RELIABILITY AND PERFORMANCE ANALYSIS OF AN UNRELIABLE RETRIAL QUEUEING SYSTEM WITH NEGATIVE ARRIVALS

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Abstract

In this paper, we foccus on a single repairable server retrial queueing system with negative arrivals and two types of breakdowns (passive and active). We assume that the customer in service is lost whenever an active breakdown occurs. Once the repair is completed, the server returns to an idle state. To obtain the system performance metrics, the matrix geometric technique is used combined with direct truncation method. From the standpoint of reliability analysis, we investigate the mean time to the first failure of the system by using the Laplace transform method. Finally, we study numerically the impact of system metrics on the mean number of customers in the orbit, the server state probabilities, the reliability function and the mean time to the first failure.

Keywords: Retrial Queue, Negative Arrivals, Passive and Active breakdowns, Reliability Analysis, Unreliable Server.

1. Introduction

Queueing systems with retrials constitutes a key component of modern teletraffic theory.A distinctive aspect of retrial queues is that when an arriving customer finds the server occupied, they enter a retrial group (orbit) and attempt service again after a random delay. The concept of retrial queues has found broad application in modeling challenges within telephone switching and computer systems. Artalejo and Gomez-Corral [3]ite and Falin and Templeton [12] have authored monographs that extensively outline the fundamental characteristics and principles of retrial queues. These comprehensive works serve as valuable resources, providing a thorough understanding of the theory behind retrial queues and their key features. Additionally, a recent survey conducted by Kim and Kim [17] provides valuable insights.

Queues involving servers that can breakdown and later undergo repair are often observed in practical scenarios. When breakdowns or repairs occur, they disrupt the service and result in increased waiting times for customers. These interruptions also provide an additional opportunity for arriving customers to enter a retrial orbit. Customers whose service is stopped by a failure have several options: they can choose to remain at the server until the repair is complete, leave the system entirely, or return to the retrial orbit. The breakdowns can be categorized as either active or passive, depending on whether they occur during the server working period or idle periods. Additionally, failures can happen randomly after a distinct amount of service time or just before service is about to start. In the paper of Taleb and Aissani [28] we can found a comprehensive source of references on retrial queues with failures, covering a vast range of studies from the earliest works to the most recent publications. Their research offers an extensive compilation of references,

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providing valuable insights into the dynamics and analysis of queues affected by failure events. See also Aissani et al [1], Zirem et al [30], Gao et al [14] and Lisovskaya et al [21] and the references therein.

Negative arrivals in queueing systems, also known as G-Networks, have inspired several researchers during the last 15 years. Models with negative arrivals are successfully used in computer Networks and Neural Networks. In computer Networks, positive claims are tasks (programs) and negative claims are computer viruses . In Neural Networks, negative and positive customers may represent inhibitory and excitatory signals, where the number of customers in the queue corresponds to the neuronal input potential. Gelenbe [15] was the first to introduce a new class of queueing processes in which customers are either negative or positive. Positive customers are primary customers who enter the system with the intention of getting served and then leaving the system. Negative customers cause damage to systems. In the simplest form, a negative customer eliminates a positive or a batch of positive customers according to a precise killing discipline. Removing policies can be classified as follows:

• Negative arrivals that eliminate all customers in the system or from a specific part of it, whether in service or waiting in the buffer (catastrophes, disaster),

• Negative arrivals that remove a positive customer at both edges of the queue (end of queues (RCE) and head of queues (RCH)),

• Negative arrivals that cause breakdowns to the service,

• Negative customers that eliminate random parts of work from the system.

The probability of completing service for positive customers can be influenced by the rate of negative arrivals. Basically, more negative arrivals join the queue, the shorter the queue becomes, which decreases the possibility of completing service and also joining the orbit in some cases.

The subject of negative arrivals in queueing systems attracted many authors. Anisimov and Artalejo [2] considered a multiserver retrial queues with negative arrivals. Two different approaches (stable and over loading) were considered. The matrix-analytic method was used in the stable case and the transient behavior was studied in the overloading case. Artalejo and Economou [4] calculated the stability condition and compared models that operate under different threshold in an $M^X/M/1$ queue controlled by batches of negative customers. Atencia and Moreno [5] studied a Geo/Geo/1 queue with negative and positive customers, under the RCE-inimmune and immune servicing killing disciplines. Muthu Canapathi Subramanian et al. [23] studied a retrial $M/M/1$ queue and negative arrivals under non-preemptive priority service. The negative customer eliminates a randomly chosen customer from the orbit. Ayyappan et al. [7] considered the same model under Pre-emptive Priority. Ayyappan et al. [8] used the matrix geometric method to study a retrial single server queue with negative customers where the service time is assumed to follow an Erlang-K distribution. Ayyappan et al. [9] analysed the influence of negative arrivals of a retrial M/M/1 queue under Coxian phase type services by using the direct truncation method. The negative arrival controls the congestion in the orbit by deleting one customer from the orbit. Dimitriou [11] investigated an $M/G/1$ retrial queue where the server is subject to breakdowns and multiple vacations with the presence of negative arrivals. The stability condition and the system performance measures in a mixed priority discipline of the system accept two types of positive customers (preemptive and non-preemptive). Krishna Kumar et al. [19] investigated an $M/G/1$ retrial queue with negative customers and two types of Bernoulli feedback, where, the negative customer deletes the positive customers in service. Kirupa and Udaya Chandrika [16] analysed a retrial $M^X/G/1$ queue with multi-types of service, negative customers and Feedback operating with a randomized vacation policy. The negative customer deletes the positive customer from the system and puts the server down. Rajadurai et al. [?] studied a retrial $M^X/G/1$ queue with feedback, two phases of service, Bernoulli vacation Schedule and negative customers, where, the server is subject to breakdown. A negative customer deletes the positive customer in service. Lee [20] studied the equilibrium behavior of customers in an unobservable and repairable $M/M/1$ queue with negative arrivals by investigating the Nash equilibrium. Rajadurai et al. [26] considered an unreliable $M/G/1$ retrial queueing system with working vacations and vacation interruption under Bernoulli Schedule with negative arrivals. Ayyappan and Thamizhselvi [6] analysed an unreliable retrial queueing system with two types of positive batch arrivals (low and high priority), non-preemptive priority service, working vacations, negative arrivals and emergency vacation. Chin et al. [10] studied an $M/M/1$ queue with two types of customers (negative and positive). The inter-arrival and service times for positive customers are assumed to follow distributions with a constant asymptotic rate. They provided a new approach to derive the stationary queue length distribution. Matalytski and Zajac [22] determined volumes of memory of information systems (IS) using a stochastic model based on the use of HM-Networks with unreliable service and negative claims. Gao et al [14] studied the behavior of customers in an $M/M/1$ G-retrial queue with complete removals in the unobservable case using game theory concepts. Recently, Koh et al [18] presented an alternative approach to find the stationary queue length distribution of an $G/M/1$ queueing system with negative customers. The inter-arrival time for positive customers is assumed to follow a distribution with a rate that converges to a constant value. Lisovskaya et al [21] focused on a retrial multi-server queue with two orbits and two classes of a positive clients. Negative customers, symbolizing breakdown events, enter the queuing system and remove all currently served customers, if any, from the system. Sun and Wang [27] studied customers' strategic behavior in an $M/M/1$ retrial queue system with negative customers and single server vacation.

In this paper, we investigate a single unreliable server retrial queueing system with negative arrivals and two types of breakdowns. Positive customers receive service as usual by the server. A negative arrival, when present, causes the removal of a customer from the orbit. Negative customers are assumed to act regardless of whether the server is idle or occupied. If the server is free the arriving positive customer is served immediately and leaves the system after service completion (if no breakdowns had occurred during his service time). Otherwise, the positive customers join the orbit and become repeated customers. The server is subject to passive and active Poisson breakdowns. Passive breakdowns can happen while the server is free while active breakdowns can occur when the server is busy. When the server experiences a failure, it is immediately sent for repair. In the case of an active breakdown, the customer being served is considered lost. Once the repair is completed, the server returns to an idle state. To our knowledge, this study is the first to investigate reliability aspects and the time to first failure in queueing models with negative arrivals.

The paper is structured as follows. Section 2 introduces the mathematical model. In Section 3, we establish the necessary and sufficient condition for system stability. Section 4 examines the steady-state probability vector, while Section 5 discusses the performance measures of the system. Section 6 focuses on reliability aspects, including the server time to first failure. Finally, Section 7 provides a numerical example to illustrate the impact of system parameters on performance indicators, the reliability function, and the mean time to the first failure.

2. Mathematical Model

We consider a Markovian single unreliable server retrial queueing system with FCFS orbit and negative arrivals. Customers arrive according to a Poisson process with rate λ for positive arrivals and δ for negative arrivals. Positive customers are served as usual by the server. A negative arrival has the effect of eliminating a customer from the orbit, if any. We assume that negative customers act when the server is busy or free and the service times are exponentially distributed with rate μ. If the server is free, an arriving positive customer is served immediately and leaves the system after service completion (if no breakdowns had occurred during his service time). Otherwise, the positive customer joins the orbit and becomes a repeated customer. The time intervals between each repeated attempt are assumed to be independent and exponentially distributed with rate ν. The server is subject to passive and active Poisson breakdowns . Passive breakdowns can occur when the server is free with rate θ_0 while active breakdowns can occur when the server is busy with rate θ_1. As soon as the server fails it is sent for repair. The repair times have an exponential distribution with rate μ_0 for passive breakdowns and μ_1 for active breakdowns. Finally, we assume that the customer whose service is disrupted due to an active breakdown is considered lost. After repair, the server becomes free. Inter-arrival times, repair times, service times and retrial times are assumed to be mutually independent.

Let $N(t)$ and $C(t)$ be, respectively, the number of customers in the orbit and the server state at time t, where

$$C(t) = \begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is busy at time } t, \\ 2 & \text{if the server is under repair during an passive breakdown at time } t, \\ 3 & \text{if the server is under repair during an active breakdown at time } t. \end{cases}$$

The state of the system is described by the Markov stochastic process:

$$\{(N(t), C(t)), t \geq 0\}$$

defined on state space $E = \mathbb{N} \times \{0, 1, 2, 3\}$.

The infinitesimal generator matrix Q of the continuous time Markov Chain (CTMC) $\{(N(t), C(t)), t \geq 0\}$ can be written as a tridiagonal block matrix as follow

$$Q = \begin{pmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_{11} & A_{12} & & \\ & A_{21} & A_{22} & A_{23} & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $A_{ij}, \; i, j \geq 0$ are square matrices of order 4 given by

$$A_{00} = \begin{pmatrix} -(\lambda+\theta_0) & \lambda & \theta_0 & 0 \\ \mu & -(\lambda+\theta_1+\mu) & 0 & \theta_1 \\ \mu_0 & 0 & -(\lambda+\mu_0) & 0 \\ \mu_1 & 0 & 0 & -(\lambda+\mu_1) \end{pmatrix},$$

$$A_{n(n+1)} = A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, n \geq 0,$$

$$A_{n(n-1)} = A_2 = \begin{pmatrix} \delta & \nu & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, n \geq 1,$$

$$A_{nn} = A_1 = \begin{pmatrix} -(\lambda+\theta_0+\delta+\nu) & \lambda & \theta_0 & 0 \\ \mu & -(\mu+\lambda+\theta_1+\delta+\nu) & 0 & \theta_1 \\ \mu_0 & 0 & -(\lambda+\mu_0+\delta) & 0 \\ \mu_1 & 0 & 0 & -(\lambda+\mu_1+\delta) \end{pmatrix}, n \geq 1.$$

If the capacity M of the orbit is finite then the CTMC $\{(N(t), C(t)), t \geq 0\}$ is defined on the state space $E_M = \{0, 1, \cdots, M\} \times \{0, 1, 2, 3\}$ whose infinitesimal generator $Q^{(M)}$ is given by

$$Q^{(M)} = \begin{pmatrix} A_{00} & A_{01} & & & & \\ A_{10} & A_{11} & A_{12} & & & \\ & A_{21} & A_{22} & A_{23} & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{(M-1)(M-2)} & A_{(M-1)(M-1)} & A_{(M-1)M} \\ & & & & A_{M(M-1)} & A'_{MM} \end{pmatrix},$$

with

$$A'_{MM} = \begin{pmatrix} -(\lambda+\theta_0+\delta+\nu) & \lambda & \theta_0 & 0 \\ \mu & -(\mu+\theta_1+\delta+\nu) & 0 & \theta_1 \\ \mu_0 & 0 & -(\mu_0+\delta) & 0 \\ \mu_1 & 0 & 0 & -(\mu_1+\delta) \end{pmatrix}.$$

3. Stability Condition

In this section, we establish the stability condition of the system.

Theorem 1. The inequality

$$\frac{\lambda\mu_0(\lambda+\nu)(\mu_1+\theta_1) + \mu_1(\mu+\theta_1)(\lambda\theta_0 - \nu\mu_0)}{\delta\left\{\mu_1(\mu+\theta_1)(\mu_0+\theta_0) + \mu_0(\lambda+\nu)(\theta_1+\mu_1)\right\}} < 1, \quad (1)$$

is the necessary and sufficient condition for the system to be stable.

Proof. The process $\{(N(t), C(t), t \geq 0)\}$ is a QBD process. According to Neuts (1981), it is stable if and only if

$$\pi A_0 e < \pi A_2 e, \tag{2}$$

where $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ is the stationary distribution of

$$A = A_0 + A_1 + A_2. \tag{3}$$

That is

$$\pi A = 0 \quad and \quad \pi e = 1, \tag{4}$$

where e is the unit vector with dimension 4.

Solving (4) and then (2), we get (1). ∎

4. Steady State Probability Vector

The goal of this section is to obtain the steady-state distribution of the system. We denote x the steady-state probability vector of the infinitesimal generator Q, which is partitioned as $x = (x(0), x(1), x(2), \dots)$ where $x(i) = (p_{i0}, p_{i1}, p_{i2}, p_{i3})$ for $i = 0, 1, 2, \dots$.
The stationary probability vector x satisfies

$$xQ = 0 \;\; and \;\; \sum_{i \geq 0} \sum_{j=0}^{3} p_{ij} = 1. \tag{5}$$

To solve system (5), we use the Direct Truncation Method, which consists of truncating the state space of the system E to a finite number of states and then computing the system performance measures based on this truncated state space. The Direct Truncation method is a widely used technique in the field of system analysis and control, as it allows for efficient computation of system performance measures without requiring a complete analysis of the system state space. This makes it particularly useful for systems with a large number of states or complex dynamics. More details can be found in (Muthu Canapathi Subramanian et al. [23], Ayyappan et al. [7], Ayyappan et al. [9]).

Let M denotes the cut-off point of the truncation method. The steady state probability vector $x^{(M)}$ is partitioned as $x^{(M)} = (x^M(0), x^M(1), x^M(2), \cdots, x^M(M))$ which satisfies $x^{(M)} Q^{(M)} = 0$ and $x^{(M)} e = 1$, where $x^M(i) = (p^M_{i0}, p^M_{i1}, p^M_{i2}, p^M_{i3})$ for $i = 0, 1, 2, \cdots, M$. To solve this system of equations, we rely on the special structure of the coefficient matrices and employ the CAUSS-SIEDEL elementary transformation method. We initiate the iterative process by setting $M = 2$ and progressively increase it until the elementary of $x^{(M)}$ exhibit negligible changes.

5. System Performance Measures

We outline in this section several important performance measures and their mathematical expressions. These measures serve to highlight the qualitative characteristics of the queueing model under consideration. A comprehensive numerical investigation is conducted to analyze the following performance indicators. Here we define in steady state

- $p(n, 0)$ the probability that there are in the orbit n customers when the server is idle, $p(n, 0) = \lim_{n\to\infty} p(N(t) = n, C(t) = 0)$,
- $p(n, 1)$ the probability that there are in the orbit n customers when the server is busy, $p(n, 1) = \lim_{n\to\infty} p(N(t) = n, C(t) = 1)$,
- $p(n, 2)$ the probability that there are in the orbit n customers when the server is under repair during a passive breakdown, $p(n, 2) = \lim_{n\to\infty} p(N(t) = n, C(t) = 2)$,
- $p(n, 3)$ the probability that there are in the orbit n customers when the server is under repair during an active breakdown. $p(n, 3) = \lim_{n\to\infty} p(N(t) = n, C(t) = 3)$.

- P_j the probability that the server is in state j, $P_j = \sum_{n\geq 0} p(n,j),\ j = 0,1,2,3,$
- $\sum_{j=0}^{3} p(n,j)$ the probability that in the orbit there are n customers , $n \geq 0$.
- MNCO the mean number of customers in the orbit

$$MNCO = \sum_{i=0}^{\infty} i(\sum_{j=0}^{3} p(i,j)).$$

6. Reliability Analysis

In what follows, we examine the time until the server first failure and its reliability, under the assumption that the system is initially empty and the server is free at time t = 0. Denote Y the time to the first failure of the server. We define as $R_Y(t) = P(Y > t)$, the reliability of the server. To evaluate this measure, the analysis is reformulated by introducing a modified queueing model in which the failure states of the server are treated as absorbing states. The infinitesimal generator matrix $\tilde{Q}$ for this model is given below

$$\tilde{Q} = \begin{pmatrix} \tilde{A}_{00} & \tilde{A}_{01} & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & \tilde{A}_{12} & & \\ & \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\tilde{A}_{ij},\ i,j \geq 0$ are square matrices of order 4 defined as follows

$$\tilde{A}_{00} = \begin{pmatrix} -(\lambda+\theta_0) & \lambda & \theta_0 & 0 \\ \mu & -(\lambda+\theta_1+\mu) & 0 & \theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{A}_{n(n+1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, n \geq 0,$$

$$\tilde{A}_{n(n-1)} = \begin{pmatrix} \delta & \nu & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, n \geq 1,$$

$$\tilde{A}_{nn} = \begin{pmatrix} -(\lambda+\theta_0+\delta+\nu) & \lambda & \theta_0 & 0 \\ \mu & -(\lambda+\theta_1+\delta+\nu) & 0 & \theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, n \geq 1.$$

If the size orbit k is finite then the infinitesimal generator $\tilde{Q}_1$ of the new system is given by

$$\tilde{Q}_1 = \begin{pmatrix} \tilde{A}_{00} & \tilde{A}_{01} & & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & \tilde{A}_{12} & & & \\ & \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde{A}_{(k-1)(k-2)} & \tilde{A}_{(k-1)(k-1)} & \tilde{A}_{(k-1)k} \\ & & & & \tilde{A}_{k(k-1)} & \tilde{A}'_{kk} \end{pmatrix},$$

$$\tilde{A}'_{kk} = \begin{pmatrix} -(\lambda+\theta_0+\delta+\nu) & \lambda & \theta_0 & 0 \\ \mu & -(\theta_1+\delta+\nu) & 0 & \theta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The truncation method is used for conducting reliability analysis.
Let $p_{ni}(t)$, be the probability that the system is in state (n, i) at time t, $i = 0, 1, 2, 3, n = 0, 1, 2 \ldots k$. We denote

- $p(t) = (p_0(t), p_1(t), \cdots, p_k(t))$, where
  $p_n(t) = (p_{n0}(t), p_{n1}(t), p_{n2}(t), p_{n3}(t)), 0 \leq n \leq k$,
- $\frac{dp(t)}{dt} = (\frac{dp_0(t)}{dt}, \frac{dp_1(t)}{dt}, \frac{dp_2(t)}{dt}, \frac{dp_3(t)}{dt})$,
- $\frac{dp_n(t)}{dt} = (\frac{dp_{n0}(t)}{dt}, \cdots, \frac{dp_{n3}(t)}{dt})$.

The state probability vector $p(t)$ is a solution of the differential equation $\frac{dp(t)}{dt} = p(t)\tilde{Q}_1$ with the normalisation equation $\sum_{n=0}^{k} \sum_{i=0}^{3} p_{ni}(t) = 1$. This leads to the following system of equations.

$$\frac{dp_{00}(t)}{dt} = -(\lambda + \theta_0)p_{00}(t) + \mu p_{01}(t) + \delta p_{10}(t), \tag{6}$$

$$\frac{dp_{n0}(t)}{dt} = -(\lambda + \delta + \nu + \theta_0)p_{n0}(t) + \mu p_{n1}(t) + \delta p_{(n+1)0}(t), \quad 1 \leq n < k, \tag{7}$$

$$\frac{dp_{k0}(t)}{dt} = -(\lambda + \delta + \nu + \theta_0)p_{k0}(t) + \mu p_{k1}(t), \tag{8}$$

$$\frac{dp_{01}(t)}{dt} = -(\lambda + \mu + \theta_1)p_{01}(t) + \lambda p_{00}(t) + \nu p_{10}(t) + \delta p_{11}(t), \tag{9}$$

$$\begin{aligned} \frac{dp_{n1}(t)}{dt} &= -(\lambda + \delta + \nu + \theta_1)p_{n1}(t) + \lambda p_{(n-1)1}(t) + \lambda p_{n0}(t) + \nu p_{(n+1)0}(t) \\ &\quad + \delta p_{(n+1)1}(t), \quad 1 \leq n < k, \end{aligned} \tag{10}$$

$$\frac{dp_{k1}(t)}{dt} = -(\delta + \mu + \theta_1)p_{k1}(t) + \lambda p_{(k-1)1}(t) + \lambda p_{k0}(t). \tag{11}$$

Using the Laplace transforms $p^*_{ni}(s) = \int_0^\infty e^{-st} p_{ni}(t)dt$, $i = 0, 1, 2, 3$, $0 \leq n \leq k$, equations (6)-(11) are transformed into

$$(\lambda + \theta_0 + s)p^*_{00}(s) - \mu p^*_{01}(s) - \delta p^*_{10}(s) = p_{00}(0), \tag{12}$$

$$(\lambda + \delta + \nu + \theta_0 + s)p^*_{n0}(s) - \mu p^*_{n1}(s) - \delta p^*_{(n+1)0}(s) = p_{n0}(0), \quad 1 \leq n < k, \tag{13}$$

$$(\lambda + \delta + \nu + \theta_0 + s)p^*_{k0}(s) - \mu p^*_{k1}(s) = p_{k0}(0), \tag{14}$$

$$(\lambda + \mu + \theta_1 + s)p^*_{01}(s) - \lambda p^*_{00}(s) - \nu p^*_{10}(s) - \delta p^*_{11}(s) = p_{01}(0), \tag{15}$$

$$\begin{aligned} (\lambda + \delta + \mu + \theta_1 + s)p^*_{n1}(s) - \lambda p^*_{(n-1)1}(s) - \lambda p^*_{n0}(s) - \nu p^*_{(n+1)0}(s) \\ - \delta p^*_{(n+1)1}(s) = p_{n1}(0), \quad 1 \leq n < k, \end{aligned} \tag{16}$$

$$(\delta + \mu + \theta_1 + s)p^*_{k1}(s) - \lambda p^*_{(k-1)1}(s) - \lambda p^*_{k0}(s) = p_{k1}(0), \tag{17}$$

where

$$p_{ni}(0) = \begin{cases} 1 & if \quad n = i = 0, \\ 0 & otherwise. \end{cases}$$

Equations (12)-(17) can be written in matrix form as

$$p^*(s)(sI_{4k \times 4k} - \tilde{Q}_1) = p(0), \tag{18}$$

where $p^*(s) = (p^*_0(s), p^*_1(s), \cdots, p^*_k(s))$, $p^*_n(s) = (p^*_{n0}(s), p^*_{n1}(s), p^*_{n2}(s), p^*_{n3}(s)), 0 \leq n \leq k$, To solve (18), we use Matlab Software. The probability that the system has failed on or before time t is given by $\sum_{n=0}^{k}\left(p_{n2}(t)+p_{n3}(t)\right)$. Hence the reliability function $R_Y(t)$ can be derived using the relation

$$R_Y(t) = 1 - \sum_{n=0}^{k}\left(p_{n2}(t)+p_{n3}(t)\right), \quad t \geq 0, \tag{19}$$

where $p_{n2}(t)$ and $p_{n3}(t)$ can be determined by taking the inverse Laplace transforms $p^*_{n2}(s)$ and $p^*_{n3}(s)$ respectively.

The mean time to the first failure MTTF is calculated as

$$\begin{aligned} MTTF &= \lim_{s\to 0} R^*_Y(s) \\ &= \lim_{s\to 0}\left[\int_0^\infty R_Y(t)e^{-st}dt\right] \\ &= \lim_{s\to 0}\left[1/s - \sum_{n=0}^{k}\left(p^*_{n2}(s)+p^*_{n3}(s)\right)\right]. \end{aligned} \tag{20}$$

7. Numerical Illustrations

In this section, we show through numerical examples the impact of some system parameters on the performance measures. We particularly show the effect of δ (negative arrival rate) and ν (retrial rate) on the mean number of customers in the orbit MNCO and the server state probabilities P_0, P_1, P_2, P_3. The values of parameters λ (positive arrival rate), δ, ν, θ_0 (passive failure rate), θ_1(active failure rate), μ (service rate), μ_0 (repair rate of passive breakdowns) and μ_1 (repair rate of active breakdowns) are selected to ensure the stability condition is satisfied. We also explore the effect of system parameters on the reliability function $R_Y(t)$ and the mean time to the first failure MTTF. To analyse $R_Y(t)$ and MTTF, we change the values of each parameter in turn while the other parameters take the following given values $\lambda = 0.2,\ \mu = 0.5,\ \nu = 0.3,\ \delta = 1,\ \theta_0 = 0.3,\ \theta_1 = 0.4,\ \mu_0 = 0.2,\ \mu_1 = 0.1, k = 3$.

7.1. Monotonicity of MNCO, P_0, P_1, P_2 and P_3

Tables 1-5 show (for different values of the cut-off point M) that

- The mean number of customers in the orbit (as expected) decreases with the increase of ν and δ. In other words, the mean number of customers in the system decreases as a result of negative customers deleting positive customers within the orbit.
- As function of ν, both probabilities P_1, P_3 (respectively P_0, P_2) increase (respectively decrease). The reason for this is that more customers retry, more the server is busier and hence more it is prone to active failures. As a result, P_0 and P_2 decrease.
- The negative arrival rate δ has a positive effect on P_0, P_2 and a negative effect on P_1, P_3. The larger δ is, the smaller the orbit size, which means that the server is free and therefore more susceptible to passive failures. Consequently, P_1 and P_3 decrease.
- P_0, P_1, P_2 and P_3 exhibit a gradual and slow monotonic trend until reaching a specific threshold.

7.2. Monotonicity of $R_Y(t)$ and MTTF

- From Figures 1(a) and 1(b), the reliability function $R_Y(t)$ is decreasing with respect to λ and increasing with respect to μ respectively. The impact of λ and μ is negligible for small values .
- Figures 1(c) and 1(d) show that the reliability function decreases with the increase of θ_0 and θ_1. As the value of θ_0 (respectively θ_1) increases, the reliability function rapidly (respectively progressively) approaches zero.

- As depicted in Figures 1(e),1(f),1(g) and 1(h) the parameters ν, δ,μ_0 and μ_1 have no influence on system reliability. This trivial observation arises from the fact that these parameters are not included in the formula used to calculate system reliability.
- Tables 6(a)-6(d) show that the mean time to the first failure MTTF decreases as λ increases.
- In Table 6(a), we remark that MTTF increases with respect to μ. The decline in MTTF is particularly steep for low values of λ but it gradually becomes less pronounced as λ increases. The impact of μ on the values of MTTF is much more substantial than that of μ.
- Tables 6(b) and 6(c) show respectively a decrease of MTTF with respect to θ_1 and θ_0. The decrease is slow for θ_1 and significant for θ_0 with small values of λ.
- The influence of δ on MTTF is negligible as shown in Table 6(d).

Table 1: *Mean number of customers in the orbit and server state probabilities for* $\lambda = 5, \mu = 20, \mu_0 = 10, \mu_1 = 15, \delta = 5, \theta_0 = 0.5, \theta_1 = 0.5, M = 2$ *and different values of* ν

ν	P_0	P_1	P_2	P_3	MNCO
10	0.7452	0.2105	0.0373	0.0070	0.1488
20	0.7391	0.2167	0.0370	0.0072	0.1185
30	0.7362	0.2196	0.0368	0.0073	0.1046
40	0.7346	0.2213	0.0367	0.0074	0.0966
50	0.7335	0.2224	0.0367	0.0074	0.0914
60	0.7328	0.2232	0.0366	0.0074	0.0878
70	0.7322	0.2237	0.0366	0.0074	0.0851
80	0.7318	0.2242	0.0366	0.0075	0.0830
90	0.7314	0.2245	0.0366	0.0075	0.0814
100	0.7311	0.2248	0.0366	0.0075	0.0800
200	0.7298	0.2261	0.0365	0.0075	0.0737
300	0.7293	0.2265	0.0365	0.0076	0.0715
400	0.7291	0.2268	0.0365	0.0076	0.0704
500	0.7290	0.2270	0.0365	0.0076	0.0697
600	0.7289	0.2271	0.0364	0.0076	0.0693
700	0.7288	0.2271	0.0364	0.0076	0.0690
800	0.7288	0.2272	0.0364	0.0076	0.0687
900	0.7288	0.2272	0.0364	0.0076	0.0685
1000	0.7288	0.2273	0.0364	0.0076	0.0684
2000	0.7286	0.2274	0.0364	0.0076	0.0677
3000	0.7285	0.2275	0.0364	0.0076	0.0675
4000	0.7285	0.2275	0.0364	0.0076	0.0673
5000	0.7285	0.2275	0.0364	0.0076	0.0673
6000	0.7285	0.2275	0.0364	0.0076	0.0672
7000	0.7285	0.2275	0.0364	0.0076	0.0672
8000	0.7285	0.2275	0.0364	0.0076	0.0672
9000	0.7285	0.2275	0.0364	0.0076	0.0671

Table 2: *Mean number of customers in the orbit and server state probabilities for* $\lambda = 5, \mu = 20, \mu_0 = 10, \mu_1 = 15, \nu = 10, \theta_0 = 0.5, \theta_1 = 0.5, M = 2$ *and different values of* δ

δ	P_0	P_1	P_2	P_3	MNCO
0.005	0.7201	0.2360	0.0360	0.0079	0.2974
0.505	0.7244	0.2317	0.0362	0.0077	0.2714
1	0.7280	0.2280	0.0364	0.0076	0.2495
1.005	0.7280	0.2280	0.0364	0.0076	0.2493
1.505	0.7312	0.2248	0.0366	0.0075	0.2304
2	0.7339	0.2220	0.03670	0.0074	0.2142
2.005	0.7340	0.2219	0.0367	0.0074	0.2140
2.505	0.7364	0.2194	0.0368	0.0073	0.1977
3	0.7386	0.2173	0.0369	0.0073	0.1872

Table 3: *Mean number of customers in the orbit and server state probabilities for* $\lambda = 5, \mu = 20, \mu_0 = 10, \mu_1 = 15, \delta = 2, \theta_0 = 0.5, \theta_1 = 0.5, M = 6$ *and different values of* ν

ν	P_0	P_1	P_2	P_3	MNCO
10	0.7304	0.2256	0.0365	0.0075	0.2865
20	0.7255	0.2306	0.0363	0.0077	0.1971
30	0.7235	0.2326	0.0362	0.0078	0.1644
40	0.7225	0.2336	0.0361	0.0078	0.1475
50	0.7218	0.2343	0.0361	0.0078	0.1371
60	0.7214	0.2347	0.0361	0.0078	0.1301
70	0.7211	0.2350	0.0361	0.0078	0.1251
80	0.7208	0.2353	0.0360	0.0078	0.1213
90	0.7206	0.2355	0.0360	0.0079	0.1183
100	0.7205	0.2356	0.0360	0.0079	0.1159
200	0.7198	0.2364	0.0360	0.0079	0.1051
300	0.7195	0.2366	0.0360	0.0079	0.1014
400	0.7194	0.2367	0.0360	0.0079	0.0996
500	0.7193	0.2368	0.0360	0.0079	0.0985
600	0.7193	0.2367	0.0360	0.0079	0.0978
700	0.7192	0.2369	0.0360	0.0079	0.0973
800	0.7192	0.2369	0.0360	0.0079	0.0970
900	0.7192	0.2369	0.0360	0.0079	0.0966

Table 4: *Mean number of customers in the orbit and server state probabilities for* $\lambda = 5, \mu = 20, \mu_0 = 10, \mu_1 = 15, \nu = 10, \theta_0 = 0.5, \theta_1 = 0.5, M = 6$ *and different values of* δ

δ	P_0	P_1	P_2	P_3	MNCO
0.005	0.7207	0.2431	0.0240	0.0121	0.4323
0.505	0.7263	0.2376	0.0242	0.0119	0.3762
1	0.7304	0.2335	0.0243	0.0117	0.3363
1.005	0.7309	0.2331	0.0244	0.0117	0.3324
1.505	0.7348	0.2293	0.0245	0.0115	0.2973
2	0.7378	0.2262	0.0246	0.0113	0.2714
2.005	0.7381	0.2260	0.0246	0.0113	0.2687
2.505	0.7410	0.2231	0.0247	0.0112	0.2450
3	0.7433	0.2209	0.0248	0.0110	0.2268

Table 5: *Mean number of customers in the orbit and server state probabilities for $\lambda = 5, \mu = 20, \mu_0 = 10, \mu_1 = 15, \delta = 2, \theta_0 = 0.5, \theta_1 = 0.5, M = 16$ and different values of v*

v	P_0	P_1	P_2	P_3	MNCO
10	0.7303	0.2256	0.0365	0.0075	0.2929
20	0.7254	0.2306	0.0363	0.0077	0.1997
30	0.7235	0.2326	0.0362	0.0078	0.1662
40	0.7225	0.2336	0.0361	0.0078	0.1489
50	0.7218	0.2343	0.0361	0.0078	0.1384
60	0.7214	0.2347	0.0361	0.0078	0.1313
70	0.7211	0.2351	0.0361	0.0078	0.1261
80	0.7208	0.2353	0.0360	0.0078	0.1223
90	0.7206	0.2355	0.0360	0.0079	0.1193
100	0.7205	0.2357	0.0360	0.0079	0.1169
200	0.7198	0.2364	0.0360	0.0079	0.1059
300	0.7195	0.2366	0.0360	0.0079	0.1022
400	0.7194	0.2367	0.0360	0.0079	0.1003
500	0.7193	0.2368	0.0360	0.0079	0.0992
600	0.7193	0.2369	0.0360	0.0079	0.0985
700	0.7192	0.2369	0.0360	0.0079	0.0980
800	0.7192	0.2369	0.0360	0.0079	0.0976
900	0.7192	0.2369	0.0360	0.0079	0.0973

Table 6: *MTTF vs system parameters.*

(a) *MTTF for different values of λ and μ*

$\lambda \setminus \mu$	0.1	0.5	1	2	3
0.1	17.376	29.580	40.563	54.703	63.414
0.2	11.610	19.835	26.829	38.603	47.123
0.3	9.506	14.853	20.724	30.352	37.907
0.4	8.415	12.568	17.273	25.335	31.981
0.5	7.748	11.138	15.055	21.963	27.849

(b) *MTTF for different values of λ and θ_1*

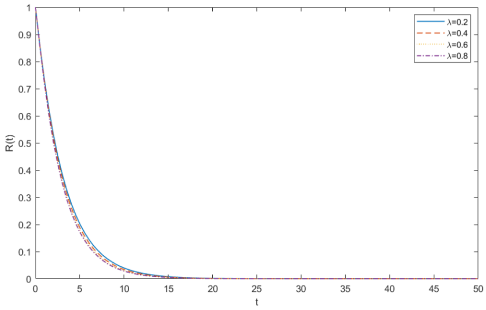
$\lambda \setminus \theta_1$	0.5	0.9	1.5	2	3
0.1	4.400	4.054	3.818	3.714	3.600
0.2	3.999	3.477	3.142	2.999	2.846
0.3	3.712	3.089	2.705	2.545	2.374
0.4	3.497	2.810	2.399	2.230	2.052
0.5	3.330	2.600	2.172	1.999	1.817

(c) *MTTF for different values of λ and θ_0*

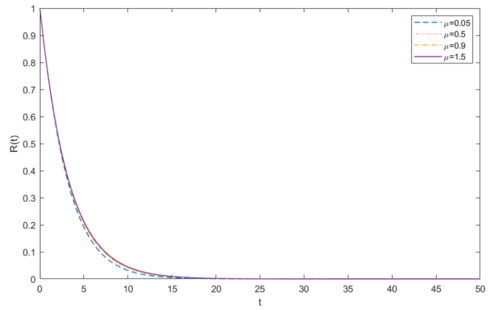
$\lambda \setminus \theta_0$	0.03	0.05	0.07	0.09	0.1
0.1	14.916	11.759	9.705	8.262	7.690
0.2	10.266	8.790	7.685	6.827	6.466
0.3	8.146	7.260	6.548	5.963	5.708
0.4	6.933	6.327	5.818	5.385	5.192
0.5	6.148	5.698	5.310	4.972	4.818

(d) *MTTF for different values of λ and δ*

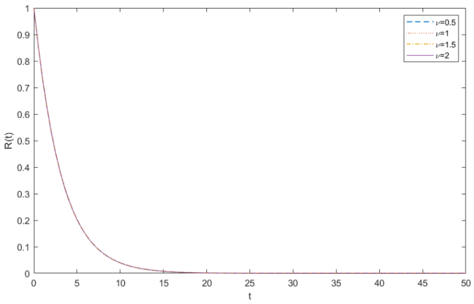
$\lambda \setminus \delta$	1	2	3	4
0.1	4.543	4.545	4.545	4.545
0.2	4.225	4.228	4.230	4.230
0.3	3.991	3.996	3.998	3.999
0.4	3.811	3.818	3.821	3.822
0.5	3.669	3.678	3.681	3.682



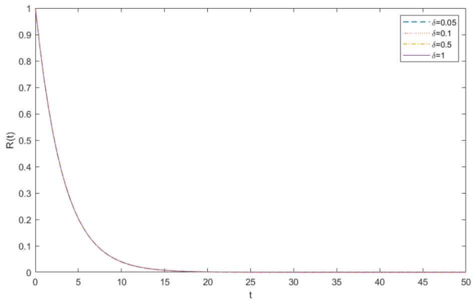
(a) $R(t)$ *versus t for different values of* λ



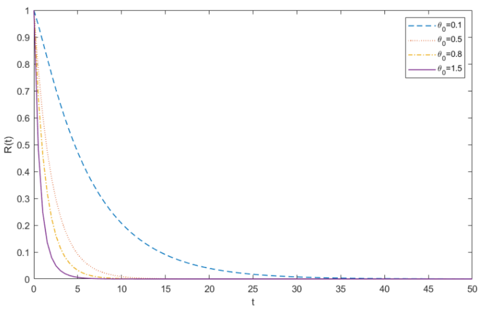
(b) $R(t)$ *versus t for different values of* μ



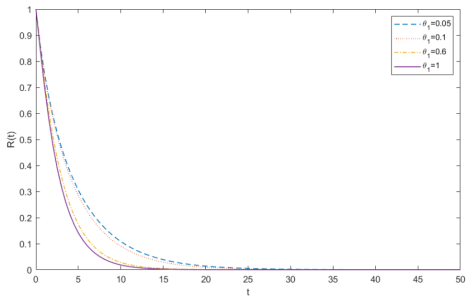
(c) $R(t)$ *versus t for different values of* ν



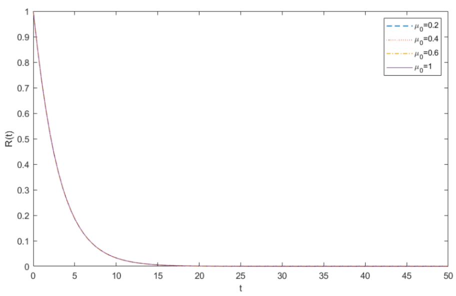
(d) $R(t)$ *versus t for different values of* δ



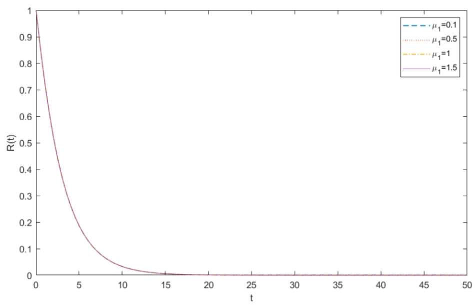
(e) $R(t)$ *versus t for different values of* θ_0



(f) $R(t)$ *versus t for different values of* θ_1



(g) $R(t)$ *versus t for different values of* μ_0



(h) $R(t)$ *versus t for different values of* μ_1

Figure 1: $R(t)$ *vs system parameters.*

8. Conclusion

This paper focused on analyzing from queueing and reliability view points an $M/M/1$ constant retrial queue that involves negative arrivals and two types of breakdowns.

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