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SOME STATISTICAL PROPERTIES AND APPLICATIONS OF LOMAX GENERATED DISTRIBUTION

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Abstract

In this paper, a probability distribution has been proposed using the concept of induced distribution, with the classical Lomax distribution as the baseline. Various statistical properties including moments, entropy measures, reliability functions, and generating functions are derived. The maximum likelihood estimation (MLE) method is used for parameter estimation. Real data application is performed to validate the suitability of the proposed model in comparison to existing distributions.

Keywords: Induced distribution, Lomax distribution, Entropy, Reliability analysis, MLE, Lifetime data.

1. Introduction

The Lomax distribution, also known as the Pareto Type II distribution, is a fundamental model for representing heavy-tailed data in a wide range of applied fields, including reliability engineering, survival analysis, actuarial science, biostatistics, and economics. Originally proposed by Lomax [1] to model business failure times, the distribution has proven to be a flexible and analytically convenient alternative to exponential and Weibull models for modeling right-skewed and extreme-valued phenomena, see Atkinson [2], Bryson [3]. Its ability to capture power-law behavior and monotonically decreasing hazard rates makes it suitable for data where the failure risk declines with time such as warranty or mechanical failure datasets see, Balkema & De Haan [4], Ahsanullah [5].

In recent years, several generalizations of the Lomax model have been proposed to enhance its adaptability to more complex data structures. Ghitany et al. [6] introduced the Marshall–Olkin Lomax distribution, enabling more control over the hazard shape. Kilany [7] proposed the Weighted Lomax distribution, which incorporates length-biased and size-biased sampling techniques often encountered in ecology and epidemiology. The Beta-Lomax, Kumaraswamy-Lomax, and Exponentiated Lomax distributions further extend the parent model by introducing additional shape parameters to improve the fit for datasets with non-monotonic hazard behavior, Gupta & Kundu [8], Nagarjuna et al. [9], El-Monsef et al. [10].

In network science, Chattopadhyay et al. [11] developed the Modified Lomax (MLM) model, which more accurately represents the degree distribution of large-scale real-world complex networks by addressing non-linearity in the tail behavior. Most recently, Aljohani [12] introduced

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the New Extended Heavy-Tailed Lomax (NEHTLx) distribution, which embeds the Lomax into a more general family capable of modeling bathtub, unimodal, and reversed-J-shaped hazard rates. These extensions have demonstrated superior performance across applications in medicine, internet traffic, economics, and failure-time analysis, with model selection based on AIC, BIC, and K–S tests Pak & Mahmoudi [13], Afify et al. [14], Ahsanullah & Nevzorov [15] .

While these parameter-driven generalizations have enriched the Lomax family, many rely on augmenting the baseline model directly through algebraic manipulation. An alternative and structurally appealing strategy is offered by the induced distribution approach recently formalized by Singh & Das [16]. This method constructs a new distribution by transforming the survival function of a given model. The induced distribution ensures that the resulting density function integrates to unity, producing a valid probability model with distinct analytical properties and often improved flexibility. The effectiveness of this approach has been recently demonstrated through the $i-$Garima distribution, which emerged from applying the induced technique to the Garima distribution—a mixture of exponential and gamma components Singh & Das [16]. In statistical literature there are many flexible distribution which are derived from transformation technique, see Singh et al. [17], Singh & Das [18]. Building upon this foundational work, we propose a new probability model termed the Induced Lomax ($i-$Lomax) distribution. This model arises by applying the induced distribution technique to the survival function of the classical Lomax distribution, yielding a tractable yet more flexible model that retains the heavy-tailed nature of the original while enabling a wider variety of hazard rate behaviors.

In this study, we develop a comprehensive theoretical treatment of the $i-$Lomax distribution. We derive its probability density and cumulative distribution functions, hazard and survival functions, raw and central moments, moment generating function, and entropy measures including Shannon and Rényi entropies. The quantile function is obtained in closed form, facilitating Monte Carlo simulation and random variate generation. Parameters are estimated using maximum likelihood estimation (MLE), and their behavior is investigated through a simulation study assessing bias and Root mean square error. We then apply the model to real-world datasets to demonstrate its goodness-of-fit performance using , $-2\log$-likelihood, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the Kolmogorov–Smirnov (K–S) test.

2. Genesis of the Distribution

The construction of new distributions via transformations of existing models is a well-established approach in statistical literature. In particular, the *induced distribution framework*, initially introduced in the context of weighted distributions by Patil & Rao [19], provides a structured method for generating flexible models. This framework was later formalized and extended by Singh & Das [16], who proposed a method for inducing new distributions based on the survival function of a baseline distribution.

Let X be a non-negative continuous random variable with cumulative distribution function $F(x)$ and finite mean $E[X]$. The *induced distribution* is defined by the probability density function:

$$g(x) = \frac{1 - F(x)}{E(X)}, \quad x > 0 \tag{1}$$

This transformation emphasizes the right tail of the original distribution and is closely related to length-biased and size-biased sampling. The induced density can also be seen as a normalized survival function scaled by the baseline mean.

In this paper, we apply the induced transformation method given by Singh & Das [16] to the Lomax distribution, resulting in the Induced Lomax in short ($i-$Lomax) distribution. This new model preserves the heavy-tailed nature of the Lomax distribution while offering greater shape flexibility and modified hazard behavior. It is particularly suited for modeling survival and reliability data with mathematical simplicity and enhanced modeling.

3. Proposed Distribution

3.1. Lomax as a Mixture of Exponential and Gamma Distributions

The two-parameter Lomax distribution is defined for a non-negative random variable X, with probability density function (PDF) and cumulative distribution function (CDF) given by:

$$f(x;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)}, \quad x>0, \quad \alpha,\lambda>0 \tag{2}$$

$$F(x;\alpha,\lambda) = 1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}, \quad x>0 \tag{3}$$

Here, α is the shape parameter, controlling tail heaviness, and λ is the scale parameter. For $\alpha > k$, the $k-$th raw moment exists, allowing flexible modeling of skewness and kurtosis. The Lomax distribution is aheavy tailed distribution used in many real life problems. The hazard function is strictly decreasing, which is appropriate for modeling lifetimes of systems that exhibit decreasing failure rates over time Singh et al. [20]. Due to its hierarchical formulation, the Lomax distribution also admits a Bayesian interpretation, as it arises as a gamma mixture of exponential distributions Chattopadhyay et al. [11], Dubey [21].

Lomax distribution can be represented as a mixture of an Exponential distribution and a Gamma distribution. This hierarchical form provides a useful structural interpretation that highlights the source of its heavy-tailed behavior introduced by Lomax [1]. This representation justifies the interpretation of the Lomax distribution as a compound Exponential–Gamma model and motivates its use as a baseline distribution in the proposed induced framework.

3.2. The Induced Lomax Distribution

Given the flexibility and mathematical tractability of the Lomax distribution, it serves as a suitable baseline model for developing new distributions using the induced framework. The concept of the induced distribution, formalized by Singh & Das [16], transforms the survival function of a non-negative random variable into a new probability density function, normalized by the mean of the original distribution.

Let $X \sim \text{Lomax}(\alpha,\lambda)$, with cumulative distribution function:

$$F(x) = 1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}, \quad x>0,\ \alpha>1,\ \lambda>0 \tag{4}$$

and expectation of the Lomax distribution is

$$E(x) = \frac{\lambda}{\alpha-1}, \quad \alpha>1 \tag{5}$$

Substituting these into the equation (1), we obtain the PDF of the proposed distribution as:

$$g(x) = \frac{(\alpha-1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\alpha}, \quad x>0 \tag{6}$$

This defines the $i-$Lomax distribution a two-parameter model that generalizes the tail behavior of the classical Lomax through an induced mechanism. In fact this distribution is a re-parameterized form of the Lomax distribution, thus, we may use either Lomax or $i-$Lomax distribution. The support of the distribution is $x \in (0,\infty)$, with shape parameter $\alpha > 1$ and scale parameter $\lambda > 0$.

For a fixed λ, increasing the value of α leads to a faster decay in the tail, thereby reducing the heaviness of the distribution.

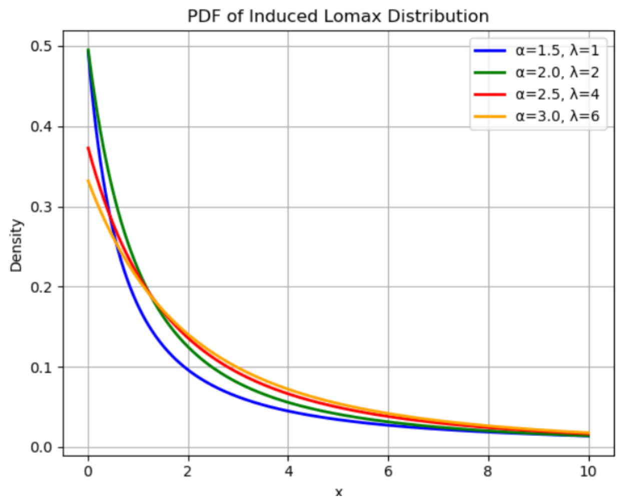


Figure 1: *PDF of the $i-$Lomax distribution for different values of α and λ.*

3.2.1 Cumulative Distribution Function (CDF)

The cumulative distribution function is derived by integrating equation (6):

$$G(x) = \int_0^x g(t)\,dt \tag{7}$$

$$G(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{1-\alpha}, \quad x > 0 \tag{8}$$

This is a well-defined function for $\alpha > 1$, and it approaches 1 as $x \to \infty$, ensuring completeness of the distribution. Since it is constructed using the induced distribution framework and the Lomax distribution as a baseline, we refer to it as the $i-$Lomax distribution.

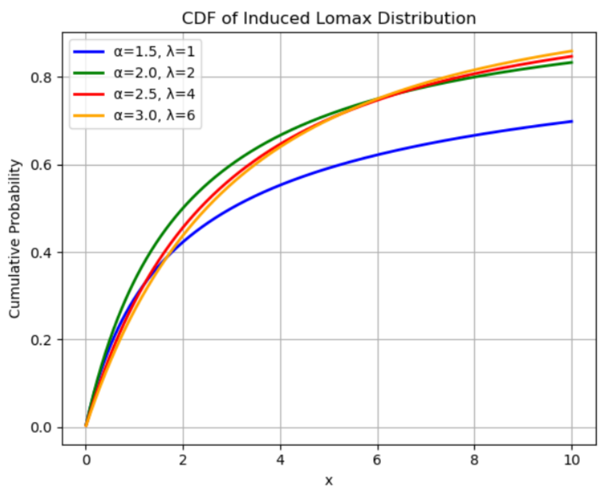


Figure 2: *CDF of the $i-$Lomax distribution for different values of α and λ.*

4. Statistical Properties

4.1. Moments

The rth raw moment of X is defined as:

$$\mu'_r = E[X^r] = \int_0^\infty x^r g(x)\,dx = \frac{(\alpha-1)}{\lambda}\int_0^\infty x^r \left(1+\frac{x}{\lambda}\right)^{-\alpha} dx \tag{9}$$

Letting $u = 1 + \frac{x}{\lambda} \Rightarrow x = \lambda(u-1), dx = \lambda du$, the integral becomes:

$$\mu'_r = (\alpha-1)\lambda^r \int_1^\infty (u-1)^r u^{-\alpha} du \tag{10}$$

This is a known integral (a beta-type form), yielding:

$$\mu'_r = \lambda^r(\alpha-1)B(r+1, \alpha-r-1), \quad \text{for } \alpha > r+1 \tag{11}$$

Using the identity $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, we have:

$$\mu'_r = \lambda^r(\alpha-1)\cdot\frac{\Gamma(r+1)\Gamma(\alpha-r-1)}{\Gamma(\alpha)} \quad \text{for } \alpha > r+1 \tag{12}$$

Hence the first four raw moments are obtained as:

$$\mu'_1 = \frac{\lambda}{\alpha-2}, \quad \text{for } \alpha > 2 \tag{13}$$

$$\mu'_2 = \frac{2\lambda^2}{(\alpha-2)(\alpha-3)}, \quad \text{for } \alpha > 3 \tag{14}$$

$$\mu'_3 = \frac{6\lambda^3}{(\alpha-2)(\alpha-3)(\alpha-4)}, \quad \text{for } \alpha > 4 \tag{15}$$

$$\mu'_4 = \frac{24\lambda^4}{(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)}, \quad \text{for } \alpha > 5 \tag{16}$$

Using the raw moments we can obtain the central moments as follows:

$$\mu_1 = \frac{\lambda}{\alpha-2}, \quad \text{for } \alpha > 2 \tag{17}$$

$$\mu_2 = \frac{\lambda^2(\alpha-1)}{(\alpha-2)^2(\alpha-3)}, \quad \alpha > 3 \tag{18}$$

$$\mu_3 = \frac{2\lambda^3\alpha(\alpha-1)}{(\alpha-2)^3(\alpha-3)(\alpha-4)}, \quad \alpha > 4 \tag{19}$$

$$\mu_4 = \frac{3\lambda^4(3\alpha^3-8\alpha^2-9\alpha-4)}{(\alpha-2)^4(\alpha-3)(\alpha-4)(\alpha-5)}, \quad \alpha > 5 \tag{20}$$

4.2. Coefficient-Based Measures

The coefficient of variation (CV), skewness $\sqrt{\beta_1}$, and kurtosis β_2 are defined as:

$$\text{CV} = \frac{\sigma}{\mu}, \quad \sqrt{\beta_1} = \frac{\mu_3}{\sigma^3}, \quad \beta_2 = \frac{\mu_4}{\sigma^4} \tag{21}$$

Based on the central moments, the following expressions for the coefficient of variation (CV), skewness $\sqrt{\beta_1}$, and kurtosis β_2 are obtained for the Induced Lomax distribution.

$$\text{CV} = \sqrt{\frac{\alpha-1}{\alpha-3}}, \quad \alpha > 3 \tag{22}$$

$$\sqrt{\beta_1} = \frac{2\alpha\sqrt{\alpha-3}}{(\alpha-1)^{1/2}(\alpha-4)}, \quad \alpha > 4 \tag{23}$$

$$\beta_2 = \frac{3(3\alpha^3-8\alpha^2-9\alpha-4)(\alpha-3)}{(\alpha-1)^2(\alpha-4)(\alpha-5)}, \quad \alpha > 5 \tag{24}$$

The coefficient-based measures further highlight the flexibility and suitability of the $i-$Lomax distribution in modeling asymmetric and highly variable data. CV is greater than unity for small values of the shape parameter α, indicating significant relative variability, and decreases as α increases. The skewness $\sqrt{\beta_1}$ remains strictly positive for all admissible values of α, confirming the right-skewed nature of the distribution, which is particularly relevant in lifetime and actuarial applications. Additionally, the kurtosis β_2 exceeds 3, indicating a leptokurtic structure characterized by heavy tails and a sharper peak compared to the normal distribution.

4.3. Index of Dispersion

The index of dispersion γ is defined as:

$$\gamma = \frac{\text{Var}(X)}{E[X]} = \frac{\mu_2}{\mu_1} \tag{25}$$

The index of dispersion is given as:

$$\gamma = \frac{\lambda(\alpha-1)}{(\alpha-2)(\alpha-3)}, \quad \alpha > 3 \tag{26}$$

The expression for the index of dispersion γ reveals that the $i-$Lomax distribution exhibits overdispersion when the shape parameter α is relatively small. As α increases, γ decreases, indicating a reduction in dispersion. This behavior demonstrates the flexibility of the distribution in modeling both highly variable and moderately dispersed data, which is particularly useful in reliability and lifetime analysis.

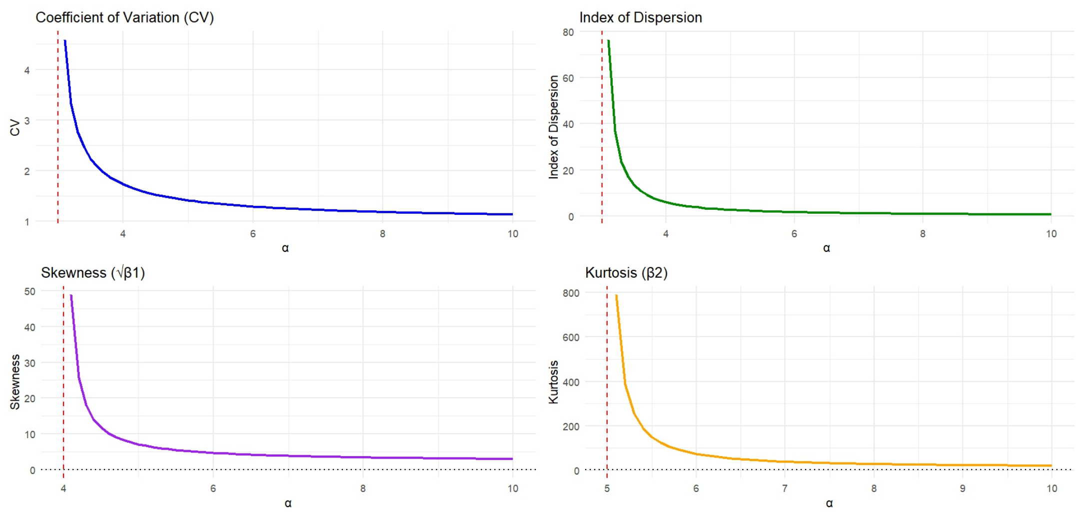


Figure 3: *Graph of the CV, γ, $\sqrt{\beta_1}$ and β_2 for different values of α.*

The figure reveals how the $i-$Lomax distribution's behavior transforms as its shape parameter α grows with constant value of λ Initially, when α is just above 3, the distribution shows extreme variability and heavy tails, with undefined moments. As α increases beyond 4 and 5, key properties stabilize - the distribution becomes less variable, transitions from left to right skewness, and develops finite kurtosis. Higher α values result in a more moderate distribution that is still right-skewed but less likely to have extreme values. This distribution is especially useful for simulating phenomena with infrequent but significant occurrences, such as insurance claims and financial collapses, because of its regulated heaviness. Three critical thresholds (α=3,4,5) determine the mathematical boundaries of successive moments, resulting in discrete statistical regimes.

5. Generating Functions

5.1. Moment Generating Function (MGF)

The moment generating function (MGF) of the $i-$Lomax distribution is defined as

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} g(x) dx \tag{27}$$

where $g(x) = \frac{\alpha-1}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\alpha}$ is the PDF of the $i-$Lomax distribution. Substituting into the integral:

$$M_X(t) = \frac{\alpha-1}{\lambda}\int_0^\infty e^{tx}\left(1+\frac{x}{\lambda}\right)^{-\alpha} dx \tag{28}$$

$$(\alpha-1)\int_1^\infty e^{t\lambda(u-1)}u^{-\alpha}du = (\alpha-1)e^{-t\lambda}\int_1^\infty e^{t\lambda u}u^{-\alpha}du \tag{29}$$

This integral does not yield a closed-form expression for general t, as it diverges for $t > 0$, and the integrand involves an exponential growth term combined with a heavy-tailed decay. Therefore, an alternative representation based on raw moments is considered.

Using the Taylor series expansion of the exponential function, the MGF is expressed as:

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!}\mu'_r \tag{30}$$

Using the first four raw moments derived earlier, the expansion becomes:

$$M_X(t) = 1 + \frac{t\lambda}{\alpha-2} + \frac{t^2\lambda^2}{(\alpha-2)(\alpha-3)} + \frac{t^3\lambda^3}{(\alpha-2)(\alpha-3)(\alpha-4)} + \cdots \tag{31}$$

This series is valid for values of t in a neighborhood of zero, and confirms the existence of all finite moments under the condition $\alpha > r+1$ for the $r-$th moment.

5.2. Cumulant Generating Function (CGF)

The cumulant generating function $\kappa_X(t)$ is the natural logarithm of the MGF:

$$\kappa_X(t) = \log M_X(t) = \log\left(\sum_{r=0}^\infty \frac{t^r}{r!}\mu'_r\right) \tag{32}$$

Using the Taylor expansion of the logarithm:

$$\log(1+x) = \sum_{k=1}^\infty (-1)^{k+1}\frac{x^k}{k}, \quad |x| < 1 \tag{33}$$

the CGF can be expressed as a series:

$$\kappa_X(t) = \sum_{r=1}^{\infty} \frac{\kappa_r t^r}{r!} \tag{34}$$

where κ_r denotes the $r-$th cumulant. The first few cumulants are:

$$\kappa_1 = \mu_1' = \frac{\lambda}{\alpha - 2}, \quad \kappa_2 = \mu_2 = \frac{\lambda^2(\alpha-1)}{(\alpha-2)^2(\alpha-3)} \tag{35}$$

$$\kappa_3 = \mu_3 = \frac{2\lambda^3\alpha(\alpha-1)}{(\alpha-2)^3(\alpha-3)(\alpha-4)} \tag{36}$$

$$\kappa_4 = \mu_4 - 3\mu_2^2 = \frac{3\lambda^4(3\alpha^3 - 8\alpha^2 - 9\alpha - 4)}{(\alpha-2)^4(\alpha-3)(\alpha-4)(\alpha-5)} - 3\left(\frac{\lambda^2(\alpha-1)}{(\alpha-2)^2(\alpha-3)}\right)^2 \tag{37}$$

5.3. Characteristic Function (CF)

The characteristic function $\Phi_X(t)$ is defined as:

$$\Phi_X(t) = E[e^{itX}] = \int_0^{\infty} e^{itx} g(x) dx \tag{38}$$

This integral also does not admit a closed-form expression for general t, due to the oscillatory exponential and the non-elementary behavior of the integrand. Hence, we use a series expansion based on raw moments:

$$\Phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r' \tag{39}$$

This power series is valid for all $t \in R$ and describes the characteristic function as the Fourier transform of the PDF.

6. Reliability Functions

Reliability analysis plays a key role in lifetime and survival modeling. In this section, we derive and discuss key reliability functions associated with the $i-$Lomax distribution.

6.1. Survival Function

The survival function $\bar{G}(x)$ is the complement of the cumulative distribution function:

$$\bar{G}(x) = 1 - G(x) = \left(1 + \frac{x}{\lambda}\right)^{1-\alpha}, \quad x > 0, \ \alpha > 1 \tag{40}$$

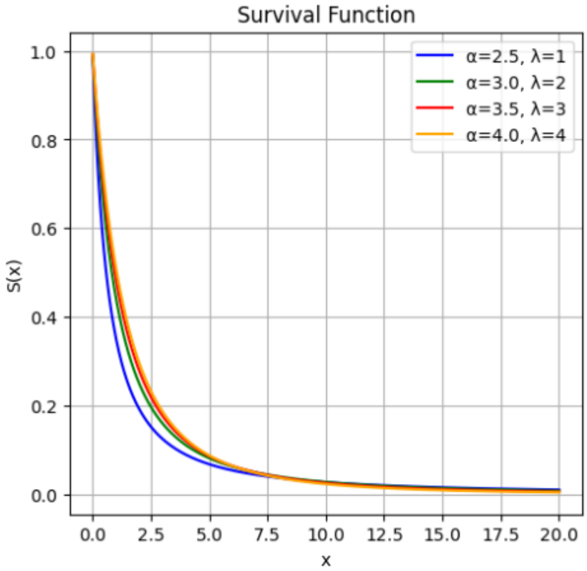


Figure 4: *Survival function of i−Lomax distribution*

6.2. Hazard Rate Function

The hazard rate function $h(x)$ is given by the ratio of the probability density function to the survival function:

$$h(x) = \frac{g(x)}{\bar{G}(x)} = \frac{\frac{(\alpha-1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\alpha}}{\left(1+\frac{x}{\lambda}\right)^{1-\alpha}} = \frac{(\alpha-1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}, \quad x > 0 \tag{41}$$

This expression shows that the hazard rate function is monotonically decreasing.

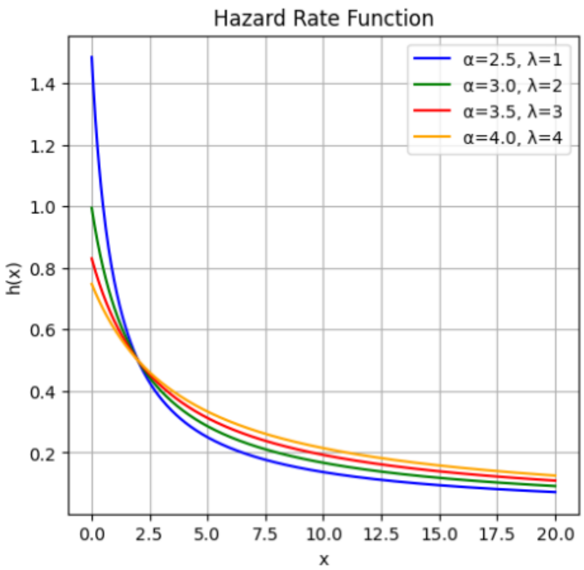


Figure 5: *Hazard function of i−Lomax distribution*

6.3. Mean Residual Life Function

The mean residual life (MRL) function $m(x)$ is defined as the expected remaining lifetime given survival up to time x:

$$m(x) = E[X - x \mid X > x] = \frac{1}{\bar{G}(x)} \int_x^\infty (t - x) g(t) dt \tag{42}$$

Substituting $g(t)$ and simplifying, the MRL function becomes:

$$m(x) = \frac{\lambda}{(\alpha - 2)} \left(1 + \frac{x}{\lambda}\right)^{-1}, \quad \alpha > 2 \tag{43}$$

This function also decreases with x, indicating that the expected remaining lifetime becomes shorter over time consistent with decreasing hazard rate behavior.

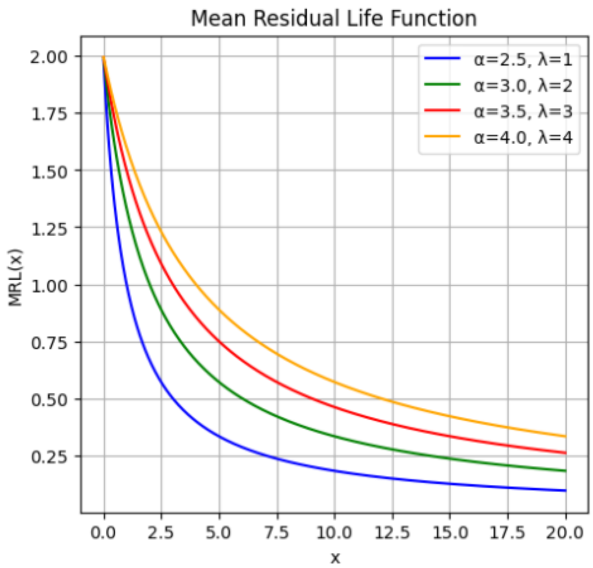


Figure 6: *Mean Residual Life function of i−Lomax distribution*

7. Quantile Function

The quantile function is the inverse of the cumulative distribution function (CDF), and is useful for generating random variates and analyzing distributional behavior.

The CDF of the $i-$Lomax distribution is:

$$G(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{1-\alpha}$$

Let $u = G(x)$ Then:

$$u = 1 - \left(1 + \frac{x}{\lambda}\right)^{1-\alpha} \quad \Rightarrow \quad 1 - u = \left(1 + \frac{x}{\lambda}\right)^{1-\alpha}$$

Taking logarithms and solving for x:

$$\left(1 + \frac{x}{\lambda}\right) = (1 - u)^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad x = \lambda \left[(1 - u)^{\frac{1}{1-\alpha}} - 1\right]$$

Hence, the quantile function $Q(u)$ is given by:

$$Q(u) = \lambda \left[(1 - u)^{\frac{1}{1-\alpha}} - 1\right], \quad 0 < u < 1,\ \alpha > 1 \tag{44}$$

8. Stochastic Ordering

Stochastic ordering of continuous random variables is an important tool for comparing their relative behaviors in terms of risk, reliability, or strength. Let X and Y be two continuous random variables. Then the following orderings are defined as:

1. Usual Stochastic Order: $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x.
2. Hazard Rate Order: $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)$ for all x.
3. Mean Residual Life Order: $X \leq_{mrl} Y$ if $m_X(x) \leq m_Y(x)$ for all x.
4. Likelihood Ratio Order: $X \leq_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ is decreasing in x.

According to Shaked and Shanthikumar (2007), the following implications hold:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow X \leq_{st} Y \tag{45}$$

Orderings in the Induced Lomax Distribution

Let $X \sim i - Lomax(\alpha_1, \lambda)$ and $Y \sim i - Lomax(\alpha_2, \lambda)$ be two $i-$Lomax random variables with the same scale parameter λ but different shape parameters α_1 and α_2.

The PDF of the $i-$Lomax distribution is:

$$f(x;\alpha,\lambda) = \frac{\alpha-1}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\alpha}$$

and the survival function is:

$$\bar{F}(x) = \left(1+\frac{x}{\lambda}\right)^{1-\alpha} \tag{46}$$

Stochastic Order

For $\alpha_1 < \alpha_2$, we have:

$$\bar{F}_X(x) > \bar{F}_Y(x) \quad \Rightarrow \quad F_X(x) < F_Y(x) \Rightarrow X \geq_{st} Y \tag{47}$$

Hazard Rate Order

The hazard rate function is:

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\alpha-1}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1} \tag{48}$$

Since $h(x)$ is decreasing in x and increases with α, we observe that:

$$\alpha_1 > \alpha_2 \Rightarrow h_X(x) > h_Y(x) \Rightarrow X \leq_{hr} Y \tag{49}$$

Mean Residual Life Order

The mean residual life function is:

$$m(x) = \frac{\lambda}{\alpha-2}\left(1+\frac{x}{\lambda}\right)^{-1}, \quad \alpha > 2 \tag{50}$$

As $m(x)$ is decreasing in α, for $\alpha_1 > \alpha_2$ we have:

$$m_X(x) < m_Y(x) \Rightarrow X \leq_{mrl} Y$$

Likelihood Ratio Order (Optional)

The likelihood ratio:

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1 - 1}{\alpha_2 - 1}\left(1 + \frac{x}{\lambda}\right)^{\alpha_2 - \alpha_1} \tag{51}$$

is decreasing in x if $\alpha_1 > \alpha_2$, thus:

$$X \leq_{lr} Y \text{ if } \alpha_1 > \alpha_2 \tag{52}$$

Thus, the $i-$Lomax distribution satisfies all four stochastic orderings with respect to the shape parameter α, when scale parameter λ is fixed. This makes it suitable for modeling data with increasing or decreasing reliability characteristics, depending on the parameter values.

9. Order Statistics

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m drawn from the $i-$Lomax distribution, and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ denote the corresponding order statistics. The PDF and CDF of the r^{th} order statistic $X_{(r)}$, say Y, are given respectively by:

$$f_{(r:m)}(y) = \frac{m!}{(r-1)!(m-r)!}\left[G(y)\right]^{r-1}\left[1 - G(y)\right]^{m-r} g(y)$$
$$F_{(r:m)}(y) = \sum_{j=r}^{m}\binom{m}{j}\left[G(y)\right]^{j}\left[1 - G(y)\right]^{m-j}$$

where $G(y)$ and $g(y)$ are the CDF and PDF of the $i-$Lomax distribution, respectively:

$$G(y) = 1 - \left(1 + \frac{y}{\lambda}\right)^{1-\alpha}, \quad g(y) = \frac{\alpha - 1}{\lambda}\left(1 + \frac{y}{\lambda}\right)^{-\alpha}$$

We may also express the PDF using a binomial expansion involving powers of the CDF:

$$f_{(r:m)}(y) = \frac{m!}{(r-1)!(m-r)!}\sum_{l=0}^{m-r}\binom{m-r}{l}(-1)^l [G(y)]^{r+l-1} g(y) \tag{53}$$

and the CDF of $Y = X_{(r)}$ as:

$$F_{(r:m)}(y) = \sum_{j=r}^{m}\sum_{l=0}^{m-j}\binom{m}{j}\binom{m-j}{l}(-1)^l [G(y)]^{j+l} \tag{54}$$

These representations are useful in computing the distribution of extremes (minimum or maximum) and in reliability applications such as system lifetimes and quantile estimation. For instance, the minimum $X_{(1)}$ has $r = 1$, and the maximum $X_{(m)}$ has $r = m$.

10. Stress-Strength Reliability

Let $X \sim$ i-Lomax(α_1, λ_1) denote the strength and $Y \sim$ i-Lomax(α_2, λ_2) the stress, assumed to be independent. The stress-strength reliability is defined as:

$$R = P(X > Y) = \int_0^{\infty} fY(y) \cdot \bar{F}X(y) dy \tag{55}$$

Substituting the expressions for the PDF and survival function of the $i-$Lomax distribution yields:

$$R = \int_0^\infty \frac{\alpha_2 - 1}{\lambda_2} \left(1 + \frac{y}{\lambda_2}\right)^{-\alpha_2} \cdot \left(1 + \frac{y}{\lambda_1}\right)^{1-\alpha_1} dy \tag{56}$$

In the special case where $\lambda_1 = \lambda_2 = \lambda$, the expression simplifies to:

$$R = \frac{\alpha_2 - 1}{\alpha_1 + \alpha_2 - 2}, \quad \text{for } \alpha_1 + \alpha_2 > 2 \tag{57}$$

This provides a closed-form solution for R, representing the probability that the strength exceeds the stress under identical scale parameters.

11. Bonferroni and Lorenz Curves

Bonferroni Curve

The Bonferroni curve $B(p)$ for the $i-$Lomax distribution is defined as:

$$B(p) = \frac{1}{p\mu} \int_0^{Q(p)} x g(x) dx \tag{58}$$

where $\mu = \frac{\lambda}{\alpha-2}$ is the mean and $Q(p) = \lambda \left[(1-p)^{\frac{1}{1-\alpha}} - 1\right]$ is the quantile function.

After algebraic simplification using the transformations $1 + \frac{x}{\lambda} = t$, the Bonferroni curve for the $i-$Lomax distribution becomes:

$$B(p) = \frac{1}{p}\left[1 - \frac{q}{\lambda}(\alpha - 2)\left(1 + \frac{q}{\lambda}\right)^{1-\alpha} + \left(1 + \frac{q}{\lambda}\right)^{2-\alpha}\right] \tag{59}$$

where $q = Q(p)$ is the quantile value for a given $p \in (0, 1)$.

Lorenz Curve

The Lorenz curve $L(p)$ is obtained as:

$$L(p) = \frac{1}{\mu} \int_0^{Q(p)} x g(x) dx = B(p) \cdot p \tag{60}$$

which simplifies to:

$$L(p) = 1 - \frac{q}{\lambda}(\alpha - 2)\left(1 + \frac{q}{\lambda}\right)^{1-\alpha} + \left(1 + \frac{q}{\lambda}\right)^{2-\alpha} \tag{61}$$

Gini Index

The Gini index G is derived using:

$$G = 1 - 2\int_0^1 L(p) dp \tag{62}$$

Substituting the form of $L(p)$, and changing variable to $x = Q(p)$, the Gini index becomes:

$$G = (\alpha - 2) \int_0^\infty \left[1 - \left(1 + \frac{x}{\lambda}\right)^{1-\alpha}\right] \left(1 + \frac{x}{\lambda}\right)^{1-\alpha} dx \tag{63}$$

With the substitution $t = 1 + \frac{x}{\lambda}$, this evaluates to:

$$G = 1 - \frac{\alpha - 2}{2\alpha - 3}, \quad \text{for } \alpha > \frac{3}{2} \tag{64}$$

This closed-form expression clearly shows that the Gini index increases as $\alpha \to \frac{3}{2}^{+}$ reflecting the heavy-tail and skewness of the i-Lomax distribution.

12. Entropy Measures

12.1. Rényi Entropy

Let X be a continuous random variable with PDF $g(x)$. The Rényi entropy of order $\eta > 0, \eta \neq 1$ is defined as:

$$\mathcal{E}(\eta) = \frac{1}{1-\eta} \log \left[\int_0^\infty g^\eta(x) dx \right]$$

Substituting the PDF of the $i-$Lomax distribution:

$$g(x) = \frac{\alpha - 1}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$$

we get:

$$\begin{aligned}\mathcal{E}(\eta) &= \frac{1}{1-\eta} \log \left[\int_0^\infty \left(\frac{\alpha-1}{\lambda}\right)^\eta \left(1 + \frac{x}{\lambda}\right)^{-\alpha\eta} dx \right] \\ &= \frac{1}{1-\eta} \log \left[\left(\frac{\alpha-1}{\lambda}\right)^\eta \int_0^\infty \left(1 + \frac{x}{\lambda}\right)^{-\alpha\eta} dx \right]\end{aligned}$$

Using the substitution $t = 1 + \frac{x}{\lambda} \Rightarrow x = \lambda(t-1) dx = \lambda dt$, we get:

$$\int_0^\infty \left(1 + \frac{x}{\lambda}\right)^{-\alpha\eta} dx = \lambda \int_1^\infty t^{-\alpha\eta} dt = \frac{\lambda}{\alpha\eta - 1}, \quad \text{for } \alpha\eta > 1$$

Hence, the Rényi entropy becomes:

$$\mathcal{E}(\eta) = \frac{1}{1-\eta} \left[\eta \log(\alpha - 1) + (1-\eta) \log \lambda - \log(\eta\alpha - 1) \right], \quad \eta > \frac{1}{\alpha}$$

12.2. Shannon Entropy

The Shannon entropy is defined as:

$$H(X) = -\int_0^\infty g(x) \log g(x)\, dx \tag{65}$$

Substituting; $g(x) = \frac{\alpha-1}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$, we get:

$$H(X) = -\int_0^\infty \left(\log\left(\frac{\alpha-1}{\lambda}\right) - \alpha \log\left(1 + \frac{x}{\lambda}\right) \right) g(x) dx \tag{66}$$

Split the integral:

$$H(X) = -\log\left(\frac{\alpha-1}{\lambda}\right) \int_0^\infty g(x) dx + \alpha \int_0^\infty \log\left(1 + \frac{x}{\lambda}\right) g(x) dx \tag{67}$$

Since $\int_0^\infty g(x)dx = 1$, we simplify:

$$H(X) = -\log\left(\frac{\alpha-1}{\lambda}\right) + \alpha \int_0^\infty \log\left(1+\frac{x}{\lambda}\right) \cdot \frac{\alpha-1}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\alpha} dx \tag{68}$$

Substitute $t = 1 + \frac{x}{\lambda} \Rightarrow dx = \lambda dt$, so:

$$H(X) = -\log\left(\frac{\alpha-1}{\lambda}\right) + \alpha(\alpha-1)\int_1^\infty \log(t)\cdot t^{-\alpha}dt \tag{69}$$

Using the known result:

$$\int_1^\infty \log(t)\cdot t^{-\alpha}dt = \frac{1}{(\alpha-1)^2} \tag{70}$$

we obtain:

$$H(X) = -\log(\alpha-1) + \log\lambda - \frac{\alpha}{1-\alpha}, \quad \alpha > 1$$

13. Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample from the $i-$Lomax distribution with PDF:

$$g(x;\alpha,\lambda) = \frac{\alpha-1}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\alpha}, \quad \alpha > 1,\ \lambda > 0$$

The log-likelihood function is given by:

$$\ell(\alpha,\lambda) = n\log(\alpha-1) - n\log\lambda - \alpha\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right) \tag{71}$$

Taking partial derivatives with respect to the parameters and equating to zero, we obtain the likelihood equations:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha-1} - \sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right) = 0$$

$$\hat{\alpha} = 1 + \frac{n}{\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)} \tag{72}$$

$$\frac{\partial\ell}{\partial\lambda} = -\frac{n}{\lambda} + \alpha\sum_{i=1}^{n}\frac{x_i}{\lambda(\lambda+x_i)} = 0 \tag{73}$$

$$\sum_{i=1}^{n}\frac{x_i}{\lambda+x_i} = \frac{n}{\alpha} \tag{74}$$

Equations (72) and (74) must be solved simultaneously to obtain the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$. Since closed-form solutions do not exist, iterative numerical techniques such as the Newton–Raphson method or optimization routines are required.

14. Simulation Study

The behaviour of the parameters of the $i-$Lomax distribution was investigated by conducting simulation studies with the aid of R software. Data sets were generated from the $i-$Lomax distribution with a replication number $m = 1000$; random samples of sizes $n = 25, 50, 100$, and 200 were further selected. The simulation was conducted for three (3) different cases using varying true parameter values. The selected true parameter values are $(\alpha, \lambda) = \{(1.5, 0.5), (2.5, 1.0), (3.5, 2.0)\}$ for the first, second, and third cases, respectively.

The MLEs of the true parameters were obtained including the Bias and the Root Mean Square Error (RMSE). The results for the simulation studies are as shown in Tables 1.

Table 1: *Simulation study results for MLE for different values of α and λ.*

Parameter	Sample Size	α_{MLE}			λ_{MLE}		
(α, λ)	(n)	Mean	Bias	RMSE	Mean	Bias	RMSE
(1.5,0.5)	25	1.5899	0.0899	0.2564	0.7676	0.2676	0.7377
	50	1.5420	0.0420	0.1410	0.6409	0.1409	0.3954
	100	1.5267	0.0267	0.0828	0.6087	0.1087	0.2396
	200	1.5250	0.0250	0.0577	0.6261	0.1261	0.2161
(2.5, 1.0)	25	3.7628	1.2628	2.7427	2.3191	1.3191	2.8984
	50	3.1427	0.6427	1.7433	1.6464	0.6464	1.7329
	100	2.6780	0.1780	0.6314	1.1922	0.1922	0.6613
	200	2.5791	0.0791	0.3400	1.0737	0.0737	0.3398
(3.5, 2.0)	25	5.5954	2.0954	3.6252	4.1678	2.1678	3.8431
	50	4.7552	1.2552	2.7302	3.3398	1.3398	2.9123
	100	4.2206	0.7206	1.8767	2.7705	0.7705	2.0378
	200	3.7857	0.2857	0.9847	2.3019	0.3019	1.0648

It can be deduced from Tables 1 that the root mean square error (RMSE) reduces for all the selected parameter values as the sample size increases. Also, the bias posed by the estimates is closer to the true parameter values and the absolute bias reduces as the sample size increases. Hence, as sample size increases, the estimates tend towards the true parameter values.

15. Goodness-of-Fit of the $i-$Lomax Distribution

The goodness-of-fit of the $i-$Lomax distribution has been demonstrated using three real-life datasets from the literature. The first dataset contains vinyl chloride concentrations (in μg/L) measured in groundwater monitoring wells, provided by Bhaumik & Gibson [22]. The second dataset includes the insulation breakdown times (in minutes) of an electrical insulating fluid under increasing voltage levels, as reported by Lawless [23] and the last dataset represents the failure times of the air conditioning system of an aircraft, reported by Linhart & Zucchini [24].

The $i-$Lomax distribution was fitted to each dataset and compared with the $i-$Garima, Weibull, and Gamma distributions using their respective probability density functions. For model comparison, the values of $-2 \log L$, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov–Smirnov (K–S) statistics with their associated $p-$values were computed. The following formulas were used:

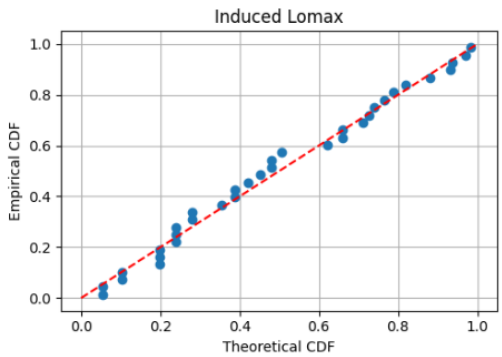
$$\text{AIC} = -2\log L + 2k$$
$$\text{BIC} = -2\log L + k\log n$$
$$D = \sup_x |F_n(x) - F_0(x)|$$

where k is the number of model parameters, n is the sample size, $F_n(x)$ is the empirical distribution function, and $F_0(x)$ is the fitted cumulative distribution function.

Table 2: *Estimates of the Parameters, -2LL, AIC, BIC and K–S Statistics of the Fitted Distributions for Different data Sets*

Data Set	Estimate	Distribution	–2LL	AIC	BIC	K–S	$p-$value
	(31.115,54.728)	$i-$Lomax	110.88	114.88	117.93	0.081	0.978
	0.674	$i-$Garima	111.19	113.19	114.71	0.104	0.857
1st	(1.010,1.888)	Weibull	110.90	114.90	117.95	0.092	0.937
	(0.565,1.063)	Gamma	110.83	114.83	117.88	0.097	0.904
	$(1.565, 2.609)$	$i-$Lomax	653.63	657.63	662.29	0.063	0.921
	0.012	$i-$Garima	875.45	877.45	879.78	0.518	0.000
2nd	$(0.434, 25.072)$	Weibull	670.84	674.84	679.50	0.106	0.365
	$(0.014, 0.812)$	Gamma	697.30	701.30	705.97	0.184	0.013
	(4.296,141.261)	$i-$Lomax	303.68	307.68	310.48	0.142	0.585
	0.022	$i-$Garima	306.75	308.75	310.15	0.240	0.063
3rd	(0.853,54.613)	Weibull	303.87	307.87	310.68	0.153	0.481
	(0.014,0.812)	Gamma	304.34	308.34	311.14	0.169	0.356

From Table 2, it is evident that the $i-$Lomax distribution provides a superior fit in several cases, particularly for the insulation breakdown, vinyl chloride, and air conditioning datasets. It substantially improves over the classical Lomax and exponential distributions in terms of AIC, BIC, and K–S with respective $p-$values. This demonstrates reasonable flexibility in modeling lifetime data with heavy tails.



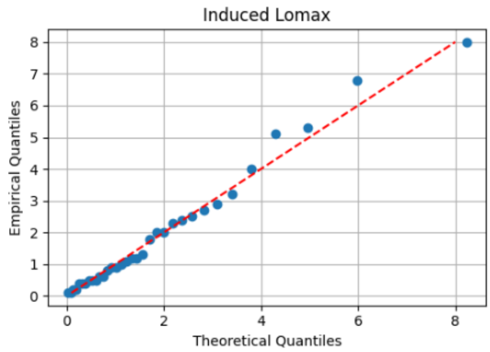


Figure 7: *P-P and Q-Q plots for $i-$Lomax distributions to the 1st dataset.*

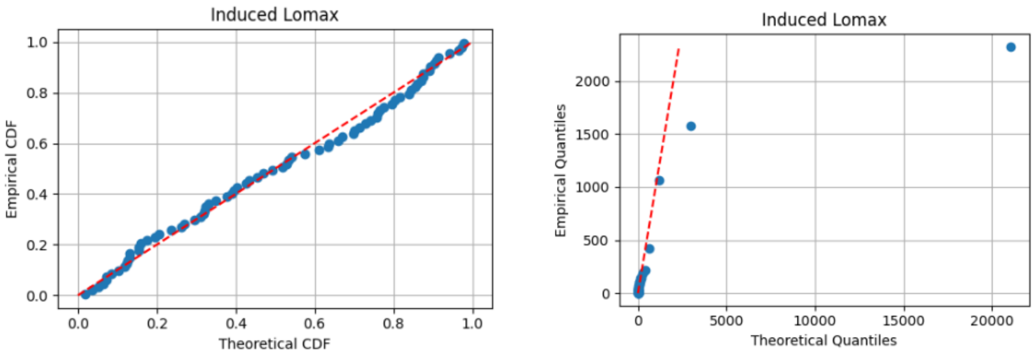


Figure 8: *P-P and Q-Q plots for $i-$Lomax distributions to the 2nd dataset.*

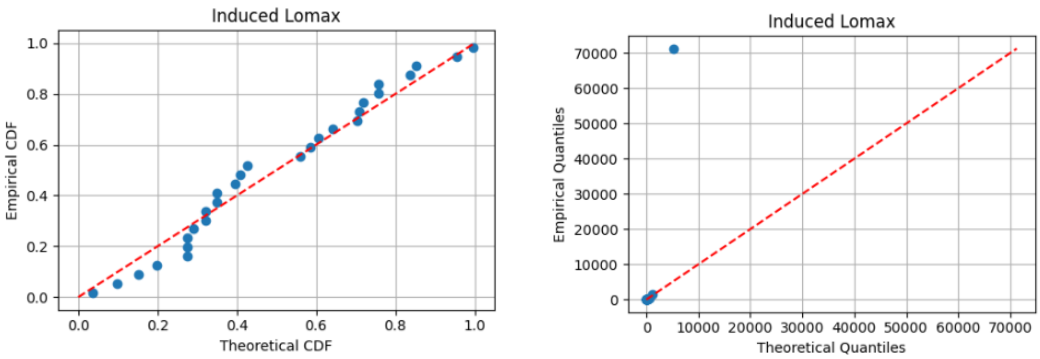


Figure 9: *P–P and Q-Q plots for $i-$Lomax distributions to the 3rd dataset.*

16. Conclusion

Better modeling of lifetime and reliability data is a significant concern for statisticians and applied researchers. Considerable progress has been made toward extending classical lifetime models to improve their flexibility and real-world applicability. In the present study, a technique based on the induced distribution framework has been employed to propose an alternative two-parameter distribution called the $i-$Lomax distribution. Various statistical properties such as raw and central moments, quantile function, survival and hazard rate functions, mean residual life function, coefficient-based measures, entropy measures, stress-strength reliability, and order statistics have been thoroughly derived and discussed.The model parameters have been estimated using the method of maximum likelihood, and a simulation study was conducted to examine the performance of the estimators in terms of bias and root mean square error (RMSE). The simulation results confirm that the estimators become increasingly accurate with larger sample sizes.

To assess the practical utility of the $i-$Lomax distribution, three real-life datasets were analyzed. Comparative goodness-of-fit analysis was carried out against classical models such as $i-$Garima, Weibull and Gamma distributions. The evaluation was based on the values of $-2LL$, AIC, BIC and K–S statistics with its $p-$value. Results demonstrate that the $i-$Lomax distribution outperforms other competing models in several cases and offers comparable fit in others. It is expected that the proposed model will serve as a useful tool in lifetime data analysis and reliability modeling. Future extensions of this work may consider censored data scenarios and Bayesian approaches under different loss functions. Also, this model may be used to develop new model using other transformation techniques of generation of new probability model.

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