is an alternative to the exponential, Weibull and gamma distributions [8], Atkinson and Harrison used

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it for modelling the business failure data [6]. The characterization of the Lomax distribution is described in numerous ways, such as a special form of Pearson type VI distribution and also as a mixture of gamma and exponential distributions. Some details on the Lomax and Pareto families are in Arnold [5] and Johnson et al. [17].

A random variable X has the Lomax distribution with two parameters β and λ if it has a cumulative distribution function (cdf) given by

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\beta} \quad ; \text{for } x > 0 \tag{1}$$

where $\beta > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively.
The corresponding probability density function (pdf) is given by

$$f(x) = \frac{\beta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\beta+1)} ; \text{for } x > 0,\ \beta, \lambda > 0 \tag{2}$$

In the literature, some extensions of the Lomax distribution are available, such as follows: Marshall-Olkin–Olkin extended-Lomax (MOEL) by Ghitany et al. [14], Exponentiated Lomax (EL) by Abdul-Moniem [1], Gamma-Lomax (GL) by Cordeiro et al. [9], Exponential Lomax by El-Bassiouny [13], and Half-logistic Lomax(HLL) by Anwar [4]. Pappas et al. [23] introduced a new Alpha Logarithmic Transformed (ALT) family of distributions with the cdf and pdf as follows:

$$F_{ALT}(x;\alpha) = \begin{cases} 1 - \frac{log(\alpha-(\alpha-1)G(x))}{\log\alpha} & ; if\ \alpha > 0, \alpha \neq 1, x \in R \\ G(x) & ; \ if\ \alpha = 1,\ x \in R \end{cases} \tag{3}$$

$$f_{ALT}(x,\alpha) = \begin{cases} \frac{(\alpha-1)g(x)}{log(\alpha-(\alpha-1)G(x))} & ; \ if\ \alpha > 0,\ \alpha \neq 1,\ x \in R \\ g(x) & ; \ if\ \alpha = 1,\ x \in R \end{cases} \tag{4}$$

Note that when $\alpha = 2$, the cdf and pdf in (3) and (4) reduce to the cdf and pdf of the logarithmic transformed method proposed by Maurya et al. [20]. Dey et al.[10] used the method of Pappas et al [23] introduced an ALT generalized exponential distribution and ALT Fréchet Distribution discussed by Dey et al. [12].

The main aim of this paper is to propose and study a new lifetime model called Alpha Logarithmic Transformed Lomax (ALTLx) distribution based on the ALT-G family. The purpose of this new model is that the additional parameter can give several properties and be more flexible in the form of the hazard and density functions. In Section .2, we derive the expansion of the Alpha Logarithmic Transformed G family of distribution, and we introduce the ALTLx distribution. In Section 3, we study some of its statistical properties, including the quantile function, moments, moment generating function, entropy, order statistics, and stress strength reliability. In Section 4, we discuss the maximum likelihood estimates (MLEs) of the model parameter. In Section 5, we perform a simulation to evaluate the effectiveness of a newly proposed ALTLx distribution. In Section 6, the analysis of real data sets is presented to illustrate the potential of the new model. Section.7, we conclude the study.

2. ALPHA LOGARITHMIC TRANSFORMED-G FAMILY OF DISTRIBUTION

Let G(x) be an absolutely continuous distribution function with pdf g(x), then the cdf and pdf of the ALT family of distributions and its expansions are as follows:

$$F_{ALT}(x;\alpha) \ = \left(1 - \frac{log(\alpha-(\alpha-1)G(x))}{\log\alpha}\right); \ if\ \alpha > 0,\ \alpha \neq 1$$

$$F_{ALT}(x;\alpha) = \left(\frac{\log\alpha - \log(\alpha+1)}{\log\alpha}\right) - \frac{1}{\log\alpha}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1+(\alpha-1)G(x)}{\alpha+1}\right)^{k}$$

$$F_{ALT}(x;\alpha) = C(\alpha) - \frac{1}{\log\alpha}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1+(\alpha-1)G(x)}{\alpha+1}\right)^{k},$$

where, $C(\alpha) = \frac{\log\alpha - \log(\alpha+1)}{\log\alpha}$

$$F_{ALT}(x;\alpha) = C(\alpha) - \frac{1}{\log\alpha}\sum_{k=1}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(\alpha-1)^j}{k(\alpha+1)^k}\left(G(x)\right)^j \; ; if\ \alpha > 0,\ \ \alpha \neq 1$$

$$f_{ALT}(x,\alpha) = \frac{(\alpha-1)g(x)}{log\left(\alpha-(\alpha-1)G(x)\right)} \;\; ; if\ \alpha > 0,\ \ \alpha \neq 1,\ \ x \in R$$

$$= \frac{(\alpha-1)g(x)}{\log\alpha\ (\alpha+1)}\left[\frac{(\alpha+1)-1-(\alpha-1)G(x)}{\alpha+1}\right]^{-1}$$

$$f_{ALT}(x,\alpha) = \frac{g(x)}{\log\alpha}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}}[G(x)]^j \; ; if\ \alpha > 0,\ \ \alpha \neq 1,\ \ x \in R$$

The random variable X is said to follow the three-parameter ALTLx distribution with the shape parameter $\beta > 0$ and scale parameter $\lambda > 0$ if the cumulative cdf of $x > 0$ is given by:

$$F_{ALTLx}(x;\alpha) = \begin{cases} 1 - \dfrac{log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}{\log\alpha} & ;\ if\ \alpha > 0, \alpha \neq 1, x \in R \\ \left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right) & ;\quad if\ \alpha = 1,\ \ x \in R \end{cases} \tag{5}$$

The corresponding probability density function (pdf) is given by

$$f_{ALTLx}(x,\alpha) = \begin{cases} \dfrac{(\alpha-1)\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)}{\log\alpha\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)} & ;\ \ if\ \alpha > 0,\ \ \alpha \neq 1,\ \ x \in R \\ \left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right) & ;\ \ if\ \alpha = 1,\ \ x \in R \end{cases} \tag{6}$$

The hazard rate function of the ALTLx distribution is as follows

$$h_{ALTLx}(x;\alpha) = \begin{cases} \dfrac{(\alpha-1)\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)}{\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)\log\alpha\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)} & ; if\alpha > 0, \alpha \neq 1 \\ \dfrac{\beta}{\lambda+x} & ;\ if\ \alpha = 1 \end{cases} \tag{7}$$

The reversed hazard rate function of the ALTLx distribution is as follows

$$\tau_{ALTLx}(x) = \begin{cases} \dfrac{(\alpha-1)\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)}{\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)\times\left(\log\alpha - \log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)\right)} & ; if\ \alpha > 0,\ \ \alpha \neq 1 \\ \dfrac{\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}}{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)} & ; if\ \alpha = 1 \end{cases} \tag{8}$$

The survival rate function of the ALTLx distribution is given by

$$S_{ALTLx}(x;\alpha) = \begin{cases} \dfrac{log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}{\log\alpha} & ;\ if\ \alpha>0,\ \ \alpha\neq 1 \\ \left(1+\frac{x}{\lambda}\right)^{-\beta} & ;if\ \alpha=1 \end{cases} \tag{9}$$

The graphical representation of the ALTLx distribution is as follows.

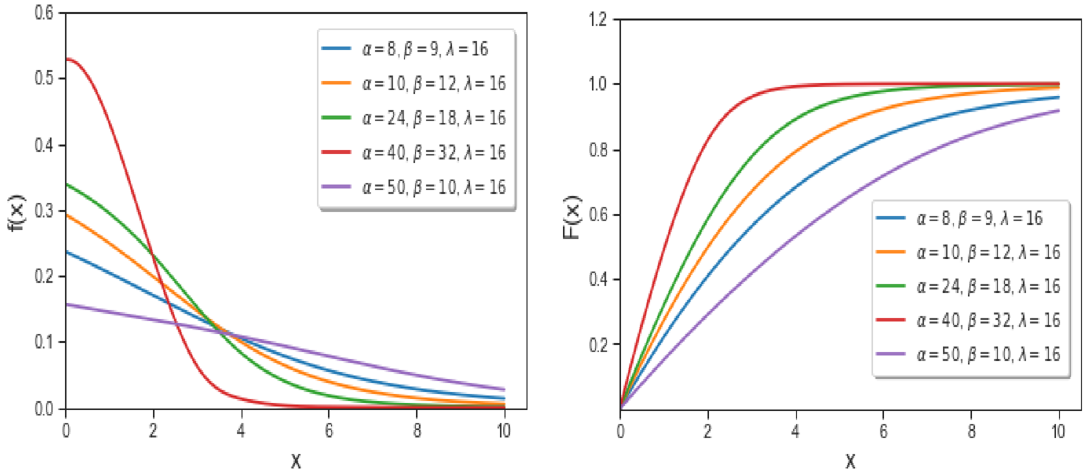

Figure 1: The pdf and cdf plot of the ALTLx distribution

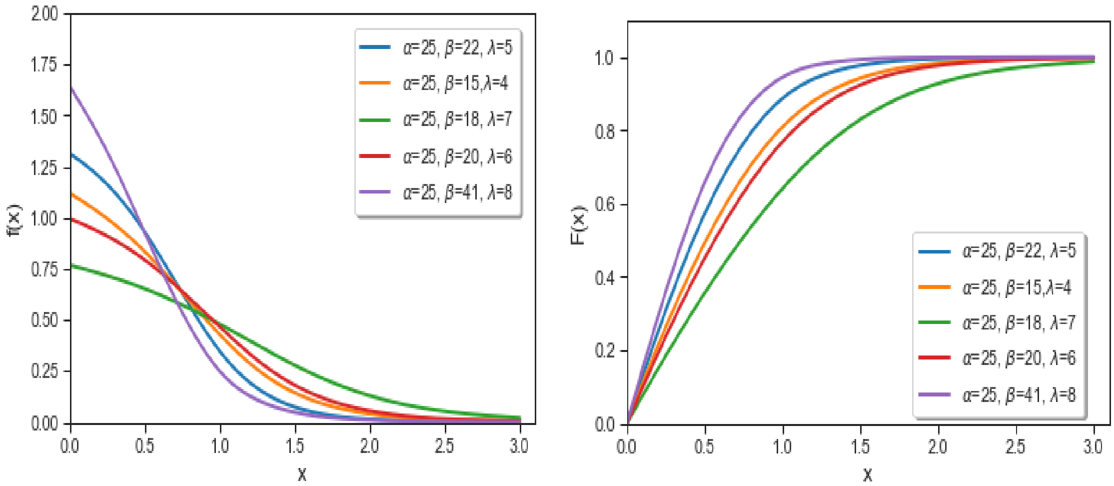

Figure 2: The pdf and cdf plot of the ALTLx distribution

Figures 1 and 2 present the pdf and cdf of the ALTLx distribution for a fixed value of λ. In Figure 1, we fix the value of the parameter $\lambda = 16$ and vary α and β, the pdf exhibits a decreasing or reversed J-shaped pattern, while the cdf is strictly increasing towards unity as x increases. It is observed that variations in the parameters α, and β influence the shapes of the pdf and the speed of convergence in the cdf. In Figure 2, we fix the parameter $\alpha = 25$ and vary the shape parameters λ, and β. The cdf curves demonstrate the expected monotonic increasing behavior, asymptotically approaching unity. Furthermore, changes in β and λ alter the steepness of the cdf, with higher values leading to faster convergence towards 1.

3. STATISTICAL PROPERTIES

In this section, the statistical properties of the APTLx distribution are derived for the case of $\alpha \neq 1$, since for the case of $\alpha = 1$ it is simply the properties of the Lomax distribution.

3.1. Quantile Function

The quantile function plays an important role when simulating random variates from a statistical distribution. The ALTLx distribution can be simulated by using equation (3).

$$X = \lambda \left[\frac{(\alpha^{(1-U)} - \alpha)}{(\alpha - 1)}\right]^{-\frac{1}{\beta}} - \lambda \tag{10}$$

where U follows a uniform (0, 1) distribution.
The p^{th} quantile function of the ALTLx distribution is

$$x_p = Q(p) = \lambda \left[\frac{(\alpha^{(1-p)} - \alpha)}{(\alpha - 1)}\right]^{-\frac{1}{\beta}} - \lambda \tag{11}$$

In particular, the first three quantiles Q_1, Q_2, Q_3 can be obtained by using $p = 0.25, p = 0.5$, $p = 0.75$ in equation (4.9) respectively.

3.2. Moments and Moment Generating Function

The rth moment of the random variable X having ALTLx distribution is obtained as follows:

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx$$

$$\mu_r' = \frac{\beta \lambda^r}{\log \alpha} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{j} (-1)^l \binom{k}{j} \binom{j}{l} \frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}} \times B[r+1,\ \beta l + \beta - r] \tag{12}$$

The first moment and mean of the ALTLx distribution is

$$\mu_1' = \frac{\beta \lambda}{\log \alpha} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{j} (-1)^l \binom{k}{j} \binom{j}{l} \frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}} \times B[2,\ \beta l + \beta - 1] \tag{13}$$

The Moment generating function of ALTLx is

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx$$

$$M_x(t) = \frac{\beta}{\log \alpha} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{j} (-1)^l \binom{k}{j} \binom{j}{l} \frac{\lambda^i t^i (\alpha-1)^{j+1}}{i!(\alpha+1)^{k+1}} \times B[i+1,\ \beta l + \beta - i] \tag{14}$$

The rth incomplete moment of X, denoted by $\varphi_r(t)$, is

$$\varphi_r(t) = \int_0^t x^r f(x) dx$$

$$\phi_r(t) = \frac{\beta \lambda^r}{\log \alpha} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{j} (-1)^l \binom{k}{j} \binom{j}{l} \frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}} \times B_t[r+1, \beta l + \beta - r] \tag{15}$$

The first incomplete moment X, denoted by $\varphi_1(t)$, is

$$\phi_1(t) = \frac{\beta \lambda}{\log \alpha} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{l=0}^{j} (-1)^l \binom{k}{j} \binom{j}{l} \frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}} \times B_t[2, \beta l + \beta - 1] \tag{16}$$

3.3. Lorenz curve and Bonferroni curve

The Lorenz curves, say $LO(x)$, is defined as

$$LO(x) = \frac{\varphi_1(x)}{E(x)}$$

$$LO(x) = \left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^l \binom{k}{j}\binom{j}{l}\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}} \times B_t[2, \beta l + \beta - 1]\right) \times \left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^l \binom{k}{j}\binom{j}{l}\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}} \times B[2, \beta l + \beta - 1]\right)^{-1} \quad (17)$$

Bonferroni curve $BO(x)$ is defined as

$$BO(x) = \frac{LO(x)}{F_{ALTLx}(x)}$$

$$BO(x) = \frac{\left(\begin{array}{c}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^l\binom{k}{j}\binom{j}{l}\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}}.B_t[2,\beta l+\beta-1] \times \\ \left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^l\binom{k}{j}\binom{j}{l}\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}}.B[2,\beta l+\beta-1]\right)^{-1}\end{array}\right)}{1-\frac{log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}{\log\alpha}} \quad (18)$$

3.4. Mean Residual Life Time and Mean Inactivity Time

The mean residual lifetime of the ALTLx distribution is given by

$$m_x(t) = E(X - t/X > t),\ t > 0$$

$$m_x(t) = \frac{\mu-\varphi_1(t)}{S(t)} - t \quad \text{where, } \mu = \mu_1'$$

$$m_x(t) = \frac{\mu-\left(\frac{\beta\lambda}{\log\alpha}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^l\binom{k}{j}\binom{j}{l}\times\left(\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}}\right)\times B_t[2,\beta l+\beta-1]\right)}{\frac{log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}{\log\alpha}} - t \quad (19)$$

The mean inactivity time is given by

$$\psi_x(t) = E(X - t/X < t)$$

$$\psi_x(t) = t - \frac{\varphi_1(t)}{F(t)}$$

$$\psi_x(t) = t - \frac{\left(\frac{\beta\lambda}{\log\alpha}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^l\binom{k}{j}\binom{j}{l}\times\left(\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}}\right)\times B_t[2,\beta l+\beta-1]\right)}{1-\frac{log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}{\log\alpha}} \quad (20)$$

3.5. Rényi entropy and δ- entropy

Entropy is a measure of randomness of systems and it is widely used in areas like physics, molecular imaging of tumors. The entropy of the random variable X measures the variation of uncertainty.

The Rényi entropy is defined as

$$RE_X(v) = \frac{1}{1-v}\log\left(\int_{-\infty}^{\infty} f(x)^v dx\right), v > 0, v \neq 1$$

Rényi entropy of the ALTLx distribution is given below

$$RE_x(v) = \frac{1}{1-v} log\left\{\int_0^{\infty}\left(\frac{(\alpha-1)\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)}{\log\alpha\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}\right)^v dx\right\}$$

$$RE_x(v) = \frac{v}{1-v}\left(log\left(\frac{(\alpha-1)\beta}{\alpha\lambda.\log\alpha}\right)\right) + \frac{1}{1-v}\ log\left\{\begin{matrix}\left(\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\binom{v+j-1}{j}\left[\frac{\alpha-1}{\alpha}\right]^j\right) \\ \times\left(\frac{\lambda}{\beta(k+v)+v-1}\right)\end{matrix}\right\} \quad (21)$$

The $\delta -$ entropy can be defined as

$$I_x(\delta) = \frac{1}{1-\delta}\ log\{1 - \int_{-\infty}^{\infty} f(x)^{\delta} dx\},\ \delta > 0$$

The $\delta -$ entropy of the ALTLx distribution is given below

$$I_x(\delta) = \frac{1}{1-\delta}\ log\left\{1 - \int_0^{\infty}\left(\frac{(\alpha-1)\left(\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)}{\log\alpha\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}\right)^{\delta} dx\right\}$$

$$I_x(\delta) = \frac{1}{1-\delta}\ log\left\{1 - \left(\frac{(\alpha-1)\beta}{\alpha.\log\alpha}\right)^{\delta} \times \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\binom{\delta+j-1}{j}\left[\frac{\alpha-1}{\alpha}\right]^j \times \left(\frac{1}{\beta(k+\delta)+\delta-1}\right)\right\} \quad (22)$$

3.6. Stress strength reliability

The stress strength parameter, $S = P(X_1 < X_2)$, in the lifetime model, describes the lifetime component, which has a random stress X_1 that is subjected to a random strength X_2.

Let X_1 and X_2 be the two continuous and independent random variables, where $X_1 \sim ALTLx(\alpha_1, \beta_1, \lambda)$ and $X_2 \sim ALTLx(\alpha_2, \beta_2, \lambda)$ then the stress strength parameter, say S is defined as follows:

$$S = \int_{-\infty}^{\infty} f_1(x)F_2(x)dx$$

$$= \int_0^{\infty}\left(\frac{(\alpha_1-1)\left(\frac{\beta_1}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta_1+1)}\right)}{\log\alpha_1\left[\alpha_1-(\alpha_1-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta_1}\right)\right]}\right) \times \left(1 - \frac{log\left[\alpha_2-(\alpha_2-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta_2}\right)\right]}{\log\alpha_2}\right)dx$$

$$S = \frac{\beta_1}{\log\alpha_1\log\alpha_2}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{n=1}^{\infty}\sum_{j=0}^{n}\binom{k}{i}\binom{n}{j}\frac{1}{n}\left[\frac{(\alpha_1-1)}{\alpha_1}\right]^{k+1}\left[\frac{(\alpha_2-1)}{\alpha_2}\right]^{n} \times \left(\frac{(-1)^{i+j}}{\beta_1+i\beta+j\beta_2}\right) \quad (23)$$

3.7. Order statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n, and let $X_{k:n}$ denotes that kth order statistic, then the pdf of $X_{k:n}$ is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\ F(x)^{k-1} f(x)(1-F(x))^{n-k}$$

We can rewrite the above equation as follows:

$$f_{k:n}(x) = \frac{1}{B(k,n-k+1)}\ F(x)^{k-1} f(x)(1-F(x))^{n-k}$$

$$f_{k:n}(x) = \frac{1}{B(k,n-k+1)}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\left(\frac{log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}{\log\alpha}\right)^{m+n-k}\left(\frac{\beta(\alpha-1)\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}}{\lambda\log\alpha\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}\right) \quad (24)$$

4. MAXIMUM LIKELIHOOD ESTIMATION

Let $x_1, x_2, \dots, x_n$ be a random sample from the ALTLx distribution with unknown parameters (α, β, λ) then the likelihood function is given by

$$l = l(\alpha, \beta, \lambda) = \log\big(L(\alpha, \beta, \lambda)\big)$$

$$l = \log\left(\prod_{i=1}^{n}\left(\frac{(\alpha-1)\left(\frac{\beta}{\lambda}\left(1+\frac{x_i}{\lambda}\right)^{-(\beta+1)}\right)}{\log\alpha\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)\right)}\right)\right)$$

$$= n\log\left(\frac{(\alpha-1)\beta}{\lambda.\log\alpha}\right) - (\beta+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n} log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)\right)$$

$$l = n\log(\alpha-1) + n\log\beta - n\log\lambda - n\log(\log\alpha) - (\beta+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n} log\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)\right) \quad (25)$$

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha-1} - \frac{n}{\alpha\log\alpha} + \sum_{i=1}^{n}\left(\frac{\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}{\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)}\right) = 0 \quad (26)$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\left(\log\left(1+\frac{x_i}{\lambda}\right)\right)}{\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)\right)}\right) = 0 \quad (27)$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} - (\beta+1)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda(\lambda+x_i)}\right) + \sum_{i=1}^{n}\frac{(\alpha-1)\left(\frac{\beta x_i}{\lambda^2}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\right)}{\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x}{\lambda}\right)^{-\beta}\right)\right)} = 0 \quad (28)$$

Then the Maximum likelihood estimations of the parameters α, λ, and β can be obtained by solving the system of Equations (26)–(28).

Fisher information I_{ij} matrix for ALTLx distribution given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

$$I_{11} = E\left[-\frac{\partial^2\log L}{\partial\alpha^2}\right], I_{22} = E\left[-\frac{\partial^2\log L}{\partial\beta^2}\right], I_{33} = E\left[-\frac{\partial^2\log L}{\partial\lambda^2}\right]$$

$$I_{12} = I_{21} = E\left[-\frac{\partial^2\log L}{\partial\alpha\partial\beta}\right], I_{13} = I_{31} = E\left[-\frac{\partial^2\log L}{\partial\alpha\partial\lambda}\right], I_{23} = I_{32} = E\left[-\frac{\partial^2\log L}{\partial\beta\partial\lambda}\right]$$

$$\frac{\partial^2\log L}{\partial\alpha^2} = -\frac{n}{(\alpha-1)^2} + \frac{n(\log\alpha+1)}{\alpha^2(\log\alpha)^2} + \sum_{i=1}^{n}\left(\frac{\left(\left(1+\frac{x_i}{\lambda}\right)^{-2\beta}\right)}{\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)\right)}\right)$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} \left(\frac{(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\left(\log\left(1+\frac{x_i}{\lambda}\right)\right)^2}{\left(\alpha-(\alpha-1)\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)\right)^2} \right)$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{n}{\lambda^2} - (\beta+1)\sum_{i=1}^{n} \frac{x_i\left(\lambda(1+x_i)+(\lambda+x_i)\right)}{\left(\lambda(\lambda+x_i)\right)^2}$$
$$-\beta(\alpha-1)\sum_{i=1}^{n} \left\{ \frac{\left(\frac{-(\beta+1)x_i}{\lambda^4}\left(1+\frac{x_i}{\lambda}\right)^{-(\beta+2)} + \frac{x_i}{\lambda^3}\left(1+\frac{x_i}{\lambda}\right)^{-(\beta+1)}\right)}{\left(1-(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)} - \frac{\left(\frac{(\alpha-1)\beta x_i}{\lambda^4}\left(1+\frac{x_i}{\lambda}\right)^{-2(\beta+1)}\right)}{\left(1-(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)^2} - \right\}$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} \left(\frac{\left(\left(1+\frac{x_i}{\lambda}\right)^{-\beta} \log\left(1+\frac{x_i}{\lambda}\right)\right)}{\left(1-(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)^2} \right)$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \lambda} = -\sum_{i=1}^{n} \left(\frac{\left(\beta x_i \left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)}{\lambda^2\left(1-(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)^2} \right)$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \lambda} = \sum_{i=1}^{n} \left(\frac{x_i}{\lambda(\lambda+x_i)}\right) - \sum_{i=1}^{n} \frac{(\alpha-1)x_i}{\lambda^2}\left(1+\frac{x_i}{\lambda}\right)^{-\beta}$$
$$\times \left\{ \frac{\left(1-(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)\left(-\beta\log\left(1+\frac{x_i}{\lambda}\right)+1\right) + \left(\beta\left(1+\frac{x_i}{\lambda}\right)^{-(\beta+1)}\log\left(1+\frac{x_i}{\lambda}\right)\right)}{\left(1-(\alpha-1)\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right)^2} \right\}$$

So we obtain the asymptotic $100(1-\alpha)\%$ confidence intervals of the unknown parameters can be easily obtained for α, β and λ as given in equation below

$$\alpha \in \left[\hat{\alpha} - z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}}, \hat{\alpha} + z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}},\right]$$

$$\beta \in \left[\hat{\beta} - z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}, \hat{\beta} + z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}},\right]$$

$$\lambda \in \left[\hat{\lambda} - z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}, \hat{\lambda} + z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}},\right]$$

where $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ quantile of the standard normal distribution.

5. MONTE CARLO SIMULATION

In this section, we perform a simulation study to assess the performance and examine the mean estimate, average bias (AB), root mean square error (RMSE) of the maximum likelihood estimators, coverage probability (CP), and average width (AW) of the confidence interval for each parameter. We study the performance of the ALTLx distribution by conducting various simulations for varying parameter values and different sample sizes. Quantile function is used to generate random data from the APTLx distribution. The simulation study is repeated for N = 1000 times each with sample size n = 25, 50, 75, 100, and parameter values Case *I:* $\alpha = 1.3, \beta = 0.2, \lambda = 0.7$, and Case *II:* $= 0.3, \beta = 1.5, \lambda = 1.2$.

Table 1: Monte Carlo Simulation Results.

Parameter	n	*Case I:* ($\alpha = 1.3, \beta = 0.2, \lambda = 0.7$)					*Case II:* ($\alpha = 0.3, \beta = 1.5, \lambda = 1.2$)				
		Mean	AB	RMSE	CP	AW	Mean	AB	RMSE	CP	AW
α	25	5.7653	4.4653	14.0604	0.753	73.7764	4.2369	3.9369	13.1231	0.96	81.6628
	50	4.6714	3.3714	12.7215	0.782	41.2183	3.6477	3.3477	17.6423	0.917	56.3358
	75	2.9959	1.6959	7.5258	0.773	19.7701	3.1569	2.8569	11.3905	0.936	41.7517
	100	2.1572	0.8572	5.9026	0.747	11.9081	2.2167	1.9167	6.7837	0.927	25.7067
β	25	0.2382	0.0382	0.1057	0.975	0.4663	2.8629	1.3629	3.7192	0.971	41.4855
	50	0.2317	0.0317	0.0884	0.971	0.3223	2.7093	1.2093	3.6262	0.984	25.9823
	75	0.2215	0.0215	0.0702	0.972	0.2495	2.5082	1.0082	2.2869	0.981	12.3680
	100	0.2118	0.0118	0.0578	0.945	0.2073	2.3548	0.8548	1.9702	0.986	8.7484
λ	25	2.0392	1.3392	4.1942	0.878	14.1571	6.3315	5.1315	12.7943	0.989	148.994
	50	1.3033	0.6033	2.1134	0.869	6.9117	6.9516	5.7516	13.0639	0.995	100.829
	75	1.1719	0.4719	1.9373	0.854	5.2348	6.3696	5.1696	10.3349	0.998	12.3680
	100	1.1389	0.4389	1.7137	0.890	4.5534	5.8881	4.6881	9.6312	0.999	44.1748

From the table, we observe that all the parametric values, the biases decrease when the sample size increases. Also, the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and the average confidence widths decrease as the sample size increases. Hence, the ML estimates of ALTLx distribution are consistent and efficient.

6. REAL LIFE APPLICATION

This section illustrates the ability of the proposed Alpha Logarithmic Transformation of the Lomax distribution by fitting this distribution to real data sets and comparing the results with other competitive distributions in the literature. The real data corresponds to the survival times (years) of 46 patients of a given chemotherapy treatment by Bekker et al. (2000) [7].

Table:2 Estimates, MLE, AIC, BIC, CAIC values

Distribution	Estimate	-log L	AIC	BIC	CAIC
ALTLx	$\alpha = 0.95559$ $\beta = 4809.6316$ $\lambda = 6570.7601$	58.2232	122.4464	127.8664	123.0318
ALTF	$\alpha = 1.085253\text{e}+04$ $\beta = 2.426293$ $\lambda = 8.841114\text{e}-02$	60.7218	127.4437	132.8637	128.0291
APTIL	$\alpha = 3.368942\text{e}-05$ $a = 1.219086$ $b = 7.218801$	58.8082	123.6166	129.0366	124.2019
Inverse Lomax	$\alpha = 3.03907$ $\beta = 0.20014$	61.9325	127.865	131.4783	128.1507

The model selection is carried out by using the maximum likelihood estimation, Akaike information criteria (AIC), the Bayesian information criteria (BIC), and the Consistent Akaike information criteria (CAIC). From Table 2, we conclude that the Alpha Logarithmic Transformation of Lomax (ALTLx) distribution is best when compared to Inverse Lomax (IL), Alpha Power Transformation Inverse Lomax (APTIL), and Alpha Power Transformation of Fréchet distribution (ALTF). Table 3 presents the results of the goodness of fit tests, namely, Kolmogorov-Smirnov (D_n), Cramér von Mises (W_n^2), and Anderson and Darling (A_n^2) tests statistics, for the proposed ALTLx distribution as well as for competitive distributions.

Table 3: Goodness-of-fit tests

Distribution	Anderson-Darling		Cramer Von Mises		Kolmogorov-Smirnov	
	A_n^2	p-value	W_n^2	p-value	D_n	p-value
ALTLx	0.4346	0.813	0.05750	0.8322	0.090714	0.8204
ALTF	1.6628	0.1421	0.3033	0.1321	0.16329	0.1622
APTIL	0.52567	0.7198	0.07635	0.7156	0.09269	0.8003
IL	0.65493	0.5967	0.07241	0.7394	0.13687	0.3370

The ALTLx distribution shows an adequate fit for the given data. Among the considered distributions, the ALTLx model consistently yields the lowest values of all three test statistics. This indicates that the discrepancy between the empirical distribution of the observed survival times and the theoretical distribution under the ALTLx model is minimal compared to the alternative models. Furthermore, the relatively high p-values obtained for the Kolmogorov-Smirnov test provide additional statistical evidence in favor of the adequacy of the ALTLx distribution. These findings confirm that the proposed model captures the underlying structure of the data more effectively. Thus, ALTLx distribution demonstrates the reliability and flexibility for survival data analysis.

7. CONCLUSION

In this paper, we proposed a new three-parameter extension of the Lomax distribution called the ALTLx distribution based on Alpha Logarithmic Transformation. The main aim behind this study is to bring more flexibility to the classical Lomax distribution in modeling lifetime and survival data. Various mathematical properties of the proposed distribution are derived, such as moments, moment generating function, quantile function, and related measures. The parameters of the distribution are estimated through the method of maximum likelihood. The usefulness of the proposed model is illustrated by means of a real-life data set consisting of the survival times of patients undergoing chemotherapy treatment. The results reveal that the ALTLx distribution provides a significantly better fit compared to competing models. These findings highlight the potential of the proposed model for broader applications in survival analysis, reliability modeling, and related fields.

REFERENCES

[1] Abdul-Moniem, I. B. (2012). Recurrence relations for moments of lower generalized order statistics from exponentiated Lomax distribution and its characterization. *Journal of Mathematics and Computer Science.*, *2*(4): 999-1011.

[2] Al-Zahrani, B., and Sagor, H. (2014). The poisson-lomax distribution. *Revista Colombiana de Estadística*, *37*(1): 225-245.

[3] Al-Zahrani, B. (2015). An extended Poisson-Lomax distribution. *Advances in Mathematics: Scientific Journal*, 4(2): 79–89.

[4] Anwar, M., and Zahoor, J. (2018). The Half-Logistic Lomax Distribution for Lifetime Modeling. *Journal of probability and Statistics*, *2018*(1): 3152807.

[5] Arnold, B. C. (1983). *Pareto Distributions.* International Cooperative Publishing House, Maryland.

[6] Atkinson A, and Harrison A (1978) Distribution of personal wealth in Britain. *Cambridge University Press, Cambridge*.

[7] Bekker, A., Roux, J. J., and Mosteit, P. J. (2000). A generalization of the compound Rayleigh distribution: using a Bayesian method on cancer survival times. *Communications in Statistics-Theory and Methods*, *29*(7): 1419-1433.

[8] Bryson, M.C. (1974). Heavy-tailed distributions: properties and tests. *Technometrics*, *16*(1), 61-68.

[9] Cordeiro, G. M., Ortega, E. M., and Popović, B. V. (2015). The gamma-Lomax distribution. *Journal of Statistical computation and Simulation*, *85*(2): 305-319.

[10] Dey, S., Nassar, M., and Kumar, D. (2017). α logarithmic transformed family of distributions with application. *Annals of Data Science*, *4*(4): 457-482.

[11] Dey, S., Alzaatreh, A., Zhang, C., & Kumar, D. (2017). A new extension of generalized exponential distribution with application to ozone data. *Ozone: Science & Engineering*, *39*(4), 273-285.

[12] Dey, S., Nassar, M., Kumar, D., and Alaboud, F. (2019). Logarithm transformed Frechet distribution: Properties and estimation. *Austrian Journal of Statistics*, *48*(1): 70-93.

[13] El-Bassiouny, A. H., Abdo, N. F., and Shahen, H. S. (2015). Exponential lomax distribution. *International journal of computer applications*, *121*(13), 24-29.

[14] Ghitany, M. E., Al-Awadhi, F. A., and Alkhalfan, L. (2007). Marshall–Olkin extended Lomax distribution and its application to censored data. *Communications in Statistics—Theory and Methods*, *36*(10): 1855-1866.

[15] Harris, C. M. (1968). The Pareto distribution as a queue service discipline. *Operations Research*, *16*(2), 307-313.

[16] Hassan, A. S., and Al-Ghamdi, A. S. (2009). Optimum step stress accelerated life testing for Lomax distribution. *Journal of applied sciences research*, *5*(12): 2153-2164.

[17] Johnson, N.L., Kotz, S. and Balakrishnan, N. *Continuous Univariate Distributions,* Wiley, New York, 1994.

[18] Mahdavi, A., and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, *46*(13): 6543-6557.

[19] Malik A. S., and Ahmad, S. P. (2017). Alpha power Rayleigh distribution and its application to life time data. *International Journal of Enhanced Research in Management & Computer Applications*, *6*(11): 212-219.

[20] Maurya, S. K., Kaushik, A., Singh, R. K., Singh, S. K., and Singh, U. (2016). A new method of proposing distribution and its application to real data. *Imperial Journal of Interdisciplinary Research*, *2*(6): 1331-1338.

[21] Nassar, M., Alzaatreh, A., Mead, M., and Abo-Kasem, O. (2017). Alpha power Weibull distribution: Properties and applications. *Communications in Statistics-theory and Methods*, *46*(20): 10236-10252.

[22] Nasiru, S., Mwita, P. N., and Ngesa, O. (2019). Alpha Power Transformed Fréchet Distribution, *Applied Mathematics and information Sciences an International Journal*. 1: 129-141.

[23] Pappas, V., Adamidis, K., and Loukas, S. (2012). A family of lifetime distributions. *Journal of Quality and Reliability Engineering*, *2012*(1): 760687.

[24] Rady, E. H. A., Hassanein, W. A., and Elhaddad, T. A. (2016). The power Lomax distribution with an application to bladder cancer data. *SpringerPlus*, *5*(1): 1838.

[25] Tahir, M. H., Cordeiro, G. M., Mansoor, M., and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications. *Hacettepe Journal of Mathematics and Statistics*, *44*(2): 455-474.

[26] Ünal, C., Cakmakyapan, S., and Özel, G. (2018). Alpha power inverted exponential distribution: Properties and application. *Gazi University Journal of Science*, *31*(3): 954-965.