A TAYLOR SERIES EXPANSION METHOD FOR SOLVING LINEAR FRACTIONAL PROGRAMMING PROBLEMS

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Abstract

This paper proposes an iterative method to solve Linear Fractional Programming (LFP) problems with inequality constraints. The core idea is to approximate the fractional objective function through a series of linear programming (LP) problems, solved successively using updated feasible points. The algorithm begins by selecting a non-zero feasible point that satisfies all inequality constraints. At this point, the fractional objective function is expanded using a first-order Taylor series, resulting in a linear approximation. This converts the LFP problem into a standard LP problem, which can be solved using common optimization techniques such as the simplex method or the graphical method, depending on the problem's size and complexity. The optimal solution of the LP problem is then used as the new feasible point for the next iteration. In each iteration, the fractional objective function is re-linearized around the current solution, and a new LP problem is formed and solved. This process continues until convergence is achieved, that is, when two successive iterations produce the same or sufficiently similar solutions. The final solution is taken as the optimal solution to the original LFP problem. The effectiveness and practical utility of the proposed method are demonstrated through numerical examples. Results indicate that the approach is computationally efficient and provides accurate solutions. Compared to traditional transformation-based techniques, the proposed method avoids the introduction of auxiliary variables or complex reformulations, offering a more intuitive and implementable solution framework. Furthermore, the approach exhibits potential for extension to more complex fractional programming models, including multi-objective and equality-constrained formulations.

Keywords: Linear fractional programming, optimal solution, Taylor series

I. Introduction

A linear fractional programming (LFP) problem is a mathematical optimization problem that involves maximizing or minimizing the ratio of two linear functions subject to linear constraints and non-negativity restrictions on the decision variables. This mathematical approach was introduced by the Hungarian mathematician Martos in 1960 and has found broad applications in scenarios where efficiency or cost-effectiveness is better modeled by ratios rather than absolute values, like health care planning, production scheduling, Markow chain decision-making, economics, and more. The growing relevance of LFP problems in various sectors has attracted substantial attention from researchers seeking efficient methods to solve these problems.

Several techniques have been developed to tackle LFP problems. However, many of these

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methods require a positive denominator in the objective function for all feasible solutions, while also ensuring the feasible region is bounded. These requirements are often necessary to avoid undefined behavior in the optimization process and ensure solution stability.

In this work, we introduce a new method for solving LFP problems with the above assumptions. Our approach is based on the concept of approximating the objective function by Taylor series about any arbitrarily chosen feasible point and transforming the LFP problem to a LP problem. Truncating the series to a finite number of terms allows our method to converge in a fixed number of iterations. This approximation strategy preserves the essential structure of the original problem while leveraging the efficiency of linear programming solvers.

II. Literature Review

Over the years, several methods have been developed to solve Linear Fractional Programming (LFP) problems. One of the earliest and most well-known approaches was introduced by Charnes and Cooper [1], who employed a variable transformation technique to convert the fractional objective function into a linear one. Bitran and Novaes [2] proposed an alternative strategy by updating the objective function iteratively. Isbell and Marlow [3], as well as Martos [4], formulated a method based on solving a sequence of linear programs to arrive at the optimal solution. Swarup [5] extended the classical simplex method used in linear programming to accommodate the structure of LFP problems. Bector [6] and Seshan [7] tackled the problem using a dual formulation, providing insights into the duality structure of LFP. Stancu-Minasian [8] comprehensively presented the foundational concepts, theoretical developments and diverse applications of fractional programming, which served as a significant reference in the field. Bajalinov [9] relied on the simplex algorithm as the primary computational tool and utilized the graphical method for problems involving two variables. Tantawy [10] introduced a novel duality framework specifically designed for LFP problems, adapting classical duality concepts to suit the fractional structure of LFP. Jahan and Islam [11] approached the problem using the complementary slackness theorem, offering a condition-based method for identifying optimality. Hasan and Acharjee [12] introduced a computer-oriented technique using the Mathematica programming language, demonstrating the applicability of symbolic computation tools in solving LFP problems. Pandian and Jayalakshmi [13] proposed a denominator objective restriction method based on the simplex algorithm to solve LFP problems. Saha et al. [14] developed a computer-based algorithmic approach tailored for solving LFP problems efficiently. Das and Mandal [15] introduced a novel homotopy perturbation method, providing an analytical framework for finding solutions to LFP problems. Simi and Talukder [16] suggested a method incorporating duality concepts to address the solution of LFP problems more effectively. Kumari and Bhardwaj [17] utilized computer programming techniques to implement solution procedures for LFP problems. Yulin and Xin [18] applied the Charnes-Cooper transformation to simplify the fractional objective function into a linear programming model, subsequently using MATLAB functions to compute the optimal solution. Gupta and Raina [19] proposed an integer solution approach for linear fractional programming problems, expanding the applicability of fractional programming to integer domains. Bennani et al. [20] employed adaptive projection methods to solve LFP problems and further refined their approach by deriving optimal solutions using Lustig's interior point method [21], demonstrating the effectiveness of modern computational techniques in fractional programming. Kumari [22] proposed an algorithm based on the efficient selection of incoming vectors. Sahoo et al. [23] addressed intuitionistic fuzzy multi-objective linear fractional programming problems by developing an alternative approach tailored to handle the inherent uncertainty and vagueness in such models. Mustafa and Sulaiman [24] tackled LFP problems with rough interval data by transforming them into linear programming problems with interval coefficients, thereby enabling the use of established LP solution techniques under uncertainty. Chauhan et al. [25] solved linear fractional programming problems with fuzzy variables and unrestricted parameters using an alpha-cut-based method. Murugan and Thamaraiselvan [26] solved the interval linear fractional programming (ILFP) problem using interval arithmetic.

In this paper, a new method to solve LFP problem is proposed. The outline of the rest of the paper is as follows: The proposed method to solve the LFP problem is described in Section III. In Section IV, we present the algorithm for the proposed method. Numerical examples are provided in Section V. The validity of our approach is verified in Section VI by comparing its results to those of existing methods. Section VII discusses the significance of our proposed method. Finally, conclusions and possible directions for future research are presented in Section VIII.

III. Proposed Method

Let the LFP problem be

$$\left.\begin{aligned} &\text{Max } z = \frac{z^1}{z^2} = \frac{c_0 + \mathbf{cx}}{d_0 + \mathbf{dx}} \\ &\text{subject to} \\ &\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \end{aligned}\right\} \tag{1}$$

where $\mathbf{c}$ and $\mathbf{d}$ are $1 \times n$ matrices, $\mathbf{A}$ is a $m \times n$ matrix, $\mathbf{b}$ is a $m \times 1$ column vector, $\mathbf{x}$ is a $n \times 1$ column vector, and c_0 and d_0 are constants.

Let the objective function of LFP problem (1) be denoted by $F(\mathbf{x})$, where $\mathbf{x} = [x_2 \quad x_2 \quad \cdots \quad x_n]^T$. We choose any non-zero feasible point $\mathbf{x}^{(0)} = [a_2 \quad a_2 \quad \cdots \quad a_n]^T$ as the initial feasible solution and expand $F(\mathbf{x})$ about $\mathbf{x}^{(0)}$ using Taylor series. Thus, we have

$$F(\mathbf{x}) = F(\mathbf{x}^{(0)}) + (x_1 - a_1)\left(\frac{\partial F}{\partial x_1}\right)_{(\mathbf{x}^{(0)})} + (x_2 - a_2)\left(\frac{\partial F}{\partial x_2}\right)_{(\mathbf{x}^{(0)})} + \ldots + (x_n - a_n)\left(\frac{\partial F}{\partial x_n}\right)_{(\mathbf{x}^{(0)})}.$$

Now, we solve the following LP problem:

$$\left.\begin{aligned} &\text{Max } F(\mathbf{x}) \\ &\text{subject to} \\ &\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \end{aligned}\right\} \tag{2}$$

Let the optimal solution to the above LP problem be $\mathbf{x}^{(1)} = [k_2 \quad k_2 \quad \ldots \quad k_n]^T$.

We have to check whether $\mathbf{x}^{(1)}$ is the optimal solution to given LFP problem (1) or not. For this, we repeat the process, taking $\mathbf{x}^{(1)}$ as the new feasible solution to LFP problem (1). Denoting the objective function of LFP problem (1) by $G(\mathbf{x})$ and expanding $G(\mathbf{x})$ about the new feasible point $\mathbf{x}^{(1)}$, we have

$$G(\mathbf{x}) = G(\mathbf{x}^{(1)}) + (x_1 - k_1)\left(\frac{\partial G}{\partial x_1}\right)_{(\mathbf{x}^{(1)})} + (x_2 - k_2)\left(\frac{\partial G}{\partial x_2}\right)_{(\mathbf{x}^{(1)})} + \ldots + (x_n - k_n)\left(\frac{\partial G}{\partial x_n}\right)_{(\mathbf{x}^{(1)})}.$$

Now, we solve the following LP problem:

$$\left.\begin{aligned} &\text{Max } G(\mathbf{x}) \\ &\text{subject to} \\ &\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \end{aligned}\right\} \tag{3}$$

Let the optimal solution to the above LP problem be $\mathbf{x}^{(2)}$.

If $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$, then it provides the optimal solution to the given LFP problem (1). Otherwise, we expand the objective function of the given LFP problem (1) about $\mathbf{x}^{(2)}$ and solve the reconstructed LP problem. The process terminates when the optional solution to two consecutive LP problems coincides.

IV. Algorithm for the Proposed Method

Step 1. Choose any non-zero feasible point $\mathbf{x}^{(0)} = [a_2 \quad a_2 \quad \cdots \quad a_n]^T$ as the initial feasible solution.

Step 2. Denote the objective function of LFP problem (1) by $F(\mathbf{x})$ and expand $F(\mathbf{x})$ about $\mathbf{x}^{(0)}$ using Taylor series.

Step 3. Solve the LP problem (2) by simplex method or graphical method. Let the optimal solution to LP problem (2) be $\mathbf{x}^{(1)} = [k_2 \quad k_2 \quad \dots \quad k_n]^T$.

Step 4. Denote the objective function of LFP problem (1) by $G(\mathbf{x})$ and expand $G(\mathbf{x})$ about $\mathbf{x}^{(1)}$ using Taylor series.

Step 5. Solve the LP problem (3) by simplex method or graphical method. Let the optimal solution to LP problem (2) be $\mathbf{x}^{(2)}$.

If $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$, then terminate the process. In this case, $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$ is the optimal solution to given LFP problem.

If $\mathbf{x}^{(1)} \neq \mathbf{x}^{(2)}$, then go to Step 4 and repeat the process, taking $\mathbf{x}^{(2)}$ in place of $\mathbf{x}^{(1)}$ until the criterion of optimality is satisfied.

V. Numerical Examples

Example 1. Max $\frac{8x_1+9x_2+4x_3+4}{2x_1+3x_2+2x_3+7}$ subject to
$x_1 + x_2 + 2x_3 \le 3,\ 2x_1 + x_2 + 4x_3 \le 4,\ 5x_1 + 3x_2 + x_3 \le 15,\ x_1, x_2, x_3 \ge 0.$

Solution. Let $F(\mathbf{x}) = \frac{8x_1+9x_2+4x_3+4}{2x_1+3x_2+2x_3+7}$, where $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$.
Let us take $\mathbf{x}^{(0)} = [0 \quad 1 \quad 0]^T$ as the initial feasible solution. Then $F(\mathbf{x}^{(0)}) = \frac{13}{10}$ and

$$\frac{\partial F}{\partial x_1} = \frac{6x_2 + 8x_3 + 48}{(2x_1 + 3x_2 + 2x_3 + 7)^2}, \quad \frac{\partial F}{\partial x_2} = \frac{-6x_1 + 6x_3 + 51}{(2x_1 + 3x_2 + 2x_3 + 7)^2}, \quad \frac{\partial F}{\partial x_3} = \frac{-8x_1 - 6x_2 + 20}{(2x_1 + 3x_2 + 2x_3 + 7)^2}.$$

Expanding $F(\mathbf{x})$ about $\mathbf{x}^{(0)}$ using Taylor series, we have

$$F(\mathbf{x}) = F(\mathbf{x}^{(0)}) + (x_1 - 0)\left(\frac{\partial F}{\partial x_1}\right)_{(\mathbf{x}^{(0)})} + (x_2 - 1)\left(\frac{\partial F}{\partial x_2}\right)_{(\mathbf{x}^{(0)})} + (x_3 - 0)\left(\frac{\partial F}{\partial x_3}\right)_{(\mathbf{x}^{(0)})}$$

i.e., $$F(\mathbf{x}) = \frac{27}{50}x_1 + \frac{51}{100}x_2 + \frac{7}{50}x_3 + \frac{79}{100}.$$

Therefore, the given LFP problem reduces to the following problem:

$$\left.\begin{array}{l} \text{Max } F(\mathbf{x}) = \frac{27}{50}x_1 + \frac{51}{100}x_2 + \frac{7}{50}x_3 + \frac{79}{100} \\ \text{subject to} \\ x_1 + x_2 + 2x_3 \le 3,\ 2x_1 + x_2 + 4x_3 \le 4,\ 5x_1 + 3x_2 + x_3 \le 15, \\ x_1, x_2, x_3 \ge 0. \end{array}\right\} \quad \text{(I)}$$

We solve the above LP problem by simplex method.
Introducing slack variables x_4, x_5, x_6, the given constraints reduce to

$$x_1 + x_2 + 2x_3 + x_4 = 3, \quad 2x_1 + x_2 + 4x_3 + x_5 = 4, \quad 5x_1 + 3x_2 + x_3 + x_6 = 15.$$

Now, using simplex method, we obtain the following simplex tables:

Table 1(a): *Initial table for LPP (I)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	3	1	1	2	1	0	0	3
$\mathbf{y}_5$	0	4	2	1	4	0	1	0	2
$\mathbf{y}_6$	0	15	5	3	1	0	0	1	3
$F(\mathbf{x}) = \frac{79}{100}$		x_j	0	0	0	3	4	15	
		c_j	27/50	51/100	7/50	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	27/50	51/100	7/50	0	0	0	
			↑				↓		

Table 1(b): *Intermediate table for LPP (I)*

$\mathbf{B}$	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	1	0	1/2	0	1	-1/2	0	2
$\mathbf{y}_1$	27/50	2	1	1/2	2	0	1/2	0	4
$\mathbf{y}_6$	0	5	0	1/2	-9	0	-5/2	1	10
$F(\mathbf{x}) = \frac{187}{100}$		x_j	2	0	0	1	0	5	
		c_j	27/50	51/100	7/50	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	0	6/25	-47/50	0	-27/100	0	
				↑		↓			

Table 1(c): *Final table for LPP (I)*

$\mathbf{B}$	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$\mathbf{y}_2$	51/100	2	0	1	0	2	-1	0
$\mathbf{y}_1$	27/50	1	1	0	2	-1	1	0
$\mathbf{y}_6$	0	4	0	0	-9	-1	-2	1
$F(\mathbf{x}) = \frac{47}{20}$		x_j	1	2	0	0	0	4
		c_j	27/50	51/100	7/50	0	0	0
		$c_j - \mathbf{c}_B\mathbf{y}_j$	0	0	-47/50	-12/25	-3/100	0

Now the optimality criterion has been met, indicating that the optimal solution to the LP problem has been found. This optimal solution is as follows:

$\mathbf{x}^{(1)} = [x_1 \;\; x_2 \;\; x_3]^T = [1 \;\; 2 \;\; 0]^T$ and $\max F(\mathbf{x}) = 47/20$.

To check that this optimal solution is the optimal solution to the given LFP problem, we repeat the process, taking $\mathbf{x}^{(1)} = [1 \quad 2 \quad 0]^T$ as the new feasible solution to the given LFP problem.

Denoting the objective function of the given LFP problem by $G(\mathbf{x})$, we have

$$G\left(\mathbf{x}^{(1)}\right) = 2, \quad \left(\frac{\partial G}{\partial x_1}\right)_{(\mathbf{x}^{(1)})} = \frac{4}{15}, \quad \left(\frac{\partial G}{\partial x_2}\right)_{(\mathbf{x}^{(1)})} = \frac{1}{5}, \quad \left(\frac{\partial G}{\partial x_3}\right)_{(\mathbf{x}^{(1)})} = 0.$$

Therefore, the reconstructed LP problem is

$$\left.\begin{array}{l} \text{Max } G(\mathbf{x}) = \frac{4}{15}x_1 + \frac{1}{5}x_2 + \frac{4}{3} \\ \text{subject to} \\ x_1 + x_2 + 2x_3 \le 3, \;\; 2x_1 + x_2 + 4x_3 \le 4, \;\; 5x_1 + 3x_2 + x_3 \le 15, \\ x_1, x_2, x_3 \ge 0. \end{array}\right\} \qquad \text{(II)}$$

Now, introducing slack variables x_4, x_5, x_6 and using simplex method, we obtain the following simplex tables:

Table 1(d): *Initial table for LPP (II)*

$\mathbf{B}$	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	3	1	1	2	1	0	0	3
$\mathbf{y}_5$	0	4	2	1	4	0	1	0	2
$\mathbf{y}_6$	0	15	5	3	1	0	0	1	3
$G(\mathbf{x}) = \frac{4}{3}$		x_j	0	0	0	3	4	15	
		c_j	4/15	1/5	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	4/15	1/5	0	0	0	0	
			↑				↓		

Table 1(e): *Intermediate table for LPP (II)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	1	0	1/2	0	1	- (1/2)	0	2
$\mathbf{y}_1$	4/15	2	1	1/2	2	0	1/2	0	4
$\mathbf{y}_6$	0	5	0	1/2	-9	0	-5/2	1	10
$G(\mathbf{x}) = \frac{28}{15}$		x_j	2	0	0	1	0	5	
		c_j	4/15	1/5	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	0	1/15	-8/15	0	-2/15	0	
				↑		↓			

Table 1(f): *Final table for LPP (II)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$\mathbf{y}_2$	1/5	2	0	1	0	2	-1	0
$\mathbf{y}_1$	4/15	1	1	0	2	-1	1	0
$\mathbf{y}_6$	0	4	0	0	-9	-1	-2	1
$G(\mathbf{x}) = 2$		x_j	1	2	0	0	0	4
		c_j	4/15	1/5	0	0	0	0
		$c_j - \mathbf{c}_B\mathbf{y}_j$	0	0	-8/15	-2/15	-1/15	0

Now the optimality criterion has been met, indicating that the optimal solution to the LP problem has been found. This optimal solution is as follows:

$\mathbf{x}^{(2)} = [x_1 \ \ x_2 \ \ x_3]^T = [1 \ \ 2 \ \ 0]^T$ and $\max G(\mathbf{x}) = 2$.

Obviously, $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$. Therefore the optimal solution to the given LFP problem is reached, which is given by $\mathbf{x} = [x_1 \ \ x_2 \ \ x_3]^T = [1 \ \ 2 \ \ 0]^T$ and $\max z = 2$.

Example 2. Max $\frac{x_1-3x_2+3x_3}{x_1+2x_2+x_3+1}$ subject to
$x_1 + x_2 - x_3 \le 6, \quad -2x_1 - x_2 + 2x_3 \le 2, \ \ 2x_1 - 3x_2 + x_3 \le 3, \ \ x_1, x_2, x_3 \ge 0.$

Solution. Let $F(\mathbf{x}) = \frac{x_1-3x_2+3x_3}{x_1+2x_2+x_3+1}$, where $\mathbf{x} = [x_1 \ \ x_2 \ \ x_3]^T$.
Let us take $\mathbf{x}^{(0)} = [1 \ \ 0 \ \ 0]^T$ as the initial feasible solution. Then $F(\mathbf{x}^{(0)}) = \frac{1}{2}$ and

$$\frac{\partial F}{\partial x_1} = \frac{5x_2 - 2x_3 + 1}{(x_1 + 2x_2 + x_3 + 1)^2}, \quad \frac{\partial F}{\partial x_2} = \frac{-5x_1 - 9x_3 - 3}{(x_1 + 2x_2 + x_3 + 1)^2}, \quad \frac{\partial F}{\partial x_3} = \frac{2x_1 + 9x_2 + 3}{(x_1 + 2x_2 + x_3 + 1)^2}.$$

Expanding $F(\mathbf{x})$ about $\mathbf{x}^{(0)}$ using Taylor series, we have

$$F(\mathbf{x}) = F(\mathbf{x}^{(0)}) + (x_1 - 1)\left(\frac{\partial F}{\partial x_1}\right)_{(\mathbf{x}^{(0)})} + (x_2 - 0)\left(\frac{\partial F}{\partial x_2}\right)_{(\mathbf{x}^{(0)})} + (x_3 - 0)\left(\frac{\partial F}{\partial x_3}\right)_{(\mathbf{x}^{(0)})}$$

i.e., $\quad F(\mathbf{x}) = \frac{1}{4}x_1 - 2x_2 + \frac{5}{4}x_3 + \frac{1}{4}.$

Therefore, the given LFP problem reduces to the following problem:

$$\left.\begin{array}{l} \text{Max } F(\mathbf{x}) = \frac{1}{4}x_1 - 2x_2 + \frac{5}{4}x_3 + \frac{1}{4} \\ \text{subject to} \\ x_1 + x_2 - x_3 \le 6, \quad -2x_1 - x_2 + 2x_3 \le 2, \ \ 2x_1 - 3x_2 + x_3 \le 3, \\ x_1, x_2, x_3 \ge 0. \end{array}\right\} \quad \text{(I)}$$

To solve the above LP problem by simplex method, we introduce slack variables x_4, x_5, x_6. Then

$$x_1 + x_2 - x_3 + x_4 = 6, \quad -2x_1 - x_2 + 2x_3 + x_5 = 2, \quad 2x_1 - 3x_2 + x_3 + x_6 = 3.$$

Now, using simplex method, we obtain the following simplex tables:

Table 2(a): *Initial table for LPP (I)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	6	1	1	-1	1	0	0	-
$\mathbf{y}_5$	0	2	-2	-1	2	0	1	0	1
$\mathbf{y}_6$	0	3	2	-3	1	0	0	1	3
$F(\mathbf{x}) = \frac{1}{4}$		x_j	0	0	0	6	2	3	
		c_j	1/4	-2	5/4	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	1/4	-2	5/4	0	0	0	
					↑		↓		

Table 2(b): *Intermediate table for LPP (I)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	7	0	1/2	0	1	1/2	0	-
$\mathbf{y}_3$	5/4	1	-1	-1/2	1	0	1/2	0	-
$\mathbf{y}_6$	0	2	3	-5/2	0	0	-1/2	1	2/3
$F(\mathbf{x}) = \frac{3}{2}$		x_j	0	0	1	7	0	2	
		c_j	1/4	-2	5/4	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	3/2	-11/8	0	0	-5/8	0	
			↑					↓	

Table 2(c): *Final table for LPP (I)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$\mathbf{y}_4$	0	7	0	1/2	0	1	1/2	0
$\mathbf{y}_3$	5/4	5/3	0	-4/3	1	0	1/3	1/3
$\mathbf{y}_1$	1/4	2/3	1	-5/6	0	0	-1/6	1/3
$F(\mathbf{x}) = \frac{5}{2}$		x_j	2/3	0	5/3	7	0	0
		c_j	1/4	-2	5/4	0	0	0
		$c_j - \mathbf{c}_B\mathbf{y}_j$	0	-1/8	0	0	-3/8	-1/2

Now the optimality criterion has been met, indicating that the optimal solution to the LP problem has been found. This optimal solution is as follows:

$\mathbf{x}^{(1)} = [x_1 \ \ x_2 \ \ x_3]^T = [2/3 \ \ 0 \ \ 5/3]^T$ and $\max F(\mathbf{x}) = 5/2$.

In order to check that this optimal solution is the optimal solution to the given LFP problem, we repeat the process, taking $\mathbf{x}^{(1)} = [2/3 \ \ 0 \ \ 5/3]^T$ as the new feasible solution to the given LFP problem.

Denoting the objective function of the given LFP problem by $G(\mathbf{x})$, we have

$$G\left(\mathbf{x}^{(1)}\right) = \frac{17}{10}, \quad \left(\frac{\partial G}{\partial x_1}\right)_{(\mathbf{x}^{(1)})} = -\frac{21}{100}, \quad \left(\frac{\partial G}{\partial x_2}\right)_{(\mathbf{x}^{(1)})} = -\frac{48}{25}, \quad \left(\frac{\partial G}{\partial x_3}\right)_{(\mathbf{x}^{(1)})} = \frac{39}{100}.$$

Therefore, the reconstructed LP problem is

$$\left.\begin{array}{l} \text{Max } G(\mathbf{x}) = -\frac{21}{100}x_1 - \frac{48}{25}x_2 + \frac{39}{100}x_3 + \frac{119}{100} \\ \text{subject to} \\ x_1 + x_2 - x_3 \le 6, \ \ -2x_1 - x_2 + 2x_3 \le 2, \ \ 2x_1 - 3x_2 + x_3 \le 3, \\ x_1, x_2, x_3 \ge 0. \end{array}\right\} \quad \text{(II)}$$

Now, introducing slack variables x_4, x_5, x_6 and using simplex method, we obtain the following simplex tables:

Table 2(d): *Initial table for LPP (II)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	6	1	1	-1	1	0	0	-
$\mathbf{y}_5$	0	2	-2	-1	2	0	1	0	1
$\mathbf{y}_6$	0	3	2	-3	1	0	0	1	3
$G(\mathbf{x}) = \frac{119}{100}$		x_j	0	0	0	6	2	3	
		c_j	-21/100	-48/25	39/100	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	-21/100	-48/25	39/100	0	0	0	
					↑		↓		

Table 2(e): *Intermediate table for LPP (II)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_4$	0	7	0	1/2	0	1	1/2	0	-
$\mathbf{y}_3$	39/100	1	-1	-1/2	1	0	1/2	0	-
$\mathbf{y}_6$	0	2	3	-5/2	0	0	-1/2	1	2/3
$G(\mathbf{x}) = \frac{79}{50}$		x_j	0	0	1	7	0	2	
		c_j	-21/100	-48/25	39/100	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	9/50	-69/40	0	0	-39/200	0	
			↑					↓	

Table 2(f): *Final table for LPP (II)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$\mathbf{y}_4$	0	7	0	1/2	0	1	1/2	0
$\mathbf{y}_3$	39/100	5/3	0	-4/3	1	0	1/3	1/3
$\mathbf{y}_1$	-21/100	2/3	1	-5/6	0	0	-1/6	1/3
$G(\mathbf{x}) = \frac{17}{10}$		x_j	2/3	0	5/3	7	0	0
		c_j	-21/100	-48/25	39/100	0	0	0
		$c_j - \mathbf{c}_B\mathbf{y}_j$	0	-63/40	0	0	-33/200	-3/50

Now the optimality criterion has been met, indicating that the optimal solution to the LP problem has been found. This optimal solution is as follows:

$\mathbf{x}^{(2)} = [x_1 \ \ x_2 \ \ x_3]^T = [2/3 \quad 0 \quad 5/3]^T$ and max $G(\mathbf{x}) = 17/10$.

Obviously, $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$. Therefore the optimal solution to the given LFP problem is reached, which is given by $\mathbf{x} = [x_1 \ \ x_2 \ \ x_3]^T = [2/3 \quad 0 \quad 5/3]^T$ and max $z = 17/10$.

Example 3. Max $\frac{x_1+2x_2+4x_3+5x_4+8x_5}{2x_1+5x_2+3x_3+4x_4+6x_5+1}$ subject to

$2x_1 + x_2 + 3x_3 + x_4 + x_5 \le 15, \quad x_1 + 2x_2 + x_5 \le 8,$
$3x_1 + 5x_2 + 2x_4 \le 10, \quad 2x_2 + 4x_3 + 3x_4 + x_5 \le 21,$
$x_1, x_2, x_3, x_4, x_5 \ge 0.$

Solution. Let $F(\mathbf{x}) = \frac{x_1+2x_2+4x_3+5x_4+8x_5}{2x_1+5x_2+3x_3+4x_4+6x_5+1}$, where $\mathbf{x} = [x_1 \ \ x_2 \ \ x_3 \ \ x_4 \ \ x_5\,]^T$.

Let us take $\mathbf{x}^{(0)} = [0 \ \ 1 \ \ 0 \ \ 1 \ \ 0]^T$ as the initial feasible solution. Then $F(\mathbf{x}^{(0)}) = \frac{7}{10}$ and

$$\frac{\partial F}{\partial x_1} = \frac{x_2 - 5x_3 - 6x_4 - 10x_5 + 1}{(2x_1 + 5x_2 + 3x_3 + 4x_4 + 6x_5 + 1)^2}, \quad \frac{\partial F}{\partial x_2} = \frac{-x_1 - 14x_3 - 17x_4 - 28x_5 + 2}{(2x_1 + 5x_2 + 3x_3 + 4x_4 + 6x_5 + 1)^2},$$

$$\frac{\partial F}{\partial x_3} = \frac{5x_1 + 14x_2 + x_4 + 4}{(2x_1 + 5x_2 + 3x_3 + 4x_4 + 6x_5 + 1)^2}, \quad \frac{\partial F}{\partial x_4} = \frac{6x_1 + 17x_2 - x_3 - 2x_5 + 5}{(2x_1 + 5x_2 + 3x_3 + 4x_4 + 6x_5 + 1)^2},$$

$$\frac{\partial F}{\partial x_5} = \frac{10x_1 + 28x_2 + 2x_4 + 8}{(2x_1 + 5x_2 + 3x_3 + 4x_4 + 6x_5 + 1)^2}.$$

Expanding $F(\mathbf{x})$ about $\mathbf{x}^{(0)}$ using Taylor series, we have

$$F(\mathbf{x}) = F(\mathbf{x}^{(0)}) + (x_1 - 0)\left(\frac{\partial F}{\partial x_1}\right)_{(\mathbf{x}^{(0)})} + (x_2 - 1)\left(\frac{\partial F}{\partial x_2}\right)_{(\mathbf{x}^{(0)})} + (x_3 - 0)\left(\frac{\partial F}{\partial x_3}\right)_{(\mathbf{x}^{(0)})} + (x_4 - 1)\left(\frac{\partial F}{\partial x_4}\right)_{(\mathbf{x}^{(0)})} + (x_5 - 0)\left(\frac{\partial F}{\partial x_5}\right)_{(\mathbf{x}^{(0)})}$$

i.e., $$F(\mathbf{x}) = -\frac{1}{25}x_1 - \frac{3}{20}x_2 + \frac{19}{100}x_3 + \frac{11}{50}x_4 + \frac{19}{50}x_5 + \frac{63}{100}.$$

Therefore, the given LFP problem reduces to the following problem:

$$\left.\begin{aligned} &\text{Max } F(\mathbf{x}) = -\frac{1}{25}x_1 - \frac{3}{20}x_2 + \frac{19}{100}x_3 + \frac{11}{50}x_4 + \frac{19}{50}x_5 + \frac{63}{100} \\ &\text{subject to} \\ &2x_1 + x_2 + 3x_3 + x_4 + x_5 \le 15, \quad x_1 + 2x_2 + x_5 \le 8, \\ &3x_1 + 5x_2 + 2x_4 \le 10, \quad 2x_2 + 4x_3 + 3x_4 + x_5 \le 21, \\ &x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}\right\} \quad \text{(I)}$$

We solve the above LP problem by simplex method.
Introducing slack variables x_6, x_7, x_8, x_9, the given constraints reduce to

$$2x_1 + x_2 + 3x_3 + x_4 + x_5 + x_6 = 15, \quad x_1 + 2x_2 + x_5 + x_7 = 8,$$
$$3x_1 + 5x_2 + 2x_4 + x_8 = 10, \quad 2x_2 + 4x_3 + 3x_4 + x_5 + x_9 = 21.$$

Now, using simplex method, we obtain the following simplex tables:

Table 3(a): *Initial table for LPP (I)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_6$	0	15	2	1	3	1	1	1	0	0	0	15
$\mathbf{y}_7$	0	8	1	2	0	0	1	0	1	0	0	8
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0	-
$\mathbf{y}_9$	0	21	0	2	4	3	1	0	0	0	1	21
		x_j	0	0	0	0	0	15	8	10	21	
$F(\mathbf{x}) = \frac{63}{100}$		c_j	$-\frac{1}{25}$	$-\frac{3}{20}$	$\frac{19}{100}$	$\frac{11}{50}$	$\frac{19}{50}$	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{1}{25}$	$-\frac{3}{20}$	$\frac{19}{100}$	$\frac{11}{50}$	$\frac{19}{50}$	0	0	0	0	
							↑		↓			

Table 3(b): *Intermediate table for LPP (I)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_6$	0	7	1	-1	3	1	0	1	-1	0	0	7
$\mathbf{y}_5$	$\frac{19}{50}$	8	1	2	0	0	1	0	1	0	0	-
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0	5
$\mathbf{y}_9$	0	13	-1	0	4	3	0	0	-1	0	1	$\frac{13}{3}$
		x_j	0	0	0	0	8	7	0	10	13	
$F(\mathbf{x}) = \frac{367}{100}$		c_j	$-\frac{1}{25}$	$-\frac{3}{20}$	$\frac{19}{100}$	$\frac{11}{50}$	$\frac{19}{50}$	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{21}{50}$	$-\frac{91}{100}$	$\frac{19}{100}$	$\frac{11}{50}$	0	0	$-\frac{19}{50}$	0	0	
						↑					↓	

Table 3(c): *Final table for LPP (I)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$
$\mathbf{y}_6$	0	$\frac{8}{3}$	$\frac{4}{3}$	-1	$\frac{5}{3}$	0	0	1	$-\frac{2}{3}$	0	$-\frac{1}{3}$
$\mathbf{y}_5$	$\frac{19}{50}$	8	1	2	0	0	1	0	1	0	0
$\mathbf{y}_8$	0	$\frac{4}{3}$	$\frac{11}{3}$	5	$-\frac{8}{3}$	0	0	0	$\frac{2}{3}$	1	$-\frac{2}{3}$
$\mathbf{y}_4$	$\frac{11}{50}$	$\frac{13}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$
$F(\mathbf{x}) = \frac{1387}{300}$		x_j	0	0	0	$\frac{13}{3}$	8	$\frac{8}{3}$	0	$\frac{4}{3}$	0
		c_j	$-\frac{1}{25}$	$-\frac{3}{20}$	$\frac{19}{100}$	$\frac{11}{50}$	$\frac{19}{50}$	0	0	0	0
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{26}{75}$	$-\frac{91}{100}$	$-\frac{31}{300}$	0	0	0	$-\frac{23}{75}$	0	$-\frac{11}{150}$

Now the optimality criterion has been met, indicating that the optimal solution to the LP problem has been found. This optimal solution is as follows:

$\mathbf{x}^{(1)} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5]^T = [0 \;\; 0 \;\; 0 \;\; 13/3 \;\; 8]^T$ and $\max F(\mathbf{x}) = 1387/300$.

In order to check that this optimal solution is the optimal solution to the given LFP problem, we repeat the process, taking $\mathbf{x}^{(1)} = [0 \;\; 0 \;\; 0 \;\; 13/3 \;\; 8]^T$ as the new feasible solution to the given LFP problem.

Denoting the objective function of the given LFP problem by $G(\mathbf{x})$, we have

$$G(\mathbf{x}^{(1)}) = \frac{257}{199}, \quad \left(\frac{\partial G}{\partial x_1}\right)_{(\mathbf{x}^{(1)})} = -\frac{945}{39601}, \quad \left(\frac{\partial G}{\partial x_2}\right)_{(\mathbf{x}^{(1)})} = -\frac{2661}{39601},$$
$$\left(\frac{\partial G}{\partial x_3}\right)_{(\mathbf{x}^{(1)})} = \frac{75}{39601}, \quad \left(\frac{\partial G}{\partial x_4}\right)_{(\mathbf{x}^{(1)})} = -\frac{99}{39601}, \quad \left(\frac{\partial G}{\partial x_5}\right)_{(\mathbf{x}^{(1)})} = \frac{150}{39601}.$$

Therefore, the reconstructed LP problem is

$$\left.\begin{aligned} &\text{Max } G(\mathbf{x}) = -\frac{945}{39601}x_1 - \frac{2661}{39601}x_2 + \frac{75}{39601}x_3 - \frac{99}{39601}x_4 + \frac{150}{39601}x_5 + \frac{50372}{39601} \\ &\text{subject to} \\ &\quad 2x_1 + x_2 + 3x_3 + x_4 + x_5 \le 15, \quad x_1 + 2x_2 + x_5 \le 8, \\ &\quad 3x_1 + 5x_2 + 2x_4 \le 10, \quad 2x_2 + 4x_3 + 3x_4 + x_5 \le 21, \\ &\quad x_1, x_2, x_3, x_4, x_5 \ge 0. \end{aligned}\right\} \qquad \text{(II)}$$

Now, introducing slack variables $x_6, x_7, x_8, x_9,$ and using simplex method, we obtain the following simplex tables:

Table 3(d): *Initial table for LPP (II)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_6$	0	15	2	1	3	1	1	1	0	0	0	15
$\mathbf{y}_7$	0	8	1	2	0	0	1	0	1	0	0	8
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0	-
$\mathbf{y}_9$	0	21	0	2	4	3	1	0	0	0	1	21
$G(\mathbf{x}) = \frac{50372}{39601}$		x_j	0	0	0	0	0	15	8	10	21	
		c_j	$-\frac{945}{39601}$	$-\frac{2661}{39601}$	$\frac{75}{39601}$	$-\frac{99}{39601}$	$\frac{150}{39601}$	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{945}{39601}$	$-\frac{2661}{39601}$	$\frac{75}{39601}$	$-\frac{99}{39601}$	$\frac{150}{39601}$	0	0	0	0	
							↑		↓			

Table 3(e): *Intermediate table for LPP (II)*

$\mathbf{B}$	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_6$	0	7	1	-1	3	1	0	1	-1	0	0	$\frac{7}{3}$
$\mathbf{y}_5$	$\frac{150}{39601}$	8	1	2	0	0	1	0	1	0	0	-
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0	-
$\mathbf{y}_9$	0	13	-1	0	4	3	0	0	-1	0	1	$\frac{13}{4}$
$G(\mathbf{x}) = \frac{51572}{39601}$		x_j	0	0	0	0	8	7	0	10	13	
		c_j	$-\frac{945}{39601}$	$-\frac{2661}{39601}$	$\frac{75}{39601}$	$-\frac{99}{39601}$	$\frac{150}{39601}$	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{1095}{39601}$	$-\frac{2961}{39601}$	$\frac{75}{39601}$	$-\frac{99}{39601}$	0	0	$-\frac{150}{39601}$	0	0	
					↑			↓				

Table 3(f): *Final table for LPP (II)*

$\mathbf{B}$	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$
$\mathbf{y}_3$	$\frac{75}{39601}$	$\frac{7}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0
$\mathbf{y}_5$	$\frac{150}{39601}$	8	1	2	0	0	1	0	1	0	0
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0
$\mathbf{y}_9$	0	$\frac{11}{3}$	$-\frac{7}{3}$	$\frac{4}{3}$	0	$\frac{5}{3}$	0	$-\frac{4}{3}$	$\frac{1}{3}$	0	1
$G(\mathbf{x}) = \frac{51747}{39601}$		x_j	0	0	$\frac{7}{3}$	0	8	0	0	10	$\frac{11}{3}$
		c_j	$-\frac{945}{39601}$	$-\frac{2661}{39601}$	$\frac{75}{39601}$	$-\frac{99}{39601}$	$\frac{150}{39601}$	0	0	0	0
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{1120}{39601}$	$-\frac{2936}{39601}$	0	$-\frac{124}{39601}$	0	$-\frac{25}{39601}$	$-\frac{125}{39601}$	0	0

Now the optimality criterion has been met, indicating that the optimal solution to the LP problem has been found. This optimal solution is as follows:

$\mathbf{x}^{(2)} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5]^T == [0 \;\; 0 \;\; 7/3 \;\; 0 \;\; 8]^T$ and max $G(\mathbf{x}) = 51747/39601 \approx 1.3$.

In order to check that optimal solution is the optimal solution to the given LFP problem, we repeat the process, taking $\mathbf{x}^{(2)} = [0 \;\; 0 \;\; 7/3 \;\; 0 \;\; 8]^T$ as the new feasible solution to the given LFP problem.

Denoting the objective function of the given LFP problem by $H(\mathbf{x})$, we have

$$H(\mathbf{x}^{(2)}) = \frac{55}{42}, \quad \left(\frac{\partial H}{\partial x_1}\right)_{(\mathbf{x}^{(2)})} = -\frac{17}{588}, \quad \left(\frac{\partial H}{\partial x_2}\right)_{(\mathbf{x}^{(2)})} = -\frac{191}{2352},$$

$$\left(\frac{\partial H}{\partial x_3}\right)_{(\mathbf{x}^{(2)})} = \frac{1}{784}, \quad \left(\frac{\partial H}{\partial x_4}\right)_{(\mathbf{x}^{(2)})} = -\frac{5}{1176}, \quad \left(\frac{\partial H}{\partial x_5}\right)_{(\mathbf{x}^{(2)})} = \frac{1}{392}.$$

Therefore, the reconstructed LP problem is

$$\left.\begin{array}{l} \text{Max } H(\mathbf{x}) = -\frac{17}{588}x_1 - \frac{191}{2352}x_2 + \frac{1}{784}x_3 - \frac{5}{1176}x_4 + \frac{1}{392}x_5 + \frac{3025}{2352} \\ \text{subject to} \\ \quad 2x_1 + x_2 + 3x_3 + x_4 + x_5 \le 15, \quad x_1 + 2x_2 + x_5 \le 8, \\ \quad 3x_1 + 5x_2 + 2x_4 \le 10, \;\; 2x_2 + 4x_3 + 3x_4 + x_5 \le 21, \\ \quad x_1, x_2, x_3, x_4, x_5 \ge 0. \end{array}\right\} \qquad \text{(IIII)}$$

Now, introducing slack variables x_6, x_7, x_8, x_9, and using simplex method, we obtain the following simplex tables:

Table 3(g): *Initial table for LPP (III)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_6$	0	15	2	1	3	1	1	1	0	0	0	15
$\mathbf{y}_7$	0	8	1	2	0	0	1	0	1	0	0	8
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0	-
$\mathbf{y}_9$	0	21	0	2	4	3	1	0	0	0	1	21
		x_j	0	0	0	0	0	15	8	10	21	
$H(\mathbf{x}) = \frac{3025}{2352}$		c_j	$-\frac{17}{588}$	$-\frac{191}{2352}$	$\frac{1}{784}$	$-\frac{5}{1176}$	$\frac{1}{392}$	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{17}{588}$	$-\frac{191}{2352}$	$\frac{1}{784}$	$-\frac{5}{1176}$	$\frac{1}{392}$	0	0	0	0	
							↑		↓			

Table 3(h): *Intermediate table for LPP (III)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$	$\frac{x_{B_i}}{y_{ij}}$
$\mathbf{y}_6$	0	7	1	-1	3	1	0	1	-1	0	0	$\frac{7}{3}$
$\mathbf{y}_5$	$\frac{1}{392}$	8	1	2	0	0	1	0	1	0	0	-
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0	-
$\mathbf{y}_9$	0	13	-1	0	4	3	0	0	-1	0	1	$\frac{13}{4}$
		x_j	0	0	0	0	8	7	0	10	13	
$H(\mathbf{x}) = \frac{3073}{2352}$		c_j	$-\frac{17}{588}$	$-\frac{191}{2352}$	$\frac{1}{784}$	$-\frac{5}{1176}$	$\frac{1}{392}$	0	0	0	0	
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{37}{1176}$	$-\frac{203}{2352}$	$\frac{1}{784}$	$-\frac{5}{1176}$	0	0	$-\frac{1}{392}$	0	0	
					↑			↓				

Table 3(i): *Final table for LPP (III)*

B	$\mathbf{c}_B$	$\mathbf{X}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_7$	$\mathbf{y}_8$	$\mathbf{y}_9$
$\mathbf{y}_3$	$\frac{1}{784}$	$\frac{7}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0
$\mathbf{y}_5$	$\frac{1}{392}$	8	1	2	0	0	1	0	1	0	0
$\mathbf{y}_8$	0	10	3	5	0	2	0	0	0	1	0
$\mathbf{y}_9$	0	$\frac{11}{3}$	$-\frac{7}{3}$	$\frac{4}{3}$	0	$\frac{5}{3}$	0	$-\frac{4}{3}$	$\frac{1}{3}$	0	1
		x_j	0	0	$\frac{7}{3}$	0	8	0	0	10	$\frac{11}{3}$
$H(\mathbf{x}) = \frac{55}{42}$		c_j	$-\frac{17}{588}$	$-\frac{191}{2352}$	$\frac{1}{784}$	$-\frac{5}{1176}$	$\frac{1}{392}$	0	0	0	0
		$c_j - \mathbf{c}_B\mathbf{y}_j$	$-\frac{75}{2352}$	$-\frac{101}{1176}$	0	$-\frac{11}{2352}$	0	$-\frac{1}{2352}$	$-\frac{5}{2352}$	0	0

Now the optimality criterion has been met, indicating that the optimal solution to the LP problem has been found. This optimal solution is as follows:

$\mathbf{x}^{(3)} = [x_1 \ \ x_2 \ \ x_3 \ \ x_4 \ \ x_5]^T = [0 \ \ 0 \ \ 7/3 \ \ 0 \ \ 8]^T$ and $\max H(\mathbf{x}) = 55/42 \approx 1.3$.

Obviously, $\mathbf{x}^{(2)} = \mathbf{x}^{(3)}$. Therefore the optimal solution to the given LFP problem is reached, which is given by $\mathbf{x} = [x_1 \ \ x_2 \ \ x_3 \ \ x_4 \ \ x_5]^T = [0 \ \ 0 \ \ 7/3 \ \ 0 \ \ 8]^T$ and $\max z = 1.3$.

VI. Comparison of the Numerical Results

The following table provides a summary of a comparison analysis between the suggested approach and existing optimization techniques for the numerical examples.

Table 4: *Comparative analysis of the proposed method and existing methods*

Example	Reference	Number of iterations	Optimal solution	Optimal value
1	Proposed	8	$[1 \quad 2 \quad 0]^T$	2
	Reference [9]	4	$[1 \quad 2 \quad 0]^T$	2
2	Proposed	8	$[2/3 \quad 0 \quad 5/3]^T$	1.7
	Reference [8]	4	$[2/3 \quad 0 \quad 5/3]^T$	1.7
3	Proposed	9	$[0 \quad 0 \quad 7/3 \quad 0 \quad 8]^T$	1.3
	Reference [20]	9	$[0.00000 \quad 0.00003 \quad 7/3 \quad 0.00009 \quad 8]^T$	1.3
	Reference [21]	12	$[0.00001 \quad 0.00002 \quad 7/3 \quad 0.00009 \quad 8]^T$	1.3

VII. Discussion

From the current study, we made the following observations:

- The optimal solutions obtained using the proposed method are identical to those derived from existing approaches.
- Although the proposed method may require more iterations for certain LFP problems compared to existing techniques, the overall computational effort remains unaffected. This is attributed to the invariance of constraints across all reconstructed LP subproblems. This can be observed by comparing Tables 1(a), 1(b), and 1(c) with Tables 1(d), 1(e), and 1(f), respectively, in Example 1, and similarly for other examples.
- In some cases, the proposed method solves LFP problems using fewer iterations than existing methods.
- While conventional methods are typically restricted to problems with positive denominators, the proposed method extends the applicability of LFP solvers to cases involving negative denominators as well.
- The optimal solution found by the proposed method satisfies all the constraints of the given LFP problem. But it is not so in all the existing methods for solving LFP problems. For example, the optimal solution of LFP problem mentioned in Example 2 by the existing methods of Grar [20] or Lustig [21] does not satisfy all the constraints.

In light of these findings, the proposed method can be regarded as a valid and effective approach for solving LFP problems.

VIII. Conclusion

In this study, we introduced a new approach for solving linear fractional programming (LFP) problems involving inequality constraints. The performance of the proposed approach was assessed through a series of numerical examples and compared against established methods. The results demonstrate that the proposed method consistently yields the same optimal solution as existing techniques, requiring more iterations in some instances and fewer in others. Notably, even in cases involving a higher number of iterations, the overall computational effort remains largely unaffected due to the structural invariance of constraints across the reconstructed linear programs. Although the objective function is approximated via a Taylor series expansion, this does not compromise the accuracy of the optimal solution. Moreover, a significant advantage of the proposed method lies in its ability to handle LFP problems with a negative denominator in the objective function—a limitation inherent in most traditional approaches. Future work may focus on extending the proposed method to address more complex and higher-dimensional LFP problems, further enhancing its applicability and robustness in real-world scenarios.

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