THE ALDAM-LOG-LOGISTIC DISTRIBUTION AND ITS APPLICATION TO BIMODAL DATA

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Abstract

The log-logistic distribution is a commonly used lifetime probability distribution, particularly for modeling lifetime data. In this paper, a new generated (G) family of distributions is introduced, referred to as the Aldam-G family of distributions. This family is derived using the T-X family of distributions. As an application, the log-logistic distribution is generalized through the proposed family, resulting in a new distribution called the Aldam-Log-logistic distribution. The probability density function (PDF) and cumulative distribution function (CDF) of the proposed distribution are derived. It is observed that the new distribution can effectively model bimodal data. Several statistical properties of the Aldam-Log-logistic distribution are also explored. Parameter estimation is carried out using the maximum likelihood method. Finally, the proposed distribution is applied to a real-life dataset, and the results indicate that it provides a better fit compared to some existing competing distributions.

Keywords: Log - Logistic distribution, T-X family of distributions, Bimodal distribution, Lifetime data.

1. Introduction

Probability distributions play an important role in describing the behavior of real-life data. The Weibull, log-logistic, and log-normal distributions are popular classical probability distributions used for modeling non-negative or lifetime data. However, for data exhibiting bimodal, bathtub-shaped, or other complex behaviors, these classical distributions often lack the flexibility needed for accurate modeling. As a result, extending and generalizing classical distributions has become an active area of research.

Azzalini [4] proposed a skewed family of distributions by introducing an additional skewness parameter. Eugene et al. [8] introduced the Beta-G family of distributions using the beta distribution. Cordeiro and de Castro [7] developed the Kumaraswamy-G family as an alternative to the Beta-G family by employing the Kumaraswamy distribution. Other notable families of distributions include the Transmuted-G family [20], the exponentiated generalized class of distributions [6], the generalized transmuted-G family [17], and the exponentiated generalized Kumaraswamy-G class [21].

Rahman et al. [18] proposed the cubic transmuted family of distributions by adding an additional parameter to a baseline distribution. They also derived an extended version of the uniform distribution, called the cubic transmuted uniform distribution, and investigated various statistical properties such as moments, mean, variance, moment-generating function, characteristic function, mean absolute deviation, quantile function, median, reliability function, hazard function, and Shannon entropy.

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Lakibul and Tubo [13] introduced a new generated family of distributions called the T-extended Standard U-quadratic-G (TeSU-G) family. Using this family, they derived a bimodal version of the Weibull distribution. An extension of the TeSU-G family was presented in a subsequent study by Lakibul and Tubo [12]. Furthermore, a bivariate version of the extended Standard U-quadratic distribution was proposed by Lakibul, Polestico, and Supe [14]. Other generalizations of the extended Standard U-quadratic distribution are given in the study of Lakibul, Polestico, and Supe [15, 16].

Lakibul [11] introduced another generated family of distributions, named the Lakibul-G family, and derived several special cases, including the Lakibul-Uniform, Lakibul-Kumaraswamy, and Lakibul-Rayleigh distributions.

In this paper, we introduce a new generated family of distributions called the Aldam-G family of distributions. Using this family, we derive a generalized form of the log-logistic distribution, termed the Aldam-Log-logistic distribution, and study its properties. The proposed distribution is then applied to real-life data and compared with several competing distributions.

The remainder of this paper is organized as follows: Section 2 introduces the Aldam-G family of distributions. In Section 3, the Aldam-Log-logistic distribution is derived. Some properties of the Aldam-Log-Logistic distribution are discussed in Section 4. In section 5, the maximum likelihood estimates of the parameters of the Aldam-Log-Logistic distribution are presented. In Section 6, the proposed distribution is applied to real data. Finally, concluding remarks are provided in Section 7.

2. The Aldam-G Family of Distributions

Alzaatreh et al. [3] introduced the T-X family of distributions to extend any continuous distribution to a generalized distribution. The cumulative distribution function of the special case of the T-X family of distribution is given by

$$F_{T-X}(x) = \int_0^{G(x)} f(t)dt, \tag{1}$$

where $G(x)$ is any baseline cumulative distribution function and $f(t)$ is any probability density function (pdf) with support on the interval $[0,1]$. The Aldam - G family of distributions is obtained by using

$$f(t) = 1 - \rho + (b+1)\rho t^b,\ \rho \in \left[-\frac{1}{b}, 1\right]. \tag{2}$$

in (1), and the cumulative distribution function (cdf) of the proposed family of distribution is

$$F(x) = (1-\rho)G(x) + \rho(G(x))^{b+1} \tag{3}$$

with corresponding probability density function given by

$$f(x) = g(x)\left[1 - \rho + (b+1)\rho(G(x))^b\right],$$

where $g(x)$ is the probability density function associated with the cdf $G(x)$, $b > 0$ and $\rho \in [-\frac{1}{b}, 1]$.

3. The Aldam - Log - Logistic Distribution (ALL)

Let X be a random variable that follows a Log-Logistic distribution with cumulative distribution function given by

$$G(x) = \frac{e^{\mu}x^{\beta}}{1 + e^{\mu}x^{\beta}}, \tag{4}$$

where $x > 0$, $\mu \in \mathbb{R}$, and $\beta > 0$. The cumulative distribution function of the Aldam - Log-Logistic distribution is obtained by inserting (4) into (3) and is

$$F(x) = \frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}}\left[1-\rho+\rho\left(\frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}}\right)^{b}\right] \tag{5}$$

with corresponding probability density function given by

$$f(x) = \frac{e^{\mu}\beta x^{\beta-1}}{(1+e^{\mu}x^{\beta})^{2}}\left[1-\rho+(b+1)\rho\left(\frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}}\right)^{b}\right], \tag{6}$$

where $x > 0$, $\mu \in \mathbb{R}$, $\beta > 0$, $b > 0$ and $\rho \in [-\frac{1}{b}, 1]$.

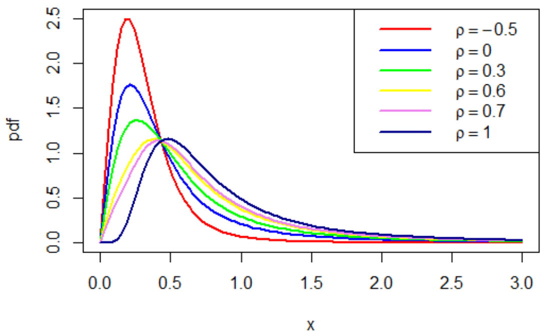

Figure 1: *PDF plots of the ALL distribution for $\mu = 2$, $\beta = 2$, $b = 2$ and varying values of ρ.*

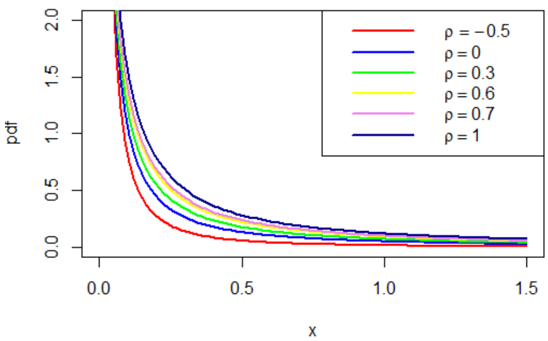

Figure 2: *PDF plots of the ALL distribution for $\mu = 2$, $\beta = 0.5$, $b = 2$ and varying values of ρ.*

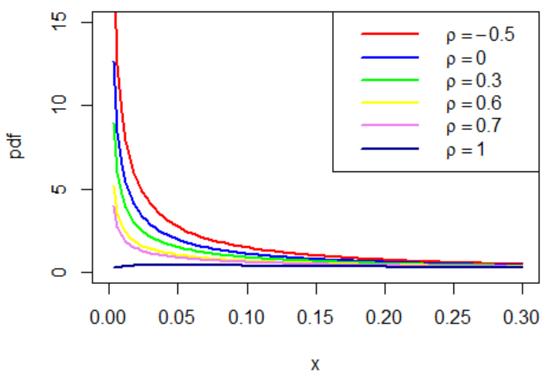

Figure 3: *PDF plots of the ALL distribution for $\mu = 0.5$, $\beta = 0.5$, $b = 2$ and varying values of ρ.*

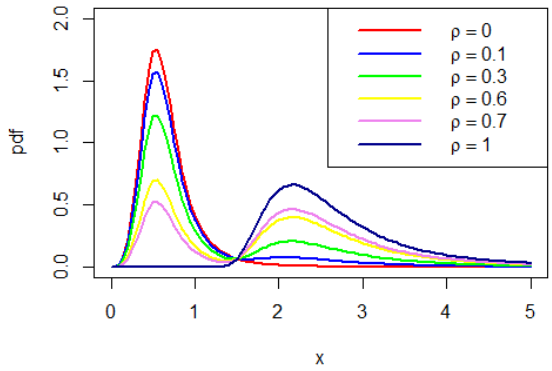

Figure 4: *PDF plots of the ALL distribution for $\mu = 2$, $\beta = 4$, $b = 200$ and varying values of ρ.*

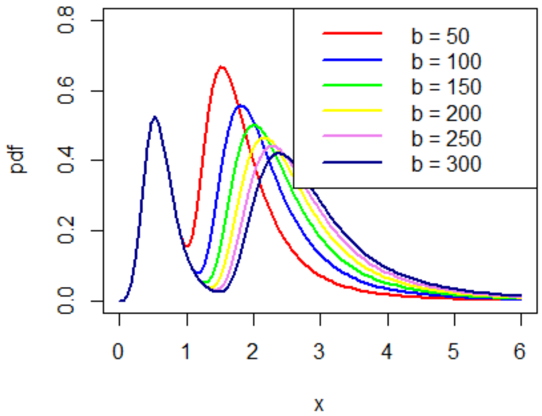

Figure 5: *PDF plots of the ALL distribution for $\mu = 2$, $\beta = 4$, $\rho = 0.7$ and varying values of b.*

Figures 1 to 5 display several possible shapes of the probability density function (PDF) of the Aldam-Log-Logistic (ALL) distribution. These figures indicate that the ALL distribution is capable of modeling various types of data behavior, including: (i) decreasing; (ii) right-skewed unimodal; and (iii) bimodal patterns.

4. Some Properties of the Aldam - Log - Logistic Distribution

This section presents some properties of the ALL distribution including the *rth* raw moments, mean, variance, moment generating function, survival and hazard functions.

4.1. Moments of the Aldam - Log - Logistic Distribution

Theorem 1. The *rth* raw moment of ALL distribution with density (6) and is

$$\mu'_r = e^{-\frac{\mu r}{\beta}} \left[(1-\rho)\mathbb{B}\left(\frac{r}{\beta}+1, 1-\frac{r}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{r}{\beta}+b+1, 1-\frac{r}{\beta}\right) \right], \tag{7}$$

where $\beta > 0$, $\mu \in \mathbb{R}$, $\rho \in \left[-\frac{1}{b}, 1\right]$, $b > 0$, $\mathbb{B}(\cdot)$ is a beta function, and $r = 1, 2, 3, \ldots$

The mean and variance are given by

$$\mu'_1 = e^{-\frac{\mu}{\beta}} \left[(1-\rho)\mathbb{B}\left(\frac{1}{\beta}+1, 1-\frac{1}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{1}{\beta}+b+1, 1-\frac{1}{\beta}\right) \right]$$

and

$$\sigma^2 = e^{-\frac{2\mu}{\beta}} \left[(1-\rho)\mathbb{B}\left(\frac{2}{\beta}+1, 1-\frac{2}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{2}{\beta}+b+1, 1-\frac{2}{\beta}\right) - \left[(1-\rho)\mathbb{B}\left(\frac{1}{\beta}+1, 1-\frac{1}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{1}{\beta}+b+1, 1-\frac{1}{\beta}\right)\right]^2 \right].$$

Proof. The rth raw moment is defined by

$$\begin{aligned}
\mu'_r &= E[X^r] \\
&= \int_0^\infty x^r f(x)dx \\
&= \int_0^\infty x^r \frac{e^\mu \beta x^{\beta-1}}{(1+e^\mu x^\beta)^2}\left(1-\rho+(b+1)\rho\left(\frac{e^\mu x^\beta}{1+e^\mu x^\beta}\right)^b\right)dx \\
&= (1-\rho)\int_0^\infty x^r \frac{e^\mu \beta x^{\beta-1}}{(1+e^\mu x^\beta)^2}dx + (b+1)\rho\int_0^\infty x^r \frac{e^\mu \beta x^{\beta-1} e^{\mu b} x^{\beta b}}{(1+e^\mu x^\beta)^{b+2}}dx \\
&= (1-\rho)e^{-\frac{\mu r}{\beta}}\int_0^\infty \frac{u^{\frac{r}{\beta}}}{(1+u)^2}du + \rho(b+1)e^{-\mu\left(\frac{r+b\beta}{\beta}\right)}\int_0^\infty \frac{u^{\frac{r+b\beta}{\beta}}}{(1+u)^{b+2}}du \\
&= e^{-\frac{\mu r}{\beta}}\left[(1-\rho)\mathbb{B}\left(\frac{r}{\beta}+1, 1-\frac{r}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{r}{\beta}+b+1, 1-\frac{r}{\beta}\right)\right]
\end{aligned}$$

The mean of ALL distribution is obtained by taking $r = 1$ in (7) and is

$$\mu'_1 = e^{-\frac{\mu}{\beta}}\left[(1-\rho)\mathbb{B}\left(\frac{1}{\beta}+1, 1-\frac{1}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{1}{\beta}+b+1, 1-\frac{1}{\beta}\right)\right].$$

The $2nd$ raw moment of ALL distribution is obtained from (7) by taking $r = 2$ and is

$$\mu'_2 = e^{-\frac{2\mu}{\beta}}\left[(1-\rho)\mathbb{B}\left(\frac{2}{\beta}+1, 1-\frac{2}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{2}{\beta}+b+1, 1-\frac{2}{\beta}\right)\right].$$

The variance of ALL distribution is obtained as

$$\begin{aligned}
\sigma^2 =& \mu'_2 - (\mu'_1)^2 \\
=& e^{-\frac{2\mu}{\beta}}\left[(1-\rho)\mathbb{B}\left(\frac{2}{\beta}+1, 1-\frac{2}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{2}{\beta}+b+1, 1-\frac{2}{\beta}\right)\right] \\
&- \left\{e^{-\frac{\mu}{\beta}}\left[(1-\rho)\mathbb{B}\left(\frac{1}{\beta}+1, 1-\frac{1}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{1}{\beta}+b+1, 1-\frac{1}{\beta}\right)\right]\right\}^2 \\
=& e^{-\frac{2\mu}{\beta}}\left[(1-\rho)\mathbb{B}\left(\frac{2}{\beta}+1, 1-\frac{2}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{2}{\beta}+b+1, 1-\frac{2}{\beta}\right)\right. \\
&\left. - \left[(1-\rho)\mathbb{B}\left(\frac{1}{\beta}+1, 1-\frac{1}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{1}{\beta}+b+1, 1-\frac{1}{\beta}\right)\right]^2\right].
\end{aligned}$$

4.2. Moment Generating Function of the Aldam - Log - Logistic Distribution

Theorem 2. Let X follows the ALL distribution then the moment generating function $M_X(t)$ is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r e^{-\frac{\mu r}{\beta}}}{r!}\left[(1-\rho)\mathbb{B}\left(\frac{r}{\beta}+1, 1-\frac{r}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{r}{\beta}+b+1, 1-\frac{r}{\beta}\right)\right], t \in \mathbb{R}.$$

Proof. By definition of moment generating function and pdf (6), we have

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tX} f(x)dx.$$

Recall that $e^{tX} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r$, then we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'.$$

Hence,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r e^{-\frac{\mu r}{\beta}}}{r!} \left[(1-\rho)\mathbb{B}\left(\frac{r}{\beta}+1, 1-\frac{r}{\beta}\right) + \rho(b+1)e^{-\mu b}\mathbb{B}\left(\frac{r}{\beta}+b+1, 1-\frac{r}{\beta}\right) \right], t \in \mathbb{R}.$$

4.3. Survival and Hazard Functions of the Aldam - Log - Logistic Distribution

Let X be a random variable with cdf (5) and pdf (6) then the survival $S(x)$ and hazard $h(x)$ functions of ALL distribution are respectively, given by

$$S(x) = 1 - \frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}} \left(1 - \rho + \rho \left(\frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}} \right)^b \right)$$

and

$$h(x) = \frac{\frac{e^{\mu}\beta x^{\beta-1}}{(1+e^{\mu}x^{\beta})^2} \left(1 - \rho + (b+1)\rho \left(\frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}} \right)^b \right)}{1 - \frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}} \left(1 - \rho + \rho \left(\frac{e^{\mu}x^{\beta}}{1+e^{\mu}x^{\beta}} \right)^b \right)},$$

where $x > 0$, $\mu \in \mathbb{R}$, $\beta > 0$, $b > 0$ and $\rho \in [-\frac{1}{b}, 1]$.

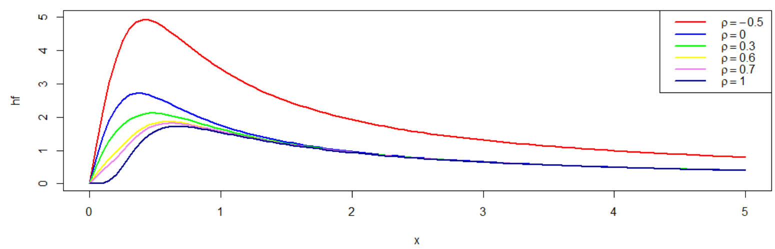

Figure 6: *HF plots of the ALL distribution for $\mu = 2$, $\beta = 2$, $b = 2$ and varying values of ρ.*

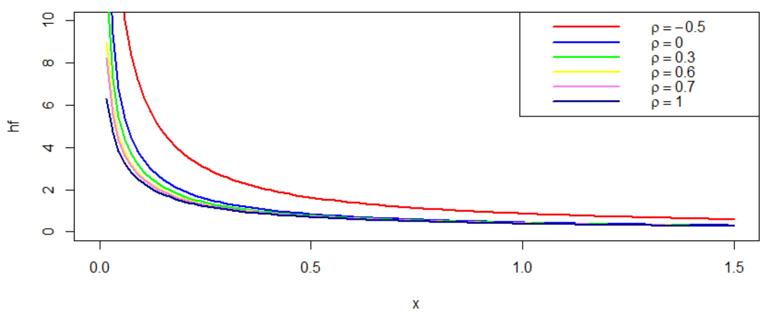

Figure 7: *HF plots of the ALL distribution for $\mu = 2$, $\beta = 0.5$, $b = 2$ and varying values of ρ.*

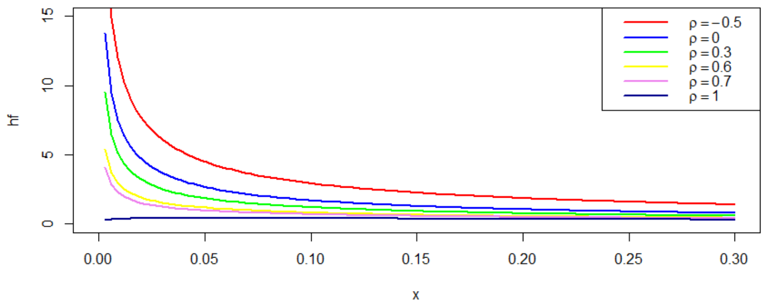

Figure 8: *HF plots of the ALL distribution for $\mu = 0.5$, $\beta = 0.5$, $b = 2$ and varying values of ρ.*

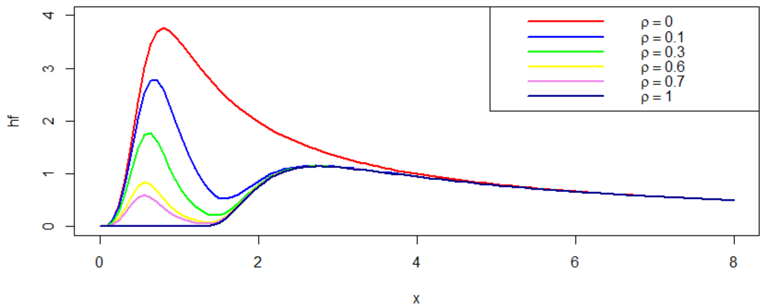

Figure 9: *HF plots of the ALL distribution for $\mu = 2$, $\beta = 4$, $b = 200$ and varying values of ρ.*

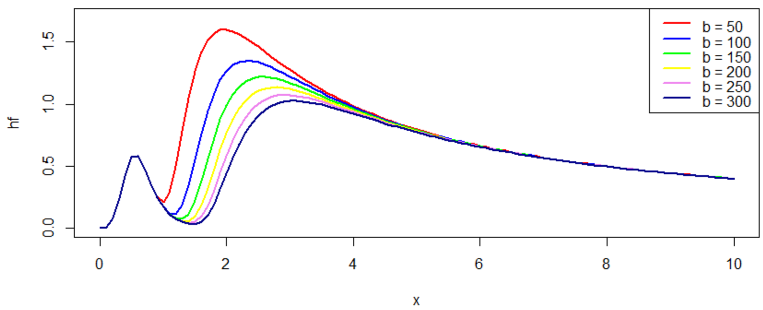

Figure 10: *HF plots of the ALL distribution for $\mu = 2$, $\beta = 4$, $\rho = 0.7$ and varying values of b.*

Figures 6 to 10 illustrate various possible shapes of the hazard function of the ALL distribution. These figures demonstrate that the hazard function of the ALL distribution is flexible enough to model a range of behaviors, including: (i) decreasing; (ii) right-skewed unimodal; and (iii) bimodal patterns.

5. Maximum Likelihood Estimates of the Aldam - Log - Logistic Distribution

Let X_1, X_2,..., X_n be a random sample of size n from ALL distribution, then the likelihood function is

$$L = e^{n\mu}\beta^n \prod_{i=1}^{n}\left(1+e^{\mu}x_i^{\beta}\right)^{-2}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left[1-\rho+(b+1)\rho\left(\frac{e^{\mu}x_i^{\beta}}{1+e^{\mu}x_i^{\beta}}\right)^{b}\right]$$

The log-likelihood function is

$$
\begin{aligned}
l =& n\mu + n\log(\beta) - 2\sum_{i=1}^{n}\log\left(1+e^{\mu}x_i^{\beta}\right) + (\beta-1)\sum_{i=1}^{n}\log(x_i) - b\sum_{i=1}^{n}\log\left(1+e^{\mu}x_i^{\beta}\right) \\
&+ \sum_{i=1}^{n}\log\left[(1-\rho)\left(1+e^{\mu}x_i^{\beta}\right)^{b} + (b+1)\rho e^{\mu b}x_i^{\beta b}\right] - b\sum_{i=1}^{n}\frac{x_i^{\beta}e^{\mu}}{1+e^{\mu}x_i^{\beta}}.
\end{aligned} \tag{8}
$$

Taking the derivative of (8) with respect to parameter μ then, we have

$$
\frac{\partial l}{\partial \mu} = n - (2+b)\sum_{i=1}^{n}\frac{x_i^{\beta}e^{\mu}}{1+e^{\mu}x_i^{\beta}} + \sum_{i=1}^{n}\frac{(1-\rho)b\left(1+e^{\mu}x_i^{\beta}\right)^{b-1}e^{\mu}x_i^{\beta} + b(b+1)\rho e^{\mu b}x_i^{\beta b}}{(1-\rho)\left(1+e^{\mu}x_i^{\beta}\right)^{b} + (b+1)\rho e^{\mu b}x_i^{\beta b}}. \tag{9}
$$

Taking the derivative of (8) with respect to parameter β then, we have

$$
\begin{aligned}
\frac{\partial l}{\partial \beta} =& \frac{n}{\beta} - 2\sum_{i=1}^{n}\frac{e^{\mu}x_i^{\beta}log(x_i)}{1+e^{\mu}x_i^{\beta}} + \sum_{i=1}^{n}\log(x_i) - b\sum_{i=1}^{n}\frac{e^{\mu}x_i^{\beta}\log(x_i)}{1+e^{\mu}x_i^{\beta}} \\
&+ \sum_{i=1}^{n}\frac{(1-\rho)b\left(1+e^{\mu}x_i^{\beta}\right)^{b-1}e^{\mu}x_i^{\beta}\log(x_i) + (b+1)\rho e^{\mu b}bx_i^{\beta b}\log(x_i)}{(1-\rho)\left(1+e^{\mu}x_i^{\beta}\right)^{b} + (b+1)\rho e^{\mu b}x_i^{\beta b}}.
\end{aligned} \tag{10}
$$

Taking the derivative of (8) with respect to parameter b then, we have

$$
\frac{\partial l}{\partial b} = \sum_{i=1}^{n}\frac{(1-\rho)\left(1+e^{\mu}x_i^{\beta}\right)^{b}\log\left(1+e^{\mu}x_i^{\beta}\right) + \rho e^{b\mu}x_i^{b\beta}\left[(b+1)(\mu+\beta\log(x_i))+1\right]}{(1-\rho)\left(1+e^{\mu}x_i^{\beta}\right)^{b} + (b+1)\rho e^{\mu b}x_i^{\beta b}} \tag{11}
$$

By simultaneously equating equations (9), (10), and (11) to zero, the numerical maximum likelihood estimates of the parameters of the ALL distribution can be obtained.

6. Application of the Aldam - Log - Logistic Distribution

In this section, we apply the ALL distribution on the Old Faithful Geyser dataset and compare with the Log-logistic distribution, and the following distributions:

- The T-extended Standard U-quadratic - Weibull Distribution (TeSU-W) [13]

$$
f(x) = b\beta x^{\beta-1}e^{-bx^{\beta}}\left(1+2\rho-12\rho e^{-bx^{\beta}}+12\rho e^{-2bx^{\beta}}\right),\ x>0, \tag{12}
$$

where $b>0$, $\beta>0$ and $\rho\in[-0.5,1]$.

- The Gumbel - Weibull Distribution (GW) [2]

$$
f(x) = \frac{\alpha\beta}{\lambda\sigma}\left(\frac{x}{\lambda}\right)^{\alpha-1}e^{\left(\frac{x}{\lambda}\right)^{\alpha}}\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}}-1\right)^{-1-\frac{1}{\sigma}}e^{-\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}}},\ x>0, \tag{13}
$$

where $\beta = e^{\frac{\nu}{\sigma}}$, $\alpha>0$, $\sigma>0$, $\lambda>0$ and $-\infty\le\nu\le\infty$

The dataset used in this alpplication is consists of 299 paired measurements representing the time intervals between the start of successive eruptions of the Old Faithful Geyser in Yellowstone National Park, Wyoming, USA. The data were collected continuously from August 1 to August

15, 1985. The observations are the following:

1. Waiting times
80 71 57 80 75 77 60 86 77 56 81 50 89 54 90 73 60 83 65 82 84 54 85 58 79 57 88 68 76 78 74 85 75 65 76 58 91 50 87 48 93 54 86 53 78 52 83 60 87 49 80 60 92 43 89 60 84 69 74 71 108 50 77 57 80 61 82 48 81 73 62 79 54 80 73 81 62 81 71 79 81 74 59 81 66 87 53 80 50 87 51 82 58 81 49 92 50 88 62 93 56 89 51 79 58 82 52 88 52 78 69 75 77 53 80 55 87 53 85 61 93 54 76 80 81 59 86 78 71 77 76 94 75 50 83 82 72 77 75 65 79 72 78 77 79 75 78 64 80 49 88 54 85 51 96 50 80 78 81 72 75 78 87 69 55 83 49 82 57 84 57 84 73 78 57 79 57 90 62 87 78 52 98 48 78 79 65 84 50 83 60 80 50 88 50 84 74 76 65 89 49 88 51 78 85 65 75 77 69 92 68 87 61 81 55 93 53 84 70 73 93 50 87 77 74 72 82 74 80 49 91 53 86 49 79 89 87 76 59 80 89 45 93 72 71 54 79 74 65 78 57 87 72 84 47 84 57 87 68 86 75 73 53 82 93 77 54 96 48 89 63 84 76 62 83 50 85 78 78 81 78 76 74 81 66 84 48 93 47 87 51 78 54 87 52 85 58 88 79.

2. Time Duration
4.0166667 2.1500000 4.0000000 4.0000000 4.0000000 2.0000000 4.3833333 4.2833333 2.0333333 4.8333333 1.8333333 5.4500000 1.6166667 4.8666667 4.3833333 1.7666667 4.6666667 2.0000000 4.7333333 4.2166667 1.9000000 4.9666667 2.0000000 4.0000000 2.0000000 4.0000000 2.8333333 4.5000000 4.0666667 3.7166667 3.5166667 4.4666667 2.2166667 4.8833333 2.6000000 4.1500000 2.2000000 4.7666667 1.8333333 4.6000000 2.2666667 4.1333333 2.0000000 4.0000000 2.0000000 4.0000000 1.8833333 4.2666667 2.0833333 4.4666667 2.5000000 4.0000000 1.7666667 4.3333333 2.1833333 4.4833333 3.8833333 3.3333333 3.7333333 4.0000000 1.9500000 5.2666667 2.0000000 4.0000000 2.0000000 4.0000000 2.0000000 4.0000000 3.5333333 2.1666667 4.5000000 2.0166667 4.1500000 4.2000000 4.3333333 1.9333333 4.6500000 3.8166667 4.0333333 4.1666667 4.6666667 1.8166667 4.0000000 3.0000000 4.0000000 2.0000000 4.4500000 2.0500000 4.2500000 1.9166667 4.6666667 1.7333333 4.3833333 1.7666667 4.6000000 1.8666667 4.4500000 1.6333333 5.0333333 1.8166667 5.1000000 1.6333333 4.2833333 2.0000000 4.0000000 2.0000000 4.5333333 2.0000000 4.0000000 2.9333333 4.7333333 3.9000000 1.9500000 4.1166667 1.8000000 4.6666667 1.8333333 4.7000000 2.1166667 4.7833333 1.8166667 4.1000000 4.6500000 4.0000000 2.0000000 4.0000000 4.0000000 4.2166667 4.1333333 3.9333333 3.7500000 4.4166667 2.4666667 4.1666667 3.8000000 4.3166667 3.8666667 4.6833333 1.7000000 4.9666667 4.2666667 4.5833333 4.0000000 4.0000000 4.0000000 4.0000000 1.9833333 4.6000000 0.8333333 4.9166667 1.7333333 4.5833333 1.7000000 4.7500000 1.8333333 4.5000000 1.8666667 4.4500000 4.4500000 4.0000000 4.8000000 4.0000000 4.0000000 2.0000000 4.0000000 1.9333333 4.5833333 2.0000000 3.7000000 2.8666667 4.8333333 3.4500000 4.3833333 1.8000000 4.4000000 2.4833333 4.5166667 2.1000000 4.3500000 4.3666667 1.7833333 4.9166667 1.8166667 4.0000000 4.0000000 4.0000000 3.8666667 1.8500000 4.7000000 2.0166667 4.4666667 1.8666667 4.1666667 1.9000000 4.2500000 3.2500000 4.2166667 1.8833333 4.9833333 1.8500000 4.0000000 1.9666667 4.7666667 4.0000000 2.0000000 4.0000000 4.0000000 2.3833333 4.4166667 4.2166667 4.3666667 2.0000000 4.4500000 1.7500000 4.5000000 1.6166667 4.7000000 2.5666667 3.7000000 4.2333333 1.9333333 4.3500000 4.0000000 4.0000000 4.0000000 4.2166667 4.0000000 4.1333333 1.8833333 4.4666667 1.9500000 4.2166667 1.7166667 4.4500000 4.2500000 3.9666667 4.3833333 1.9666667 4.4500000 4.2666667 1.9166667 4.4166667 3.0000000 4.0000000 2.0000000 4.0000000 3.2833333 1.8333333 4.6166667 1.8333333 4.6166667 4.6000000 4.2500000 1.9333333 4.9833333 1.9666667 4.3000000 4.2000000 4.5333333 4.4000000 4.6166667 2.0000000 4.0000000 4.0000000 3.9166667 2.0000000 4.5000000 1.8000000 4.0000000 2.7500000 4.7333333 3.9666667 1.9500000 4.9666667 1.8500000 4.8000000 4.0000000 4.0000000 4.0000000 4.0000000 4.0000000 4.0000000 4.0000000 2.0000000 4.0000000 1.9333333 4.3333333 1.6666667 4.7666667 1.9500000 4.6833333 1.9333333 4.4166667 2.1333333 4.0833333 2.0666667 4.0000000 4.0000000 2.0000000.

Here, we use a package "fitdistrplus" in R software. Moreover, we utilize the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for comparison.

Table 1: *MLEs of the fitted models for the Waiting time of the Geyser dataset.*

	Parameters			
Distributions	$\hat{\mu}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{b}$
ALL	−57.4616339	14.297131	0.6309489	132.38172
LL	−35.448639	8.279677		
TeSU-W		4.236527	−0.5000000	0.00000000888
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{v}$
GW	0.000050714	3.091603	14.31577	23.65889

Table 1 presents the maximum likelihood estimates (MLEs) of the parameters for four candidate distributions fitted to the waiting time variable of the Geyser dataset: the ALL (Aldam-Log-Logistic), LL (Log-Logistic), TeSU-W (T-extended Standard U-quadratic-Weibull), and GW (Gumbel - Weibull) distributions.

The ALL distribution produced parameter estimates of $\hat{\mu} = -57.4616$, $\hat{\beta} = 14.2971$, $\hat{\rho} = 0.6309$, and $\hat{b} = 132.3817$. The LL distribution, a simpler two-parameter model, yielded $\hat{\mu} = -35.4486$ and $\hat{\beta} = 8.2797$. For the TeSU-W distribution, the estimates were $\hat{\beta} = 4.2365$, $\hat{\rho} = -0.5$, and $\hat{b} = 8.88 \times 10^{-9}$. Lastly, the GW distribution had four estimated parameters: $\hat{\alpha} = 5.07 \times 10^{-5}$, $\hat{\beta} = 3.0916$, $\hat{\sigma} = 14.3158$, and $\hat{v} = 23.6589$.

Table 2: *Numerical values of the AIC and BIC of the fitted models for waiting time of the Geyser dataset.*

Distributions	AIC	BIC
ALL	**2338.192**	**2352.994**
LL	2469.652	2477.053
TeSU-W	2420.931	2432.032
GW	2441.881	2456.683

Table 2 shows the corresponding Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values for model comparison. Among the four fitted distributions, the ALL distribution obtained the lowest AIC (2338.192) and BIC (2352.994), indicating the best model fit among the candidates, taking into account both goodness-of-fit and model complexity. This suggests that the ALL distribution provides the most appropriate model for the waiting time variable in the Geyser dataset.

In contrast, the LL, TeSU-W, and GW distributions showed higher AIC and BIC values, with the LL model performing the worst in terms of both criteria. Although the TeSU-W and GW distributions slightly outperformed the LL, they still fall short of the performance of ALL model. These results provide strong evidence in favor of using the ALL distribution for modeling the waiting time data in this context. Further, same result is observed from Figure 11.

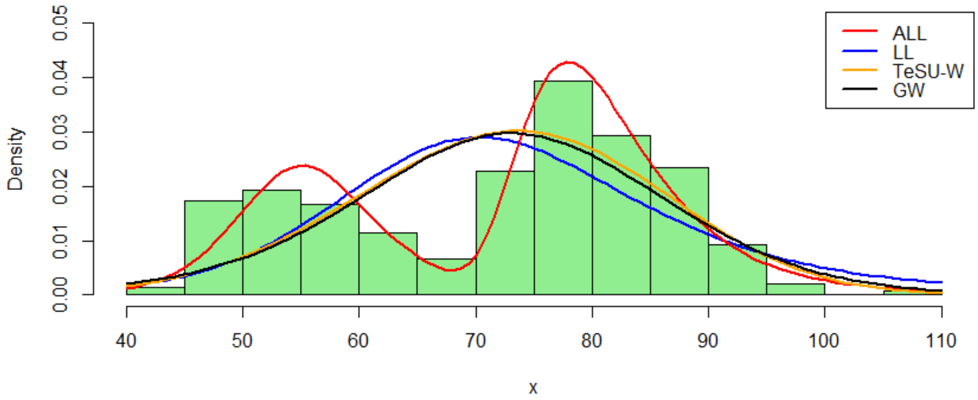

Figure 11: *Estimated PDF of the fitted models for the waiting time of the Geyser dataset.*

Table 3: *MLEs of the fitted models for Time Duration of the Geyser dataset.*

	Parameters			
Distributions	$\hat{\mu}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{b}$
ALL	−8.9151564	13.1175181	0.6297066	15667.9303
LL	−5.243472	4.294778		
TeSU-W		4.310113470	0.982276578	0.004548
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{v}$
GW	0.0000036545	9.903009	7.159973	−0.6311149

Table 3 presents the maximum likelihood estimates (MLEs) of the parameters for four candidate distributions fitted to the duration variable of the Geyser dataset: the Aldam-Log-Logistic (ALL), Log-Logistic (LL), T-extended Standard U-Weibull (TeSU-W), and Gumbel-Weibull (GW) distributions.

The ALL distribution provided the following parameter estimates: $\hat{\mu} = -8.9152$, $\hat{\beta} = 13.1175$, $\hat{\rho} = 0.6297$, and $\hat{b} = 15667.9303$. The simpler LL distribution had parameter estimates of $\hat{\mu} = -5.2435$ and $\hat{\beta} = 4.2948$. For the TeSU-W distribution, the estimated parameters were $\hat{\beta} = 4.3101$, $\hat{\rho} = 0.9823$, and $\hat{b} = 0.004548$. Meanwhile, the GW distribution yielded parameter estimates of $\hat{\alpha} = 3.6545 \times 10^{-6}$, $\hat{\beta} = 9.9030$, $\hat{\sigma} = 7.1600$, and $\hat{v} = -0.6311$.

Table 4: *Numerical values of AIC and BIC of the fitted models for Time Duration of the Geyser dataset.*

Distribution	AIC	BIC
ALL	**601.5399**	**616.3416**
LL	1010.538	1017.939
TeSU-W	705.2641	716.3655
GW	753.9445	768.7462

To evaluate the relative performance of the fitted models, Table 4 presents the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values. Among all considered models, the ALL distribution achieved the lowest AIC (601.540) and BIC (616.342), indicating the best fit to the data while balancing model complexity and goodness-of-fit.

In contrast, the LL distribution recorded the highest AIC and BIC values (1010.538 and 1017.939, respectively), indicating poor model performance. The TeSU-W and GW distributions performed better than LL but were still outperformed by the ALL distribution by a considerable margin. These findings demonstrate that the ALL distribution provides the most appropriate model for the duration variable of the Geyser dataset, further emphasizing its flexibility in capturing the distributional characteristics of the data. In addition, same observation is drawn from the following figure.

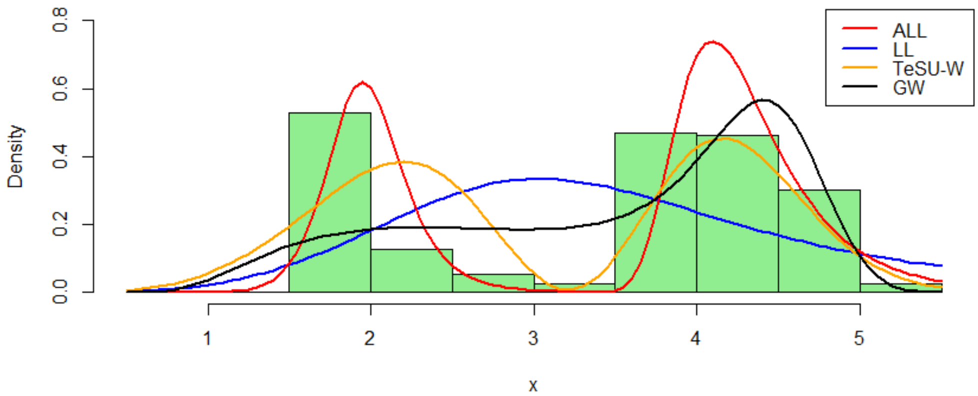

Figure 12: *Estimated PDF of the fitted models for the time duration of the Geyser dataset.*

7. Concluding Remarks

In this paper, a new generated family of distributions called the Aldam-G Family of Distributions has been introduced. In particular, the Aldam - Log-Logistic distribution was derived. Some properties of the derived distribution, such as its moments, mean, variance, and moment generating function, were obtained. The maximum likelihood method was used to estimate the parameters of the proposed distribution. Real data sets were utilized to examine the flexibility and applicability of the proposed distribution. It was observed that the Aldam-Log- Logistic distribution gives better estimates for the given bimodal data than the log-logistic, Gumbel - Weibull and the T - extended Standard U-quadratic Weibull distributions.

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