SIX SIGMA BASED CONTROL CHARTS FOR MEAN UNDER TWO-PARAMETER EXPONENTIAL DISTRIBUTION

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Abstract

The statistical process control (SPC) method is widely regarded as the most efficient approach for evaluating production processes, balancing sampling costs with chart performance. Recent research has explored this problem in the context of economic planning for control charts, particularly focusing on adaptive control charts. Traditionally, control chart design relies on a fixed structure to determine key parameters such as sample size, sampling interval, and control limits to meet economic or statistical requirements. However, under the exponential distribution, optimizing control charts based on Six Sigma principles can ensure process stability while minimizing costs. This approach helps reduce either the total expected costs over a finite time horizon or the long-term average expected costs.

Keywords: Exponential distribution, Mean, Process capability and Six sigma

I. Introduction

In today's highly competitive marketplace, quality has emerged as one of the most critical factors influencing consumer decisions when choosing among various products and services. This emphasis on quality transcends market segments and applies universally whether the consumer is a private individual, a large-scale industrial enterprise, a retail chain, a financial institution, or even a government entity such as a military defense program. The widespread nature of this trend underscores the vital importance of both understanding and continuously improving quality across all levels of operation.

Businesses that prioritize quality as a fundamental component of their strategic approach are more likely to achieve sustained success, drive organizational growth, and maintain a strong competitive edge in their respective industries. Moreover, investments in enhancing quality tend to yield substantial returns, not only in terms of customer satisfaction and loyalty but also through increased operational efficiency and profitability. As such, embedding quality into the core of business strategy is no longer optional but essential for long-term viability and excellence.

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1.1 Control Charts

A control chart is a powerful graphical tool used in quality management to determine whether a process is operating within its expected range of variation or if it is being influenced by unusual, assignable factors. It visually displays data collected from process samples over time, enabling the identification of patterns, trends, and deviations that might indicate changes in process behaviour. The chart is characterized by two critical boundary lines: the Upper Control Limit (UCL) and the Lower Control Limit (LCL). These limits are statistically calculated based on historical process data and represent the threshold between common cause variation the natural, random fluctuations inherent to any stable process and assignable cause variation, which points to specific, identifiable problems that may disrupt normal operations.

The area between the upper and lower control limits is considered the normal range of variation. As long as the data points fall within this region, the process is deemed to be "in control," implying that it is functioning as expected. However, when one or more data points fall outside of these control limits, it signals that the process may be out of control, likely due to an assignable cause. Such an occurrence prompts further investigation to identify and correct the issue before it impacts product quality or process performance.

1.2 Concept of Six Sigma

The Six Sigma methodology offers a structured and data-driven approach comprising a set of tools and techniques aimed at enhancing process capability and minimizing defects in both manufacturing and service-oriented operations. Originally conceptualized as a means to optimize industrial manufacturing processes and eliminate defects, Six Sigma has since evolved into a versatile framework applicable across a wide range of business processes and service sectors. Within the Six Sigma paradigm, a defect is broadly defined as any deviation from customer expectations that may result in dissatisfaction.

The foundation of Six Sigma is deeply rooted in decades of prior advancements in quality management methodologies, including Quality Control, Total Quality Management, and the Zero Defects philosophy. These movements were significantly influenced by the pioneering contributions of quality experts such as Shewhart [10], Deming [2], [3], Juran [4], and Taguchi [11], [12], whose work laid the groundwork for modern quality improvement strategies.

1.3 Process Capability (CP)

Process capability refers to the ability of a manufacturing process to consistently produce products that meet predefined customer requirements, particularly in relation to the specified tolerance or specification limits of a product parameter. It reflects both the repeatability and stability of the process over time and serves as an objective benchmark to assess whether the process can reliably deliver outputs within acceptable limits.

To quantify and visually interpret this performance, capability indices have been developed. These indices provide a statistical representation of how well the natural variability of a process aligns with the upper and lower specification limits defined by the customer or engineering requirements. In essence, capability indices help to determine how centred the process is within the allowed range and how much of the variation falls within those limits. The most common indices, such as CP, CPK, PP, and PPK, are used to evaluate whether the process has sufficient capability to produce conforming products without excessive rework or rejection. These values also assist in comparing processes, making improvement decisions, and guiding process design.

1.4 Construction of control charts under lifetime distributions

The development of control charts specifically designed for lifetime distributions plays a crucial role in accurately tracking and managing the reliability and performance of products and systems throughout their operational lifespan. Conventional control charts are typically based on the assumption that the underlying data follows a normal distribution. However, this assumption does not hold true in many practical scenarios, particularly when dealing with lifetime or reliability data, which often conform to non-normal distributions such as the Exponential, Gamma and Weibull distributions. By tailoring control charts to these particular lifetime distributions, organizations can significantly improve the effectiveness of their quality control efforts. These customized charts offer greater sensitivity to changes in process behaviour and provide more precise insights into potential failures or deviations, thereby supporting more informed decision-making and fostering enhanced product reliability.

Processes with a constant hazard rate where the interval between subsequent occurrences or failures stays constant throughout time are specifically monitored by Shewhart-type control charts that are adapted to the exponential distribution. Because it assumes a constant failure rate, the exponential distribution is frequently used in survival analysis and reliability engineering. This makes it especially well-suited for estimating the lifespans of specific systems and components. The primary parameter of the distribution, which is closely related to the mean time between failures (MTBF), is what these control charts are intended to identify as it changes. Shewhart-type charts based on the exponential distribution are a useful and practical tool for assessing process reliability and assisting in the early detection of possible quality problems or failures in maintenance and production environments by identifying deviations from the expected process behaviour.

1.5 Importance of the study

This study definitely will help the floor engineers working in the different fields to select their six-sigma based control chart based on the requirements and environments. In addition, for providing the procedures for construction of six sigma-based control chart under Exponential distribution, readymade tables are also provided examples followed by practical applications which will help the readers and the users to have a comprehensive knowledge on the theory and applications of six sigma-based control charts. The following conditions for application

- When the objective is to analyse the time intervals between consecutive events directly
- When there is a need to utilize standard Six Sigma control charts for quality improvement.
- When the process requires a more gradual and stable response to change

II. Methods and Materials

According to Ross [9], the probability density function (PDF) of the two-parameter exponential distribution is expressed as:

$$f(x) = (x, \lambda, \theta) = \begin{cases} \lambda e^{-\lambda(x-\theta)}, x \geq \theta \\ 0, x < 0 \end{cases} \quad (1)$$

where,

x is the random variable,

$\lambda > 0$ is the rate parameter2

θ is the location parameter (shift)

The mean of a distribution, also known as the expected value, represents the long-term average outcome of a random variable over numerous observations. As described by Kin Lam et al. [5], the mean for the two-parameter exponential distribution is determined based on its probability density function and reflects the central tendency of the distribution:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \tag{2}$$

Let u=x-θ, so x=u+θ and dx=du. When x=θ, u=0, and when $x \to \infty$, $u \to \infty$ and substituting into the integral (Monjed H. Samuh and Areen, [6] then it shows as follows:

$$E(X) = \theta + \frac{1}{\lambda},\ \ E(X^2) = \theta^2 + \frac{2\theta}{\lambda} + \frac{2}{\lambda^2} \text{ and } V(x) = E(x^2) - \{E(x)\}^2 = \frac{1}{\lambda^2} \Rightarrow \sigma = \frac{1}{\lambda} \tag{3}$$

For a given tolerance level (TL) and a specified process capability (PC), the standard deviation 'σ' (termed as for exponential distribution) often referred to as the process variation is determined using a predefined mathematical relationship . This calculation is performed for various combinations of tolerance levels and process capability values to assess the consistency and precision of the manufacturing process (Pavithra and Balamurugan), [8]. By analysing these values, organizations can better understand the extent of variation within the process and ensure that it remains within acceptable limits to meet quality and performance requirements.To determine the process standard deviation , first, establish the tolerance level (TL) and the desired process capability cp=2, as outlined by Montgomery [7], Once the values of is obtained, it is applied to the control limit equations for mean as follows:

$$\left(\theta_{6\sigma}^{\bar{X}} + \frac{1}{\lambda_{6\sigma}^{\bar{X}}}\right) \pm \left[\left(\frac{Z_{6\sigma}}{\sqrt{n}}\right)\left(\frac{1}{\lambda_{6\sigma}^{\bar{X}}}\right)\right] \tag{4}$$

III. Structuring and organizing data tables

3.1 Illustration 1

The example presented by Acheson J. Duncan [1], is referenced in this analysis. In this example, a dataset consisting of inside diameter measurements is examined are shown in the table 1. These measurements are collected in sample groups, with each group containing five observations. This structured sampling approach allows for the assessment of process variation and control using statistical quality control techniques.

Table 1: *Measurements of inside diameters*

S.No	X_1	X_2	X_3	X_4	X_5	Mean	Range	Standard deviation
1	15	11	8	15	6	11.00	9	4.06
2	14	16	11	14	7	12.40	9	3.51
3	13	6	9	5	10	8.60	8	3.21

4	15	15	9	15	7	12.20	8	3.90
5	9	12	9	8	8	9.20	4	1.64
6	11	14	11	12	5	10.60	9	3.36
7	13	12	9	6	10	10.00	7	2.74
8	10	15	12	4	6	9.40	11	4.45
9	8	12	14	9	10	10.60	6	2.41
10	10	10	9	14	14	11.40	5	2.41
11	13	16	12	15	18	14.80	6	2.39
12	7	10	9	11	16	10.60	9	3.36
13	11	7	16	10	14	11.60	9	3.51
14	11	7	10	10	7	9.00	4	1.87
15	13	9	12	13	17	12.80	8	2.86
16	17	10	11	9	8	11.00	9	3.54
17	4	14	5	11	11	9.00	10	4.30
18	8	9	6	13	9	9.00	7	2.55
19	9	10	7	10	13	9.80	6	2.17
20	15	10	12	12	16	13.00	6	2.45
						$\overline{X} = 10.80$	$\overline{R} = 7.50$	$\overline{S} = 3.03$

Source: Acheson J. Duncan [1]

The rate parameter $(\lambda_{3\sigma}^{\overline{X}})$ and the location parameter $(\theta_{3\sigma}^{\overline{x}})$ of the two-parameter exponential distribution are estimated based on the 3-Sigma criterion for the mean. These parameters play a crucial role in defining the distribution's characteristics, where the rate parameter determines the spread of the distribution, and the location parameter represents the threshold or minimum value. The estimation process ensures that the control limits align with the principles of statistical quality control, allowing for accurate process monitoring and performance evaluation. The estimated values for these parameters are given as follows:

$$\lambda_{3\sigma}^{\overline{x}} = \frac{1}{\sigma_{3\sigma}^{\overline{x}}} = \frac{1}{3.03} \Rightarrow \lambda_{3\sigma}^{\overline{x}} = 0.3296 \ and \ \theta_{3\sigma}^{\overline{x}} = x - \sigma_{3\sigma}^{\overline{x}} = 10.80 - 3.03 \Rightarrow \theta_{3\sigma}^{\overline{x}} = 7.7659 \tag{5}$$

Shewhart [10], proposed the 3-Sigma control limits for the mean, which are a key instrument in statistical process control (SPC). Under typical operating conditions, the process mean is intended to fluctuate within the allowable range defined by these control limits. These constraints aid in identifying variances that might point to possible process instability or departures from expected performance by taking into account the process mean and standard deviation. According to Shewhart's technique, the mean's control boundaries are determined as

$$UCL_{3\sigma.\overline{X}}^{Exp} = \left(\theta_{3\sigma}^{\overline{X}} + \frac{1}{\lambda_{3\sigma}^{\overline{X}}}\right) + \left[\left(\frac{3}{\sqrt{n}}\right)\left(\frac{1}{\lambda_{3\sigma}^{\overline{X}}}\right)\right] = \left(7.7659 + \frac{1}{0.3296}\right) + \left(\frac{3}{\sqrt{5}}\right)\left(\frac{1}{0.3269}\right) = 10.80 \tag{6}$$

$$LCL_{3\sigma.\overline{X}}^{Exp} = \left(\theta_{3\sigma}^{\overline{X}} + \frac{1}{\lambda_{3\sigma}^{\overline{X}}}\right) - \left[\left(\frac{3}{\sqrt{n}}\right)\left(\frac{1}{\lambda_{3\sigma}^{\overline{X}}}\right)\right] = \left(7.7659 + \frac{1}{0.3296}\right) - \left(\frac{3}{\sqrt{5}}\right)\left(\frac{1}{0.3269}\right) = 6.73 \tag{7}$$

The Six Sigma methodology for the mean is used to estimate the rate parameter $(\lambda_{6\sigma}^{\bar{x}})$and location parameter $(\theta_{6\sigma}^{\bar{x}})$of the two-parameter exponential distribution. These characteristics are crucial for describing the distribution; the location parameter establishes the lowest value that can exist, while the rate parameter affects how the data spreads. By ensuring that the control limits are in line with Six Sigma concepts, the estimating process improves quality control and process monitoring. The following are the estimated values for these parameters:

$$\lambda_{6\sigma}^{\overline{X}} = \frac{1}{\eta_{6\sigma}^{\overline{X}}} = \frac{1}{0.23} \Rightarrow \lambda_{6\sigma}^{\overline{X}} = 4.28 \text{ and } \theta_{6\sigma}^{\overline{X}} = \overline{X} - \eta_{6\sigma}^{\overline{X}} = 10.80 - 0.23 \Rightarrow \theta_{6\sigma}^{\overline{X}} = 10.57 \tag{8}$$

The Six Sigma methodology-derived control limits for the mean offer a strong framework for keeping an eye on process performance and guaranteeing quality control. Under stable operating conditions, these limitations specify the acceptable range in which the process mean is anticipated to fluctuate. These control limits improve the ability to identify deviations, reduce faults, and preserve process consistency by implementing Six Sigma principles. The following is the mean's Six Sigma-based control limits are as follows:

$$UCL_{6\sigma.\overline{X}}^{Exp} = \left(\theta_{6\sigma}^{\overline{X}} + \frac{1}{\lambda_{6\sigma}^{\overline{X}}}\right) + \left[\left(\frac{Z_{6\sigma}}{\sqrt{n}}\right)\left(\frac{1}{\lambda_{\backslash 6\sigma}^{\overline{X}}}\right)\right] = \left(10.57 + \frac{1}{4.28}\right) + \left(\frac{4.831}{\sqrt{5}}\right)\left(\frac{1}{4.28}\right) = 11.31 \tag{9}$$

$$LCL_{6\sigma.\overline{X}}^{Exp} = \left(\theta_{6\sigma}^{\overline{X}} + \frac{1}{\lambda_{6\sigma}^{\overline{X}}}\right) - \left[\left(\frac{Z_{6\sigma}}{\sqrt{n}}\right)\left(\frac{1}{\lambda_{\backslash 6\sigma}^{\overline{X}}}\right)\right] = \left(10.57 + \frac{1}{4.28}\right) - \left(\frac{4.831}{\sqrt{5}}\right)\left(\frac{1}{4.28}\right) = 11.29 \tag{10}$$

An in-depth analysis of the data presented in Table-2 and Figure-1 reveals that the process under consideration is operating outside of acceptable control boundaries when assessed using the Six Sigma methodology for monitoring the mean, specifically under the assumption of a two-parameter exponential distribution. This conclusion is supported by the observation that only five samples fall within the established Six Sigma control limits, suggesting considerable process variability and signaling a deviation from statistical control. Furthermore, the Control Limit Interval (CLI) calculated for the Six Sigma-based control chart is 1.01, which is notably narrower compared to the CLI derived from the conventional Three Sigma control chart for the same sample size (n = 5). This quantitative comparison underscores the heightened sensitivity and more stringent monitoring capability of the Six Sigma approach. By narrowing the control boundaries, the Six Sigma methodology facilitates earlier detection of process shifts or anomalies, thereby enhancing the ability to implement timely corrective actions. This distinction highlights the practical advantages of Six Sigma-based control charts in maintaining high standards of quality and reliability in process monitoring.

Table 2: *Comparison of 3-sigma and six sigma control limits for the mean under two-parameter exponential distribution*

S.No	Type of control chart	LCL	CL	UCL	CLI
1	Shewhart 3σ	6.73	10.80	14.87	8.14
2	Six sigma	10.29	10.80	11.31	1.01

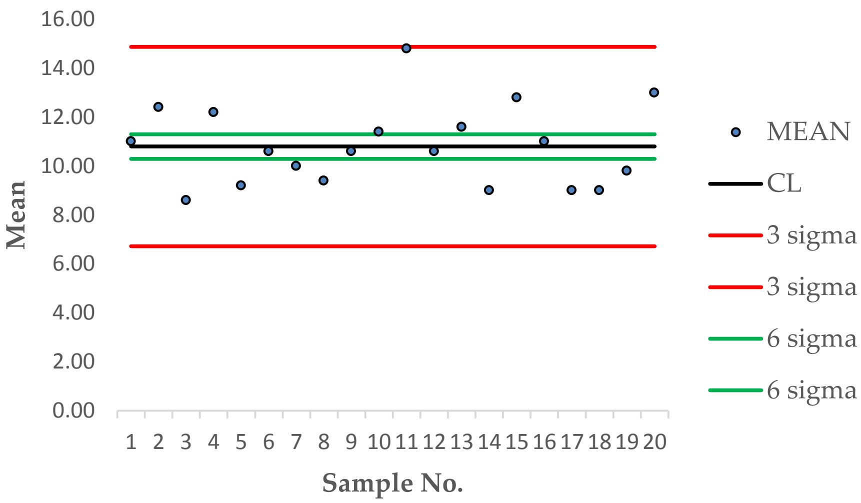

Figure 1: *Shewhart 3σ and 6σ control charts for mean under two-parameter exponential distribution*

In addition, the findings illustrated in Table 3 and Figure 2 provide compelling evidence that the traditional Shewhart 3-Sigma control chart for monitoring the process mean is comparatively less effective when assessed using the Average Run Length (ARL) criterion. When juxtaposed with the proposed Six Sigma-based control chart formulated under the two-parameter exponential distribution—the conventional approach demonstrates reduced sensitivity in detecting shifts or subtle deviations in the process.

Table 3: *Average run length (ARL) of 3σ and six sigma control limits for mean under two-parameter exponential distribution*

S.No	Multiple of σ	Shewhart 3σ	Six sigma
1	0.31	293	32
2	0.32	289	26
3	0.33	285	21
4	0.34	281	18
5	0.35	277	15
6	0.36	273	12

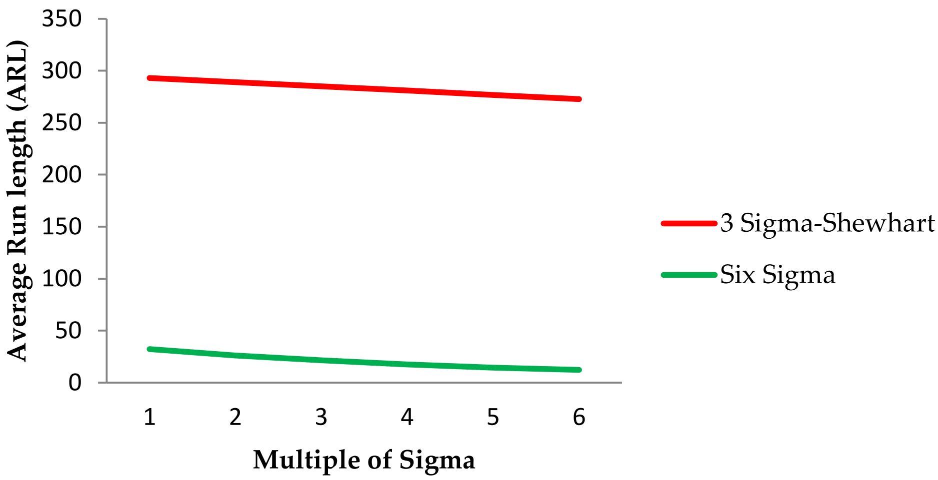

Figure 2: *ARL of Shewhart 3σ and 6σ mean control charts under two-parameter exponential distribution*

The Six Sigma control chart offers a more rigorous and refined monitoring framework, which significantly enhances the early identification of process anomalies. This tighter level of control contributes to more accurate detection of out-of-control signals and facilitates the prompt implementation of corrective measures, thereby supporting higher standards of quality assurance.

This comparative analysis highlights the inherent limitations of the Shewhart 3-Sigma chart in environments where detecting minor but critical process shifts is essential. It also reinforces the strategic advantages of adopting the Six Sigma methodology, particularly its capacity for improved process precision, reduced variability, and substantial defect minimization key goals in both manufacturing and service-oriented quality management systems

Conclusion

The results discussed above clearly indicate that when both the traditional Shewhart 3-Sigma control chart and the Six Sigma-based control limits for the mean are applied under the exponential distribution, the process under evaluation is found to be statistically out of control. This conclusion is drawn from the observed inconsistencies in sample points relative to their respective control limits, signaling that the process is experiencing variations beyond acceptable limits. Notably, the control limit interval (CLI) derived from the Six Sigma-based control chart is considerably narrower than that of the Shewhart 3-Sigma chart. This reduced interval reflects a more stringent and precise monitoring capability, which enhances the chart's effectiveness in detecting minor process shifts and emerging irregularities at an earlier stage.

The presence of such deviations suggests that the quality of the product or service fails to meet the expected standards, underscoring the urgent need for system or process improvement. Implementing Six Sigma-based control strategies enables organizations to more effectively identify and rectify sources of variation, ultimately contributing to enhanced process stability and performance. Furthermore, this methodological shift holds significant economic value. By improving detection accuracy and enabling timely interventions, the Six Sigma-based approach contributes to a reduction in the total expected costs within a finite operational period, as well as a decrease in the average long-term costs associated with sustained process inefficiencies or quality failures. As a result, organizations can achieve both improved quality outcomes and optimized cost-efficiency through the adoption of this enhanced control charting technique.

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