PERFORMANCE MEASURE OF OPTIMAL RESERVE INVENTORY MODEL BETWEEN TWO MACHINES WITH REFERENCE TO TRUNCATION POINT OF THE REPAIR TIME UNDER FUZZY ENVIRONMENT

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Abstract

In inventory control theory, various models have been developed to determine optimal stock levels in practical settings. In sequential production systems, reserve inventory is essential to minimize idle time when upstream machine failures occur. This study assumes that machine repair times are exponentially distributed and satisfy the Setting the Clock Back to Zero (SCBZ) property, with the truncation point itself modeled as a random variable following a mixed exponential distribution. Under these assumptions, an optimal reserve inventory model is derived and analyzed in a fuzzy environment. Uncertainty in system parameters is incorporated using fuzzy numbers, which are defuzzified through the Trisectional Fuzzy Trapezoidal ranking method. Numerical examples validate the applicability of the model in real-life scenarios.

Keywords: Reserve Inventory, SCBZ property, Repair time and Truncation point, Fuzzy Numbers, Trisectional Fuzzy Trapezoidal ranking.

I. Introduction

Mathematical modeling of inventory systems provides valuable insights into determining optimal inventory policies in real-world industrial environments. In sequential machine systems, reserve inventory is critical for reducing the idle costs of downstream machines during breakdowns of upstream machines.

Consider two machines, M1 and M2, where the output of M1 serves as the input for M2. If M1 fails, M2 remains idle, incurring significant costs. Maintaining reserve inventory between them minimizes such costs. However, excess reserves lead to holding costs, while insufficient stock increases idle time costs. Therefore, the challenge lies in determining an optimal reserve inventory that balances these conflicting costs. This problem was first formalized by Hanssmann [6], with subsequent contributions from Ramanarayanan et al. [12], Sachithanantham et al. [16, 17], and later Sachithanantham and Jagatheesan [18], who extended the model under the SCBZ property with truncation points as random variables. Further modifications were made by Ramathilagam et al. [13], and applications were discussed in series machine models by Srinivasan et al. [20], Sehik

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Udhuman et al. [19], Nandakumar et al. [9, 10], and others.

Parallel developments included extensions to three-machine systems by Venkatesan et al. [21] and models using generalized distributions developed by Henry et al. [7]. Studies have also incorporated order statistics done by Govindhan et al. [5] and different change-point assumptions. Incorporating fuzzy set theory by Kaufmann [8] has enhanced realism in handling uncertain environments. Several fuzzy ranking methods have been proposed, including those by Chu [3], Chen et al. [2], Deng [4], Wang et al. [22], and Yager [23]. More recently, Ramesh et al. [14, 15] employed fuzzy ranking in queueing models.

This article enhances the model of Sachithanantham and Jagatheesan [18] by combining stochastic processes with fuzzy set theory, applying the Trisectional Fuzzy Trapezoidal ranking method for defuzzification. Numerical examples validate the model's robustness and applicability in reserve inventory management.

II. Preliminaries

Optimal Reserve Inventory - A system in which there are two machines M_1 and M_2 are in series. The output of the machine M_1 is the input of the machine M_2. The breakdown of M_1 causes the idle time of M_2, since there is no input to M_2 from M_1. The idle time of M2 is very costly and hence to avoid it, a reserve inventory is maintained in between M_1 and M_2. If the reserve inventory is surplus in quantity, there is an inventory holding cost. If the reserve inventory slacks in its quantity, then it assumes the high idle time cost. An inventory which is balancing out the above two costs is known as Optimal Reserve Inventory.

Fuzzy Number - A fuzzy number is a generalization of a regular real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1.

Trapezoidal Fuzzy Number - A trapezoidal fuzzy number $\tilde{A}(a_1, a_2, a_3, a_4; 1)$ is dictated by a membership function

$$\mu_{\tilde{A}}(z) = \begin{cases} \dfrac{z-a_1}{a_2-a_1}, a_1 \le z \le a_2 \\ 1, \qquad a_1 \le z \le a_2 \\ \dfrac{z-a_4}{a_3-a_4}, a_3 \le z \le a_4 \\ 0, \qquad otherwise \end{cases}$$

III. Model Description

The model which is assumed that the lead time random variable follows an exponential distribution, which satisfies the so called SCBZ property and it is also assumed that the truncation point itself a random variable which has the mixed exponential distribution with parameter θ_1 and θ_2. Based on the above stated assumptions the optimal reserve inventory model considered by Sachithanantham et al. [18] the derived model is

$$\frac{h\mu r}{d} - \left[\beta e^{-s/_r(\theta+\theta_1)} \left(\frac{\theta-\theta^*}{\theta_1+\theta-\theta^*} \right) + (1-\beta) e^{-s/_r(\theta+\theta_2)} \left(\frac{\theta-\theta^*}{\theta_2+\theta-\theta^*} \right) \right. \\ \left. + e^{-\theta^*(s/_r)} \left(\frac{\beta\theta_1}{(\theta_1+\theta-\theta^*)} + \frac{(1-\beta)\theta_2}{(\theta_2+\theta-\theta^*)} \right) \right] = 0 \tag{1}$$

where,

h	:	Inventory holding cost/ unit/ unit time.
d	:	Idle time cost due to M_2 / unit time.
μ	:	Mean time interval between successive breakdown of machine M_1, assuming exponential distribution of inter-arrival times.
t	:	Continuous random variable denoting the repair time of M_1 with pdf g(.) and c.d.f G(.)
r	:	Constant consumption rate of M_2 per unit of time.
S	:	Reserve inventory between M_1 and M_2.
T	:	Random variable denoting the idle time of M_2.
θ	:	Parameter of exponential distribution before the truncation point.
θ^*	:	Parameter of exponential distribution after the truncation point.
β	:	Probability value involved in the mixed exponential distribution.
θ_1, θ_2	:	Parameters of mixed exponential distribution.

Here the performance of above optimum reserve inventory model is verifying under fuzzy environment by using Tri-sectional Fuzzy Trapezoidal ranking method.

I. Tri-Sectional Fuzzy Trapezoidal Ranking– Algorithm

Let us consider an interval data (P, Q). The trisection of this interval is taken as d $d = \frac{Q-P}{3}$ then the required trapezoidal fuzzy number is [P, P+d, P+2d, Q]. Consider a normal trapezoidal fuzzy number $\tilde{A}[a_1, a_2, a_3, a_4]$ which is represented in Figure 1.

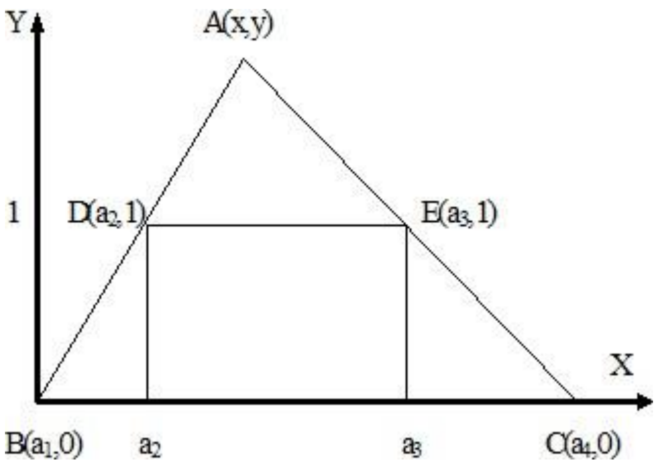

Figure1: *Normal Trapezoidal Fuzzy Number*

Extend the line joining B(a_1, 0) and D (a_2, 1) as BDA and the line joining C(a_4, 0) and E(a_3, 1) as CEA. The intersection of extended lines BD and CE is A. The coordinates for intersection point A is (x, y), where

$$x = \frac{a_1 a_3 - a_2 a_4}{a_3 - a_4 - a_2 + a_1}$$

$$y = \frac{x - a_2}{a_2 - a_1} + 1$$

The proposed ranking technique is based on the concept of in-center for triangle ABC and is noted in Figure 2

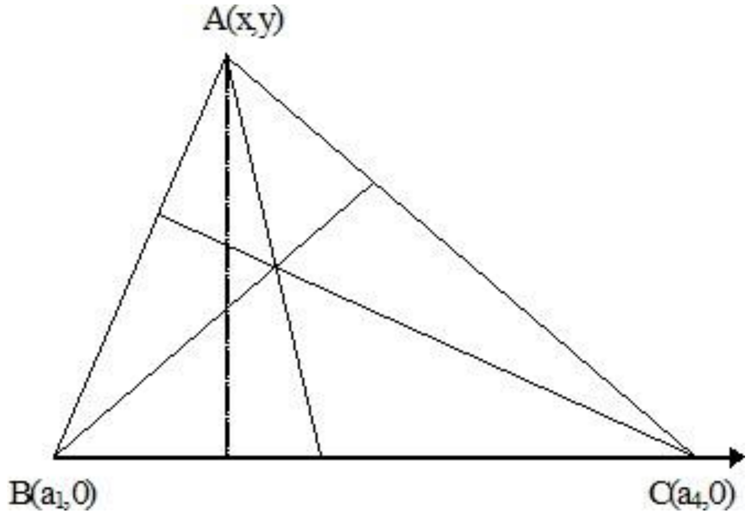

Figure 2 : *In-center*

and is defined as

$$R(\tilde{A}) = \frac{ax + ba_1 + a_4}{a + b + c}$$

where

$a = a_4 - a_1$, $b = \sqrt{y^2 + (a_4 - x)^2}$ and $c = \sqrt{y^2 + (x - a_1)^2}$

This proposed fuzzy ordering technique, which de-fuzzifies the fuzzy numerals in the optimum reserve inventory model. We have used these fuzzy numerals as the fuzzy parameters.

IV. Real – Life Cases

In tile manufacturing companies, there are at least two key steps involved in the production process. The first step is the creation of tiles using raw materials, and the second is the polishing of the tiles with colors and designs. In the event that the tile-making machine is under repair, the polishing machine would not receive any input. By applying the proposed model outlined in equation (1) and maintaining an optimal level of inventory, it is possible to avoid delays and the associated costs of waiting.

When this scenario is applied in a standard environment, the equation in (1) has already been derived and verified through numerical examples. In this case, however, the same scenario is applied in a fuzzy environment, and its performance is evaluated accordingly.

Trapezoidal fuzzy number - Let us assumes the values of various parameters in equation (1) and its performance were analyzed in Fuzzy environment.

Illustration 1.1: For $\tilde{d}$=[2500,2600,2700,2800], $\tilde{\mu}$=[4,6,8,10], $\tilde{r}$=[30,40,50,60], $\tilde{\theta}$=[2,3,4,5], $\tilde{\beta}$=[0.3,0.5,0.7,0.9], $\tilde{\theta}^*$=[0,2,4,6], $\tilde{\theta}_1 = [0,2,4,6]$, $\tilde{\theta}_2 = [1,3,5,7]$, the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of $\tilde{h}$ are listed.

Now the ranking index of $\tilde{h}$is R($\tilde{h}$) = R(3,4,5,6) = 4.5 , R($\tilde{h}$) = R(4,5,6,7) = 5.5, R($\tilde{h}$) = R(5,6,7,8) = 6.5 and R($\tilde{h}$) = R(6,7,8,9) = 7.5.

Using the de-fuzzified value of the stock holding cost $\tilde{h}$, along with the corresponding de-fuzzified values of other relevant parameters, the optimal reserve inventory $\hat{S}$ is determined by solving Equation (1). The results are summarized in the table provided below.

Table 1: *The variations in $\hat{S}$ for the changes in the value of $\tilde{h}$*

$\tilde{h}$	$\tilde{d}$	$\tilde{\mu}$	$\tilde{r}$	$\tilde{\theta}$	$\tilde{\beta}$	$\tilde{\theta}^*$	$\tilde{\theta}_1$	$\tilde{\theta}_2$	$\hat{S}$ x10-4
[3,4, 5,6]	[2500,2600, 2700,2800]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3 ,5,7]	203990
[4,5, 6,7]	[2500,2600, 2700,2800]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3, 5,7]	174900
[5,6, 7,8]	[2500,2600, 2700,2800]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3, 5,7]	151000
[6,7, 8,9]	[2500,2600, 2700,2800]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3, 5,7]	130600

The above predicted values of Ŝ are shown in the following graph.

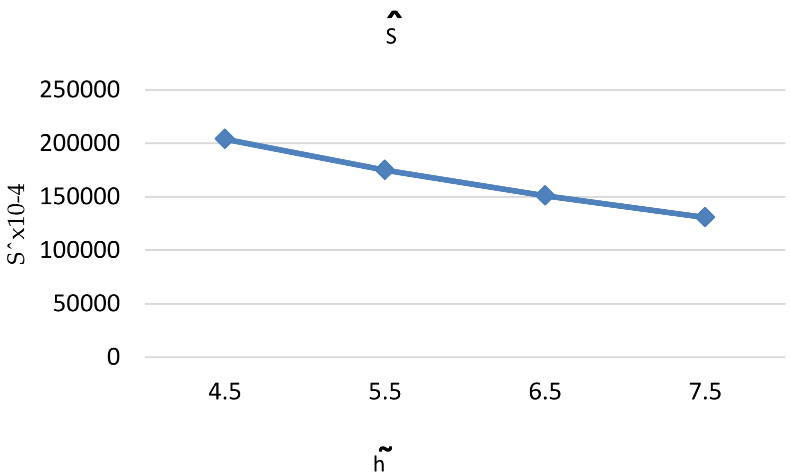

Figure 3: *Variations in $\hat{S}$ for the changes in the value of $\tilde{h}$*

From the graph (Figure 3), it is observed that if the values of the holding cost h˜increases, then the optimum base stock is reduced.

Illustration 1.2:

For $\tilde{h}$=[3, 4, 5, 6], $\tilde{\mu}$=[4,6,8,10],$\tilde{r}$=[30,40,50,60],$\tilde{\theta}$=[2,3,4,5],$\tilde{\beta}$=[0.3,0.5,0.7,0.9],$\tilde{\theta}^*$=[0,2,4,6], $\tilde{\theta}_1$ = [0,2,4,6] , $\tilde{\theta}_2$ = [1,3,5,7], the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of $\tilde{d}$ are listed.

Now the ranking index of $\tilde{d}$ is R($\tilde{d}$) = R(2500, 2600 ,2700, 2800) = 2650, R($\tilde{d}$) = R(2600, 2700, 2800, 2900) = 2750, R($\tilde{d}$) = R(2700, 2800, 2900, 3000) = 2850 and R($\tilde{d}$) = R(2800, 2900, 3000, 3100) = 2950.

By using the above de-fuzzified numbers for the idle time cost and similarly the de-fuzzified values of all other parameters, the optimal reserve inventory $\hat{S}$ is obtained by solving the equation (1) and are presented in the table below.

Table 2: *The variations in $\hat{S}$ for the changes in the value of $\tilde{d}$*

$\tilde{h}$	$\tilde{d}$	$\tilde{\mu}$	$\tilde{r}$	$\tilde{\theta}$	$\tilde{\beta}$	$\tilde{\theta}^*$	$\tilde{\theta}_1$	$\tilde{\theta}_2$	$\hat{S}$ x10-4
[3,4, 5,6]	[2500,2600, 2700,2800]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3, 5,7]	203990
[3,4, 5,6]	[2600,2700, 2800,2900]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3, 5,7]	209300

[3,4, 5,6]	[2700,2800, 2900,3000]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3, 5,7]	214500
[3,4, 5,6]	[2800,2900, 3000,3100]	[4,6, 8,10]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2, 4,6]	[0,2, 4,6]	[1,3, 5,7]	219600

The above predicted values of Ŝ are shown in the following graph.

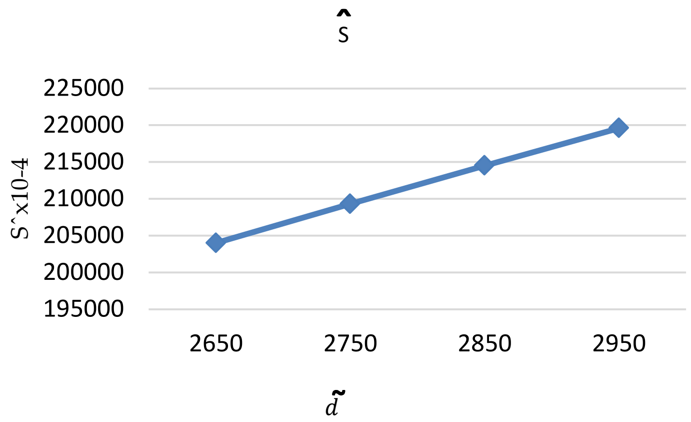

Figure 4: *Variations in $\hat{S}$ for the changes in the value of $\tilde{d}$*

From the graph (Figure 4), it is observed that if the values of the idle time cost $\tilde{d}$ increases, then the optimum base stock is increased.

Illustration 1.3: For h̃=[3, 4, 5, 6],d̃=[2500,2600,2700,2800], r̃=[30,40,50,60],$\tilde{\theta}$=[2,3,4,5], β̃=[0.3,0.5,0.7,0.9],θ̃*=[0,2,4,6], $\tilde{\theta}_1$ = [0,2,4,6] , $\tilde{\theta}_2$ = [1,3,5,7], the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of μ̃ are listed.

Now the ranking index of μ̃ is R(μ̃) = R(1.5, 2.5, 3.5, 4.5) = 3, R(μ̃) = R(2.5, 3.5, 4.5, 5.5) = 4, R(μ̃) = R(3.5, 4.5, 5.5, 6.5) = 5 and R(μ̃) = R(4.5, 5.5, 6.5, 7.5) = 6

By using the above de-fuzzified numbers for the Mean time interval between successive breakdown of machine M_1 and similarly the de-fuzzified values of all other parameters, the optimal reserve inventory $\hat{S}$ is obtained by solving the equation (1) and are presented in the table below.

Table 3: *The variations in $\hat{S}$ for the changes in the value of μ̃*

h̃	d̃	μ̃	r̃	$\tilde{\theta}$	β̃	θ̃*	$\tilde{\theta}_1$	$\tilde{\theta}_2$	$\hat{S}$ x10-4
[3,4,5,6]	[2500,2600, 2700,2800]	[1.5, 2.5, 3.5, 4.5]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2,4,6]	[0,2, 4,6]	[1,3, 5,7]	203990
[3,4,5,6]	[2500,2600, 2700,2800]	[2.5, 3.5, 4.5, 5.5]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2,4,6]	[0,2, 4,6]	[1,3, 5,7]	162450
[3,4,5,6]	[2500,2600, 2700,2800]	[3.5, 4.5, 5.5, 6.5]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2,4,6]	[0,2, 4,6]	[1,3, 5,7]	130650
[3,4,5,6]	[2500,2600, 2700,2800]	[4.5, 5.5, 6.5, 7.5]	[30,40, 50,60]	[2,3, 4,5]	[0.3,0.5, 0.7,0.9]	[0,2,4,6]	[0,2, 4,6]	[1,3, 5,7]	83580

The above predicted values of Ŝ are shown in the following graph.

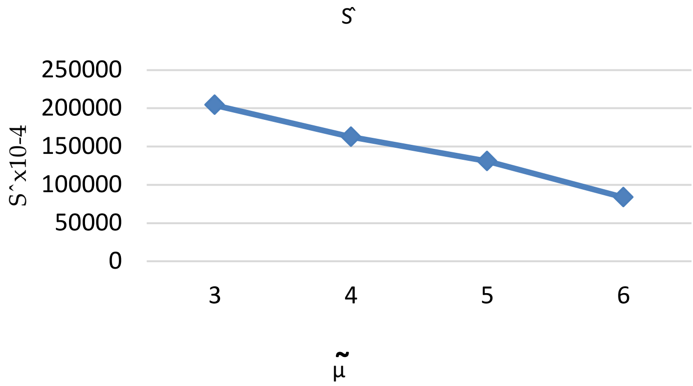

Figure 5: Variations in $\hat{S}$ for the changes in the value of $\tilde{\mu}$

From the graph (Figure 5), it is observed that if the values of $\tilde{\mu}$ increases, then the optimum base stock is increased.

V. Findings

The performance of the proposed optimal reserve inventory model has been examined under a fuzzy environment using the Trisectional Fuzzy Trapezoidal ranking method. The numerical illustrations, supported by tables and graphs, reveal important managerial and theoretical insights into the behavior of the system:

Effect of Holding Cost (h): When the inventory holding cost increases, the optimal reserve inventory $\hat{S}$ decreases. This implies that organizations are encouraged to adopt a leaner reserve policy in high holding cost environments, minimizing storage expenses even at the risk of higher idle time.

Effect of Idle Time Cost (d): As the idle time cost of machine M2 increases, the optimal reserve inventory $\hat{S}$ increases. This result emphasizes the importance of ensuring a sufficient buffer stock when production downtime is highly penalized, as it directly safeguards against costly interruptions.

Effect of Mean Breakdown Interval (μ): An increase in the mean time between successive breakdowns leads to a decrease in $\hat{S}$. This is logical, as longer intervals between breakdowns reduce the likelihood of M2 experiencing input shortages, thereby lowering the need for larger reserves.

These findings highlight the trade-off between holding costs and downtime costs, illustrating that the reserve inventory acts as a balancing mechanism. The fuzzy environment introduces flexibility by accommodating uncertainty in parameters, ensuring that the derived solutions are not rigid but adaptable to real-world variability.

From a broader perspective, the results establish that the fuzzy-based model is robust, as the optimal solutions derived under fuzziness are consistent with those obtained in the crisp setting. This validates the practical reliability of the model and demonstrates its capability to integrate stochastic variability with decision-making under uncertainty.

VI. Concluding Remarks and Future Scope

This study successfully integrates stochastic processes, SCBZ property-based repair time analysis, and fuzzy set theory into a unified framework for reserve inventory optimization. The combination of mixed exponential distributions for truncation points with fuzzy defuzzification techniques results in a more comprehensive and practical model.

Future extensions may include:

Extending the model to multi-machine or multi-stage systems, capturing more complex industrial workflows.

Considering time-varying parameters, such as age-dependent breakdown rates or dynamic consumption rates of M2.

Incorporating hybrid fuzzy-stochastic approaches, which may further refine accuracy in environments where both randomness and vagueness coexist strongly.

In conclusion, the proposed fuzzy-based reserve inventory model not only corroborates the findings of its crisp counterparts but also broadens their applicability to uncertain environments. This dual validation enhances confidence in the model, making it a valuable decision-support tool for modern manufacturing systems seeking to balance cost efficiency with operational reliability.

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