A NEW MODIFIED POWER GENERALIZED WEIBULL DISTRIBUTION: PROPERTIES AND APPLICATION

[1]A. M. Rangoli, [2]A. S. Talawar and [3]Vijaya Sorganvi
•
[1] Lecturer Cum Statistician, Department of Community Medicine, BLDE (DU), Shri B. M. Patil Medical College, Hospital and Research Centre, Vijayapura, India
ajmr1008@gmail.com
[2]Senior Professor, Department of Statistics, Karnatak University, Dharwad. India.
astalawar@kud.ac.in
[3]Associate Professor, Department of Community Medicine, BLDE (DU), Shri B. M. Patil Medical College, Hospital and Research Centre, Vijayapura, India.
vijaya.sorganvi@bldedu.ac.in

Abstract

Introduction: Survival analysis has become increasing important for various fields like clinical-trails, public health industrial reliability. A major challenge is accurately modelling of failure time data. Using lifetime distribution, we can accurately model the failure time data and can estimate hazard and survival function. In this study we introduced new modified power generalized Weibull (NMPGW) distribution which exhibits bimodal as well as unimodal density patterns, and also, various patterns of hazard curves like increasing, bathtub and decreasing-increasing-decreasing patterns.
Methodology: The proposed NMPGW distribution's various density and hazard patterns were studied. Various statistical properties of the model were derived. Estimation of the parameters were done using maximum likelihood estimation (via Expectation Maximization algorithm) and Bayes technique (via Metropolis Hasting algorithm). Also, standard error and confidence limit were estimated. Comparison of various lifetime distributions were done using information criterion.
Results: The model shows good fit for hospital dataset in comparison to various lifetime distributions. Estimated the hazard rate which shows decreasing-increasing-decreasing patterns. In comparison to MLE, Bayes estimation gives lower standard error. Kaplan-Meier survival and NMPGW distribution survival curve shows nearer.
Conclusion: The NMPGW distribution introduced in this study offers a versatile tool for modeling different hazard rate patterns. The model's strong performance, validated through real hospital data, suggests it could be a valuable addition to survival analysis, outperforming other modified Weibull models in terms of fit and flexibility.

Keywords: New Modified power generalized Weibull distribution, MCMC, hazard, survival, EM algorithm.

I. Introduction

Recently, survival analysis has gained significant attention due to the increasing complexity of data that tracks the time until an event occurs, such as in clinical trials, public health studies, and industrial reliability testing. This method is particularly valuable because it can handle incomplete

DOI: https://doi.org/10.24412/1932-2321-2025-489-434-448

data (referred to as censored data) and helps identify the factors influencing the duration of survival for individuals or items within a group. One of the main challenges in survival analysis is accurately modeling failure rates, which necessitates the use of lifetime distributions capable of capturing various hazard rate patterns over time.

Among the commonly used models, the Weibull distribution (WD) has proven effective in lifetime and reliability studies due to its ability to model failure rates that either increase or decrease steadily. Its flexibility in handling right- or left-skewed data makes it applicable across a wide range of fields. However, the standard Weibull distribution has limitations when dealing with complex, non-monotonic hazard rate patterns, such as the bathtub-shaped or upside-down bathtub-shaped hazard functions frequently observed in real-world data. To address these limitations, researchers have proposed several extensions and modifications of the Weibull distribution, incorporating additional parameters that enhance the flexibility and accuracy of the model in representing real-world failure behaviors.

For instance, the Additive Weibull Distribution was introduced to capture the bathtub-shaped hazard rate [1]. A power generalized Weibull distribution with three parameters was proposed to offer greater flexibility [2]. A three-parameter Weibull model specifically designed to accommodate bathtub-shaped hazard functions was also developed [3]. Another variation, the modified Weibull distribution, was created by incorporating an additional scale parameter into the standard Weibull model [4]. The Flexible Weibull Distribution, capable of modeling both increasing and bathtub-type failure behaviors, was also introduced [5]. Furthermore, a modification of the Bebbington et al. [5] model was developed to better handle bimodal datasets using three parameters [6]. A modified power generalized Weibull distribution with four parameters, capable of modeling various hazard patterns, was later proposed [7]. More recently, a new modified exponentiated Weibull distribution comprising four parameters has been introduced, demonstrating utility in modeling different hazard patterns and proving especially effective for bimodal datasets [8].

In this article, we introduce new modified power generalized Weibull distribution (NMPGW) with having five parameters and estimate hazard and survival functions. This paper is organized as follows: We introduce the new modified power generalized Weibull distribution in Section 2. Statistical properties were considered in section 3. The maximum likelihood estimation and Bayes estimation were provided in Section 4. In Section 5, we discuss with real-life application. Finally, Section 6 deals with concluding remarks.

II. Model Formulation

Let $T_1, T_2, \dots T_i., T_n$ are failure times, we assume that T_i to follows new modified power generalized Weibull random variables with parameters $(\alpha,\ \beta,\ \gamma,\ \lambda, \theta)$, $i = 1,2, \dots n$.

Let $f(t)$, $F(t)$, $S(t)$ and $h(t)$ be density, cumulative distribution, survival and hazard functions of t.

Let the cumulative distribution function (CDF) is,

$$F(t) = 1 - e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha}-(\beta t^{\gamma})} \tag{1}$$

where $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$, $\lambda > 0$, $t \geq 0$, β and λ are scale parameters, α, θ and γ are shape parameters.

The probability density function (PDF) is,

$$f(t) = \left(\alpha\lambda\,\theta\, t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\, t^{\gamma-1})\right) e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha}-(\beta t^{\gamma})} \tag{2}$$

Hazard Function is,

$$h(t) = \left(\alpha\lambda\,\theta\, t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\, t^{\gamma-1})\right) \tag{3}$$

Survival Function is,

$$S(t) = e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha}-\left(\beta t^{\gamma}\right)} \tag{4}$$

This modified Weibull is widely used in reliability, human mortality, etc.

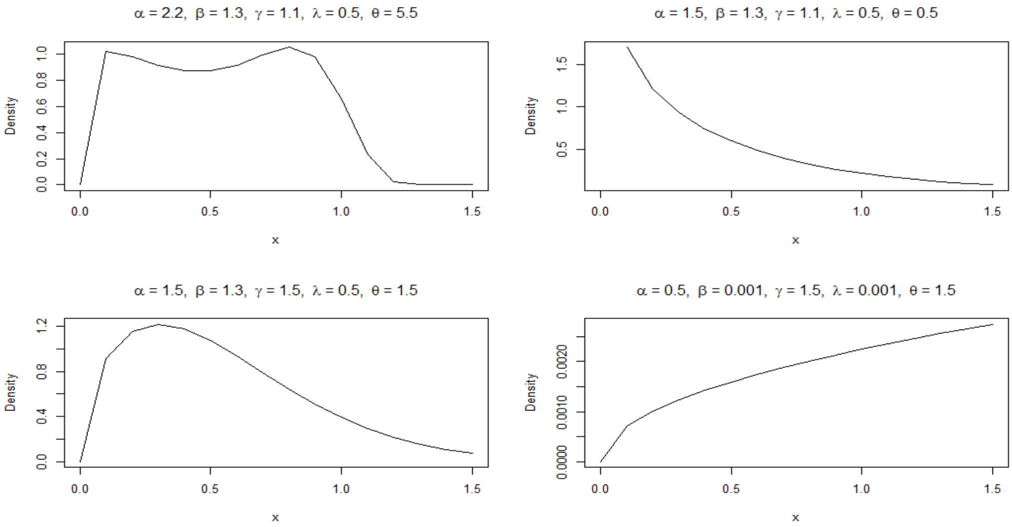

Figure 1: *Density Pattern for different values of parameter.*

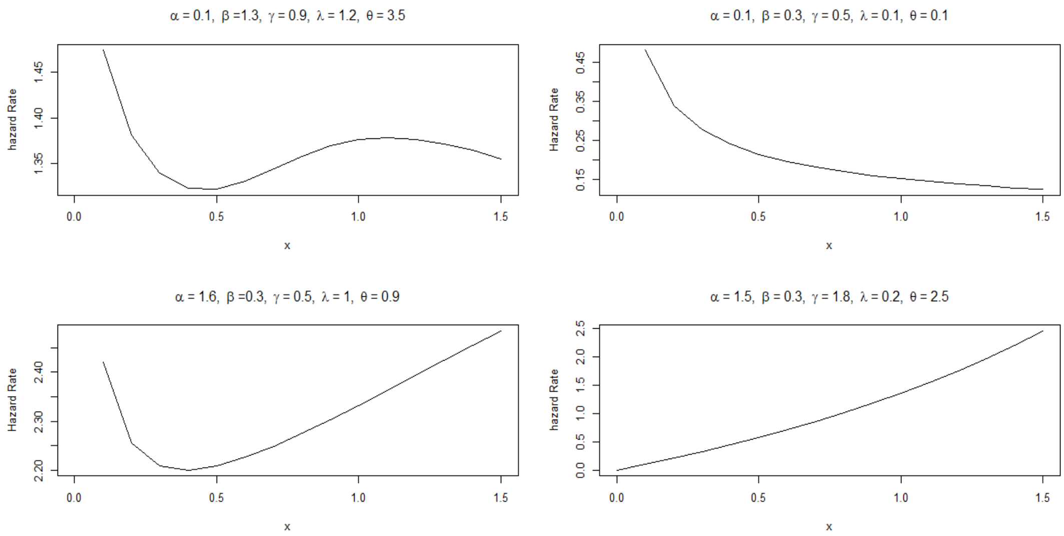

Figure 2: *Hazard Pattern for different values of parameters.*

Figure 1 illustrates various patterns of the density plot. It is evident that the proposed density function is particularly effective in capturing bimodal distributions. Figure 2 displays different patterns of the hazard rate, demonstrating the flexibility of the newly modified distribution. It is capable of modeling increasing, decreasing, constant, bathtub-shaped, as well as decreasing-increasing-decreasing hazard rate patterns.

Hazard patterns: For all $\lambda, \beta < 1$
As $\alpha, \theta, \gamma < 1$ the hazard is decreasing and $\alpha, \theta, \gamma > 1$ the hazard is increasing.
As $\alpha, \theta < 1, \gamma > 1$ the hazard is increasing and $\alpha, \theta > 1, \gamma < 1$ the hazard is bathtub in shape.
As $\alpha, \gamma < 1, \theta > 1$ the hazard is decreasing and $\alpha, \gamma > 1, \theta < 1$ the hazard is bathtub in shape.
As $\alpha, \theta, \gamma = 1$ gives the constant hazard rate.
As $\theta, \gamma < 1, \alpha > 1$ the hazard is decreasing and $\theta, \gamma > 1, \alpha < 1$ the hazard is increasing in shape

I. Sub distributions

a. $\gamma = 1$ tends to modified power generalized Weibull distribution given by [7]

$$F(t) = 1 - e^{1-(1+\lambda t^{\theta})^{\alpha} - \beta t}$$

b. $\beta = 0$ tends to power generalized Weibull distribution given by, [2]

$$F(t) = 1 - e^{1-(1+\lambda t^{\theta})^{\alpha}}$$

c. $\alpha = 1$ tends to Additive Weibull distribution given by [1]

$$F(t) = 1 - e^{-(\lambda t^{\theta} + \beta t^{\theta})}$$

d. $\alpha = 1, \gamma = 1$ tends to modified Weibull distribution given by [4]

$$F(t) = 1 - e^{-(\lambda t^{\theta} + \beta t)}$$

e. $\alpha = 1, \beta = 0$ tends to Weibull distribution

$$F(t) = 1 - e^{-\lambda t^{\theta}}$$

f. $\beta = 0, \alpha = 1, \theta = 1$ tends to Exponential distribution

$$F(t) = 1 - e^{\lambda t}$$

III. Statistical Properties

Mean: The mean of the random variable T with pdf $f(t)$ is given by,

$$E(T) = \int_0^{\infty} t\, f(t) dt$$

$$E(T) = \int_0^{\infty} t\, \left(\alpha\lambda\, \theta\, t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\, \gamma\, t^{\gamma-1}\,)\right) e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})} dt$$

Since this integral cannot be solved in a simple closed form, a numerical method must be used to estimate the mean, which represents the average time until the event happens.

Variance: The variance measures the spread of the time to failure.

$$V(T) = E(T^2) - (E(T))^2$$

$$E(T^2) = \int_0^{\infty} t^2\, f(t) dt$$

$$E(T^2) = \int_0^{\infty} t^2 \left(\alpha\lambda\, \theta\, t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\, \gamma\, t^{\gamma-1}\,)\right) e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})} dt$$

This integral is also computed using numerical method.

Moment Generating Function (MGF): The r^{th}moment of T denotes $E(T^r)$ is given by,

$$E(T^r) = \int_0^{\infty} t^r\, f(t) dt$$

The MGF is defined as,

$$M_T(s) = E(e^{sT}) = \int_0^{\infty} e^{sT}\, f(t) dt$$

$$M_T(s) = \int_0^{\infty} \left(\alpha\lambda\, \theta\, t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\, \gamma\, t^{\gamma-1}\,)\right) e^{sT+1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})}\, dt$$

The MGF provides a way to compute all moments of the distribution by taking derivates,

$$E(T^r) = \left.\frac{d^r M_T(s)}{ds^r}\right|_{s=0}$$

Here if $r = 1$ we get first moment (Mean, $E(T)$), and $r = 2$ we get second moment ($E(T^2)$).

Cumulative Hazard Function: The cumulative hazard function is given by,

$$H(t) = \int_0^t \lambda(u)\, du = \int_0^t \left(\alpha\lambda\, \theta\, t^{\theta-1}\left(1 + \lambda t^{\theta}\right)^{\alpha-1} + (\beta\, \gamma\, t^{\gamma-1})\right) du$$

$$H(t) = \alpha\lambda\theta \int_0^t u^{\theta-1}\left(1 + \lambda u^{\theta}\right)^{\alpha-1} du + \gamma\, \beta \int_0^t u^{\lambda-1}\, du$$

$$H(t) = \alpha\lambda\theta \int_0^t u^{\theta-1}\left(1 + \lambda u^{\theta}\right)^{\alpha-1} du + \gamma \frac{\beta}{\lambda} t^{\lambda}$$

This integral does not have a general closed-form solution for arbitrary α, λ, θ. It is typically solved numerically method.
So, we write $S(t) = e^{-H(t)}$.

Quantile: The quantile function is the inverse of the CDF. For a given probability p the quantile function $Q(p)$ gives the value at t such that probability of failure by that time is p.

$$Q(p) = F^{-1}(p)$$
$$F(t) = p$$
$$1 - e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})} = p$$
$$1 - \left(1 + \lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma}) = -\ln(1-p)$$
$$\left(1 + \lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma}) = 1 - \ln(1-p)$$

This equation can be solved using numerical method. If we substitute $p = 0.25,\ 0.5,\ 0.75$, we get first, second (median) and third quartiles respectively.

Order Statistics:

Let $T_1, T_2, \dots, T_n$ be a random sample of size n from new modified power generalized Weibull distribution with PDF and CDF defined in (2) and (1) respectively. We give the density of r^{th} order statistic as,

$$f_r(t) = \frac{n!}{(r-1)!(n-r)!}\left(1 - e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})}\right)^{r-1}\left(e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})}\right)^{n-r+1}\left(\alpha\lambda\, \theta\, t^{\theta-1}\left(1 + \lambda t^{\theta}\right)^{\alpha-1} + (\beta\, \gamma\, t^{\gamma-1})\right)$$

The density of I^{st} order statistic is given by,

$$f_1(t) = n\left(e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})}\right)^{n}\left(\alpha\lambda\, \theta\, t^{\theta-1}\left(1 + \lambda t^{\theta}\right)^{\alpha-1} + (\beta\, \gamma\, t^{\gamma-1})\right)$$

And the density of n^{th}order statistic is given by,

$$f_n(t) = n\left(1 - e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})}\right)^{n-1}\left(\alpha\lambda\, \theta\, t^{\theta-1}\left(1 + \lambda t^{\theta}\right)^{\alpha-1} + (\beta\, \gamma\, t^{\gamma-1})\right) e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})}$$

IV. Estimation Methods

I. Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) is widely used among the statistical inference because of its desirable properties like consistency, asymptotic efficiency. Use of maximum likelihood estimation is illustrated below.
The Likelihood function is given by,

$$L = \prod_{i=1}^{n} (f(t_i))$$

The log likelihood can be written as,

$$logL = \sum_{i=1}^{n} \log\left(f(t_i)\right)$$

$$logL = \sum_{i=1}^{n} \log\left(\left(\alpha\lambda\,\theta\, t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\, t^{\gamma-1})\right) * e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha}-(\beta t^{\gamma})}\right)$$

$$logL = \sum_{i=1}^{n} \log\left(\left(\alpha\lambda\,\theta\, t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\, t^{\gamma-1})\right)\right) - \sum_{i=1}^{n}\left(1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})\right) \quad (5)$$

To estimate the parameters, we take the partial derivatives of $logL$ with respect to each parameter and set them equal to zero. The resulting first-order derivatives are listed in Appendix. However, because these equations do not lead to direct solutions, we use numerical methods, such as Expectation-Maximization (EM) algorithm to find the parameter estimates.

II. Expectation-Maximization (EM) Algorithm

The EM algorithm is a powerful iterative method used for finding maximum likelihood estimates or maximum a posteriori estimates in statistical models where the data is incomplete or contains latent variables. The EM algorithm consists of an expectation step (E-step), and a maximization step (M-step). The E-step only needs to compute the conditional expectation of the log-likelihood with respect to the incomplete data, given the observed data. The M-step needs to find the maximize of this expected likelihood [9]. These two steps are then alternated until some convergence criterion is met [10]. The E-step of the EM algorithm consists of computing the conditional expectation of the complete data likelihood, given the observed data. That is, the objective function at iteration k is given by$Q(\Theta|\Theta_{k-1}) = E_{\Theta_{k-1}}(l_c(\Theta; y, X)|Y = y)$, where Y be the observed data and X be the missing data , l_c for the log-likelihoods based on the complete and Θ_{k-1} is the parameter estimate obtained from the previous iteration.

The M-step of the EM algorithm consists of maximizing the objective function constructed in the previous E-step. That is, we define $\Theta_k = argmax_{\Theta} Q(\Theta|\Theta_{k-1})$. We can combine the E and M steps of the EM algorithm into a single "update function". Thus, we write $M(\Theta_{k-1}) = argmax_{\Theta} Q(\Theta|\Theta_{k-1})$.

Asymptotic Confidence Bounds

When the maximum likelihood estimates do not have a closed form, making it difficult to determine the distribution directly, we instead rely on the asymptotic distribution of the MLEs to calculate confidence intervals [11]. It is known that the asymptotic distribution of the MLE $\widehat{\Theta}$ is given by,

$$\left(\widehat{\Theta} - \Theta\right) \to N_5(0, I^{-1}(\Theta))$$

Where $I(\Theta) \to$ Fisher information matrix of the unknown parameters $\Theta = (\alpha, \beta, \gamma, \lambda, \theta)$.
The elements of the 5 X 5 matrix of $I(.)$, are approximated by $I_{ij}\left(\widehat{\Theta}\right)$, $i, j = 1, 2, 3, 4, 5$
where, $I_{ij}\left(\widehat{\Theta}\right) = -\frac{\partial^2 l(\Theta)}{\partial\Theta_i \partial\Theta_j}\Big|_{\Theta=\widehat{\Theta}}$
$\widehat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda}, \hat{\theta})$ estimated parameters.
Now information matrix can be written as,

$$\mathrm{I}\left(\widehat{\Theta}\right) = -\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha_j^2} & \frac{\partial^2 l}{\partial\alpha_j\partial\beta_j} & \frac{\partial^2 l}{\partial\alpha_j\partial\gamma_j} & \frac{\partial^2 l}{\partial\alpha_j\partial\lambda_j} & \frac{\partial^2 l}{\partial\alpha_j\partial\theta_j} \\ \frac{\partial^2 l}{\partial\alpha_j\partial\beta_j} & \frac{\partial^2 l}{\partial\beta_j^2} & \frac{\partial^2 l}{\partial\beta_j\partial\gamma_j} & \frac{\partial^2 l}{\partial\beta_j\partial\lambda_j} & \frac{\partial^2 l}{\partial\beta_j\partial\lambda_j} \\ \frac{\partial^2 l}{\partial\alpha_j\partial\gamma_j} & \frac{\partial^2 l}{\partial\beta_j\partial\gamma_j} & \frac{\partial^2 l}{\partial\gamma_j^2} & \frac{\partial^2 l}{\partial\gamma_j\partial\lambda_j} & \frac{\partial^2 l}{\partial\gamma_j\partial\lambda_j} \\ \frac{\partial^2 l}{\partial\alpha_j\partial\lambda_j} & \frac{\partial^2 l}{\partial\beta_j\partial\lambda_j} & \frac{\partial^2 l}{\partial\gamma_j\partial\lambda_j} & \frac{\partial^2 l}{\partial\lambda_j^2} & \frac{\partial^2 l}{\partial\gamma_j\partial\lambda_j} \\ \frac{\partial^2 l}{\partial\alpha_j\partial\theta_j} & \frac{\partial^2 l}{\partial\beta_j\partial\theta_j} & \frac{\partial^2 l}{\partial\gamma_j\partial\theta_j} & \frac{\partial^2 l}{\partial\gamma_j\partial\theta} & \frac{\partial^2 l}{\partial\theta_j^2} \end{bmatrix} \quad (6)$$

The elements of the fisher information matrix are given in Appendix. Therefore, the approximate $100(1-\gamma)\%$ two-sided, confidence interval for Θ is given by

$$\widehat{\Theta} \pm Z_{\gamma/2}\sqrt{I^{-1}(\widehat{\Theta})} \tag{7}$$

Here $Z_{\gamma/2}$is the upper $\gamma/2$ th percentile of a standard normal distribution.

III. Bayes Estimates using MCMC

In this section we consider the estimation of the parameters using Bayesian techniques. Many authors have estimated the parameters of the distribution using Bayesian techniques [12, 13]. Here we estimated the parameters $(\alpha,\beta,\gamma,\lambda,\theta)$ using MCMC techniques. We have density function,

$$f(t) = \left(\alpha\lambda\,\theta\,t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\,t^{\gamma-1}\,)\right)e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha}-(\beta t^{\gamma}\,)}$$

Here beta (β), lambda (λ) is scale and alpha (α), theta (θ), gamma (γ) is scale parameters.
It is assumed that the priors of the parameters $(\alpha,\beta,\gamma,\lambda,\theta)$ were following below distributions.
$\alpha \sim Gamma\ (a_1,b_1)$ where a_1 and b_1 are the respective shape and scale hyperparameters of α.
$\beta \sim Gamma\ (a_2,b_2)$ where a_2 and b_2 are the respective shape and scale hyperparameters of β.
$\gamma \sim Gamma\ (a_3,b_3)$ where a_3 and b_3 are the respective shape and scale hyperparameters of γ.
$\lambda \sim Exp\ (a_4)$ where a_4 is scale hyperparameters of λ.
$\theta \sim Gamma\ (a_5,b_5)$ where a_5 and b_5 are the respective shape and scale hyperparameters of θ.
Prior distributions were given by,

$$\pi(\alpha) = \frac{b_1^{a_1}}{\Gamma a_1}\alpha^{a_1-1}e^{-\alpha b_1} \quad a_1, b_1 > 0$$
$$\pi(\beta) = \frac{b_2^{a_2}}{\Gamma a_2}\beta^{a_2-1}e^{-\beta b_2} \quad a_2, b_2 > 0$$
$$\pi(\gamma) = \frac{b_3^{a_3}}{\Gamma a_3}\gamma^{a_3-1}e^{-\gamma b_3} \quad a_3, b_3 > 0$$
$$\pi(\lambda) = a_4 e^{-\lambda a_4} \quad a_4 > 0$$
$$\pi(\theta) = \frac{b_5^{a_5}}{\Gamma a_5}\theta^{a_5-1}e^{-\theta b_5} \quad a_5, b_5 > 0$$

Hence joint prior distribution is given by

$$\pi(\Theta) = \pi(\alpha)\,\pi(\beta)\,\pi(\gamma)\,\pi(\lambda)\,\pi(\theta) \ \text{ where } \Theta = (\alpha,\beta,\gamma,\lambda,\theta)$$

$$\pi(\Theta) \propto \alpha^{a_1-1}e^{-\alpha b_1}\,\beta^{a_2-1}e^{-\beta b_2}\,\gamma^{a_3-1}e^{-\gamma b_3}\ \ e^{-\lambda a_4}\,\theta^{a_5-1}e^{-\theta b_5} \tag{8}$$

Then posterior distribution is given by,

$$f(\Theta \mid \mathrm{x_i}) = \frac{L(\Theta|x_i)\,\pi(\Theta)}{\int_{\Theta} L(\Theta|x_i)\ \pi(\Theta)}$$

Where $L(\Theta|x_i)$ is a likelihood function and $\pi(\Theta)$ is joint prior density.

$$f(\Theta|\ \mathrm{x_i}) \propto \prod_{i=1}^{n}\left(\alpha\lambda\,\theta\,t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\,t^{\gamma-1}\,)\right)e^{1-\left(1+\lambda t^{\theta}\right)^{\alpha}-(\beta t^{\gamma}\,)}\ \alpha^{a_1-1}e^{-\alpha b_1}\,\beta^{a_2-1}e^{-\beta b_2}\,\gamma^{a_3-1}e^{-\gamma b_3}\ \ e^{-\lambda a_4}\,\theta^{a_5-1}e^{-\theta b_5} \tag{9}$$

Therefore, the Bayes estimate of any function of $\alpha,\beta,\gamma,\lambda,\theta$ say $g(\alpha,\beta,\gamma,\lambda,\theta)$ under squared error loss (SEL) function is,

$$\hat{g}(\alpha,\beta,\gamma,\lambda,\theta) = E_{\Theta}g(\Theta|\ \mathrm{x_i}) = \frac{\int_{\Theta} g(\Theta|\ \mathrm{x_i})L(\Theta|x_i)\,\pi(\Theta)}{\int_{\Theta} L(\Theta|x_i)\ \pi(\Theta)} \tag{10}$$

Generally, the ratio of two integrals given in above equation cannot be obtained in a closed form. Hence, we considered one of the MCMC methods i.e. Metropolis-Hastings (MH) algorithm to estimate the parameters from posterior distribution and then computed the Bayes estimates under SEL function.

IV. Metropolis-Hasting Algorithm

Metropolis-Hastings is a Markov Chain Monte Carlo (MCMC) algorithm used for sampling from a probability distribution [14]. The algorithm is as follows,

- Set initial value Θ^0.
- For $t = 1,2, \dots T$ repeat the III to VI steps
- Set $\Theta = \Theta^{t-1}$
- Generate new candidate parameter values, here we have used normal distribution with parameter $(\Theta, 1)$, say it $q(\Theta') \sim N(\Theta, 1)$.
- Calculate acceptance probability, $\gamma = \min\left(1, \frac{f(\Theta' \mid x_i) q(\Theta')}{f(\Theta \mid x_i) q(\Theta)}\right)$

Update $\Theta^t = \Theta'$ with probability γ, otherwise set $\Theta^t = \Theta$ with probability $1 - \gamma$.

V. Information Criterion

The Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion methods are used to know which of the following distributions fits the data well [11]. The distribution fits the data well, whose information Criterion values are less.

$$AIC = 2K - 2\ lnL$$
$$CAIC = -2logL + K(logn + 1)$$
$$BIC = Kln(n) - 2\ lnL$$
$$HQIC = -2\ logL + 2\ K\ log\ (logn)$$

Where L is the likelihood function, n is the sample size and K is the number of parameters estimated.

VI. Kaplan-Meier (K-M) Estimator

The Kaplan-Meier estimator of survival function [11] is defined as,

$$\hat{S}(t) = \prod_{i:t_i<t} (1 - \frac{d_i}{n_i})$$

Where $t_i$is the failure time, $d_i$is the number of events that occurs at time $t_i$and
n_i is the number individuals at risk of experiencing the event immediately prior to t_i.

V. Results and Discussion

To validate the newly proposed new modified power generalized Weibull distribution, we analyzed the failure times of patients admitted to BLDE Hospital, Vijayapura. The dataset comprises 577 patients who passed away between January 1, 2024, and August 30, 2024.

Table 1 presents the estimated parameters obtained using the Maximum Likelihood Estimation (MLE) method for various lifetime distributions.

Table 2 provides a model comparison based on the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan–Quinn Information Criterion (HQIC). The results indicate that the new modified power generalized Weibull distribution offers the best fit, as it records the lowest values across all four criteria compared to the competing models.

Table 3 shows the parameter estimates for the new modified power generalized Weibull distribution using both MLE and Bayesian methods. Due to the complexity of the first-order partial derivatives under MLE, the Expectation-Maximization (EM) algorithm was employed for parameter estimation. For Bayesian estimation, we utilized the Metropolis-Hastings algorithm, a Markov Chain Monte Carlo (MCMC) technique, with Gamma and Exponential priors. It was observed that the standard errors under Bayesian estimation were lower than those from MLE, suggesting better precision.

Table 4 lists the 95% confidence intervals for the estimated parameters of the new modified power generalized Weibull distribution. These intervals were computed at a 5% level of significance and support the statistical reliability of the estimates.

Figure 3 displays the trace plots and histograms for the posterior distributions of the parameters after the burn-in period of the MCMC iterations, indicating good mixing and convergence.

Figure 4 shows the histogram of the observed data with the fitted density curves of various lifetime distributions. The curve corresponding to the new modified power generalized Weibull distribution (represented in purple) aligns most closely with the histogram, confirming a better visual fit.

Figure 5 illustrates the hazard functions for all compared distributions. The hazard curve for the new modified power generalized Weibull distribution initially increases, then decreases suddenly, and eventually rises again over time capturing the complex nature of the real-life hazard trend. This indicates that the initial days of hospital admission were associated with a higher risk for these patients, which implies most of the patients admitted to the hospital at the time study of study period are in emergency cases.

Figure 6 compares the Kaplan-Meier survival curve with the survival curves of all other fitted distributions. The new modified power generalized Weibull distribution survival curve is found to be the closest to the Kaplan-Meier estimate. Notably, approximately 50% of the patients experienced failure (death) within 5 days of hospital admission. As mentioned earlier, this provides strong evidence that the initial days of hospitalization carried the greater risk.

Table 1: *Estimated parameters using MLE for different distributions.*

Distribution\ Parameters	α	β	γ	λ	θ
Exponential	0.09563282	-	-	-	-
Weibull	0.22864357	0.7016423	-	-	-
Lai et al (2003) [3]	0.2376663	0.5949811		0.01	-
Xie and Lai (1996) [1]	0.2276905	0.7015965	0.001	0.701521	-
Sarhan and Zaindin (2009) [4]	0.227689	0.6994166	0.001	-	-
Rangoli et al (2025) [8]	0.001	0.4723638	0.2374116	0.6618276	-
Mustafa et al (2023) [7]	0.001	0.0956337	-	0.0406266	0.052673947
New modified power generalized Weibull	0.17924108	0.002460846	1.641992384	2.37270958	1.351152555

Table 2: *Distribution comparison using AIC, CAIC, BIC and HQIC.*

Distribution\ Parameters	$logl$	AIC	$CAIC$	BIC	$HQIC$
Exponential	-1931.357	3864.714	3870.071842	3869.072	3866.413378
Weibull	-1858.96	3721.921	3732.635685	3730.636	3725.318756
Lai et al (2003) [3]	-1868.952	3743.903	3759.977527	3756.977	3749.002134
Xie and Lai (1996) [1]	-1858.96	3725.921	3747.351369	3743.352	3732.717512
Sarhan and Zaindin (2009) [4]	-1859.048	3724.096	3740.169527	3737.169	3729.194134
Rangoli et al (2025) [8]	-1854.67	3717.34	3738.771369	3734.772	3724.137512
Mustafa et al (2023) [7]	-1931.358	3870.715	3892.147369	3888.147	3877.513512
New modified power generalized Weibull	-1834.69	3679.379	3706.169211	3701.168	3687.876891

Table 3: *Parameter estimation using MLE and Bayes method for the good fit model.*

New modified power generalized Weibull distribution	
EM estimates	Bayes estimate
$\boldsymbol{\alpha = 0.179241081}$ (0.02865731) *	0.1665027 (0.0160304)
$\boldsymbol{\beta = 0.002460846}$ (0.000682678)	0.01573168 (0.005635611)
$\boldsymbol{\gamma = 1.641992384}$ (0.071844198)	1.204626 (0.06549102)
$\boldsymbol{\lambda = 2.372709582}$ (0.777731161)	2.655452 (0.2526936)
$\boldsymbol{\theta = 1.351152555}$ (0.137535405)	1.326286 (0.09576754)
* Within parenthesis indicates standard error	

Table 4: *Estimated parameters with Lower confidence limit (LCL) and Upper Confidence limit (UCL).*

Parameter	Standard Error	LCL	UCL
$\boldsymbol{\alpha = 0.179241081}$	0.02865731	0.123072754	0.235409408
$\boldsymbol{\beta = 0.002460846}$	0.000682678	0.001122798	0.003798894
$\boldsymbol{\gamma = 1.641992384}$	0.071844198	1.501177755	1.782807013
$\boldsymbol{\lambda = 2.372709582}$	0.777731161	0.848356506	3.897062658

$\boldsymbol{\theta = 1.351152555}$	0.137535405	1.081583162	1.620721948

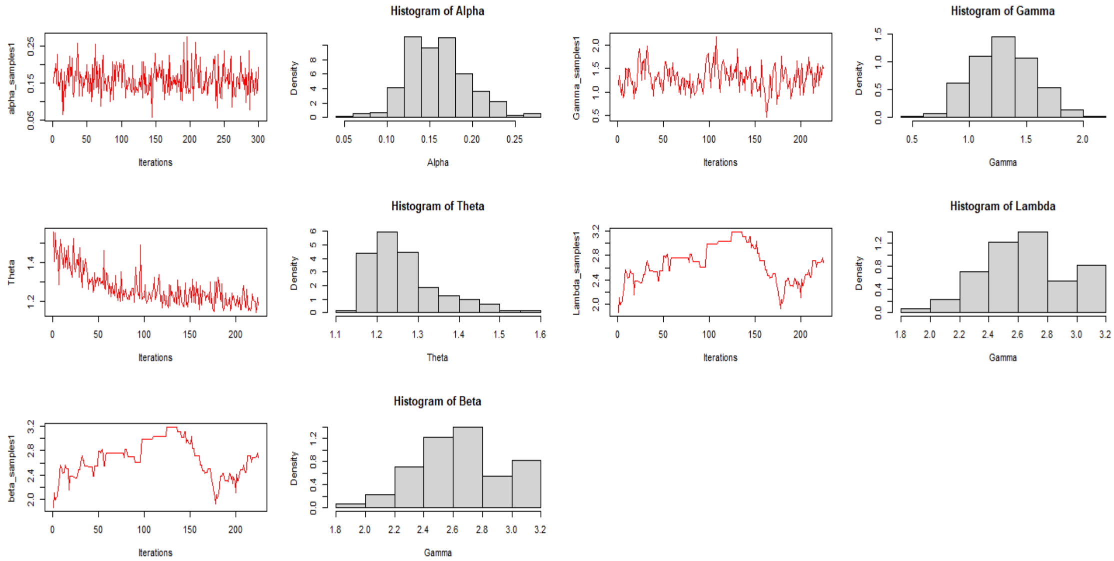

Figure 3: *Trace plot and histogram of the parameters of the good fit model using MCMC.*

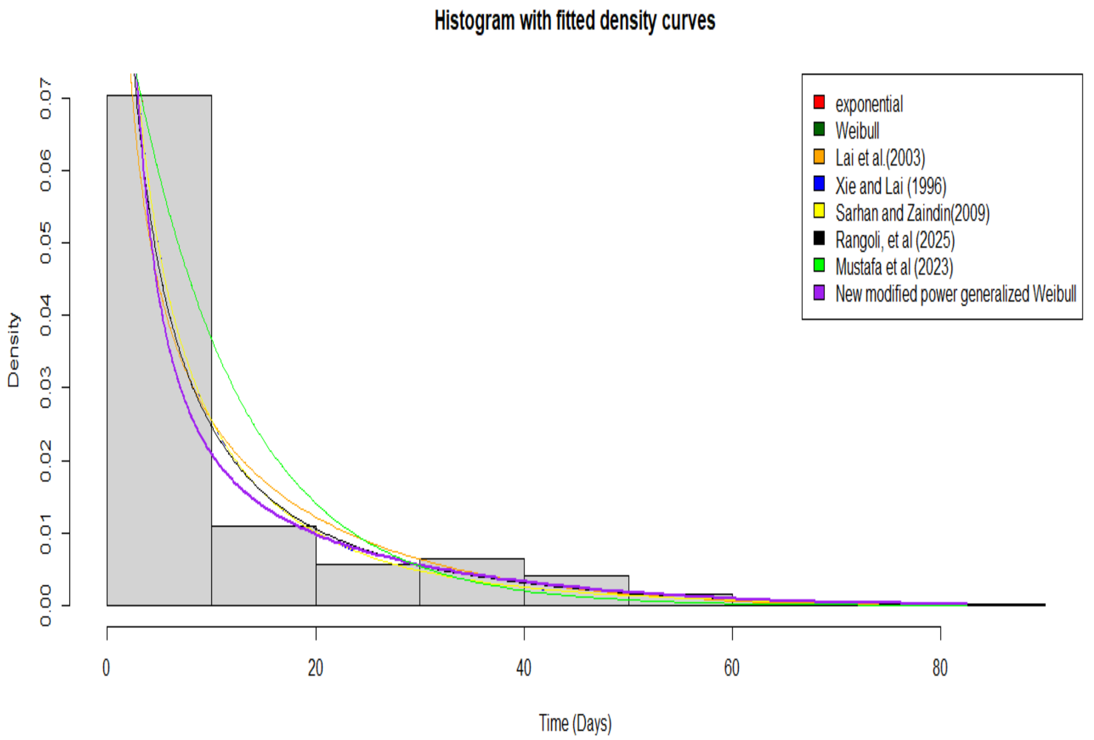

Figure 4: *Histogram and fitted density curve of different distributions for real life data.*

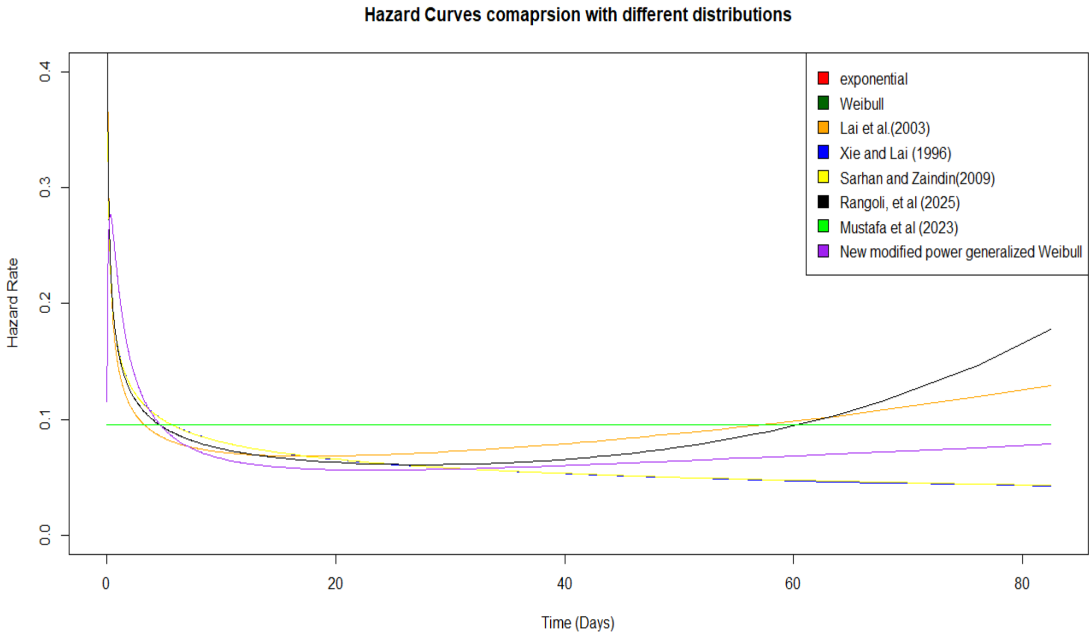

Figure 5: *Hazard Curves for different distribution for the real-life data.*

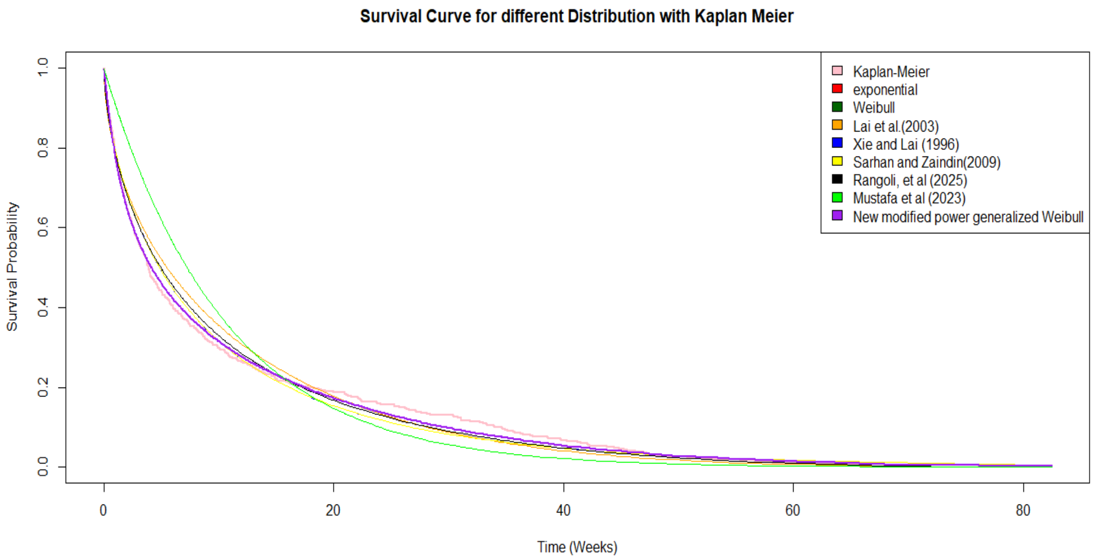

Figure 6: *Kaplan-Meier survival curve with all distribution survival curve for the real life dataset.*

VI. Conclusions

The proposed new modified power generalized Weibull (NMPGW) distribution is designed for analyzing failure time data and has proven particularly effective in modeling real-life survival scenarios. This distribution is capable of handling bimodal density functions and exhibits flexibility in representing various hazard rate patterns, including increasing, decreasing, bathtub-shaped, and decreasing–increasing–decreasing forms. We derived many statistical properties NMPGW distributions. Parameter estimation was carried out using both Maximum Likelihood Estimation via the EM algorithm and Bayesian methods through the Metropolis-Hastings algorithm with Gamma and Exponential priors. Bayesian estimates exhibited better convergence and lower standard errors, leading to more precise parameter inference. In a comparative analysis with several existing lifetime distributions, the NMPGW model consistently achieved the lowest values of AIC, CAIC, BIC, and HQIC, indicating a best fit. The model was validated using real patient failure time data from Hospital, where visual comparisons of fitted density, hazard, and survival curves showed a close match with the empirical data, especially with the Kaplan–Meier survival curve. Notably, the hazard function estimated from the model displayed an initially increasing trend, followed by a sudden decrease and then another increase, suggesting that most hospital admissions were emergency cases. It was observed that 50% of the patients failed within 5 days of admission. In conclusion, the NMPGW distribution offers a flexible and precise framework for analyzing time-to-event data, particularly in medical settings where accurate modeling of patient risk over time is needed.

Acknowledgements: The authors would like to thank BLDE Shri B. M. Patil Medical College Hospital and Research Centre, Vijayapura, for providing access to the dataset.

Conflict of Interest: The authors declare that there is no conflict of interest.

References

[1] Xie, M. and Lai, C. D. (1996). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. *Reliability Engineering and System Safety*, *52*(1), 87-93.

[2] Bagdonavicius, V., & Nikulin, M. (2001). *Accelerated life models: modeling and statistical analysis*. Chapman and Hall/CRC.

[3] Lai, C. D., Xie, M. and Murthy, D. N. P. (2003). A modified Weibull distribution. *IEEE Transactions on reliability*, *52*(1), 33-37.

[4] Sarhan, A. M. and Zaindin, M. (2009). Modified Weibull distribution. *APPS. Applied Sciences*, *11*, 123-136.

[5] Bebbington, M., Lai, C. D. and Zitikis, R. (2007). A flexible Weibull extension. *Reliability Engineering and System Safety*, *92*(6), 719-726.

[6] Rangoli, A. M., and Talawar, A. S. (2024). INFERENCE ON MODIFIED WEIBULL TYPE DISTRIBUTION AND ITS APPLICATION TO COMPETING RISKS DATA USING MCMC. *International Journal of Agricultural & Statistical Sciences*, *20*(2). 333-342.

[7] Mustafa S. S., Alharthi, A. S., Almulhim, F. A., Gemeay, A. M., Meraou, M. A., Mustafa, M. S., and Aljohani, H. M. (2023). Modified generalized Weibull distribution: theory and applications. *Scientific Reports*, *13*(1), 12828.

[8] Rangoli A M, Talawar A S, Agadi R P, et al. (2025) New Modified Exponentiated Weibull Distribution: A Survival Analysis. Cureus 17(1): e77347. DOI 10.7759/cureus.77347.

[9] Park, C. (2005). Parameter estimation of incomplete data in competing risks using the EM algorithm. *IEEE Transactions on Reliability*, 54(2), 282-290.

[10] Ruth, W. (2024). A review of Monte Carlo-based versions of the EM algorithm. *arXiv preprint arXiv:2401.00945*.

[11] Lawless, J. F. (2003). *Statistical models and methods for lifetime data*. John Wiley and Sons, New York.

[12] Upadhyay, S. K., and Gupta, A. (2010). A Bayes analysis of modified Weibull distribution via Markov chain Monte Carlo simulation. *Journal of Statistical Computation and Simulation*, *80*(3), 241-254.

[13] Pradhan, B., and Kundu, D. (2014). Analysis of interval-censored data with Weibull lifetime distribution. *Sankhya B*, *76*, 120-139.

[14] Chib, S., and Greenberg, E. (1995). Understanding the metropolis-hastings algorithm. *American statistician*, 327-335.

Appendix:

Using log likelihood function

$$logL = \sum_{i=1}^{n} \log\left(\left(\alpha\lambda\,\theta\,t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\,t^{\gamma-1})\right)\right) - \sum_{i=1}^{n}\left(1-\left(1+\lambda t^{\theta}\right)^{\alpha} - (\beta t^{\gamma})\right)$$

$$deno = \left(\alpha\lambda\,\theta\,t^{\theta-1}\left(1+\lambda t^{\theta}\right)^{\alpha-1} + (\beta\,\gamma\,t^{\gamma-1})\right)$$

$$nume\ \alpha = \lambda\,\theta\,t^{\theta-1}\left(\left(\left(1+\lambda t^{\theta}\right)^{\alpha-1}\right) + \alpha\left(\left(1+\lambda t^{\theta}\right)^{\alpha}\right)\log\left(\left(1+\lambda t^{\theta}\right)\right)\right)$$

$$nume\ \lambda = \alpha\,\theta\,t^{\theta-1}\left(\left(1+\lambda t^{\theta}\right)^{\alpha-1} + \lambda\,(\alpha-1)\left(1+\lambda t^{\theta}\right)^{\alpha-2}\,t^{\theta}\right)$$

$$nume\ \theta = \alpha\,\lambda\,t^{\theta-1}\left((\theta\ logt+1)\left(\left(1+\lambda t^{\theta}\right)^{\alpha-1}\right) + \theta\,(\alpha-1)\left(1+\lambda t^{\theta}\right)^{\alpha-2}\lambda\,\theta\,t^{\theta-1}\right)$$

$$nume\ \beta = \gamma\,t^{\gamma-1}$$

$$nume\ \gamma = \beta\,t^{\gamma-1}(\gamma logt+1)$$

$$\frac{\partial logL}{\partial\alpha} = \sum_{i=1}^{n}\frac{nume\ \alpha}{deno} - \sum_{i=1}^{n}\left(1+\lambda t^{\theta}\right)^{\alpha}\log\left(1+\lambda t^{\theta}\right)$$

$$\frac{\partial^2 logL}{\partial\alpha^2} = \sum_{i=1}^{n}\frac{deno\,\frac{\partial(nume\ \alpha)}{\partial\alpha} - (nume\ \alpha)^2}{deno^2} - \sum_{i=1}^{n}\left(1+\lambda t^{\theta}\right)^{\alpha}\left(\log\left(1+\lambda t^{\theta}\right)\right)^2$$

$$\frac{\partial logL}{\partial\lambda} = \sum_{i=1}^{n}\frac{nume\ \lambda}{deno} - \sum_{i=1}^{n}\alpha\left(1+\lambda t^{\theta}\right)^{\alpha-1}\mathrm{t}^{\theta}$$

$$\frac{\partial^2 logL}{\partial\lambda^2} = \sum_{i=1}^{n}\frac{deno\,\frac{\partial\ nume\lambda}{\partial\lambda} - (nume\ \lambda)^2}{deno^2} - \sum_{i=1}^{n}\alpha\,(\alpha-1)\left(1+\lambda t^{\theta}\right)^{\alpha-2}\mathrm{t}^{2\theta}$$

$$\frac{\partial logL}{\partial\theta} = \sum_{i=1}^{n}\frac{nume\ \theta}{deno} - \sum_{i=1}^{n}\alpha\left(1+\lambda t^{\theta}\right)^{\alpha-1}\lambda\,\theta\,\mathrm{t}^{\theta-1}$$

$$\frac{\partial^2 logL}{\partial\theta^2} = \sum_{i=1}^{n}\frac{deno\,\frac{\partial\ nume\theta}{\partial\theta} - (nume\ \theta)^2}{deno^2}$$

$$-\sum_{i=1}^{n}\alpha\,\lambda\left(\left(\theta\mathrm{t}^{\theta-1}\log t + t^{\theta-1}\right)\left(1+\lambda t^{\theta}\right)^{\alpha-1} + \theta t^{\theta-1}\,(\alpha-1)\left(1+\lambda t^{\theta}\right)^{\alpha-1}\lambda\theta x^{\theta-1}\right)$$

$$\frac{\partial logL}{\partial\beta} = \sum_{i=1}^{n}\frac{nume\ \beta}{deno} - \sum_{i=1}^{n}t^{\gamma}$$

$$\frac{\partial^2 logL}{\partial\beta^2} = -\sum_{i=1}^{n}\frac{(nume\ \beta)^2}{deno^2}$$

$$\frac{\partial logL}{\partial\gamma} = \sum_{i=1}^{n}\frac{nume\ \beta}{deno} - \sum_{i=1}^{n}\beta\,t^{\gamma}\,logt$$

$$\frac{\partial^2 logL}{\partial\gamma^2} = \sum_{i=1}^{n}\frac{deno\,\frac{\partial\ nume\gamma}{\partial\gamma} - (nume\ \gamma)^2}{deno^2} - \sum_{i=1}^{n}\beta\,t^{\gamma}\,(logt)^2$$

$$\frac{\partial^2 logL}{\partial\alpha\partial\gamma} = \sum_{i=1}^{n}\frac{deno\,\frac{\partial\ (nume\alpha)}{\partial\ \lambda} - nume\alpha\ nume\lambda}{deno^2} - \sum_{i=1}^{n}\frac{\partial}{\partial\lambda}\left(\left(1+\lambda t^{\theta}\right)^{\alpha}\log\left(1+\lambda t^{\theta}\right)\right)$$

$$\frac{\partial^2 logL}{\partial\alpha\partial\theta} = \sum_{i=1}^{n} \frac{deno\,\frac{\partial\,(nume\alpha)}{\partial\,\theta} - nume\alpha\; nume\theta}{deno^2} - \sum_{i=1}^{n} \alpha\,\lambda \frac{\partial}{\partial\theta}\left(\left(1+\lambda t^{\theta}\right)^{\alpha-1} \theta\, t^{\theta-1}\right)$$

$$\frac{\partial^2 logL}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n} \frac{nume\alpha\; nume\beta}{deno^2}$$

$$\frac{\partial^2 logL}{\partial\alpha\partial\gamma} = -\sum_{i=1}^{n} \frac{nume\alpha\; nume\gamma}{deno^2}$$

$$\frac{\partial^2 logL}{\partial\lambda\partial\theta} = \sum_{i=1}^{n} \frac{deno\,\frac{\partial\,(nume\lambda)}{\partial\,\theta} - nume\lambda\; nume\theta}{deno^2} - \sum_{i=1}^{n} \frac{\partial}{\partial\theta}\left(\alpha\, t^{\theta}\left(1+\lambda t^{\theta}\right)^{\alpha-1}\right)$$

$$\frac{\partial^2 logL}{\partial\lambda\partial\beta} = -\sum_{i=1}^{n} \frac{nume\beta\; nume\lambda}{deno^2}$$

$$\frac{\partial^2 logL}{\partial\lambda\partial\gamma} = -\sum_{i=1}^{n} \frac{nume\gamma\; nume\lambda}{deno^2}$$

$$\frac{\partial^2 logL}{\partial\theta\partial\beta} = -\sum_{i=1}^{n} \frac{nume\beta\; nume\theta}{deno^2}$$

$$\frac{\partial^2 logL}{\partial\theta\partial\gamma} = -\sum_{i=1}^{n} \frac{nume\gamma\; nume\theta}{deno^2}$$

$$\frac{\partial^2 logL}{\partial\beta\partial\gamma} = \sum_{i=1}^{n} \frac{deno\,\frac{\partial\,(nume\beta)}{\partial\,\gamma} - nume\beta\; nume\gamma}{deno^2} - \sum_{i=1}^{n} t^{\gamma}\, logt$$