OPTIMIZATION OF A DETERIORATING INVENTORY MODEL WITH TIME-DEPENDENT HEAVISIDE DEMAND AND RETURN POLICY UNDER TRIANGULAR FUZZY DECISION-MAKING

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Abstract

The objective of this work is to present an improved inventory system with fuzzy constraints dealing with two warehouse systems, own and rented. In the present model, we analyze the system under the consideration of two warehouses without shortages with the development of a fuzzy inventory model with time-varying heaviside demand function, deterioration with returns policy. The demand , deterioration and returns rate parameters are taken as triangular fuzzy numbers. Numerical example is given to validate the proposed mathematical model which has been developed for determining the optimal cycle time and optimal total inventory cost and profit. Sensitivity analysis is also carried out to explore the effect of changes in the optimal solution with respect to change in various parameters. The aim of this paper is to develop inventory policies that minimize the total cost to get the maximum total profit in both crisp and fuzzy modelling and comparison of crisp and fuzzy models. Our study focuses on defuzzifying the total cost using the signed distance method and comparing it with the crisp model. We assume that the return product will be sold at the same price. Customers are allowed to return the product during any phase of the length of the replenishment cycle. The consumers get the 85% of the price on the returned items and the number of returns are assumed to be depend on demand.

Keywords: Optimization, Inventory, Deterioration, Holding Cost, Returns Rate, Warehouse, Triangular Fuzzy Number, Signed Distance Method.

1. Introduction and literature review

This study aims to develop a two-warehouse inventory model under both crisp and fuzzy environments, incorporating a return policy. In high-demand retail settings such as supermarkets and corporate markets, storage capacity is often limited. When suppliers offer attractive price discounts for bulk purchases or when the item is seasonal such as agricultural harvest outputs-large quantities are procured. However, these quantities may exceed the capacity of the primary warehouse (own warehouse or OW). In such situations, an additional storage facility, referred to as the rented warehouse (RW), is hired. This RW is typically located at a distance and incurs higher holding costs compared to OW. Consequently, inventory is stored in OW first, and only the surplus is placed in RW. To minimize costs, items from RW are transferred continuously to OW following a release pattern. Return policies offered by sellers serve as an incentive for customers. These policies permit buyers to return unsatisfactory products, with a refund typically amounting

to 85 percent of the purchase price. In this model, the return rate is considered proportional to the demand rate. In inventory systems, uncertainty often arises, which is effectively captured through fuzzy set theory. Fuzziness reflects real-world ambiguity more accurately than crisp values and offers an improved framework for optimizing inventory models. Therefore, fuzzy logic presents a valuable approach in refining the results of inventory control systems. In the proposed two-warehouse inventory framework, we model the demand rate, deterioration rate, and return rate as trapezoidal fuzzy numbers. The structure of the paper is as follows: initially, we present the key assumptions and notations. This is followed by the development of the inventory model under a crisp environment. Thereafter, we extend the model to a fuzzy environment using the signed distance method and perform a sensitivity analysis through a numerical example.

In recent years, several researchers have examined the integration of time-dependent parameters, partial backlogging, and return policies within two-warehouse inventory models under fuzzy environments. For example, Sahoo et al. [1] proposed a fuzzy inventory model with exponential demand and time-varying deterioration. Jaggi et al. [2] investigated inventory decisions for deteriorating items of imperfect quality with exponentially declining demand, incorporating trade credit and partially backlogged shortages. Malik and Garg [3] presented an improved fuzzy inventory model involving two warehouses. Chou [4] formulated a fuzzy economic order quantity (EOQ) inventory model. Chakraborty et al. [5] developed a two-warehouse inventory model with partial backlogging, ramp-type demand, and three-parameter Weibull deterioration under inflation and permissible payment delays. Chang et al. [6] introduced a fuzzy inventory model that incorporates fuzzy random variables, lead time, and total demand. Dutta et al. [7] worked on a continuous review inventory system combining fuzzy and stochastic elements. Lee and Yao [8] analyzed economic production quantity models under fuzzy demand and fuzzy production quantities. Vujosevic et al. [9] extended the EOQ model to incorporate fuzzy inventory costs. Malik and Singh [10] proposed a fuzzy mixture model for two-warehouse systems with linear demand. Malik et al. [11] studied inventory with time-dependent demand for non-instantaneous deteriorating items considering maximum shelf life. Pakkala and Achary [12] developed a deterministic two-warehouse inventory model for deteriorating items with a finite replenishment rate. Recent research has focused on incorporating environmental factors and technology investments into inventory models with deterioration. Yadav and Kumar [13] proposed a model that considers selling price, time-sensitive demand, and carbon emissions under green technology investment. Yadav et al. [14] introduced a two-warehouse model using an interval approach, including preservation technology to manage uncertainty. In another study, Yadav et al. [15] addressed deterioration during storage within a two-warehouse setup, aiming to reduce total costs. They further extended this by including reliability and carbon emission constraints with time-based demand [16]. Similarly, Mahata and Debnath [17] analyzed a price-sensitive inventory system with preservation investment to handle deterioration. Later, they explored the impact of green technology and flexible production on economic production models with carbon emissions [18], showing that cleaner and more flexible systems help reduce environmental costs. Debnath et al. [19] developed an EOQ model under uncertainty using a generalized intuitionistic fuzzy Laplace transform. Their approach helps handle imprecise data more effectively in inventory systems. Rana and Kumar [20] developed a two-warehouse inventory model for deteriorating items with hybrid and stock-dependent demand under partial backlogging, reflecting real-world storage and demand situations. Nwoba et al. [21] presented a model with exponential demand and Weibull deterioration, considering shortage-dependent partial backlogging. Patra et al. [22] proposed a retailer's model using power pattern demand with partial backlogging, under both crisp and fuzzy environments to handle uncertainty. Gupta and Mishra [23] introduced a multi-item stochastic inventory model with power demand, partial backlogging, and joint replenishment, helping manage uncertain demand and coordination. All these studies are applicable to industries dealing with perishable or deteriorating items such as food and beverages, pharmaceuticals, chemicals, and seasonal goods. These sectors often face challenges related to limited storage, variable demand, product decay, and customer return behavior.

2. Methods

2.1. Definitions and Preliminaries

1. **Fuzzy set:** A fuzzy set allows for partial membership in which an element can belong to a set to a certain degree or degree of membership, which is represented by a value between 0 and 1. Mathematically, A fuzzy set defined on a universe of discourse $X = \{x_1, x_2, x_3, \ldots, x_n\}$ is given by $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A(x)$ is the membership function of A given by $\mu_A(x) : X \to [0, 1]$.

2. **Fuzzy Number:** A fuzzy number is a fuzzy set on the real line $\mathbb{R}$, if its membership function $\mu_A(x)$ has the following properties:

   (a) $\mu_A(x)$ is upper semi-continuous.

   (b) There exist some real numbers a_2 and a_3, $a_1 \leq a_2 \leq a_3 \leq a_4$ such that $\mu_A(x)$ is increasing on $[a_1, a_2]$, decreasing on $[a_3, a_4]$ and $\mu_A(x) = 1$ for each x on $[a_2, a_3]$.

   (c) $\mu_A(x) = 0$ outside the interval $[a_1, a_4]$.

3. **Triangular Fuzzy Number:** A triangular fuzzy number is specified by the triplet (a_1, a_2, a_3) and defined by its membership function $\mu_A(x) : X \to [0, 1]$ as follows:

$$\mu_A(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & x \geq a_3 \end{cases}$$

2.2. Assumptions and notations

The following assumptions have been adopted while developing the fuzzy inventory model:

1. Single item is used in this developed fuzzy inventory model.

2. The system involves two-warehouse system, one is own warehouse (OW) with limited capacity and other is rented warehouse (RW) with infinite capacity.

3. Demands are first met directly from RW and after all items are released from RW, the demands will be met from OW.

4. The time-dependent Heaviside demand rate function is defined as:

$$\gamma(t) = a + bt + c[t - (t - \mu)H(t - \mu)]t$$

5. The inventory deteriorates as a linear function of time and given by:

$$\eta(t) = \alpha t$$

6. It is assumed that customers return in proportional to demand so return rate is:

$$R(t) = \sigma\gamma(t), \quad \text{where } 0 \leq \sigma < 1$$

7. The holding cost per item is constant.

8. Time horizon is finite.

9. Transportation cost and rent of warehouse is assumed to be added in holding cost at RW.

10. The lead time is assumed to be zero.

11. Shortages are not allowed.

The following notations are used for the proposed fuzzy inventory model:

Table 1: *Notations used in the proposed fuzzy inventory model*

Symbol	Description
OW	Own warehouse
RW	Rented warehouse
W_o	Maximum inventory level in OW
$\eta(t)$	Deterioration rate
$\gamma(t)$	Time dependent demand
T	Length of one cycle
$Q_1(t)$	Inventory level at time $t \in [0, t_1]$
$Q_2(t)$	Inventory level at time $t \in [t_1, T]$
Q_{o1}	Inventory level at OW at time $t \in [0, t_1]$
Q_{o2}	Inventory level at OW at time $t \in [t_1, T]$
Q_r	Inventory level at RW at time $t \in [0, t_1]$
A	Ordering cost
c_s	Price of an item
p	Selling price of an item
x	Deterioration rate in RW
y	Deterioration rate in OW
d_c	Deterioration cost per unit
α	Deterioration coefficient parameter
$\hat{\alpha}$	Fuzzy deterioration coefficient parameter
h_c	Holding cost coefficients
$R(t)$	Return rate
T_{CC}	Optimal cost in crisp model

3. Mathematical Formulation of Model

Here, we consider an inventory system under two warehouses system for seasonal products. For RW, the inventory level Q_r reaches zero level after the time t_1. After that, the customer demand is fulfilled by OW during time period (t_1, T). The initial inventory level for own warehouse is W_o. During period $(0, t_1)$, the demand of the customer is fulfilled from RW, so in between $(0, t_1)$ some product deteriorates in OW. Let $Q_1(t)$ be the inventory level at time $t \in [0, t_1]$, $Q_2(t)$ be the inventory level at time $t \in [t_1, T]$ and Q_{o1} inventory level at OW at time $t \in [0, t_1]$ and Q_{o2} be the inventory level at OW at time $t \in [t_1, T]$. Therefore, the governing equations of the system under the conditions

$$Q_r(t_1) = 0, \quad Q_{o1}(0) = W_o, \quad Q_{o2}(T) = 0$$

are:

$$\frac{dQ_r}{dt} = -\alpha t \, Q_r - \gamma(t) + \sigma \, \gamma(t) \quad \text{for } 0 \le t \le t_1 \tag{1}$$

$$\frac{dQ_{o1}}{dt} = -\alpha t \, Q_{o1} \quad \text{for } 0 \le t \le t_1 \tag{2}$$

$$\frac{dQ_{o2}}{dt} = -\alpha t \, Q_{o2} - \gamma(t) + \sigma \, \gamma(t) \quad \text{for } t_1 \le t \le T \tag{3}$$

3.1.   Case-I: Crisp Inventory Model, When $t < \mu$

: The solutions of differential equations (i), (ii) and (iii) with boundary conditions are:

$$\begin{aligned}
Q_r(t) = (\sigma - 1)\Big[& a(t - t_1) + 0.34a\alpha(t^3 - t_1^3) + 0.5b(t^2 - t_1^2) - 0.125b\alpha(t^4 - t_1^4) \\
& + 0.33c(t^3 - t_1^3) - 0.067c\alpha(t^5 - t_1^5) + 0.01c\alpha^2(t^7 - t_1^7) - 0.5\alpha t^2(at - T) \\
& - 0.17\alpha^2 t^2(t^3 - t_1^3) - 0.25\alpha t^2 b(t^2 - t_1^2) + 0.0625 t^2 b\alpha^2(t^4 - t_1^4) \\
& - 0.0165\alpha t^2 c(t^3 - t_1^3) + 0.125\alpha^2 t^4 a(t - t_1) + 0.0625\alpha^2 t^4 b(t^2 - t_1^2) \\
& + 0.021b\alpha^2(t^6 - t_1^6) + 0.041\alpha^2 t^4 c(t^3 - t_1^3) - 0.125b\alpha(t^4 - t_1^4)\Big]
\end{aligned} \tag{4}$$

$$Q_{o1}(t) = W_0 \left(1 - 0.5\alpha t^2 + 0.125\alpha^2 t^4 - 0.021\alpha^3 t^6 \right) \tag{5}$$

$$\begin{aligned}
Q_{o2}(t) = (\sigma - 1)\Big[& a(t - T) + 0.34a(t^3 - T^3) + 0.5b(t^2 - T^2) - 0.125b\alpha(t^4 - T^4) \\
& + 0.33c(t^3 - T^3) - 0.067c\alpha(t^5 - T^5) - 0.5\alpha t(a - T) - 0.17\alpha^2 t^2(t^3 - T^3) \\
& - 0.034 t^2 \alpha^3(t^5 - T^5) - 0.25\alpha t^2 b(t^2 - T^2) - 0.0165\alpha t^2 c(t^3 - T^3) \\
& + 0.125\alpha^2 t^4 a(t - T) + 0.041\alpha^2 t^4 c(t^3 - T^3) - 0.125b\alpha(t^4 - T^4)\Big]
\end{aligned} \tag{6}$$

The total cost of the system is the sum of all associated inventory costs calculated as above. Therefore, the total cost per unit time (or total average cost per cycle) is given by:

1. **Ordering cost:**
$$OC = A$$

2. **Purchase cost:**
$$PC = cQ_0$$

3. **Deterioration Cost:** Inventory deterioration cost in RW is
$$DC_{\text{RW}} = d_c \int_0^{t_1} x Q_r(t) \, dt$$

   Inventory deterioration cost in OW is
$$DC_{\text{OW}} = d_c \int_0^{t_1} y Q_{o1}(t) \, dt + d_c \int_{t_1}^{T} y Q_{o2}(t) \, dt$$

4. **Storage Cost:**
$$SC = \left[h_r \int_0^{t_1} Q_r(t) \, dt + h_o \int_0^{t_1} Q_{o1}(t) \, dt + h_o \int_{t_1}^{T} Q_{o2}(t) \, dt \right]$$

Total cost per unit time of inventory system is:

$$TC = \frac{1}{T}[OC + PC + DC + HC]$$

$$
\begin{aligned}
TC = \frac{1}{T}\Big[& A + cQ_0 + d_c[(x - y)(\sigma - 1)(0.5\alpha t_1^2 - 0.29\alpha x t_1^4 - 0.33bt_1^3 \\
& + 0.233\,t_1^5 b\alpha - 0.049b\alpha^2 t_1^7 - 0.25ct_1^4 + 0.0666\alpha ct_1^6)] + yW_0(t_1 - 0.166\alpha t_1^2 \\
& + 0.025\alpha^2 t_1^5) + y(\sigma - 1)((0.5\alpha T^2 - 0.29\alpha T^4 - 0.085\,\alpha^2 T^6 - 0.33bT^3 + 0.233\,T^5 b\alpha \\
& - 0.25cT^4 + 0.0666\alpha cT^6 - 0.019\,\alpha^2 T^8)] + (h_r - h_o)(\sigma - 1)(0.5\alpha t_1^2 - 0.29\alpha t_1^4 \\
& - 0.33bt_1^3 + 0.233\,t_1^5 b\alpha - 0.25ct_1^4 + 0.0666\alpha ct_1^6)] + h_o W_0[t_1 - 0.166\alpha t_1^2 + 0.025\alpha^2 t_1^5 \\
& + h_o(\sigma - 1)(0.5\alpha T^2 - 0.29\alpha T^4 - 0.085\,\alpha^2 T^6 - 0.33bT^3 + 0.233\,T^5 b\alpha \\
& \qquad\qquad\qquad\qquad - 0.25cT^4 + 0.0666\alpha cT^6 - 0.019\,\alpha^2 T^8)]\Big] \quad (7)
\end{aligned}
$$

Sales revenue collected over the cycle (SR):

$$\text{SR} = pQ_0 + 0.15p\int_0^T R(t)\,dt = pQ_0 + 0.15p\sigma\left(aT + 0.5bT^2 + 0.33cT^3\right) \tag{8}$$

The total profit per unit time is:

$$\text{TP}^0 = \frac{1}{T}\left[\text{SR} - \text{OC} - \text{PC} - \text{HC} - \text{DC} - \text{SC}\right] \tag{vi}$$

3.2. Case-I: Fuzzy Inventory Model, When $t < \mu$

In this section, the fuzzy inventory model is presented through the signed distance method. Let

$$\hat{a} = (a_1, a_2, a_3), \quad \hat{b} = (b_1, b_2, b_3), \quad \hat{c} = (c_1, c_2, c_3), \quad \hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \quad \hat{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

be triangular fuzzy numbers. By the signed distance method, the total cost per unit time is:

$$TC^* = \frac{1}{4}(TC_1 + 2TC_2 + TC_3) \tag{9}$$

where

$$
\begin{aligned}
TC_1 = \frac{1}{T}\Big[& A + cQ_0 + d_c[(x - y)(\sigma_1 - 1)(0.5a_1 t_1^2 - 0.29a_1\alpha_1 t_1^4 - 0.085\,a_1\alpha_1^2 t_1^6 \\
& - 0.33b_1 t_1^3 + 0.233\,t_1^5 b_1\alpha_1 - 0.049b_1\alpha_1^2 t_1^7 - 0.25c_1 t_1^4 + 0.0666\alpha_1 c_1 t_1^6 \\
& - 0.019\,c_1\alpha_1^2 t_1^8)] + yW_0[t_1 - 0.166\alpha_1 t_1^2 + 0.025\alpha_1^2 t_1^5] \\
& + y(\sigma_1 - 1)[(0.5a_1 T^2 - 0.29a_1\alpha_1 T^4 - 0.085\,a_1\alpha_1^2 T^6 - 0.33b_1 T^3 + 0.233\,T^5 b_1\alpha_1 \\
& - 0.049b_1\alpha_1^2 T^7 - 0.25c_1 T^4 + 0.0666\alpha_1 c_1 T^6 - 0.019\,c_1\alpha_1^2 T^8)] \\
& + (h_r - h_o)(\sigma_1 - 1)(0.5a_1 t_1^2 - 0.29a_1\alpha_1 t_1^4 - 0.085\,a_1\alpha_1^2 t_1^6 - 0.33b_1 t_1^3 \\
& + 0.233\,t_1^5 b_1\alpha_1 - 0.25c_1 t_1^4 + 0.0666\alpha_1 c_1 t_1^6 - 0.019\,c_1\alpha_1^2 t_1^8)] \\
& + h_o W_0[t_1 - 0.166\alpha_1 t_1^2 + 0.025\alpha_1^2 t_1^5] \\
& + h_o(\sigma_1 - 1)[(0.5a_1 T^2 - 0.29a_1\alpha_1 T^4 - 0.085\,a_1\alpha_1^2 T^6 - 0.33b_1 T^3 + 0.233\,T^5 b_1\alpha_1 \\
& - 0.049b_1\alpha_1^2 T^7 - 0.25c_1 T^4 + 0.0666\alpha_1 c_1 T^6 - 0.019\,c_1\alpha_1^2 T^8)]\Big]
\end{aligned}
$$

$$TC_2 = \frac{1}{T}\Big[A + cQ_0 + d_c[(x - y)(\sigma_2 - 1)(0.5a_2t_1^2 - 0.29a_2\alpha_2t_1^4 - 0.085\,a_2\alpha_2^2t_1^6$$
$$- 0.33b_2t_1^3 + 0.233\,t_1^5b_2\alpha_2 - 0.049b_2\alpha_2^2t_1^7 - 0.25c_2t_1^4 + 0.0666\alpha_2c_2t_1^6$$
$$- 0.019\,c_2\alpha_2^2t_1^8)] + yW_0[t_1 - 0.166\alpha_2t_1^2 + 0.025\alpha_2^2t_1^5]$$
$$+ y(\sigma_2 - 1)[(0.5a_2T^2 - 0.29a_2\alpha_2T^4 - 0.085\,a_2\alpha_2^2T^6 - 0.33b_2T^3 + 0.233\,T^5b_2\alpha_2$$
$$- 0.049b_2\alpha_2^2T^7 - 0.25c_2T^4 + 0.0666\alpha_2c_2T^6 - 0.019\,c_2\alpha_2^2T^8)]$$
$$+ (h_r - h_o)(\sigma_2 - 1)(0.5a_2t_1^2 - 0.29a_2\alpha_2t_1^4 - 0.085\,a_2\alpha_2^2t_1^6 - 0.33b_2t_1^3$$
$$+ 0.233\,t_1^5b_2\alpha_2 - 0.25c_2t_1^4 + 0.0666\alpha_2c_2t_1^6 - 0.019\,c_2\alpha_2^2t_1^8)]$$
$$+ h_oW_0[t_1 - 0.166\alpha_2t_1^2 + 0.025\alpha_2^2t_1^5]$$
$$+ h_o(\sigma_2 - 1)[(0.5a_2T^2 - 0.29a_2\alpha_2T^4 - 0.085\,a_2\alpha_2^2T^6 - 0.33b_2T^3 + 0.233\,T^5b_2\alpha_2$$
$$- 0.049b_2\alpha_2^2T^7 - 0.25c_2T^4 + 0.0666\alpha_2c_2T^6 - 0.019\,c_2\alpha_2^2T^8)]\Big]$$

$$TC_3 = \frac{1}{T}\Big[A + c\,Q_0 + d_e\big((x - y)(\sigma_3 - 1)(0.5a_3t_1^2 - 0.29a_3t_1^4 - 0.085\,a_3\alpha_3^2t_1^6$$
$$- 0.33b_3t_1^3 + 0.233\,t_1^5b_3\alpha_3 - 0.049b_3\alpha_3^2t_1^7 - 0.25c_3t_1^4 + 0.0666\alpha_3c_3t_1^6 - 0.019\,c_3\alpha_3^2t_1^8)\big)$$
$$+ yW_0[t_1 - 0.166\alpha_3t_1^2 + 0.025\alpha_3^2t_1^5]$$
$$+ y(\sigma_3 - 1)\big(0.5a_3T^2 - 0.29a_3\alpha_3T^4 - 0.085\,\alpha_3^2T^6 - 0.33b_3T^3$$
$$+ 0.233\,T^5b_3\alpha_3 - 0.049b_3\alpha_3^2T^7 - 0.25c_3T^4 + 0.0666\alpha_3c_3T^6 - 0.019\,c_3\alpha_3^2T^8\big)$$
$$+ (h_r - h_o)(\sigma_3 - 1)(0.5a_3t_1^2 - 0.29a_3t_1^4 - 0.085\,a_3\alpha_3^2t_1^6 - 0.33b_3t_1^3 + 0.233\,t_1^5b_3\alpha_3 - 0.25c_3t_1^4$$
$$+ 0.0666\alpha_3c_3t_1^6 - 0.019\,c_3\alpha_3^2t_1^8) + h_oW_0[t_1 - 0.166\alpha_3t_1^2 + 0.025\alpha_3^2t_1^5]$$
$$+ h_o(\sigma_3 - 1)\big(0.5a_3T^2 - 0.29a_3\alpha_3T^4 - 0.085\,\alpha_3^2T^6 - 0.33b_3T^3$$
$$+ 0.233\,T^5b_3\alpha_3 - 0.049b_3\alpha_3^2T^7 - 0.25c_3T^4 + 0.0666\alpha_3c_3T^6 - 0.019\,c_3\alpha_3^2T^8\big)\Big]$$

Sales revenue collected over the cycle (SR):

$$SR = pQ_0 + 0.15\sigma p\left(at + 0.5b\,T^2 + 0.33c\,T^3\right) \tag{10}$$

The total profit per unit time is calculated as follows:

$$TP^* = \frac{1}{T}\left[SR - OC - PC - HC - DC - SC\right] \tag{11}$$

3.3. Case-II: Crisp Inventory Model, When $t > \mu$

The solutions of differential equations 1, 2 and 3 with boundary conditions are:

$$Q_r(t) = (\sigma - 1)\big[a(t - t_1) + 0.34a\alpha(t^3 - t_1^3) + 0.067a\alpha^2(t^5 - t_1^5) + 0.5\,b(t^2 - t_1^2)$$
$$- 0.125(b + c\mu)\alpha(t^4 - t_1^4) + 0.021(b + c\mu)\alpha^2(t^6 - t_1^6)\big] + 0.039(b + c\mu)\alpha^3(t^8 - t_1^8) - 0.5\,\alpha t^2$$
$$- 0.17a\alpha^2t^2(t^3 - t_1^3) - 0.034\,t^2a\alpha^3(t^5 - t_1^5) - 0.25\,\alpha\,t^2(b + c\mu)(t^2 - t_1^2)$$
$$+ 0.0625\,t^2(b + c\mu)\alpha^2(t^4 - t_1^4) - 0.004\,t^2(b + c\mu)\alpha^3(t^6 - t_1^6)$$
$$+ 0.1252\,t^2a(t - t_1) + 0.0425\,t^4a\alpha^3(t^3 - t_1^3) + 0.067\,a\alpha^2(t^5 - t_1^5)$$
$$+ 0.0625\,t^2\alpha^4(b + c\mu)(t^2 - t_1^2) - 0.0153\,t^4(b + c\mu)\alpha(t - t_1) + 0.021(b + c\mu)\alpha^2(t^6 - t_1^6)$$
$$- 0.0213\,t^6\alpha^6(t - t_1) - 0.011\,t^3\alpha^6(b + c\mu)(t^2 - t_1^2) - 0.125\,(b + c\mu)\alpha(t^4 - t_1^4)\big] \tag{12}$$

$$Q_{01}(t) = W_0 \left(1 - 0.5\,\alpha\,t + 0.125\alpha^2 t^4 - 0.021\alpha^3 t^6\right) \tag{13}$$

$$
\begin{aligned}
Q_{02}(t) = (\sigma - 1)\big[&a(t-T) + 0.34a\alpha(t^3 - T^3) + 0.067a\alpha^2(t^5 - T^5) + 0.5(b+c\mu)(t^2 - T^2) \\
&- 0.125(b+c\mu)\alpha(t^4 - T^4) + 0.021(b+c\mu)\alpha^2(t^6 - T^6) + 0.039(b+c\mu)\alpha^3(t^8 - T^8) - 0.5\,\alpha t^2 \\
&- 0.17a\alpha^2 t^2(t^3 - T^3) - 0.034\,t^2 a\alpha^3(t^5 - T^5) - 0.25\,\alpha\,t^2(b+c\mu)(t^2 - T^2) \\
&+ 0.0625\,t^2(b+c\mu)\alpha^2(t^4 - T^4) - 0.004\,t^2(b+c\mu)\alpha^3(t^6 - T^6) + 0.1252\,t^2 a(t-T) \\
&+ 0.0425\,t^4 a\alpha^3(t^3 - T^3) + 0.067\,a\alpha^2(t^5 - T^5) + 0.0625\,t^2\alpha^4(b+c\mu)(t^2 - T^2) \\
&- 0.0153\,t^4(b+c\mu)\alpha(t-T) + 0.021(b+c\mu)\alpha^2(t^6 - T^6) \\
&- 0.0213\,t^6\alpha^6(t-T) - 0.011\,t^3\alpha^6(b+c\mu)(t^2 - T^2) - 0.125\,(b+c\mu)\alpha(t^4 - T^4)\big] \tag{14}
\end{aligned}
$$

The total cost of the system is the sum of all associated inventory costs calculated as above. Therefore, the total cost per unit time (or total average cost per cycle) is given by:

1. **Ordering cost:**

$$OC = A$$

2. **Purchase cost:**

$$PC = cQ_0$$

3. **Deterioration Cost:** Inventory deterioration cost in RW is

$$DC_{RW} = d_c \int_0^{t_1} xQ_r(t)\,dt$$

Inventory deterioration cost in OW is

$$DC_{OW} = d_c \int_0^{t_1} yQ_{o1}(t)\,dt + d_c \int_{t_1}^{T} yQ_{o2}(t)\,dt$$

4. **Storage Cost:**

$$SC = \left[h_r \int_0^{t_1} Q_r(t)\,dt + h_o \int_0^{t_1} Q_{o1}(t)\,dt + h_o \int_{t_1}^{T} Q_{o2}(t)\,dt \right]$$

The total cost per unit time of inventory system is:

$$TC = \frac{1}{T}\left[OC + PC + DC + HC\right]$$

$$
\begin{aligned}
= \frac{1}{T}\Big[&A + c\,Q_0 + d_e\big[(x-y)(\sigma-1)(0.5at_1^2 - 0.29a\alpha t_1^4 - 0.085\,a\,\alpha^2 t_1^6 - 0.33(b+c\mu)t_1^3 \\
&+ 0.233\,t_1^5(b+c\mu)\alpha - 0.049(b+c\mu)\alpha^2 t_1^7)\big] + yW_0\left[t_1 - 0.166\alpha t_1^2 + 0.025\alpha^2 t_1^5\right] \\
&+ y(\sigma-1)\left(0.5aT^2 - 0.29a\alpha T^4 - 0.085\alpha^2 T^6 - 0.33(b+c\mu)T^3 + 0.233\,T^5(b+c\mu)\alpha \right. \\
&- 0.049(b+c\mu)\alpha^2 T^7 + \big[(h_r - h_o)(\sigma-1)(0.5at_1^2 - 0.29a\alpha t_1^4 - 0.085\,a\alpha^2 t_1^6 - 0.33(b+c\mu)t_1^3 \\
&+ 0.233\,t_1^5(b+c\mu)\alpha - 0.049(b+c\mu)\alpha^2 t_1^7) + h_o W_0\left[t_1 - 0.166\alpha t_1^2 + 0.025\alpha^2 t_1^5\right] + h_o(\sigma-1)\left(0.5aT^2 \right. \\
&- 0.29a\alpha T^4 - 0.085\alpha^2 T^6 - 0.33(b+c\mu)T^3 + 0.233\,T^5(b+c\mu)\alpha - 0.049(b+c\mu)\alpha^2 T^7\Big] \tag{15}
\end{aligned}
$$

Sales revenue collected over the cycle (SR):

$$\text{SR} = pQ_0 + 0.15\sigma p \left(aT + 0.5b\,T^2 + 0.5c\,\mu T^2 \right) \tag{16}$$

The total profit per unit time is:

$$\text{TP}^o = \frac{1}{T}\left[\text{SR} - \text{OC} - \text{PC} - \text{HC} - \text{DC} - \text{SC}\right] \tag{17}$$

3.4. Case-II: Fuzzy Inventory Model, When $t > \mu$

In this section the fuzzy inventory model is presented through the signed distance method. Let

$$\hat{a} = (a_1, a_2, a_3), \quad \hat{b} = (b_1, b_2, b_3), \quad \hat{c} = (c_1, c_2, c_3), \quad \hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \quad \hat{\mu} = (\mu_1, \mu_2, \mu_3)$$

and

$$\hat{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

are triangular fuzzy numbers. By the signed distance method, the total cost per unit time is:

$$\text{TC}^* = \frac{1}{4}\left(\text{TC}_1 + 2\,\text{TC}_2 + \text{TC}_3\right) \tag{18}$$

$$
\begin{aligned}
\text{TC}_1 = \frac{1}{T}\Big[& A + c\,Q_0 + d_e\big[(x-y)(\sigma_1 - 1)(0.5a_1 t_1^2 - 0.29a_1\alpha_1 t_1^4 - 0.085\,a_1\,\alpha_1^2 t_1^6 \\
& - 0.33(b_1 + c_1\mu)t_1^3 + 0.233\,t_1^5(b_1 + c_1\mu)\alpha_1 - 0.049(b_1 + c_1\mu)\alpha_1^2 t_1^7)\big] \\
& + y\,W_0[t_1 - 0.166\alpha_1 t_1^2 + 0.025\alpha_1^2 t_1^5] \\
& + y(\sigma_1 - 1)\big(0.5a_1 T^2 - 0.29a_1\alpha_1 T^4 - 0.085\,a_1\alpha_1^2 T^6 - 0.33(b_1 + c_1\mu)T^3 \\
& \quad + 0.233\,T^5(b_1 + c_1\mu)\alpha_1 - 0.049(b_1 + c_1\mu)\alpha_1^2 T^7\big) \\
& + (h_r - h_o)(\sigma_1 - 1)\big(0.5a_1 t_1^2 - 0.29a_1\alpha_1 t_1^4 - 0.085\,a_1\alpha_1^2 t_1^6 - 0.33(b_1 + c_1\mu)t_1^3 \\
& \quad + 0.233\,t_1^5(b_1 + c_1\mu)\alpha_1 - 0.049(b_1 + c_1\mu)\alpha_1^2 t_1^7\big) \\
& + h_o\,W_0[t_1 - 0.166\alpha_1 t_1^2 + 0.025\alpha_1^2 t_1^5] \\
& + h_o(\sigma_1 - 1)\big(0.5a_1 T^2 - 0.29a_1\alpha_1 T^4 - 0.085\,\alpha_1^2 T^6 - 0.33(b_1 + c_1\mu)T^3 \\
& \quad + 0.233\,T^5(b_1 + c_1\mu)\alpha_1 - 0.049(b_1 + c_1\mu)\alpha_1^2 T^7\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\text{TC}_2 = \frac{1}{T}\Big[& A + c\,Q_0 + d_e\big[(x-y)(\sigma_2 - 1)(0.5a_2 t_1^2 - 0.29a_2\alpha_2 t_1^4 - 0.085\,a_2\,\alpha_2^2 t_1^6 \\
& - 0.33(b_2 + c_2\mu)t_1^3 + 0.233\,t_1^5(b_2 + c_2\mu)\alpha_2 - 0.049(b_2 + c_2\mu)\alpha_2^2 t_1^7)\big] \\
& + y\,W_0[t_1 - 0.166\alpha_2 t_1^2 + 0.025\alpha_2^2 t_1^5] \\
& + y(\sigma_2 - 1)\big(0.5a_2 T^2 - 0.29a_2\alpha_2 T^4 - 0.085\,a_2\alpha_2^2 T^6 - 0.33(b_2 + c_2\mu)T^3 \\
& \quad + 0.233\,T^5(b_2 + c_2\mu)\alpha_2 - 0.049(b_2 + c_2\mu)\alpha_2^2 T^7\big) \\
& + (h_r - h_o)(\sigma_2 - 1)\big(0.5a_2 t_1^2 - 0.29a_2\alpha_2 t_1^4 - 0.085\,a_2\alpha_2^2 t_1^6 - 0.33(b_2 + c_2\mu)t_1^3 \\
& \quad + 0.233\,t_1^5(b_2 + c_2\mu)\alpha_2 - 0.049(b_2 + c_2\mu)\alpha_2^2 t_1^7\big) \\
& + h_o\,W_0[t_1 - 0.166\alpha_2 t_1^2 + 0.025\alpha_2^2 t_1^5] \\
& + h_o(\sigma_2 - 1)\big(0.5a_2 T^2 - 0.29a_2\alpha_2 T^4 - 0.085\alpha_2^2 T^6 - 0.33(b_2 + c_2\mu)T^3 \\
& \quad + 0.233\,T^5(b_2 + c_2\mu)\alpha_2 - 0.049(b_2 + c_2\mu)\alpha_2^2 T^7\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{TC}_3 = \frac{1}{T}\Big[& A + c\,Q_0 + d_e\big[(x-y)(\sigma_3-1)(0.5a_3t_1^2 - 0.29a_3\alpha_3t_1^4 - 0.085\,a_3\,\alpha_3^2t_1^6 \\
& - 0.33(b_3+c_3\mu)t_1^3 + 0.233\,t_1^5(b_3+c_3\mu)\alpha_3 - 0.049(b_3+c_3\mu)\alpha_3^2t_1^7)\big] \\
& + y\,W_0[t_1 - 0.166\alpha_3t_1^2 + 0.025\alpha_3^2t_1^5] \\
& + y(\sigma_3-1)\big(0.5a_3T^2 - 0.29a_3\alpha_3T^4 - 0.085\,a_3\alpha_3^2T^6 - 0.33(b_3+c_3\mu)T^3 \\
& \quad + 0.233\,T^5(b_3+c_3\mu)\alpha_3 - 0.049(b_3+c_3\mu)\alpha_3^2T^7\big) \\
& + (h_r-h_o)(\sigma_3-1)\big(0.5a_3t_1^2 - 0.29a_3\alpha_3t_1^4 - 0.085\,a_3\alpha_3^2t_1^6 - 0.33(b_3+c_3\mu)t_1^3 \\
& \quad + 0.233\,t_1^5(b_3+c_3\mu)\alpha_3 - 0.049(b_3+c_3\mu)\alpha_3^2t_1^7\big) \\
& + h_o\,W_0[t_1 - 0.166\alpha_3t_1^2 + 0.025\alpha_3^2t_1^5] \\
& + h_o(\sigma_3-1)\big(0.5a_3T^2 - 0.29a_3\alpha_3T^4 - 0.085\alpha_3^2T^6 - 0.33(b_3+c_3\mu)T^3 \\
& \quad + 0.233\,T^5(b_3+c_3\mu)\alpha_3 - 0.049(b_3+c_3\mu)\alpha_3^2T^7\big)\Big]
\end{aligned}
$$

Sales revenue collected over the cycle (SR):

$$
\mathrm{SR} = pQ_0 + 0.15\sigma p\left(aT + 0.5\,b\,T^2 + 0.5\,c\,\mu T^2\right) \tag{19}
$$

The total profit per unit time is calculated as follows:

$$
\mathrm{TP}^* = \frac{1}{T}\left[\mathrm{SR} - \mathrm{OC} - \mathrm{PC} - \widehat{\mathrm{HC}} - \widehat{\mathrm{DC}} - \widehat{\mathrm{SC}}\right] \tag{20}
$$

4. Optimal Solution Procedure

The primary aim of the model is to find the values of t_1 and T, which can minimize the total cost per unit time TC^* and TC° throughout a complete cycle over time T, which maximize the total profit. The conditions required to minimize the total cost per unit time during the cycle time T are:

$$
\frac{\partial(\mathrm{TC})}{\partial t_1} = 0 \quad , \quad \frac{\partial(\mathrm{TC})}{\partial T} = 0
$$

$$
\frac{\partial(\mathrm{TC}^*)}{\partial t_1} = 0 \quad , \quad \frac{\partial(\mathrm{TC}^*)}{\partial T} = 0
$$

The condition for sufficiency is that TC and TC^* are convex functions over the cycle time T. The necessary and sufficient conditions to minimize the total cost function $TC(t_1, T)$ attains a local minimum at a critical point (t_1^*, T^*) if:

$$
\boxed{D > 0, \quad \text{and} \frac{\partial^2 TC}{\partial t_1^2} > 0}
$$

Where

$$
D = f_{11}f_{22} - (f_{12})^2
$$

and

$$
f_{11} = \frac{\partial^2 TC}{\partial t_1^2}, \quad f_{22} = \frac{\partial^2 TC}{\partial T^2}, \quad f_{12} = \frac{\partial^2 TC}{\partial t_1 \partial T}
$$

5. NUMERICAL ILLUSTRATION AND SENSITIVITY ANALYSIS

The numerical illustration and sensitivity analysis of the proposed fuzzy inventory model developed satisfying the necessary and sufficient conditions, and a numerical example has been solved. Data have been assumed randomly from the literature in appropriate units to obtain optimal values of t_1, T, TC°, TC^*, TP° and TP^*, and to perform the sensitivity analysis. A numerical example has been presented to illustrate the proposed inventory model. The main objective is to determine the optimal values of t_1, T which minimize total cost per unit time and maximize the total profit. The values of different parameters are as follows:

$$A = 500, \quad Q_0 = 3000, \quad \sigma = 0.15, \quad \alpha = 0.07, \quad x = 0.07, \quad y = 0.05, \quad W_0 = 2000,$$
$$h_r = 0.17, \quad h_o = 0.12, \quad a = 100, \quad b = 5, \quad c = 0.5,$$
$$c_s = 100, \quad d = 0.07, \quad p = 125$$

1. **Case-I: When $t < \mu$**
   The optimal values of t_1, T and TC in the crisp model, which are $t_1^0 = 9.16$, $T^0 = 50.51$, $TC^0 = 7853$ and $TP^0 = 1392$ respectively and the optimal values of t_1, T and TC are obtained in the fuzzy model, which are $t_1^* = 9.15$, $T^* = 50.62$, $TC^* = 7830$ and $TP^0 = 2621$.

2. Case-II: When $t > \mu$
   The optimal values of t_1, T and TC in the crisp model, which are $t_1^0 = 9.04$, $T^0 = 62.65$, $TC^0 = 6196$ and $TP^0 = 1117$ respectively and the optimal values of t_1, T and TC are obtained in the fuzzy model, which are $t_1^* = 9.04$, $T^* = 62.53$, $TC^* = 6225$ and $TP^0 = 1438$.

5.1. Sensitivity Analysis for Case-I

Table 2: *Comparison of Crisp and Fuzzy Models for Different Values of a*

a	Crisp Model				Fuzzy Model						
	t_1°	T°	TC°	TP°	t_1^*	T^*	a_1	a_2	a_3	TC^*	TP^*
30	9.02	50.10	8044	1045	9.02	50.24	30	70	100	8013	2461
70	9.10	50.31	7935	1242	9.05	50.45	30	70	100	7918	2542
100	9.16	50.51	7853	1392	9.10	50.52	30	100	130	7803	2621
125	9.24	50.90	7785	1496	9.21	50.78	100	125	150	7762	2681
150	9.34	51.25	7716	1596	9.27	50.94	125	150	175	7693	2741

Table 3: *Comparison of Crisp and Fuzzy Models for Different Values of b*

b	Crisp Model				Fuzzy Model						
	t_1°	T°	TC°	TP°	t_1^*	T^*	b_1	b_2	b_3	TC^*	TP^*
1.0	9.13	52.05	7480	1336	9.13	52.07	0.2	1	3	7485	2893
3.0	9.15	51.50	7669	1342	9.14	51.40	1	3	5	7645	2763
5.0	9.16	50.60	7853	1392	9.15	50.62	3	5	7	7830	2621
7.0	9.17	49.78	8033	1421	9.16	49.90	5	7	9	8011	2485
10.0	9.18	48.75	8290	1471	9.18	48.61	7	10	13	8269	2315

Table 4: *Comparison of Crisp and Fuzzy Models for Different Values of c*

2*c	Crisp Model				Fuzzy Model						
	t_1°	T°	TC°	TP°	t_1^*	T^*	c_1	c_2	c_3	TC^*	TP^*
0.2	9.07	59.05	6494	1190	9.07	59.20	0.1	0.2	0.3	6472	2393
0.3	9.11	55.60	7093	1260	9.12	54.90	0.2	0.3	0.4	7094	2514
0.5	9.16	50.60	7853	1392	9.15	50.62	0.4	0.5	0.6	7830	2621
0.7	9.19	47.07	8475	1541	9.18	47.76	0.6	0.7	0.8	8455	2730
0.9	9.24	44.30	8987	1709	9.20	45.00	0.8	0.9	1.0	8485	2236

Table 5: *Comparison of Crisp and Fuzzy Models for Different Values of σ*

σ	Crisp Model					Fuzzy Model						
	t_1°	T°	TC°	TP°		t_1^*	T^*	σ_1	σ_2	σ_3	TC^*	TP^*
0.05	9.19	49.23	8072	1129		9.19	49.38	0.01	0.05	0.10	8047	2451
0.10	9.18	49.84	7996	749		9.18	50.01	0.05	0.10	0.15	7954	2541
0.15	9.17	50.10	7853	1392		9.16	50.62	0.10	0.15	0.20	7830	2621
0.20	9.14	51.09	7733	2063		9.14	51.05	0.15	0.20	0.25	7727	2704
0.25	9.12	51.70	7617	2778		9.13	52.08	0.20	0.25	0.30	7592	2781

5.2. Sensitivity Analysis for Case-II

Table 6: *Comparison of Crisp and Fuzzy Models for Different Values of a*

a	Crisp Model					Fuzzy Model						
	t_1°	T°	TC°	TP°		t_1^*	T^*	a_1	a_2	a_3	TC^*	TP^*
30	8.87	61.50	6450	787		8.87	61.70	70	80	90	6476	1249
70	8.97	60.29	6206	956		8.97	62.18	90	100	110	6333	1356
100	9.04	62.65	6306	1006		9.04	62.65	100	120	140	6225	1438
130	9.13	62.90	6190	1171		9.13	62.90	150	170	190	6115	1521
170	9.24	63.15	6070	1578		9.21	63.15	200	230	250	5993	1596

Table 7: *Comparison of Crisp and Fuzzy Models for Different Values of b*

b	Crisp Model					Fuzzy Model						
	t_1°	T°	TC°	TP°		t_1^*	T^*	b_1	b_2	b_3	TC^*	TP^*
1	9.00	66.53	5498	1215		9.02	66.53	2	3	4	5924	1376
3	9.02	64.55	5851	1185		9.03	64.33	2	3	4	6061	1536
5	9.06	63.00	6196	1171		9.04	62.70	4	5	6	6645	1438
7	9.08	60.75	6501	1193		9.10	60.14	5	7	9	7120	1336
10	9.08	58.70	6931	1185		9.10	58.87	8	10	12	6958	1023

Table 8: *Comparison of Crisp and Fuzzy Models for Different Values of c*

c	Crisp Model					Fuzzy Model						
	t_1°	T°	TC°	TP°		t_1^*	T^*	c_1	c_2	c_3	TC^*	TP^*
0.2	8.97	67.06	5402	1225		8.99	66.52	0.1	0.2	0.4	5506	1887
0.4	8.99	65.04	5684	1172		9.03	65.02	0.2	0.4	0.5	5735	1776
0.6	9.04	62.65	5854	1170		9.10	62.53	0.3	0.5	0.7	6225	1438
0.8	9.05	60.16	6653	1174		9.12	60.18	0.4	0.6	0.8	6669	1181
1.0	"	"	"	"		9.12	57.00	0.5	0.8	1.2	7256	1005

Table 9: *Comparison of Crisp and Fuzzy Models for Different Values of σ*

σ	**Crisp Model**				**Fuzzy Model**						
	t_1°	T°	TC°	TP°	t_1^*	T^*	σ_1	σ_2	σ_3	TC^*	TP^*
0.05	9.00	61.52	6334	165	9.00	61.34	0.00	0.05	0.10	6423	1331
0.10	9.01	62.08	6292	663	9.01	61.93	0.05	0.10	0.15	6312	1397
0.15	9.04	62.65	6196	1171	9.03	62.51	0.10	0.15	0.20	6216	1300
0.20	9.05	63.25	6100	1663	9.03	63.10	0.15	0.20	0.25	6100	1483
0.25	8.99	63.84	5997	2215	9.03	63.72	0.21	0.25	0.29	6020	1531

6. Graphical Illustration

The graphical representation depicts the variation of total cost (TC) and total profit (TP) with respect to changes in model parameters such as a, b, c, σ, α, and μ. The comparative plots between crisp and fuzzy models illustrate that the fuzzy model yields more flexible and cost-effective results under parameter uncertainty.

6.1. For Case-I and Case-II

Convexity of total cost: To demonstrate the convexity of total cost per unit of inventory system, graphical representations of the convexity of TC^o and TC^* with respect to decision variables are presented in Figure 1a, Figure 1b, Figure 2a, and Figure 2b respectively.
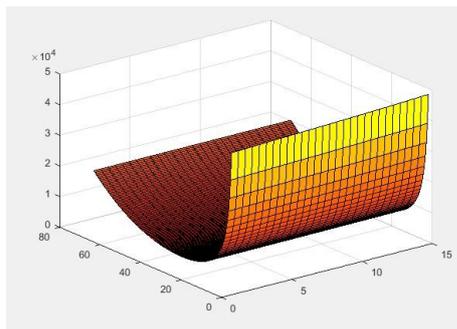
(a) *Total cost in crisp model.*
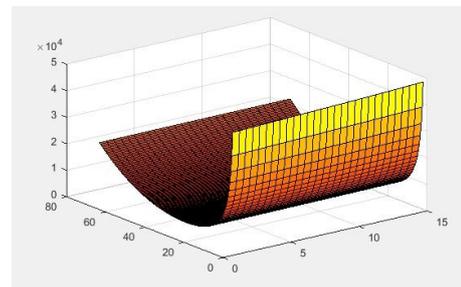
(b) *Total cost in fuzzy model.*

Figure 1: *Total cost variation in case-I.*
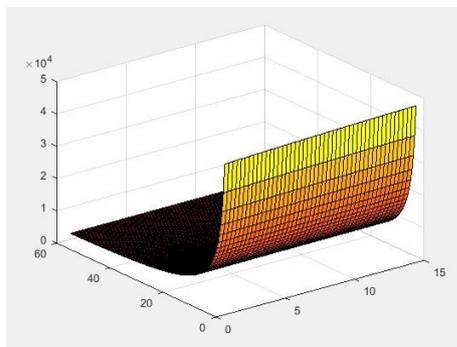
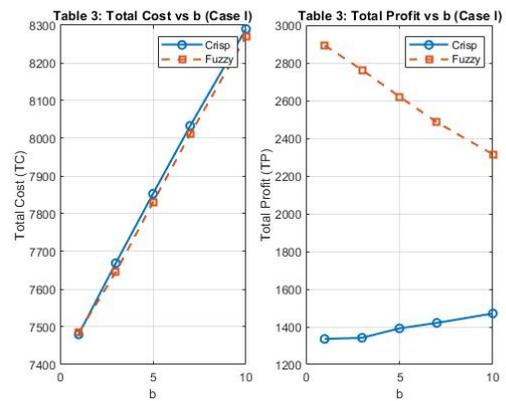**(a)** *Total cost in crisp model.*
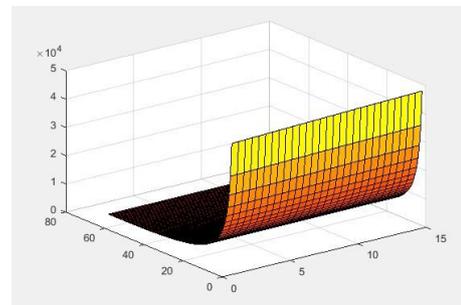
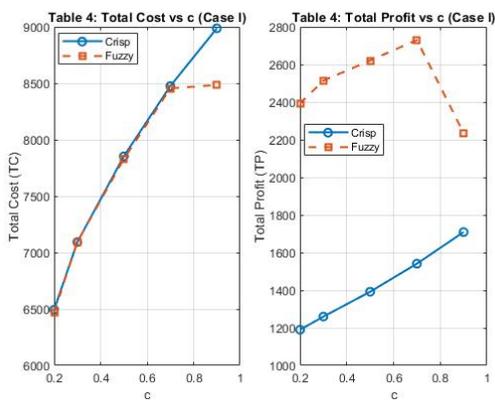**(b)** *Total cost in fuzzy model.*

Figure 2: *Total cost variation in case-II.*
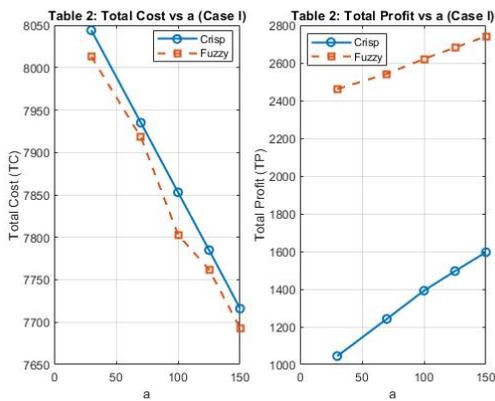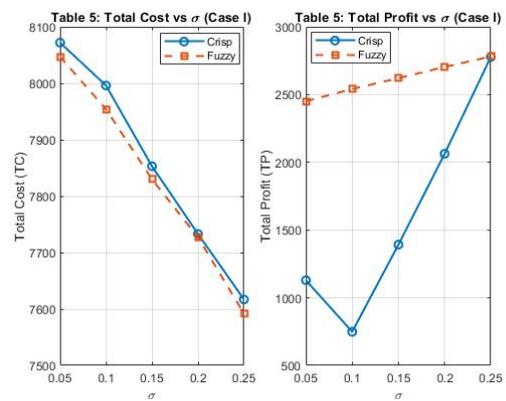
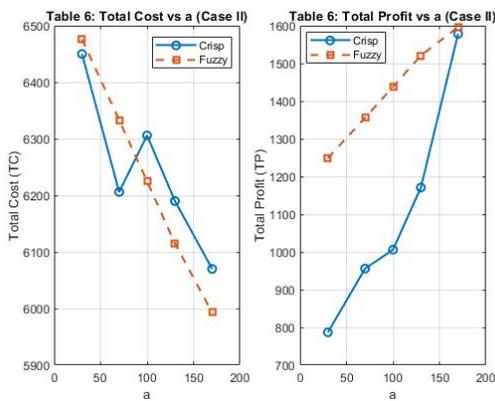**(a)** *Total cost & profit in crisp & fuzzy model.*

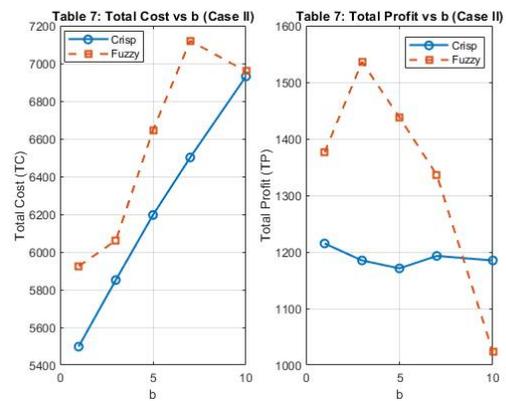**(b)** *Total cost & profit in crisp & fuzzy model.*

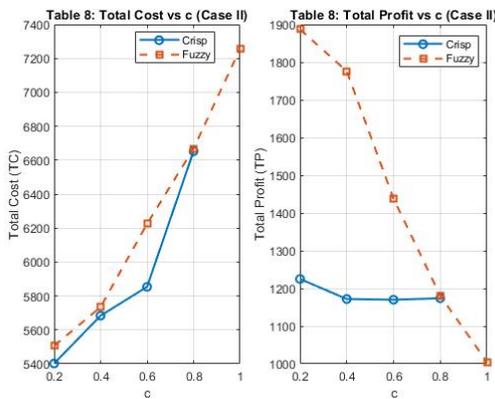**(c)** *Total cost & profit in crisp & fuzzy model.*
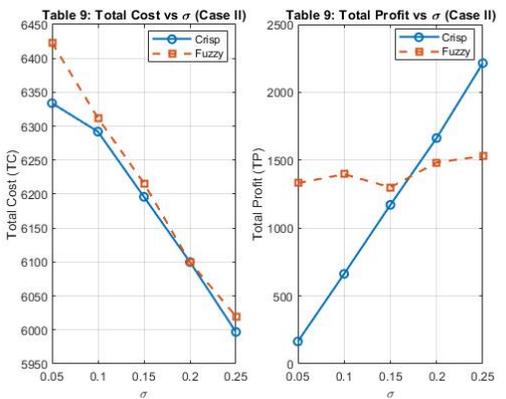
(d) *Total cost & profit in crisp & fuzzy model.*

(e) *Total cost & profit in crisp & fuzzy model.*

(f) *Total cost & profit in crisp & fuzzy model.*

(g) *Total cost & profit in crisp & fuzzy model.*

(h) *Total cost & profit in crisp & fuzzy model.*

Figure 3: *Total cost & profit variation for both the cases.*

The following observations were made:

1. As the demand parameters b and c increase, optimal value of total cost per unit time TC^o and TC^* increases and the corresponding profit decreases. As the demand parameters a increases, optimal value of total cost per unit time TC^o and TC^* decreases; the corresponding profit increases.

2. As the parameters σ increase, the optimal value of total cost per unit time TC^o and TC^* decreases and there is an increase in the corresponding profit. As the parameters d increase, the optimal value of total cost per unit time TC^o and TC^* increases and there is a decrease in the corresponding profit.

3. As the parameters μ increase, the optimal value of total cost per unit time TC^o and TC^* decreases and there is an increase in the corresponding profit.

4. The optimal value of total cost per unit time in the fuzzy model is less than total cost per unit time in the crisp model and the profit is more in fuzzy model compared to crisp model.

Figure 3(a) shows that increasing the base demand parameter a lowers the total cost and increases profit, with the fuzzy model performing better than the crisp model. Figure 3(b) indicates that as the demand growth rate b increases, both models experience higher costs, while profits decline more sharply in the fuzzy case. Figure 3(c) highlights that rising nonlinear demand parameter c leads to higher costs and reduced profits, although the fuzzy model maintains relative advantage at moderate values. Figure 3(d) demonstrates that higher return rates σ decrease costs and substantially boost profits, particularly under the fuzzy model. Similarly, Figures 3(e)-3(h) (Case II) confirm the same trends: total cost decreases and profit rises with higher a and σ [Figures 3(e), 3(h)], while increasing b and c escalates costs and reduces profit [Figures 3(f), 3(g)], again with the fuzzy model consistently outperforming the crisp one.

7. Conclusion

In this paper, an inventory model has been described under two warehouses with fuzzy constraints. The objective of the work is to minimize the total inventory cost which includes ordering, storage and deterioration cost for both own and rented warehouses to get maximum profit. The study includes a numerical example and a sensitivity analysis of the model in both crisp and fuzzy modelling. The analysis shows that in the crisp model, the total cost per unit and corresponding total profit are highly sensitive to changes in b, σ and c, fairly sensitive to changes μ and α, less sensitive to changes in parameter a. In the fuzzy model, the total cost per unit and corresponding total profit are highly sensitive to changes in b, σ and c, fairly sensitive to changes μ and α, less sensitive to changes in a.

Additionally, the total cost per unit in the fuzzy model is less than in the crisp model and hence corresponding total profit is more in fuzzy model than in the crisp model.

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