RELIABILITY AND RISK MODELLING IN ECONOMIC, ENVIRONMENTAL SUSTAINABILITY, AND FINANCIAL SYSTEMS WITH THE ARCTAN MARSHALL-OLKIN WEIBULL DISTRIBUTION

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Abstract

The Arctan Marshall-Olkin family is presented as a novel and versatile class of heavy-tailed distributions for modelling extreme occurrences in the fields of economics, finance, and the Environmental sustainability. This family is produced by combining the Arctan-X method, which makes use of the arctangent inverse trigonometric function, with the Marshall-Olkin framework. A specific example, the Arctan-Marshall-Olkin-Weibull (ATMOW) distribution, is thoroughly examined. By adding a third parameter, ATMOW improves its capacity to handle heavy-tailed data in contrast to the traditional two-parameter Weibull distribution. Additionally, to evaluate severe financial hazards, closed-form formulas for important actuarial risk measures such as value at risk and tail value at risk are generated. ATMOW performs better than a number of rival multiparameter distributions, according to empirical applications to actual financial, environmental sustainability, and economic datasets.

Keywords: Financial Systems, Economic Modelling, Environmental Sustainability, ATMOW Distribution, Value at Risk.

I. Introduction

Distribution frameworks are essential to data analysis in economics, finance, and environmental sustainability because they make it possible to effectively describe and comprehend how random variables like interest rates, financial asset returns, economic indices, and environmental indicators perform. Because of intrinsic features including heavy tails, skewness, non-normality, and heterogeneity, it might be difficult to effectively capture real-world data in these domains. The dynamics of income, inflation, and growth show asymmetry and persistence in economics; extreme market movements and volatility clustering in finance necessitate flexible models; and in environmental systems, variables such as pollutant concentrations, resource depletion rates, and climate variability frequently exhibit intricate, non-linear patterns. As a matter of fact, exceptional events that are more often than those expected by a normal distribution and may also be asymmetric are often seen in financial and economic series [1]. These features put traditional models to the test and could result in erroneous asset appraisal or inadequate risk assessment.

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Therefore, for accurate forecasting, realistic modeling, and well-informed decision-making across all three domains, the application of sophisticated distributional frameworks is essential.

These limitations are addressed by modeling various kinds of heavy-tailed data using a number of classical distributions, including Weibull [2], Gumbel [3], Pareto [4], Cauchy [5], and log-normal [6]. These distributions do not, however, provide the flexibility needed to handle massive volumes of data. New techniques for developing new distributions and families of distributions have been investigated in an effort to improve this circumstance. These techniques include the T-X transformer method, beta-generated techniques, generating asymmetric distributions, adding parameters to an existing distribution, quantiles, differential equations, transformations, and composition techniques [7]. For example, families such as Arctan-X [8], Arcsine Exponentiated-X [9], Exponential T–X [10], New Arctan-G [11], New Topp-Leone Kumaraswamy Marshall-Olkin generated family of distributions [12], a novel family of distributions [13], two-parameter family of distributions [14], Truncated Inverse Lomax Generated Family [15], Sine Inverse Lomax Generated Family [16] as well as distributions like the New Topp Leone exponentiated Exponential Model [17], Kumaraswamy Weibull (KW) [18], Kumaraswamy Burr XII [19], gamma-Lomax distribution [20], Marshall Olkin Alpha Power Extended Weibull Distribution [21], Topp-Leone Alpha Power Weibull Distribution [22], A New Extension of the Odd Inverse Weibull-G Family of Distributions [23], the New Versions of Bivariate Inverse Weibull Distribution based on Progressive Type II Censoring [24], the Discrete Weibull Marshall-Olkin Family of Distributions [25] and the Marshall-Olkin Weibull-Burr XII distribution [26] have been developed to better fit the data.

The current research addresses these gaps by presenting a novel family of statistical distributions created to meet the particular requirements of heavy-tailed data in economics and finance. In particular, the study aims to: (i) integrate the Arctan and Marshall-Olkin families to form the Arctan-Marshall-Olkin family; (ii) derive and analyze the mathematical properties of the ATMOW distribution; (iii) estimate the parameters of the ATMOW distribution using the maximum likelihood method; and (iv) illustrate the ATMOW distribution's applicability using real-world datasets in economics, finance and environmental sustainability. A strong framework for modeling complicated datasets is produced by combining the flexibility of the Arctan family with the adaptability of the Marshall-Olkin family of distributions. In order to improve model flexibility and risk assessment accuracy, the study focuses on the Arctan-Marshall-Olkin-Weibull (ATMOW) distribution, a specialized member of this family. This research presents a novel family of distributions that blend flexibility and robustness in order to enhance risk management, asset valuation, and sustainability assessments as well as to better represent the complex features of financial, economic, and environmental data.

II. ATMO-G Distribution

In the current investigation, we combine the Marshall-Olkin family [27] with the Arctan-X family [8], which has already shown success in modeling insurance data, to propose a new family of distributions. The following expression provides the cumulative distribution function (or CDF) of a random variable X that is a member of the Arctan-X family:

$$F_{arctan}(x;\zeta) = \frac{4}{\pi} arctan\ (G(x;\zeta)), G(x;\zeta) \in (0,1) \tag{1}$$

In the present scenario, $\zeta \in \mathbb{R}$, $\mathbf{G(x;\boldsymbol{\zeta})}$ is thought to be the CDF of the baseline random variable given a parameter vector. Similarly, a CDF defines the Marshall-Olkin family:

$$F_{MO}(x;\theta,\zeta) = \frac{G(x;\zeta)}{\theta+(1-\theta)G(x;\zeta)}, \theta > 0 \tag{2}$$

Therefore, the CDF of the ATMO-G family is defined as follows:

$$F(x;\theta,\zeta) = \frac{4}{\pi} arctan\left(\frac{G(x;\zeta)}{\theta+(1-\theta)G(x;\zeta)}\right) \tag{3}$$

It is essential to examine some of the fundamental characteristics, parameters, and practical uses of the Arctan Marshall Olkin family of distributions after becoming acquainted with its concept.

I. Probability Distribution Function (PDF)

we distinguish the CDF in (3) with relation to x. As a conclusion, we are given:

$$f(x;\theta,\zeta) = \frac{4}{\pi}\frac{\theta\cdot g(x;\zeta)}{G(x;\zeta)^2+[\theta+(1-\theta)G(x;\zeta)]^2}, \theta > 0, \tag{4}$$

where $g(x;\zeta)$ is the derivative of $G(x;\zeta)$.

II. Hazard Rate Function

The instantaneous failure rate at a specific moment is described by the hazard rate function (hrf), which is the ratio of the PDF to the survival function. It is essential for simulating and examining the occurrence of extreme events and natural disasters.

$$h(x;\zeta) = \frac{f(x;\zeta)}{1-F(x;\zeta)} \tag{5}$$

$$h(x;\zeta) = \frac{\frac{4}{\pi}\frac{\theta g(x;\zeta)}{G(x;\zeta)^2+[\theta+(1-\theta)G(x;\zeta)]^2}}{1-\frac{4}{\pi}arctan\left(\frac{G(x;\zeta)}{\theta+(1-\theta)G(x;\zeta)}\right)} \tag{6}$$

III. Quantile Function

The quantile function of a statistical distribution divides the distribution into two sections at a given probability level by providing the value of the variable at which the cumulative probability reaches a given proportion. Analyzing the new family of distributions is essential for risk assessment, scenario modeling, distribution comparison, and confidence interval creation.
The quantile function Q can be written as follows:

$$Q = G^{-1}\left(\frac{\theta\tan\left(\frac{\pi}{4}u\right)}{1-(1-\theta)\tan\left(\frac{\pi}{4}u\right)}\right) \tag{7}$$

III. ATMOW Distribution and Actuarial Measures

I. ATMOW Distribution

Specifically, financial, environmental sustainability and economic data can be represented using the Weibull distribution. Because of its adaptability, flexibility, and capacity to simulate a wide range of data behaviors and failure rates, the base distribution was chosen as the model of choice for risk assessment, failure modelling, and lifetime analysis.

The CDF for the Weibull distribution is explained by:

$$G(x;\lambda,k) = 1 - e^{-\left(\frac{x}{\lambda}\right)^k} \; x \in \mathbb{R}^+ \tag{8}$$

The PDF is as follows for the specific member:

$$g(x;\lambda,k) = \frac{k}{\lambda} \cdot \left(\frac{x}{\lambda}\right)^{k-1} \cdot e^{-\left(\frac{x}{\lambda}\right)^k} \tag{9}$$

The Weibull distribution is used to determine the CDF of this special member, as shown in [8]:

$$F(x;\theta,\lambda,k) = \frac{4}{\pi} arctan\left(\frac{1-e^{-\left(\frac{x}{\lambda}\right)^k}}{\theta+(1-\theta)\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)}\right), x \in \mathbb{R}^+ \tag{10}$$

The associated PDF and hrf are expressed as follows:

$$f(x;\theta,\lambda,k) = \frac{4}{\pi}\frac{\theta \cdot \frac{k}{\lambda} \cdot \left(\frac{x}{\lambda}\right)^{k-1} \cdot e^{-\left(\frac{x}{\lambda}\right)^k}}{\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^2 + \left[\theta+(1-\theta)\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)\right]^2}, \theta > 0, \lambda > 0, k > 0. \tag{11}$$

II. Value at Risk

The maximum losses that could transpire with a chance of q are evaluated by the Value at Risk (VaR). $\text{i.e.}, VaR_q = Q_X(q).$, or the quantile function of the distribution, can be used to define the VaR. Therefore, the following is the definition of the quantile function for the ATMO-W distribution:

$$VaRq = \lambda\left(-log\left(1 - \frac{\theta tan\left(\frac{\pi}{4}q\right)}{1-(1-\theta)tan\left(\frac{\pi}{4}q\right)}\right)\right)^{1/k} \tag{12}$$

III. Tail Value at Risk and Tail Variance

The losses over the Value at Risk (VaR) are measured by the Tail Value at Risk (TVaR). With probability $1-q$, it denotes the predicted loss in the distribution's tail, which corresponds to the worst-case situations. TVaR is therefore defined as follows:

$$TVaR_q = \frac{1}{1-q}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{2i+j+1} - C_{ijm} \cdot \lambda\left(\frac{1}{m}\right)^{1+\frac{1}{k}} \Gamma\left(\frac{1}{k}+1, m\left(\frac{VaRq}{\lambda}\right)^k\right) \tag{13}$$

The variability of losses above the VaR is measured by the Tail Variance (TV). It is essential for measuring a company's overall risk. The TV has the following definition:

$$TVq(X) = \frac{1}{1-q}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{2i+j+1} - C_{ijm} \cdot \lambda^2\left(\frac{1}{m}\right)^{1+\frac{2}{k}} \Gamma\left(\frac{2}{k}+1, m\left(\frac{VaRq}{\lambda}\right)^k\right) - \left(TVaR_q\right)^2 \tag{14}$$

IV. Results and Discussion

I. Real life Applications

We evaluate the suggested Arctan Marshall-Olkin Weibull (ATMOW) distribution against a number of rival distributions that have proven successful in heavy-tailed data modeling. We evaluate the ATMOW model's fitting performance in comparison to alternative flexible distributions using actual datasets from the financial, economic, and environmental realms. The distributions chosen for comparison are the Arctan-Weibull, New Modified Weibull, Exponentiated Weibull, Arcsine-Weibull, and Cosine Topp-Leone Weibull distributions. The results demonstrate that the ATMOW distribution continuously performs better in capturing the underlying patterns and extreme behaviors found in real-world data across all three domains when these models are assessed using common goodness-of-fit metrics.
These distributions' respective CDFs are as follows:

Arctan Weibull (ArctanW) [8]

$$F(x;\theta,\lambda,k)=\frac{4}{\pi}arctan\left(1-exp\left(-\left(\frac{x}{\lambda}\right)^{k}\right)\right)$$

New Modified Weibull (NMW) [28]

$$F(x;\eta,\kappa)=\frac{\left(1-e^{-\lambda x^{k}}\right)\left(e^{-\left(e^{-\lambda x^{k}}\right)}\right)^{\eta}}{e^{\eta}}$$

Exponentiated Weibull Weibull (EWW) [29]

$$F(x)=\left\{1-e^{-\alpha\left(e^{\lambda x^{k}}-1\right)^{m}}\right\}^{a}$$

Arcsine Weibull (ArcsineW) [30]

$$F(x;\alpha,\lambda,k)=\frac{2}{\pi}arcsin\left(1-e^{-\lambda x^{\alpha}}\right)$$

Cosinus Topp Leone Weibull (CTLW) [31]

$$F(x)=1-cos\left[\frac{\pi}{2}\left(1-e^{-2\lambda x^{k}}\right)^{\alpha}\right]$$

Four popular analytical metrics are employed to identify the best-fitting distribution among the candidates: the Hannan–Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), and Akaike Information Criterion (AIC). These criteria strike a balance between complexity and model fit; a model that fits better is indicated by lower numbers. The model that captures the underlying data structure the best is the one with the lowest AIC, BIC, HQIC, and CAIC values among all the competing models.

II. Dataset 1: Services related to insurance and finance (% of service exports, BoP)

The various insurance products that resident insurance companies provide to non-residents and vice versa, as well as the services of financial intermediaries and auxiliary services that residents and non-residents exchange (apart from insurance companies and pension funds), are all considered insurance and financial services in India. The information was gathered from the World Bank website and spans the years 1975–2022.

The estimated parameters and competing distributions of the ATMOW model are shown in Table 1. Because of its scale (λ), shape (k), and transformation (θ) characteristics, the ATMOW model exhibits great adaptability in capturing skewed and heavy-tailed data. Similar to this, models such as EWW and CTLW may depict more intricate distributional shapes by incorporating more shape factors ($m, \alpha, and\ a$). On the other hand, ArctanW and ArcsineW provide less flexibility due to their smaller parameter counts. Model evaluation should account for variables such as insurance and finance, which may play a critical role in shaping the statistical distribution of development-related data.

Table 1: *Estimated Parameters for insurance and finance*

Model	θ	k	λ	η	m	α	a
ATMOW	141.67	374.06	0.1503	-	-	-	-
ArctanW	-	401.775	0.0887	-	-	-	-
NMW	-	1.3294	14.8770	158.5600	-	-	-
EWW	-	1.9057	1.001	-	4.6957	0.1048	0.8310
ArcsineW	-	0.587	0.3318	-	-	-	-
CTLW	-	0. 27188	3.9283	-	-	100.460	-

Table 2 shows that ATMOW has the greatest overall fit among the models, achieving the lowest AIC, BIC, CAIC, and HQIC values. Simpler models like ArctanW, NMW, and ArcsineW exhibit noticeably worse fit, although EWW does rather well. Consequently, ATMOW is the best model for the provided finance and insurance data. Figure 1 provides a visualization of the empirical PDF for finance and insurance, highlighting the fit of the ATMOW distribution to the data.

Table 2: *Outcomes of the data criteria*

Model	AIC	CAIC	BIC	HQIC
ATMOW	-38958.01	-38952.33	-38952.4	-38955.89
ArctanW	-38262.91	-38259.13	-38259.17	-38261.5
NMW	-33296.94	-33291.27	-33291.33	-33294.82
EWW	171.8946	179.4619	179.3794	174.7231
ArcsineW	243.2807	247.0644	247.0231	244.695
CTLW	-113.4197	-107.7442	-107.8061	-111.2983

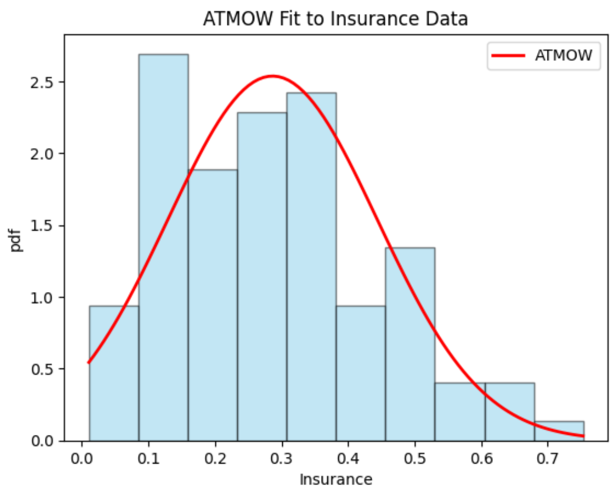

Figure 1: *Empirical PDF visualization for insurance and finance*

III. Dataset 2: The Percentage of GNI that is net ODA received

According to World Bank records, this dataset includes India's net official development aid (ODA) received between 1975 and 2022. Financial flows given by non-DAC states, multilateral organizations, and DAC member countries' official agencies to advance economic growth and welfare are referred to as official development assistance (ODA). Concessional loans (net of principal repayments) and grants are included; each loan has a minimum grant component of 25%, which is determined by applying a 10% discount rate.

In Table 3, the estimated parameters of the ATMOW model and its rivals are displayed. ATMOW offers great versatility for modelling skewed, heavy-tailed data because of its transmutation (θ), shape (k), and scale (λ) characteristics. Additionally, EWW and CTLW incorporate shape parameters ($m, \alpha, and\ a$) to enable them to represent intricate patterns. As opposed to this, ArctanW and ArcsineW employ fewer parameters, which restricts their versatility. The variable Net ODA received (as a percentage of GNI) may further influence the distributional behavior observed in development-related datasets.

Table 3: *Estimated Parameters for Net ODA received*

Model	θ	k	λ	η	m	α	a
ATMOW	37395.021	5.7318	0.007	-	-	-	-
ArctanW	-	0.0104	4.699	-	-	-	-
NMW	-	1.2309	2.252	2.076	-	-	-
EWW	-	3.2405	0.0001	-	0.0789	4.8401	2.5835
ArcsineW	-	0.5475	0.747	-	-	-	-
CTLW	-	5.3307	20.280	-	-	2.5106	-

Table 4 shows that ATMOW has the greatest overall fit among the models, achieving the lowest AIC, BIC, CAIC, and HQIC values. Simpler models like ArctanW, NMW, and ArcsineW exhibit noticeably worse fit, although EWW does rather well. Consequently, ATMOW is the best model for the provided percentage of GNI that is net ODA received data. Figure 2 provides a visualization of the fitted PDF of the ATMOW model for net ODA received data.

Table 4: *Outcomes of the data criteria*

Model	AIC	CAIC	BIC	HQIC
ATMOW	-5284.1007	-5280.3170	-5280.3583	-5282.6864
ArctanW	-1684.723	-1679.048	-1679.11	-1682.602
NMW	54.5954	60.2708	60.2090	56.7168
EWW	-1699.9293	-1692.3621	-1692.4445	-1697.1008
ArcsineW	98.7869	102.5705	102.5293	100.2011
CTLW	-273.3468	-267.6713	-267.7332	-271.2254

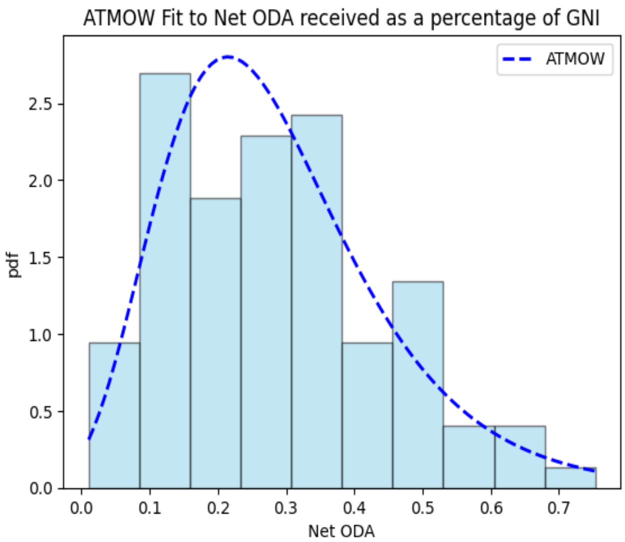

Figure 2: *Empirical PDF visualization for percentage of GNI that is net ODA received*

IV. Dataset 3: USD/GBP foreign exchange rate

This dataset covers daily official USD to GBP exchange rates from 1975 to 2022, relevant to India's financial context. Data are sourced from World Bank records.

The predicted parameters for the ATMOW model and its rivals are shown in Table 5. ATMOW has great adaptability for modelling skewed, heavy-tailed data with transmutation (θ), shape (k), and scale (λ). To capture more intricate patterns, EWW and CTLW can incorporate more form parameters (m, α, a). However, the flexibility of ArctanW and ArcsineW is limited due to their reliance on fewer parameters. In datasets centred on development, the variable Foreign Direct Investment (as a percentage of GDP) may impact the distributional behaviour and should be taken into consideration while choosing a model.

Table 5: *Estimated Parameters for Foreign Direct Investment*

Model	θ	k	λ	η	m	α	a
ATMOW	5.8188	4.3599	0.003	-	-	-	-
ArctanW	-	0.0168	3.564	-	-	-	-
NMW	-	0.4667	1.304	1.1363	-	-	-
EWW	-	0.0277	1.969	-	42.5246	150857.38	6.765
ArcsineW	-	0.2525	7.226	-	-	-	-
CTLW	-	0.4335	8.367	-	-	5.5075	-

Table 6 shows that ATMOW has the greatest overall fit among the models, achieving the lowest AIC, BIC, CAIC, and HQIC values. Simpler models like ArctanW, NMW, and ArcsineW exhibit noticeably worse fit, although EWW does rather well. Consequently, ATMOW is the best model for the provided USD/GBP foreign exchange rate data. Figure 3 visualises the empirical PDF for Foreign direct investment to show how well the ATMOW distribution fits the data.

Table 6: *Outcomes of the data criteria*

Model	AIC	CAIC	BIC	HQIC
ATMOW	-4043.2719	-4039.4883	-4039.5295	-4041.8577
ArctanW	-1235.774	-1230.288	-1230.354	-1233.754
NMW	124.4395	130.1150	130.0531	126.5609
EWW	-1078.4609	-1070.8936	-1070.9761	-1075.6323
ArcsineW	59.6221	63.4057	63.3645	61.0364
CTLW	-293.1883	-287.5128	-287.5747	-291.0669

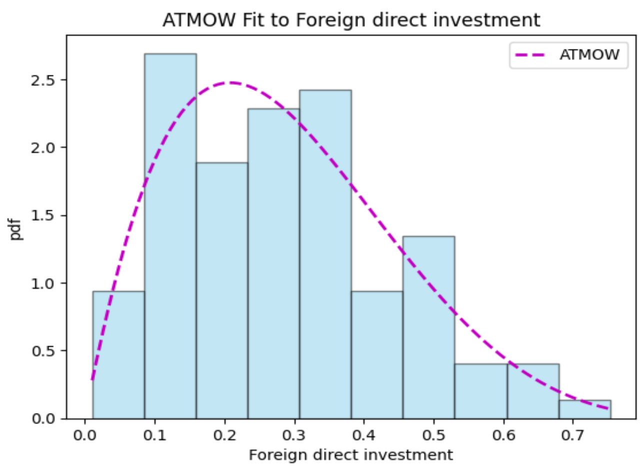

Figure 3: *Empirical PDF visualization for Foreign direct investment*

V. Dataset 4: Environmental Sustainability

This dataset covers annual CO_2 emissions in India from 1975 to 2022, sourced from the World Bank, and reflects trends in environmental sustainability over time.

Table 7 presents the estimated parameters of the ATMOW model and its competing distributions. The ATMOW model demonstrates strong flexibility in capturing skewed and heavy-tailed data, owing to its transmutation (θ), shape (k), and scale (λ) parameters. Similarly, models like EWW and CTLW incorporate additional shape parameters (m, α, a), allowing them to represent more complex distributional forms. In contrast, ArctanW and ArcsineW, with fewer parameters, offer more limited adaptability. The variable Environmental Sustainability may also play a key role in shaping the statistical distribution of development-related indicators and should be accounted for in model evaluation.

Table 7: *Estimated Parameters for environmental sustainability*

Model	θ	k	λ	η	m	α	a
ATMOW	1.5558	29.620	0.2001	-	-	-	-
ArctanW	-	8.1243	0.4953	-	-	-	-
NMW	-	2.1690	0.7727	1.0082	-	-	-
EWW	-	3.4902	0.0001	-	0.0419	1.6940	0.94678
ArcsineW	-	0.5729	0.6749	-	-	-	-
CTLW	-	940.8311	16.6031	-	-	42.1481	-

Table 8 shows that ATMOW has the greatest overall fit among the models, achieving the lowest AIC, BIC, CAIC, and HQIC values. Simpler models like ArctanW, NMW, and ArcsineW exhibit noticeably worse fit, although EWW does rather well. Consequently, ATMOW is the best model for the provided environmental sustainability data. Figure 4 illustrates how well the ATMOW distribution fits the data by visualising the empirical PDF for environmental sustainability.

Table 8: *Outcomes of the data criteria*

Model	AIC	CAIC	BIC	HQIC
ATMOW	-2221.286	-2215.61	-2215.672	-2219.164
ArctanW	17.7765	21.5602	21.5189	19.1908
NMW	162.919	168.5952	168.5334	165.0412
EWW	-2022.906	-2015.338	-2015.4212	-2020.077
ArcsineW	126.335	130.1188	130.0775	127.7494
CTLW	-244.776	-239.1014	-239.163	-242.655

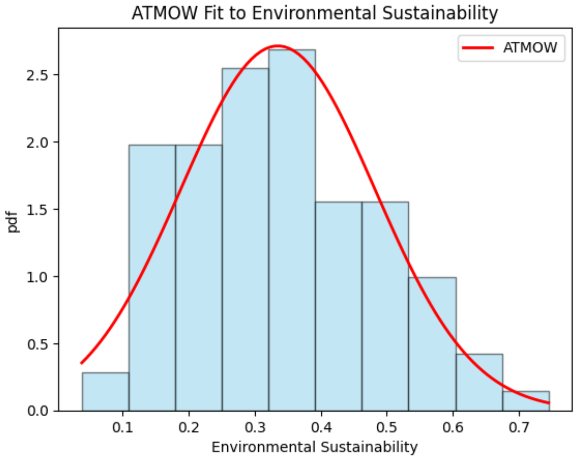

Figure 4: *Empirical PDF visualization for environmental sustainability*.

V. Conclusion

In this research, we constructed a novel family of Arctan Marshall Olkin distributions by combining the Arctan-X and Marshall-Olkin families. Because of this, the suggested family of distributions enhances its ability to describe real data while also accounting for the benefits of the families from which it originates. Throughout the investigation, the special member Arctan Marshall-Olkin Weibull demonstrated dependable mathematical properties (entropy, moments, quantiles, etc.), acceptable actuarial measures (VaR, TV, etc.), and an excellent ability to model heavy-tailed data. We have verified its superiority in modeling heavy-tailed data through simulations using actual data and performance comparisons with three rival models. Therefore, in the fields of finance, economics, and environmental sustainability, the ATMOW distribution is an effective tool for risk modeling and management. Its adaptability in capturing heavy tails, asymmetry, and non-normality enables more precise modeling of extreme occurrences in environmental phenomena including resource depletion, pollution spikes, and climate extremes, as well as in financial markets and economic systems. In terms of financial planning, economic forecasting, and environmental policy, this empowers decision-makers to make more responsible and knowledgeable decisions. The ATMOW distribution supports data-driven governance and sustainable development by increasing the accuracy of predictive models across various interrelated domains.

Future studies may explore new members of the Arctan Marshall-Olkin distribution family by incorporating alternative baseline distributions such as Gamma, Log-Normal, Burr, or Gumbel. Extending the distribution to multivariate, time series, and regression frameworks can broaden its use. Applications to environmental sustainability and improved numerical methods also offer promising directions for development.

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