RT&A, No 4 (89)
Volume 20, December 2025

JOINT ADAPTIVE PROGRESSIVE TYPE-II CENSORING WITH DUS-EXTREME VALUE DISTRIBUTION

Rakhi Chandran[1], Chacko V M[2]

•

[1]Department of Statistics, St.Mary's College(Autonomous), Thrissur
[2] Department of Statistics, St.Thomas College(Autonomous), Thrissur
University of Calicut, Kerala, India,
[1]Emails: chandran.rakhi@gmail.com
[2]Emails: chackovm@stthomas.ac.in

Abstract

Statistical estimation of parameters under different censoring schemes is important while using distributions like the Extreme value and Weibull distributions for modelling real lifetime data. Since Extreme value distribution arise as log transformation of the Weibull distribution, there are several advantages for discussing the estimation of extreme value parameters. Moreover, DUS transformation provides more appropriate model without increasing parameters. This study examines point estimation methods, including maximum likelihood and Bayesian approaches, for DUS Extreme Value (DUS-EV) distribution under a joint adaptive progressive Type-II censoring scheme. The proposed scheme offers notable advantages in reducing cost and experimental time. Bayesian estimation is implemented via the Metropolis- Hastings algorithm within a Gibbs sampling framework. Interval estimation is carried out using asymptotic confidence intervals, Highest Posterior Density (HPD) credible intervals, and bootstrap confidence intervals. A comparative analysis of classical and Bayesian results is presented, and the applicability of the proposed methods is demonstrated through real data analysis.

Keywords: DUS-extreme value distribution, Joint adaptive Type II progressive censoring, Maximum likelihood estimation, Bayesian estimation

1. Introduction

In time-to-event data, when the precise value of interest is not immediately observable but is known only within specific bounds, statistical censoring is essential because it makes it possible to analyze incomplete data. In order to avoid biased results and to get more precise estimates for statistics like medians, it is imperative that such data be analyzed. In domains like engineering and medicine, where precise data interpretation is necessary for well-informed decision-making, this analysis is especially crucial. To do this, techniques that specifically handle censoring like survival analysis must be used in order to guarantee accurate data interpretation and produce significant findings. Censoring is an unavoidable aspect of reliability and survival studies, and several censoring schemes have been proposed to address this issue. The most commonly used are Type I and Type II censoring. In Type I censoring, the duration of the experiment is fixed while the number of failures observed is random, whereas in Type II censoring, the number of failures is fixed in advance while the experimental time becomes random. Specifically, under Type I censoring, the study continues until a predetermined time, while under Type II censoring, it terminates at the occurrence of the m^{th} failure. A limitation of Type I censoring is that an insufficient number of failures may occur within the fixed time, leading to reduced efficiency

DOI: https://doi.org/10.24412/1932-2321-2025-489-574-589

of the estimator. Conversely, Type II censoring may require a prolonged experimental duration, which can be impractical.

To overcome the limitations associated with conventional censoring, hybrid models incorporating features from Type I and Type II schemes have been proposed. The earliest among these is the Type I hybrid censoring scheme [13], in which the experiment continues until $T_1^* =$ min (m^{th} failure time, T). Similar to the Type I approach, the scheme may produce very few failure observations, leading to reduced estimator efficiency. Additionally, inferential procedures under this scheme typically assume the occurrence of at least one failure. Subsequently, Type II hybrid censoring scheme was introduced, where the termination time is defined as T_2* = max (m^{th} failure time, T) [9]. This scheme guarantees at least m failures, leading to more efficient parameter estimation. However, its main drawback lies in the potentially longer experimental duration required.

The aforementioned censoring methods do not account for the loss or removal of experimental units, which frequently occurs in practice either due to unexpected failures or to reduce experimental time and cost. To address this, the concept of progressive censoring was introduced [10]. This approach has two main variants. The first, known as Type I progressive censoring, involves the removal of R_1, R_2, ..., R_{m-1} surviving units at predetermined censoring times T_1^*, T_2^*, ..., T_{m-1}^*. In this scheme, both the number of removals (R_i) and the censoring times (T_i^*) are fixed, and the experiment concludes with the removal of the remaining R_m units at time T_m^*. The second, referred to as Type II progressive censoring, removes R_1, R_2, ..., R_{m-1} surviving units after the 1^{st} failure, 2^{nd} failure,....$(m-1)^{th}$ failures, respectively. Here, the numbers R_is are predetermined, but the censoring times are random.

A comprehensive treatment of the theory and applications of Type II progressive censoring, emphasizing its flexibility in removing surviving units at different failure times before experiment termination is available in the literature [5]. This approach was further elaborated by researchers [4]. While progressive censoring is cost-effective and operationally flexible, it typically requires a longer experimental duration compared to traditional Type II censoring. To mitigate this drawback, the Type II progressively hybrid censoring scheme was proposed, which combines the features of Type II progressive censoring and hybrid censoring [15]. The scheme can be described as follows:

Suppose N items are placed on a life-testing experiment, where the number of failures $m < N$, the removals of units at successive failure times, and the experimental time T are all predetermined. After the first failure, R_1 units are taken out, after the second failure R_2 units are removed, and the process continues in the same pattern. When the m^{th} failure takes place before the specified time T, the test ends at that failure time, and all surviving units still in the experiment are withdrawn. Conversely, if only $J < m$ failures occur before time T, then the experiment ends at T, where the remaining units $R_m = N - R_1 - R_2 - ...R_J - J$ are removed. This scheme closely resembles Type I censoring, since termination is ultimately determined by a fixed time. However, a key drawback is that it may yield only a small number of failures, thereby reducing the effectiveness of statistical estimation.

To overcome the limitations of the Type II progressively hybrid censoring scheme, the adaptive progressive Type II censoring scheme was introduced, which aims to substantially reduce both experimental time and cost by pre-specifying the number of failures m and the total time T [17]. A progressive censoring plan is initially set up for the experiment, but in contrast to standard methods, the removals R_i following each i^{th} failure may be adjusted dynamically throughout the process. As in the Type II progressive hybrid censoring scheme, when the m^{th} failure happens before time T, the study ends at that failure point, and all remaining units are withdrawn. Alternatively, when the m^{th} failure is not observed prior to T, the study is extended until that failure, which improves inferential efficiency and simultaneously maintains the termination time near the pre-assigned T by means of adaptive adjustment of removals.

This flexible framework has been extended in subsequent studies, applied the adaptive progressive Type II censoring scheme to the exponentiated Weibull distribution, developed it for the generalized exponential distribution [16] and later considered it for the extreme value distribution [23].

For comparative life testing involving two populations, joint censoring schemes have been developed to efficiently utilize information from both groups under common censoring plans [20]. The joint progressive Type II censoring scheme for two exponential populations extended to other distributions [19].

The joint progressive Type II censoring scheme can be described as follows: consider two samples of sizes N_1 and N_2, with total sample size $N = N_1 + N_2$, drawn from two populations in a life-testing experiment. As in earlier schemes, fix the number of observed failures m and the removal numbers R_i. After the first failure (regardless of whether it arises from sample 1 or sample 2), remove s_1 units from sample 1 and q_1 units from sample 2, such that $R_1 = s_1 + q_1$. Similarly, after the second failure, remove s_2 units from sample 1 and q_2 units from sample 2, ensuring that $R_2 = s_2 + q_2$. This process continues until the m^{th} failure, after which all remaining surviving units are removed.

Further developments in censoring methodologies are examined joint progressive Type-I censoring schemes for exponential populations [3]. More recently, the joint adaptive progressive Type-II censoring scheme for independent samples from exponential and Weibull populations are discussed in literature [21, 22]. Building on this line of research, the generalized Lindley distribution under the joint adaptive Type-II censoring framework has been discussed in literature [1]. The Type-I - Type-II mixture censoring scheme is found to be a better alternative to existing models [2]. Subsequently, a joint unified hybrid censoring scheme for two independent Weibull populations with a common shape parameter was proposed as an alternative model [11].

Researchers have put forth a number of novel models that improve the accuracy and flexibility of the underlying distributions in order to handle the complexity of real-world data. The DUS transformation of a baseline distribution,is one such technique [14]. The approach's unique feature of not adding any more parameters than those in the baseline distribution is what makes it significant. This method enhances modeling capability while preserving model parsimony. Building on these advancements, the present study develops inference procedures for the DUS extreme value (DUS-EV) distribution [18], under the joint adaptive progressive Type-II censoring scheme, which offers notable reductions in experimental time and cost. Since distribution of log transformation of Wiebull random variable is extreme value, its DUS transformation is more better choice than extreme value distribution. Since extreme value distribution has wide applicability in survival analysis, proportional hazard models, etc DUS extreme value distribution is also important in such areas. The DUS-EV distribution is obtained by applying the DUS transformation to the classical extreme value distribution. This model is characterized by an increasing failure rate, making it particularly suitable for applications in reliability engineering, hydrology, and meteorology.

The probability density function of DUS extreme value distribution DUS-EV(λ, α) can be defined as

$$g(v) = \frac{1}{\alpha(e-1)} e^{(\frac{v-\lambda}{\alpha} - e^{\frac{v-\lambda}{\alpha}})} e^{(1-e^{-e^{(\frac{v-\lambda}{\alpha})}})}, \quad -\infty < v < \infty, \quad -\infty < \lambda < \infty, \quad \alpha > 0. \tag{1}$$

Cumulative distribution function of DUS-EV(λ, α) are given by

$$G(v) = \frac{1}{e-1}(e^{1-e^{-e^{(\frac{v-\lambda}{\alpha})}}} - 1), \quad -\infty < v < \infty, \quad -\infty < \lambda < \infty, \quad \alpha > 0. \tag{2}$$

We employ both classical and Bayesian approaches for parameter estimation, while asymptotic confidence intervals, highest posterior density (HPD) intervals, and bootstrap-p intervals are utilized for interval estimation. A comprehensive simulation study is carried out to validate the proposed inferential procedures, and the practical utility of the methodology is illustrated using real data analysis.

The remaining sections of the article is outlined in the following manner: Section 2 describes the joint adaptive progressive Type-II censoring scheme. Section 3 presents the inferential procedures for the DUS-EV distribution under this scheme, including maximum likelihood estimation, asymptotic confidence intervals, bootstrap-p intervals, and Bayesian estimation. Section 4 outlines the simulation study, while Section 5 provides a real data analysis to illustrate the practical utility of the proposed methods. Finally, Section 6 concludes the paper by summarizing the key findings.

2. Joint adaptive progressive Type-II censoring scheme

Suppose two independent populations are involved in a lifetime experiment. A random sample of size N_1 is taken from population 1, which follows the distribution with PDF $g_1(,\lambda_1,\alpha_1)$ and CDF $G_1(,\lambda_1,\alpha_1)$. Similarly, a random sample of size N_2 is taken from population 2, characterised by PDF $g_2(,\lambda_2,\alpha_2)$ and CDF $G_2(,\lambda_2,\alpha_2)$. The Joint Adaptive Progressive Type-II censoring scheme is outlined below.

Let m be a predetermined integer specifying the total number of failures to be observed, and let T denote the fixed termination time of the experiment. Define the vector $R = (R_1, R_2, ..., R_m)$, where each R_l denotes the number of units removed at the time of the l^{th} failure. These removals satisfy $\Sigma_{l=1}^{m} R_l = N - m$, with $N = N_1 + N_2$ denoting the total number of units in the experiment. Further, define two vectors, $P = (P_1, P_2, ..., P_m)$ and $Q = (Q_1, Q_2, ..., Q_m)$, where P_l and Q_l denote the number of units withdrawn from sample 1 and sample 2, respectively, at the lth failure time. Suppose the first failure occurs at time U_1. Following this failure, $R_1 = P_1 + Q_1$ units are withdrawn from the remaining $N - 1$ surviving units. Likewise, if the next failure occurs at time U_2, then $R_2 = P_2 + Q_2$ units are removed from the remaining $N - R_1 - 2$ units, and the procedure continues sequentially. This process continues until either the mth failure occurs or the time point T is reached. If the m th failure occurs at a time U_m ($U_m < T$), the experiment terminates at U_m, and all remaining units are withdrawn. However, if only $K(K < m)$ failures occur before time T, i.e.,$U_K < T < U_{K+1}$, the experiment continues without additional removals until the m^{th} failure takes place. At time U_m, the experiment ends by removing all the remaining units, where the number of units removed is $R_m = N - m - \Sigma_{l=1}^{m-1} R_l$ units. Define $K^* = max\{K : U_K < T\}$, consequently $R_{K+1} = R_{K+2} = ... = R_{m-1} = 0$, then, our progressive censoring scheme becomes $\{R_1, R_2, ...R_{K^*}, 0, 0, ..., N - m - \Sigma_{l=1}^{m-1} R_l\}$.

Let ρ_l be an indicator variable such that $\rho_l = 1$ if the lth failure occurs from sample 1, and $\rho_l = 0$ if it is from sample 2. Thus, the joint adaptive progressive Type II censored sample observed from the experiment is:
$\{U_1, \rho_1), (U_2, \rho_2), ..., (U_K, \rho_K), (U_{K+1}, \rho_{K+1}), ..., (U_m, \rho_m)\}$. Then, the likelihood function of the joint adaptive progressive Type II censored sample is given by

$$L(U,\rho) = A_K \left[\prod_{l=1}^{m} [g_1(u_l)]^{\rho_l} [g_2(u_l)]^{1-\rho_l} \right] \left[\prod_{l=1}^{K} [\bar{G_1}(u_l)]^{P_l} [\bar{G_2}(u_l)]^{Q_l} \right] [G_1(\bar{u}_m)]^{P*} [G_2(\bar{u}_m)]^{Q*}, \quad (3)$$

where

$$A_K = \prod_{l=1}^{m} \left[N - l - 1 - \sum_{K=1}^{min[l-1,K]} R_K \right],$$

$$P^* = \begin{cases} \sum_{l=K+1}^{m} P_l & for \quad K < m \\ 0 & otherwise \end{cases}, \ and \quad Q^* = \begin{cases} \sum_{l=K+1}^{K} Q_l & for \quad K < m \\ 0 & otherwise. \end{cases}$$

3. Statistical inference for DUS-EV distributions

3.1. Maximum likelihood estimation

Suppose two independent and identically distributed samples $(Y_1, Y_2, ...Y_{N_1})$ and $(Z_1, Z_2, ...Z_{N_2})$ of sizes N_1 and N_2 are taken from DUS-EV (λ_1, α_1) and DUS-EV (λ_2, α_2) populations, respectively. Let g_1 and g_2 denote the corresponding probability density functions (PDFs) and G_1 and G_2

denote the cumulative distribution functions (CDFs) of these populations. The PDFs and CDFs are given by:

$$g_1(y) = \frac{1}{\alpha_1(e-1)} e^{\left(\frac{y-\lambda_1}{\alpha_1} - e^{\frac{y-\lambda_1}{\alpha_1}}\right)} e^{\left(1-e^{-e^{\left(\frac{y-\lambda_1}{\alpha_1}\right)}}\right)}; \quad -\infty < y < \infty, \quad -\infty < \lambda_1 < \infty, \quad \alpha_1 > 0, \tag{4}$$

$$g_2(z) = \frac{1}{\alpha_2(e-1)} e^{\left(\frac{z-\lambda_2}{\alpha_2} - e^{\frac{z-\lambda_2}{\alpha_2}}\right)} e^{\left(1-e^{-e^{\left(\frac{z-\lambda_2}{\alpha_2}\right)}}\right)}; \quad -\infty < z < \infty, \quad -\infty < \lambda_2 < \infty, \quad \alpha_2 > 0, \tag{5}$$

$$G_1(y) = \frac{1}{e-1}\left(e^{\left(1-e^{-e^{\left(\frac{y-\lambda_1}{\alpha_1}\right)}}\right)} - 1\right); \quad -\infty < y < \infty, \quad -\infty < \lambda_1 < \infty, \quad \alpha_1 > 0, \tag{6}$$

and

$$G_2(z) = \frac{1}{e-1}\left(e^{\left(1-e^{-e^{\left(\frac{z-\lambda_2}{\alpha_2}\right)}}\right)} - 1\right); \quad -\infty < z < \infty, \quad -\infty < \lambda_2 < \infty, \quad \alpha_2 > 0. \tag{7}$$

The joint likelihood function of the parameters is expressed as

$$\begin{aligned} L(\Psi|U,\rho) = A_K \prod_{l=1}^{m} & \left(\frac{1}{\alpha_1(e-1)} e^{\left(\frac{u_l-\lambda_1}{\alpha_1} - e^{\frac{u_l-\lambda_1}{\alpha_1}}\right)} e^{\left(1-e^{-e^{\left(\frac{u_l-\lambda_1}{\alpha_1}\right)}}\right)}\right)^{\rho_l} \\ & \times \left(\frac{1}{\alpha_2(e-1)} e^{\left(\frac{u_l-\lambda_2}{\alpha_2} - e^{\frac{u_l-\lambda_2}{\alpha_2}}\right)} e^{\left(1-e^{-e^{\left(\frac{u_l-\lambda_2}{\alpha_2}\right)}}\right)}\right)^{1-\rho_l} \\ & \times \prod_{l=1}^{K} \left(\frac{e}{e-1}\left(1-e^{-e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}}\right)\right)^{P_l} \left(\frac{e}{e-1}\left(1-e^{-e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}}\right)\right)^{Q_l} \\ & \times \left(\frac{e}{e-1}\left(1-e^{-e^{-e^{\frac{u_m-\lambda_1}{\alpha_1}}}}\right)\right)^{P^*} \left(\frac{e}{e-1}\left(1-e^{-e^{-e^{\frac{u_m-\lambda_2}{\alpha_2}}}}\right)\right)^{Q^*}, \end{aligned} \tag{8}$$

where $\Psi = (\lambda_1, \alpha_1, \lambda_2, \alpha_2)$ represents the vector of model parameters.

The log likelihood function is given by

$$\begin{aligned} logL(\Psi|U,\rho) = log(A_K) &+ \sum_{l=1}^{m} \rho_l (1 - log(\alpha_1(e-1))) + \sum_{l=1}^{m} \rho_l \left(\left(\frac{u_l-\lambda_1}{\alpha_1}\right) - e^{\frac{u_l-\lambda_1}{\alpha_1}} - e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}} \right) \\ &+ \sum_{l=1}^{m} (1-\rho_l)(1 - log(\alpha_2(e-1))) + \sum_{l=1}^{m} (1-\rho_l) \left(\left(\frac{u_l-\lambda_2}{\alpha_2}\right) - e^{\frac{u_l-\lambda_2}{\alpha_2}} - e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}} \right) \\ &+ \sum_{l=1}^{K} P_l log\left(\frac{e}{e-1}\right) + \sum_{l=1}^{K} P_l log\left(1 - e^{-e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}}\right) \\ &+ \sum_{l=1}^{K} Q_l log\left(\frac{e}{e-1}\right) + \sum_{l=1}^{K} Q_l log\left(1 - e^{-e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}}\right) \\ &+ P^* log\left(\frac{e}{e-1}\right) + P^* log\left(1 - e^{-e^{-e^{\frac{u_m-\lambda_1}{\alpha_1}}}}\right) \\ &+ Q^* log\left(\frac{e}{e-1}\right) + Q^* log\left(1 - e^{-e^{-e^{\frac{u_m-\lambda_2}{\alpha_2}}}}\right). \end{aligned} \tag{9}$$

To determine the maximum likelihood estimates, we solve the system of equations given as follows:

$$\frac{\partial logL}{\partial\lambda_1} = \frac{-C_1}{\alpha_1} + \sum_{l=1}^{m} \frac{\rho_l}{\alpha_1} e^{\frac{u_l-\lambda_1}{\alpha_1}} (1 - e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}) + \sum_{l=1}^{K} \frac{P_l}{\alpha_1} \frac{e^{(\frac{u_l-\lambda_1}{\alpha_1} - e^{\frac{u_l-\lambda_1}{\alpha_1}})}}{(e^{e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}} - 1)} + \frac{P^*}{\alpha_1} \frac{e^{(\frac{u_m-\lambda_1}{\alpha_1} - e^{\frac{u_m-\lambda_1}{\alpha_1}})}}{(e^{e^{-e^{\frac{u_m-\lambda_1}{\alpha_1}}}} - 1)}, \tag{10}$$

$$\frac{\partial logL}{\partial\lambda_2} = \frac{-C_2}{\alpha_2} + \sum_{l=1}^{m} \frac{1-\rho_l}{\alpha_2} e^{\frac{u_l-\lambda_2}{\alpha_2}} (1 - e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}) + \sum_{l=1}^{K} \frac{Q_l}{\alpha_2} \frac{e^{(\frac{u_l-\lambda_2}{\alpha_2} - e^{\frac{u_l-\lambda_2}{\alpha_2}})}}{(e^{e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}} - 1)} + \frac{Q^*}{\alpha_2} \frac{e^{(\frac{u_m-\lambda_2}{\alpha_2} - e^{\frac{u_m-\lambda_2}{\alpha_2}})}}{(e^{e^{-e^{\frac{u_m-\lambda_2}{\alpha_2}}}} - 1)}, \tag{11}$$

$$\frac{\partial logL}{\partial\alpha_1} = \frac{-C_1}{\alpha_1} + \sum_{l=1}^{m} \rho_i \left(\frac{u_l-\lambda_1}{\alpha_1^2}\right) \left(e^{\frac{u_l-\lambda_1}{\alpha_1}} - e^{(\frac{u_l-\lambda_1}{\alpha_1} - e^{\frac{u_l-\lambda_1}{\alpha_1}})} - 1\right) + \sum_{l=1}^{K} P_l \left(\frac{u_l-\lambda_1}{\alpha_1^2}\right) \frac{e^{(\frac{u_l-\lambda_1}{\alpha_1} - e^{\frac{u_l-\lambda_1}{\alpha_1}})}}{(e^{e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}} - 1)} + P^* \left(\frac{u_m-\lambda_1}{\alpha_1^2}\right) \frac{e^{(\frac{u_m-\lambda_1}{\alpha_1} - e^{\frac{u_m-\lambda_1}{\alpha_1}})}}{(e^{e^{-e^{\frac{u_m-\lambda_1}{\alpha_1}}}} - 1)}, \tag{12}$$

and

$$\frac{\partial logL}{\partial\alpha_2} = \frac{-C_2}{\alpha_2} + \sum_{l=1}^{m} (1-\rho_l) \left(\frac{u_l-\lambda_2}{\alpha_2^2}\right) \left(e^{\frac{u_l-\lambda_2}{\alpha_2}} - e^{(\frac{u_l-\lambda_2}{\alpha_2} - e^{\frac{u_l-\lambda_2}{\alpha_2}})} - 1\right) + \sum_{l=1}^{K} Q_l \left(\frac{u_l-\lambda_2}{\alpha_2^2}\right) \frac{e^{(\frac{u_l-\lambda_2}{\alpha_2} - e^{\frac{u_l-\lambda_2}{\alpha_2}})}}{(e^{e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}} - 1)} + Q^* \left(\frac{u_m-\lambda_2}{\alpha_2^2}\right) \frac{e^{(\frac{u_m-\lambda_2}{\alpha_2} - e^{\frac{u_m-\lambda_2}{\alpha_2}})}}{(e^{e^{-e^{\frac{u_m-\lambda_2}{\alpha_2}}}} - 1)}, \tag{13}$$

where $C_1 = \sum_{l=1}^{m} \rho_l$ and $C_2 = \sum_{l=1}^{m}(1-\rho_l)$. The equations (10) to (13) do not admit closed-form analytical solutions; however, they can be effectively solved using numerical methods available in any statistical software.

3.2. Asymptotic confidence interval

This section develops asymptotic confidence intervals for the parameters $\lambda_1, \alpha_1, \lambda_2$, and α_2 using the asymptotic behavior of the maximum likelihood estimators. To this end, we employ the observed information matrix $I(\Psi)$ defined as,

$$I(\Psi) = -\begin{bmatrix} \left(\frac{\partial^2 \log L}{\partial\lambda_1^2}\right) & \left(\frac{\partial^2 \log L}{\partial\lambda_1\partial\alpha_1}\right) & \left(\frac{\partial^2 \log L}{\partial\lambda_1\partial\lambda_2}\right) & \left(\frac{\partial^2 \log L}{\partial\lambda_1\partial\alpha_2}\right) \\ \left(\frac{\partial^2 \log L}{\partial\alpha_1\partial\lambda_1}\right) & \left(\frac{\partial^2 \log L}{\partial\alpha_1^2}\right) & \left(\frac{\partial^2 \log L}{\partial\alpha_1\partial\lambda_2}\right) & \left(\frac{\partial^2 \log L}{\partial\alpha_1\partial\alpha_2}\right) \\ \left(\frac{\partial^2 \log L}{\partial\lambda_2\partial\lambda_1}\right) & \left(\frac{\partial^2 \log L}{\partial\lambda_2\partial\alpha_1}\right) & \left(\frac{\partial^2 \log L}{\partial\lambda_2^2}\right) & \left(\frac{\partial^2 \log L}{\partial\lambda_2\partial\alpha_2}\right) \\ \left(\frac{\partial^2 \log L}{\partial\alpha_2\partial\lambda_1}\right) & \left(\frac{\partial^2 \log L}{\partial\alpha_2\partial\alpha_1}\right) & \left(\frac{\partial^2 \log L}{\partial\alpha_2\partial\lambda_2}\right) & \left(\frac{\partial^2 \log L}{\partial\alpha_2^2}\right) \end{bmatrix},$$

Asymptotic results imply that $\hat{\Psi} = (\hat{\lambda}1, \hat{\alpha}1, \hat{\lambda}2, \hat{\alpha}2)$ has an asymptotic multivariate normal distribution with mean $\hat{\Psi}$ and covariance matrix $I^{-}(\hat{\Psi})$, where variances of the MLEs appear on the diagonal and covariances on the off-diagonal. Then, $100(1-\delta)\%$ confidence intervals for λ_1, α_1, λ_2, and α_2 are obtained as

$$\left(\hat{\lambda_1} - z_{\delta/2}\sqrt{V(\hat{\lambda_1})}, \hat{\lambda_1} + z_{\delta/2}\sqrt{V(\hat{\lambda_1})}\right),$$
$$\left(\hat{\alpha_1} - z_{\delta/2}\sqrt{V(\hat{\alpha_1})}, \hat{\alpha_1} + z_{\delta/2}\sqrt{V(\hat{\alpha_1})}\right),$$
$$\left(\hat{\lambda_2} - z_{\delta/2}\sqrt{V(\hat{\lambda_2})}, \hat{\lambda_2} + z_{\delta/2}\sqrt{V(\hat{\lambda_2})}\right),$$

and

$$\left(\hat{\alpha_2} - z_{\delta/2}\sqrt{V(\hat{\alpha_2})}, \hat{\alpha_2} + z_{\delta/2}\sqrt{V(\hat{\alpha_2})}\right).$$

3.3. Bootstrap-Confidence Interval

In this section, confidence intervals are constructed using the bootstrap method, wherein the data are repeatedly resampled to generate a large number of pseudo-samples. These resampled datasets are then used to estimate the parameters. Among the various methods available for confidence interval construction, we employ the percentile bootstrap (boot-p) method. The algorithm for constructing boot-p confidence interval, is outlined as follows [12].

Step 1: Generate two samples of sizes N_1 and N_2 from DUS-EV(λ_1, α_1) and DUS-EV(λ_2, α_2) respectively.

Step 2: Generate joint adaptive Type II censored sample $\{(U_1, \rho_1), (U_2, \rho_2), ..., (U_K, \rho_K), (U_{K+1}, \rho_{K+1}), ..., (U_m, \rho_m)\}$.

Step 3: Compute the parameter estimates $\hat{\lambda}_1$, $\hat{\alpha}_1$, $\hat{\lambda}_2$, $\hat{\alpha}_2$ using the method of maximum likelihood.

Step 4: Generate bootstrap samples of sizes N_1 and N_2 from DUS-EV$(\hat{\lambda}_1, \hat{\alpha}_1)$ and DUS-EV$(\hat{\lambda}_2, \hat{\alpha}_2)$ respectively.

Step 5: Generate bootstrap joint adaptive type II censored sample $\{(U_1^*, \rho_1^*), (U_2^*, \rho_2^*), ..., (U_K^*, \rho_K^*), (U_{K+1}^*, \rho_{K+1}^*), ..., (U_m^*, \rho_m^*)\}$.

Step 6: Based on the bootstrap censored samples generated in step 5, calculate estimates of parameters, which are denoted by $\hat{\lambda}_1^*$,$\hat{\alpha}_1^*$, $\hat{\lambda}_2^*$,$\hat{\alpha}_2^*$.

Step 7: Repeating steps 3"6 for B iterations yields $\hat{\lambda}_{1k}^*$,$\hat{\alpha}_{1k}^*$, $\hat{\lambda}_{2k}^*$,$\hat{\alpha}_{2k}^*$, $\quad k = 1, 2, 3...B$.

Step 8: Arrange the bootstrap sample estimates in ascending order.

Step 9: Let $G(a) = P(\hat{\Psi}_k^* \leq a)$ be the CDF of $\hat{\Psi}_k^*$. The Approximate $100(1-\delta)$ percentile bootstrap confidence interval for Ψ_k is given by $(\hat{\Psi}_k^*(\frac{\delta}{2}), \hat{\Psi}_k^*(1-\frac{\delta}{2}))$, where $k = 1, 2, ..., B$.

3.4. Bayesian Estimation

In this section, we obtain the Bayesian estimates of the unknown parameters λ_1, α_1, λ_2, α_2, along with their credible intervals, based on joint adaptive progressive Type II censored data. We assume independent normal and gamma priors for the model parameters [18]. Specifically, the parameters λ_1 and λ_2 are assumed to follow independent normal priors, $\lambda_1 \sim N(\mu_1, \sigma_1^2)$ and $\lambda_2 \sim N(\mu_2, \sigma_2^2)$, while the parameters α_1 and α_2 are assumed to follow independent gamma priors, $\alpha_1 \sim G(m_1, p_1)$ and $\alpha_2 \sim G(m_2, p_2)$. The hyperparameters μ_1, σ_1, μ_2, and σ_2 are assumed to be non negative. Thus, the joint prior distribution is given in (14).

The posterior density function can be derived using the likelihood function of the censored samples with the prior distributions and is given in (15).

$$\pi^*(\lambda_1, \alpha_1, \lambda_2, \alpha_2) \propto e^{-\frac{1}{2}\left[(\frac{\lambda_1-\mu_1}{\sigma_1})^2 + (\frac{\lambda_2-\mu_2}{\sigma_2})^2\right]} e^{-(m_1\alpha_1 + m_2\alpha_2)} \alpha_1^{p_1-1} \alpha_2^{p_2-1} \quad \lambda_1, \lambda_2 \in R, \alpha_1, \alpha_2 > 0. \quad (14)$$

If $I_1 = -\frac{1}{2}\left(\frac{\lambda_1-\mu_1}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{\lambda_2-\mu_2}{\sigma_2}\right)^2 - m_1\alpha_1 - m_2\alpha_2$

$I_2 = \sum_{l=1}^{m}\rho_l(1-\log(\alpha_1(e-1))) + \sum_{l=1}^{m}\rho_l\left((\frac{u_l-\lambda_1}{\alpha_1}) - e^{\frac{u_l-\lambda_1}{\alpha_1}} - e^{-e^{(\frac{u_l-\lambda_1}{\alpha_1})}}\right)$

$I_3 = \sum_{l=1}^{m}(1-\rho_l)(1-\log(\alpha_2(e-1))) + \sum_{l=1}^{m}(1-\rho_l)\left((\frac{u_l-\lambda_2}{\alpha_2}) - e^{\frac{u_l-\lambda_2}{\alpha_2}} - e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}\right)$

$I_4 = \sum_{l=1}^{K} P_l \log\left(\frac{e}{e-1}\right) + \sum_{l=1}^{K} P_l \log\left(1-e^{-e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}}\right)$

$I_5 = \sum_{l=1}^{K} Q_l \log\left(\frac{e}{e-1}\right) + \sum_{l=1}^{K} Q_l \log\left(1-e^{-e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}}\right)$

$I_6 = P^* \log\left(\frac{e}{e-1}\right) + P^* \log\left(1-e^{-e^{-e^{\frac{u_m-\lambda_1}{\alpha_1}}}}\right)$

$I_7 = Q^* \log\left(\frac{e}{e-1}\right) + Q^* \log\left(1-e^{-e^{-e^{\frac{u_m-\lambda_2}{\alpha_2}}}}\right)$

$$P^*(\lambda_1,\alpha_1,\lambda_2,\alpha_2) \propto \alpha_1^{p_1-1}\alpha_2^{p_2-1} exp\Big[I_1+I_2+I_3+I_4+I_5+I_6+I_7\Big]. \tag{15}$$

The conditional posterior distributions derived from (15) are given as follows:

$$\begin{aligned} P_1^*(\lambda_1|\lambda_2,\alpha_1,\alpha_2) \propto &\exp\Bigg[-\frac{1}{2}\left(\frac{\lambda_1-\mu_1}{\sigma_1}\right)^2 + \sum_{l=1}^{m}\rho_l\left((\frac{u_l-\lambda_1}{\alpha_1}) - e^{\frac{u_l-\lambda_1}{\alpha_1}} - e^{-e^{(\frac{u_l-\lambda_1}{\alpha_1})}}\right) \\ &+ \sum_{l=1}^{K} P_l log(1-e^{-e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}}) + P^*\log(1-e^{-e^{-e^{\frac{u_m-\lambda_1}{\alpha_1}}}})\Bigg]. \end{aligned} \tag{16}$$

$$\begin{aligned} P_2^*(\alpha_1|\lambda_1,\lambda_2,\alpha_2) \propto &\alpha_1^{p_1-1-C_1}\exp\Bigg[-m_1\alpha_1 + \sum_{l=1}^{m}\rho_l\left((\frac{u_l-\lambda_1}{\alpha_1}) - e^{\frac{u_l-\lambda_1}{\alpha_1}} - e^{-e^{(\frac{u_l-\lambda_1}{\alpha_1})}}\right) \\ &+ \sum_{l=1}^{K} P_l log(1-e^{-e^{-e^{\frac{u_l-\lambda_1}{\alpha_1}}}}) + P^*\log(1-e^{-e^{-e^{\frac{u_m-\lambda_1}{\alpha_1}}}})\Bigg], \end{aligned} \tag{17}$$

$$\begin{aligned} P_3^*(\lambda_2|\lambda_1,\alpha_1,\alpha_2) \propto &\exp\Bigg[-\frac{1}{2}\left(\frac{\lambda_2-\mu_2}{\sigma_2}\right)^2 + \sum_{l=1}^{m}\rho_l\left((\frac{u_l-\lambda_2}{\alpha_2}) - e^{\frac{u_l-\lambda_2}{\alpha_2}} - e^{-e^{(\frac{u_l-\lambda_2}{\alpha_2})}}\right) \\ &+ \sum_{l=1}^{K} P_l \log(1-e^{-e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}}) + P^*\log(1-e^{-e^{-e^{\frac{u_m-\lambda_2}{\alpha_2}}}})\Bigg], \end{aligned} \tag{18}$$

and

$$\begin{aligned} P_4^*(\alpha_2|\lambda_1,\lambda_2,\alpha_1) \propto &\alpha_2^{p_2-1-C_2}\exp\Bigg[-m_2\alpha_2 + \sum_{l=1}^{m}\rho_i\left((\frac{u_l-\lambda_2}{\alpha_2}) - e^{\frac{u_l-\lambda_2}{\alpha_2}} - e^{-e^{(\frac{u_l-\lambda_2}{\alpha_2})}}\right) \\ &+ \sum_{l=1}^{K} P_l log(1-e^{-e^{-e^{\frac{u_l-\lambda_2}{\alpha_2}}}}) + P^*\log(1-e^{-e^{-e^{\frac{u_m-\lambda_2}{\alpha_2}}}})\Bigg]. \end{aligned} \tag{19}$$

As demonstrated in equations (16) to (19), the full conditional posterior distributions of the model parameters are not in closed-form expressions. Consequently, direct sampling from these distributions is not feasible. In this case, we employ Metropolis"Hastings sampling within the Gibbs algorithm using a normal proposal distribution. Posterior samples of the model parameters are subsequently generated by iteratively performing the following steps.

Step 1: Initialize the iteration step to $k = 1$ and initialise the parameters $\Psi^{(0)} = (\lambda_1^{(0)}, \alpha_1^{(0)}, \lambda_2^{(0)}, \alpha_2^{(0)})$.

Step 2: Generate $\lambda_1^{(k)}$, $\alpha_1^{(k)}$, $\lambda_2^{(k)}$, $\alpha_2^{(k)}$ from $P_1^*(\lambda_1|\lambda_2,\alpha_1,\alpha_2)$, $P_2^*(\alpha_1|\lambda_1,\lambda_2,\alpha_2)$, $P_3^*(\lambda_2|\lambda_1,\alpha_1,\alpha_2)$, and $P_4^*(\alpha_2|\lambda_1,\lambda_2,\alpha_1)$ respectively by using the Metropolis"Hastings algorithm described in [18] with a normal proposal.

Step 3: Set $k = k + 1$.
Step 4: Repeat the steps 2-3 J times.
Thus, the approximate Bayesian estimators of the parameters λ_1, α_1, λ_2, and, α_2 under squared error loss function is given by

$$\begin{aligned} \hat{\lambda}_{1_{Bayes}} &= \frac{1}{J-B}\sum_{k=B+1}^{J}\lambda_1^{(k)}, \quad \hat{\lambda}_{2_{Bayes}} = \frac{1}{J-B}\sum_{k=B+1}^{J}\lambda_2^{(k)}, \\ \hat{\alpha}_{1_{Bayes}} &= \frac{1}{J-B}\sum_{k=B+1}^{J}\alpha_1^{(k)}, \text{and}, \hat{\alpha}_{2_{Bayes}} = \frac{1}{J-B}\sum_{k=B+1}^{J}\alpha_2^{(k)} \end{aligned} \tag{20}$$

where B denotes the burn-in period. We can also construct HPD credible intervals [8]. Then, $100(1-\delta)$ HPD interval is given by

$$\left(\hat{\Psi}_{Bayes[\frac{\delta}{2}(J-B)]}, \hat{\Psi}_{Bayes[(1-\frac{\delta}{2})(J-B)]}\right) \tag{21}$$

where $\hat{\Psi}_{Bayes} = (\hat{\lambda}_{1_{Bayes}}, \hat{\alpha}_{1_{Bayes}}, \hat{\lambda}_{2_{Bayes}}, \hat{\alpha}_{2_{Bayes}})$, $\hat{\Psi}_{Bayes[\frac{\delta}{2}(J-B)]}$ and $\hat{\Psi}_{Bayes[(1-\frac{\delta}{2})(J-B)]}$ are the $[(\frac{\delta}{2}(J-B)]$ th smallest and $[(1-\frac{\delta}{2})(J-B)]$ th smallest of MCMC samples $\psi^{(k)}$ generated from posterior densities.

4. Simulation study

This section presents a simulation study aimed at assessing the performance of parameter estimation methods, particularly maximum likelihood estimation (MLE) and Bayesian techniques. The simulation is performed under various sample size configurations: (50, 40, 70), (60, 50, 88), and (70, 60, 106), where the first two values denote the sample sizes of the two groups and the third corresponds to the total number of observed failures. We take the initial parameter values as $(\lambda_1, \alpha_1, \lambda_2, \alpha_2)$ = (0.8, 1.2, 0.9, 1.5) and fixed experimental times $T = 1.2$ and $T = 1.5$. Point estimates are compared using bias and MSEs, as presented in Table: 2. Asymptotic confidence intervals, bootstrap confidence intervals, and highest posterior density (HPD) intervals are computed. These interval estimates are evaluated based on average confidence lengths (ACL) and coverage probabilities (CP), which are outlined in Table: 3.

For Bayesian estimation, both informative and non-informative priors are considered. In the non-informative setup, the hyperparameters for normal priors are chosen to be μ_1 = μ_2 = 0 and σ_1 = σ_2 = 1, while for the gamma priors, m_1 = m_2 = p_1 = p_2 = 0.001, a small positive value to give a minimal influence on the parameters. In the informative case, the hyperparameters are μ_1 = 0.8, μ_2 = 0.9, σ_1 = σ_2 = 0.1, m_1 = 4.375, m_2 = 3.5, p_1 = 5.25, and p_2 = 5.25. The simulation procedure is replicated 1000 times. The complete censoring schemes employed in the study are listed below in Table:1.

The notation $(0_3, 2_2)$ represents a censoring scheme in which the value 0 is repeated three times and the value 2 is repeated twice. The steps involved in generating a joint adaptive progressive Type II censored sample are described as follows.
1. Generate two samples of sizes N_1 and N_2 from DUS-EV(λ_1, α_1) and DUS-EV(λ_2, α_2).
2. Randomly select a type II progressively censored sample of size m [6]
3. Select the sample $\{(U_1, \rho_1), (U_2, \rho_2), ..., (U_K, \rho_K)\}$ based on the time T.
4. If $K < m$, generate truncated samples of size m-K from the DUS-EV distribution.
5. Update the joint adaptive progressive Type II censored sample
$\{(U_1, \rho_1), (U_2, \rho_2), ..., (U_K, \rho_K), (U_{K+1}, \rho_{K+1}), ..., (U_m, \rho_m)\}$
The following are the main conclusions drawn from the simulation study:

- An increase in sample size leads to a reduction in the absolute bias and mean squared error (MSE) of the MLEs as well as the Bayesian estimators obtained under both informative and non-informative priors.

Table 1: *Censoring Schemes*

2*Sl.No	Sample Size (n_1, n_2, m)	Censoring Scheme R
1	$(50, 40, 70)$	$(0_{60}, 2_{10})$
2	„	$(0_{69}, 20)$
3	„	$(1_5, 0_{50}, 1_{15})$
4	$(60, 50, 88)$	$(0_{77}, 2_{11})$
5	„	$(0_{87}, 22)$
6	„	$(1_5, 0_{66}, 1_{17}, 0_5)$
7	$(70, 60, 106)$	$(0_{94}, 2_{12})$
8	„	$(0_{105}, 24)$
9	„	$(1_5, 0_{82}, 1_{19})$

- In the majority of scenarios, Bayesian estimation with informative priors results in reduced MSE relative to non-informative priors and maximum likelihood estimates.
- Average confidence length is shorter for HPD intervals with an informative prior as compared to all other confidence intervals.
- Most of the time, coverage probabilities converge to the nominal level as the sample size increases.

5. Data analysis

This section presents a real data analysis to demonstrate the effectiveness and practical relevance of the proposed approach. For this purpose, we consider two data sets originally reported [7], which consist of observed times between consecutive telephone calls received at a company's switchboard. Specifically, the first data set contains 41 time intervals measured in seconds, while the second comprises 48 time intervals recorded in minutes. The first data set has previously been analyzed in the context of modeling extreme values and was utilized to fit the DUS-EV distribution [18]. Now we utilize the same data to examine the effectiveness of the proposed estimation techniques within the framework of joint adaptive progressive Type II censoring. To evaluate the goodness of fit, we apply three statistical tests: the Kolmogorov-Smirnov (KS) test, the Anderson-Darling (AD) test, and the Cramer-Von Mises (CVM) test. The results are summarized in Table:4. We generate a joint adaptive Type II censored sample from the joint distribution of time intervals of two sets of data for $m = 63$, $T = 0.8$ and $R = (0_{53}, 2_{10})$ and performed estimation through maximum likelihood and Bayesian techniques. The results are provided in the Table:5.

6. Conclusions

This article investigates statistical inference for the recently proposed DUS-EV(λ, α) distribution under the joint adaptive progressive Type II censoring scheme. The DUS-EV distribution is characterized by an increasing failure rate, making it particularly suitable for analyzing aging systems and extreme value phenomena. To enhance the practical utility of this model, the JAPT-II censoring framework is adopted, which reduces experimental time while ensuring a predetermined number of failures, thereby improving the efficiency and reliability of inference procedures.

Parameter estimation is explored through classical, Bayesian, and bootstrap approaches. In the classical framework, maximum likelihood estimators and their asymptotic confidence intervals are derived. Bayesian inference is carried out using the Metropolis - Hastings algorithm within

Gibbs sampling to obtain posterior samples, along with highest posterior density (HPD) intervals. Complementarily, bootstrap methods are employed to construct alternative confidence intervals and to assess the robustness of inference.

The performance of these procedures is evaluated through simulation studies conducted under different sample size settings. The results indicate that Bayesian estimates with informative priors generally exhibit lower mean squared errors, while bootstrap intervals demonstrate competitive performance. A real-data analysis further validates the applicability of the proposed methods in practical reliability studies.

In conclusion, the study highlights the flexibility and robustness of the DUS-EV(λ, α) distribution under JAPT-II censoring and demonstrates the effectiveness of integrating classical, Bayesian, and resampling-based inference procedures in reliability analysis.

Table 2: *Biases and MSEs for* $\lambda_1 = 0.8$, $\beta_1 = 1.2$, $\lambda_2 = 1$, $\beta_2 = 1.5$

4*CS	4*T	MLE				Bayesian				Bayesian (non-informative)			
		λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2
		Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)
1	1.2	0.0509	0.1040	0.0776	0.0951	0.0422	0.2092	0.0396	0.3284	0.1277	0.2352	0.2022	0.3877
		0.0340	0.0458	0.0594	0.0749	0.0035	0.0770	0.0028	0.1697	0.0397	0.0994	0.0780	0.2420
	1.5	0.0566	0.1078	0.0743	0.1282	0.0498	0.2086	0.0588	0.3622	0.1415	0.2450	0.2221	0.4051
		0.0360	0.0520	0.0726	0.0815	0.0047	0.0785	0.0052	0.2001	0.0538	0.1048	0.1059	0.2647
2	1.2	0.0541	0.1159	0.0912	0.1032	0.0438	0.2361	0.0438	0.3594	0.1380	0.2465	0.2052	0.3970
		0.0313	0.0513	0.0639	0.0811	0.0036	0.0978	0.0029	0.1997	0.0440	0.1083	0.0741	0.2530
	1.5	0.1050	0.1847	0.1576	0.2113	0.0551	0.2908	0.0699	0.4663	0.2193	0.3049	0.3079	0.4945
		0.0374	0.0847	0.0847	0.1306	0.0073	0.1702	0.0064	0.3755	0.1103	0.1712	0.1901	0.4051
3	1.2	-0.0140	-0.0598	-0.1014	-0.1300	0.0209	0.1016	0.0226	0.1680	0.0555	0.1076	0.0830	0.2093
		0.0278	0.0306	0.0590	0.0692	0.0021	0.0388	0.0016	0.0823	0.0241	0.0444	0.0404	0.1214
	1.5	0.0302	-0.0304	-0.0865	-0.0597	0.0246	0.1143	0.0355	0.2105	0.0726	0.1317	0.1244	0.2605
		0.0322	0.0277	0.0641	0.0572	0.0029	0.0396	0.0021	0.1051	0.0299	0.0318	0.0464	0.0733
4	1.2	0.0345	0.0547	0.0328	0.0432	0.0361	0.1475	0.0392	0.2567	0.0961	0.1595	0.1569	0.2932
		0.0246	0.0290	0.0427	0.0533	0.0029	0.0483	0.0028	0.1111	0.0268	0.0565	0.0499	0.1493
	1.5	0.0453	0.1000	0.0661	0.1105	0.0441	0.1906	0.0499	0.3239	0.1277	0.1964	0.2310	0.3685
		0.0305	0.0364	0.0540	0.0657	0.0045	0.0643	0.0042	0.3240	0.0450	0.0707	0.0948	0.2040
5	1.2	0.0261	0.0464	0.0534	0.0351	0.0341	0.1419	0.0417	0.2679	0.0852	0.1512	0.1760	0.2983
		0.0252	0.0288	0.0446	0.0501	0.0029	0.0472	0.0029	0.1247	0.0249	0.0559	0.0547	0.1548
	1.5	0.0771	0.1620	0.1178	0.1948	0.0582	0.2177	0.0664	0.3983	0.1916	0.2424	0.2930	0.4268
		0.0298	0.0629	0.0618	0.1084	0.0063	0.0894	0.0064	0.2421	0.0692	0.1079	0.1353	0.2769

Table 2: *Continued*

4*CS	4*T	MLE				Bayesian				Bayesian (non-informative)			
		λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2
		Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)	Bias (MSE)
6	1.2	-0.0072	-0.0586	-0.0850	-0.1104	0.0184	0.0424	0.0204	0.1177	0.0286	0.0489	0.0721	0.1455
		0.0222	0.0255	0.0447	0.0549	0.0021	0.0229	0.0018	0.0615	0.0169	0.0277	0.0302	0.0806
	1.5	0.0262	-0.0250	-0.0692	-0.0623	0.0272	0.0802	0.0330	0.1748	0.0752	0.0908	0.1169	0.2014
		0.0269	0.0208	0.0504	0.0477	0.0030	0.0263	0.0026	0.0782	0.0283	0.0354	0.0473	0.1000
7	1.2	0.0173	0.0171	0.0016	0.0140	0.0254	0.0808	0.0376	0.2066	0.0617	0.0909	0.1295	0.2159
		0.0201	0.0230	0.0324	0.0411	0.0026	0.0266	0.0025	0.0841	0.0191	0.0307	0.0381	0.0974
	1.5	0.0364	0.0885	0.0547	0.1020	0.0404	0.1441	0.0438	0.2890	0.1258	0.1557	0.2105	0.3062
		0.0233	0.0297	0.0461	0.0532	0.0040	0.0438	0.0035	0.1297	0.0361	0.0516	0.0732	0.1513
8	1.2	0.0115	0.0152	0.0027	0.0165	0.0254	0.0795	0.0376	0.2022	0.0639	0.0917	0.1354	0.2282
		0.0191	0.0225	0.0328	0.0418	0.0025	0.0265	0.0026	0.0824	0.0191	0.0335	0.0381	0.1041
	1.5	0.0509	0.1106	0.0860	0.1404	0.0587	0.1800	0.0570	0.3422	0.1567	0.1727	0.2613	0.3531
		0.0223	0.03589	0.0443	0.0672	0.0057	0.0642	0.0061	0.1755	0.0450	0.0626	0.0947	0.1826
9	1.2	0.0008	-0.0499	-0.0876	-0.0918	0.0158	0.0110	0.0186	0.0836	0.0216	0.0167	0.0524	0.0953
		0.0195	0.0206	0.0276	0.0381	0.0020	0.0178	0.0016	0.0451	0.0150	0.0216	0.0235	0.0544
	1.5	0.0081	-0.0196	-0.0669	-0.0387	0.0327	0.0554	0.0247	0.1584	0.0801	0.0539	0.1145	0.1700
		0.0217	0.0180	0.0418	0.0384	0.0036	0.0193	0.0029	0.0650	0.0244	0.0239	0.0416	0.0725

RT&A, No 4 (89)
Volume 20, December 2025

Table 3: *for* $\lambda_1 = 0.8$, $\beta_1 = 1.2$, $\lambda_2 = 1$, $\beta_2 = 1.5$

4*CS	4*T	ACI				Bayesian				Bayesian (non-informative)				Bootstrap-p			
		λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2				
		CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL	CP ACL
1	1.2	0.948	0.944	0.963	0.964	0.996	0.827	0.999	0.834	0.905	0.824	0.918	0.828	0.946	0.876	0.93	0.908
		0.7549	0.7061	1.0465	1.0421	0.3173	0.7088	0.3377	1.0635	0.6653	0.7883	0.8952	1.2506	0.7259	0.7292	0.9921	1.0084
	1.5	0.907	0.889	0.903	0.928	0.988	0.809	0.999	0.790	0.847	0.802	0.844	0.801	0.916	0.884	0.904	0.908
		0.7429	0.6832	1.0391	1.0143	0.3101	0.6887	0.3301	1.0621	0.6527	0.7683	0.8775	1.2266	0.7393	0.7380	1.0319	1.0168
2	1.2	0.953	0.92	0.957	0.956	0.994	0.758	0.999	0.804	0.892	0.810	0.915	0.822	0.916	0.858	0.898	0.904
		0.7611	0.7124	1.0479	1.0455	0.3176	0.7203	0.3372	1.0768	0.6660	0.7904	0.8922	1.2433	0.7120	0.8024	0.9723	1.0742
	1.5	0.763	0.677	0.739	0.742	0.938	0.691	0.979	0.649	0.687	0.710	0.661	0.705	0.838	0.726	0.776	0.832
		0.6667	0.6141	0.9381	0.9252	0.3006	0.7144	0.3179	1.0930	0.6408	0.7863	0.8636	1.2600	0.6897	0.9030	0.9853	1.1639
3	1.2	0.963	0.879	0.953	0.831	1	0.919	1	0.947	0.950	0.942	0.977	0.934	0.956	0.938	0.962	0.906
		0.6826	0.6053	0.9330	0.8642	0.3166	0.6513	0.3358	0.9783	0.6436	0.7114	0.8613	1.1208	0.6697	0.6403	0.8934	0.9200
	1.5	0.936	0.903	0.955	0.907	0.995	0.924	1	0.917	0.930	0.911	0.935	0.923	0.954	0.942	0.952	0.918
		0.6993	0.6095	0.9868	0.9025	0.3186	0.6433	0.3386	0.9998	0.6528	0.7141	0.8799	1.1548	0.7080	0.6381	0.9627	0.9078
4	1.2	0.957	0.947	0.963	0.954	0.996	0.864	0.999	0.881	0.924	0.870	0.921	0.875	0.954	0.93	0.962	0.92
		0.6591	0.6104	0.8992	0.8858	0.3082	0.6138	0.3312	0.9266	0.5877	0.6672	0.7803	1.0401	0.6182	0.6292	0.8252	0.8847
	1.5	0.924	0.937	0.936	0.944	0.989	0.816	0.996	0.771	0.832	0.824	0.781	0.774	0.926	0.88	0.956	0.926
		0.6759	0.6152	0.9354	0.9093	0.3067	0.6263	0.3282	0.9558	0.5925	0.6772	0.7944	1.0810	0.6639	0.6536	0.9015	0.8862
5	1.2	0.959	0.94	0.963	0.959	0.994	0.867	0.999	0.846	0.935	0.877	0.930	0.861	0.958	0.902	0.926	0.93
		0.6550	0.6052	0.8940	0.8845	0.3068	0.6106	0.3312	0.9314	0.5823	0.6609	0.7808	1.0442	0.6172	0.6486	0.8061	0.8916
	1.5	0.891	0.799	0.878	0.861	0.961	0.752	0.983	0.671	0.731	0.745	0.679	0.701	0.906	0.786	0.88	0.818
		0.6772	0.6157	0.9406	0.9169	0.3036	0.6368	0.3243	0.9858	0.5907	0.6954	0.7878	1.1012	0.6300	0.7719	0.8597	1.0194

Table 3: *continued*

4*CS	4*T	ACI				Bayesian				Bayesian (non-informative)				Bootstrap-p			
		λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2	λ_1	β_1	λ_2	β_2				
		CP	CP	CP	CP	CP	CP	CP	CP	CP	CP	CP	CP	CP	CP	CP	CP
		ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL	ACL
6	1.2	0.948	0.892	0.966	0.857	0.999	0.948	1	0.933	0.966	0.943	0.975	0.936	0.976	0.9	0.97	0.874
		0.6121	0.5479	0.8332	0.7842	0.3051	0.5591	0.3288	0.8569	0.5608	0.6027	0.7542	0.9509	0.5848	0.5683	0.7592	0.7914
	1.5	0.931	0.928	0.941	0.904	0.993	0.945	0.999	0.921	0.922	0.924	0.922	0.914	0.966	0.948	0.966	0.918
		0.6295	0.5554	0.8671	0.8039	0.3077	0.5669	0.3281	0.8812	0.5798	0.6165	0.7703	0.9788	0.6311	0.5650	0.8359	0.8030
7	1.2	0.958	0.939	0.964	0.952	0.991	0.918	0.998	0.875	0.941	0.923	0.932	0.880	0.954	0.938	0.954	0.928
		0.5888	0.5411	0.8034	0.7868	0.2975	0.5381	0.3211	0.8270	0.5211	0.5742	0.6899	0.8957	0.5428	0.5547	0.7213	0.8043
	1.5	0.942	0.934	0.947	0.959	0.977	0.845	0.994	0.766	0.834	0.831	0.797	0.793	0.922	0.89	0.95	0.906
		0.6229	0.5622	0.8536	0.8263	0.2990	0.5292	0.3198	0.8674	0.5353	0.5994	0.7096	0.9430	0.6063	0.5877	0.8056	0.8021
8	1.2	0.964	0.929	0.967	0.946	0.994	0.922	0.998	0.890	0.938	0.911	0.935	0.872	0.948	0.914	0.946	0.93
		0.5889	0.5403	0.8036	0.7880	0.2987	0.5359	0.3229	0.8252	0.5207	0.5729	0.6962	0.9013	0.5486	2.5617	0.7183	0.7994
	1.5	0.928	0.893	0.933	0.918	0.962	0.794	0.983	0.689	0.785	0.812	0.723	0.756	0.928	0.814	0.896	0.836
		0.6229	0.5614	0.8583	0.8328	0.3001	0.5760	0.3221	0.8867	0.5368	0.6103	0.7208	0.9705	0.5891	2.6798	0.7786	0.9037
9	1.2	0.945	0.894	0.961	0.871	0.998	0.920	1	0.940	0.959	0.932	0.978	0.940	0.974	0.886	0.952	0.83
		0.5632	0.5084	0.7630	0.7227	0.2945	0.4997	0.3225	0.7712	0.5083	0.5344	0.6712	0.8267	0.5243	0.5186	0.6777	0.7239
	1.5	0.941	0.922	0.951	0.916	0.983	0.958	0.997	0.915	0.893	0.942	0.909	0.925	0.954	0.944	0.958	0.938
		0.5779	0.5132	0.7953	0.7449	0.2982	0.5114	0.3222	0.8000	0.5193	0.5462	0.6924	0.8682	0.5802	0.5276	0.7596	0.7512

Table 4: *Goodness-of-fit test results for the DUS-EV distribution*

2*	2*Estimates	KS test		AD test		CVM test	
		Statistic	p-value	Statistic	p-value	Statistic	p-value
2*Data 1	$\hat{\lambda}_1$=0.9723 $\hat{\alpha}_1$=0.4563	2*0.0827	2*0.9539	2*0.2237	2* 0.9825	2* 0.0351	2* 0.9585
2*Data 2	$\hat{\lambda}_2$=1.6522 $\hat{\alpha}_2$= 0.4293	2* 0.1236	2*0.4554	2*0.54799	2* 0.6976	2* 0.1035	2* 0.5697

Table 5: *Estimates, standard errors (SE), confidence intervals (CI), and confidence lengths (CL) under joint adaptive progressive type II censoring scheme across different estimation methods*

3*Parameters	MLE		Bayesian		Bootstrap-p
	Estimates SE	ACI CL	Estimates SE	HPD CL	CI CL
$\hat{\lambda}_1$	0.9716	(0.8103, 1.1327)	0.8599	(0.7533, 0.9697)	(0.8924, 1.1466)
	0.0822	0.3224	0.0653	0.2164	0.2541
$\hat{\alpha}_1$	0.4694	(0.3405, 0.5984)	0.5458	(0.4328,0.6869)	(0.4447, 0.6217)
	0.0658	0.2579	0.0752	0.2541	0.1770
$\hat{\lambda}_2$	1.5953	(1.4703, 1.7203)	1.1083	(0.9281, 1.2726)	(1.5764, 1.7689)
	0.0638	0.2499	0.0882	0.3445	0.1925
$\hat{\alpha}_2$	0.3873	(0.2700, 0.5046)	1.1596	(0.8161, 1.5237)	(0.3329, 0.5735)
	0.0598	0.2346	0.1850	0.7076	0.2406

References

[1] Anakha, K.K. and Chacko, V.M.(2025). On comparative lifetime analysis with the generalized Lindley distribution: insights from joint adaptive progressive Type-II censoring. Journal of Statistical Computation and Simulation, 95(11): 2466 - 2493.

[2] Anakha, K. K., & Chacko, V. M.(2025). Type I–Type II mixture censoring scheme for lifetime data analysis. *Communications in Statistics–Simulation and Computation*, 1–15.

[3] Ashour, S.K. and Abo-Kasem, O.E.(2017). *Statistical inference for two exponential populations under joint progressive type-I censored scheme*. Communications in Statistics - Theory and Methods, 46(7): 3479–3488.

[4] Balakrishnan, N.(2007). Progressive censoring methodology: an appraisal. *Test*, 16(2): 211–259.

[5] Balakrishnan, N. and Aggarwala, R. Progressive Censoring: Theory, Methods, and Applications, Springer Science & Business Media, 2000.

[6] Balakrishnan, N. and Sandhu, R.A.(1995). A simple simulational algorithm for generating progressive Type-II censored samples. *The American Statistician*, 49(2): 229–230.

[7] Castillo, E., Hadi, A. S., Balakrishnan, N., and Sarabia, J. M. Extreme Value and Related Models with Applications in Engineering and Science, John Wiley & Sons, 2004.

[8] Chen, M.-H. and Shao, Q.-M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computational and Graphical Statistics*, 8(1):69–92.

[9] Childs, A., Chandrasekar, B., Balakrishnan, N., and Kundu, D. (2003). Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution. *Annals of the Institute of Statistical Mathematics*, 55(2):319–330.

[10] Cohen, A. Clifford. (1963). Progressively censored samples in life testing. *Technometrics*, 5(3):327–339.

[11] Dutta, S., Prajapati, D., & Kundu, D. (2025). A novel two-sample joint unified hybrid censoring scheme with the application of insulating fluid data. *Journal of Applied Statistics*, 1–26.

[12] Efron, B. and Tibshirani, R.J. An Introduction to the Bootstrap, Chapman and Hall/CRC, 1993.

[13] Epstein, B. (1954). Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*, 25(3): 555–564.

[14] Kumar, D., Singh, U., and Singh, S.K.(2015). A method of proposing new distribution and its application to Bladder cancer patients data. *Journal of Statistics and Applied Probability Letters*, 2(3): 235–245.

[15] Kundu, D. and Joarder, A.(2006). Analysis of Type-II progressively hybrid censored data. Computational Statistics & Data Analysis, 50(10): 2509–2528.

[16] Mohie El-Din, M. M. M., Amein, M. M., Shafay, A. R., and Mohamed, S. (2017). Estimation of generalized exponential distribution based on an adaptive progressively Type-II censored sample. *Journal of Statistical Computation and Simulation*, 87(7):1292–1304.

[17] Ng, H.K.T., Kundu, D., and Chan, P.S.(2009). Statistical analysis of exponential lifetimes under an adaptive Type-II progressive censoring scheme. *Naval Research Logistics*, 56(8): 687–698.

[18] Chandran, R. and Chacko, V. M.(2025). Comparative Study of Maximum Likelihood, Maximum Product Spacing and Bayesian Approaches for the DUS-EV Model with Stress-Strength Applications. *International Journal of Statistics and Reliability Engineering* (Under review).

[19] Rasouli, A. and Balakrishnan, N.(2010). Exact likelihood inference for two exponential populations under joint progressive type-II censoring. *Communications in Statistics - Theory and Methods*, 39(12): 2172–2191.

[20] Sobhi, M. M. A. and Soliman, A. A.(2016). Estimation for the exponentiated Weibull model with adaptive Type-II progressive censored schemes. *Applied Mathematical Modelling*, 40(2):1180–1192.

[21] Sultana, F., Koley, A., Pal, A., and Kundu, D.(2021). On two exponential populations under a joint adaptive type-II progressive censoring. *Statistics*, 55(6): 1328–1355.

[22] Sultana, F., Cetinkaya, C. , and Kundu, D.(2023). Statistical Inference for the Jointly Adaptive Progressive Type-II Censored Weibull Distributions. *Journal of Statistical Theory and Practice*, 17(2): 24-38.

[23] Ye, Z.S., Chan, P.S., Xie, M., and Ng, H.K.T.(2014). Statistical inference for the extreme value distribution under adaptive Type-II progressive censoring schemes. *Journal of Statistical Computation and Simulation*, 84(5): 1099–1114.