ON THE TYPE II HALF-LOGISTIC GENERALIZED INVERSE WEIBULL DISTRIBUTION AND ITS APPLICATIONS

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Abstract

Inverse Weibull distribution is frequently used in the survival and reliability analysis owing to its upside down bathtub type hazard function. Generalized inverse Weibull is the flexible generalization of inverse Weibull distribution which has increasing, decreasing and unimodal hazard rates. This paper introduces a novel extension of generalized inverse Weibull distribution which is developed using new type II half logistic generator. We introduced New type II half logistic generalized inverse Weibull distributions. Mathematical and statistical features of the newly derived distribution were examined. The point estimates of the unknown parameters were computed using maximum likelihood estimation, Weighted Least Square, Anderson-Darling and Cramer-von Mises estimator. We have conducted MCMC simulation using R package to validate the estimation algorithms. Finally, we demonstrated the usefulness of the proposed distribution using two real-world failure time data sets.

Keywords: Inverse Weibull distribution, Generalized inverse Weibull distribution, New Type II half Logistic G distribution, Order Statistics, point Estimation, upside down bathtub shape failure rate.

1. Introduction

Positive skewness and thicker tails are common in real-world data sets. Many academic fields have observed this discrepancy, including finance, economics, biology, engineering, reliability modelling, and environmental studies. Modell that fails to take into account the skewness present in that data may lead to accurate results. Decision-making procedures are affected in a lot of real-world situations by the existence of skewness in datasets. Developing adaptable and precise frameworks that can accurately capture the subtleties present in real-world datasets has long motivated statistical researchers to pursue the goal of modeling complex probabilistic phenomena. The continuous half logistic G (HL-G) family was first presented as a flexible distributional framework and has proven effective in modeling a variety of phenomena, including dependability modeling and financial data. In recent years, several new distributions were created by adding new parameters in the existing model. The newly developed probability models were found useful in capturing complex real-world phenomena. Currently there is lot of interest to generate new lifetime models exhibiting monotone and non-monotone failure rate properties having

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numerous applications in the field of engineering and reliability modelling.

In this paper we concentrate on the half-logistic (HL) distribution (Balakrishnan, 1985) due to its simple structure, with single parameter and tractable characteristics.

A random variable X is said to have a one parameter HL Distribution with parameter α, if its probability density function (pdf) is given by

$$g(x;\alpha) = \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})^2} \tag{1}$$

and the cumulative distribution function (cdf) is

$$G(x) = \frac{1-e^{-\alpha x}}{1+e^{-\alpha x}} \tag{2}$$

for all $x > 0$, where $\alpha > 0$.

One of the limitations of HL distribution is that it is not appropriate for modeling bimodal and left-skewed datasets. Some scholars have suggested alternative generalizations of the HL distribution in order to eliminate the drawbacks of this distribution. Cordeiro et al. [2] invented exponentiated half-logistic-G, type I half-logistic-G by Cordeiro et al. ([3]), Amal et al. ([4]) investigated type II half logistic G (TIIHL-G), by using the half logistic generator to obtain type II half logistic family and is denoted by TIIHLG.

TIIHLG ([4]) family can be expressed as follows,

$$F(x;\alpha,\zeta) = 1 - \int_0^{-log(G(x,\zeta))} \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})^2}\, dt, \tag{3}$$

where $x > 0$, $\alpha > 0$ is the scale parameter and $G(x,\zeta)$ is the baseline cdf, which depends on a parameter vector ζ. The cdf in (3) gives the type II half logistic generated distribution.

After simplification (3) can be expressed as follows,

$$F(x) = \frac{2\,(G(x,\zeta))^{\alpha}}{1+(G(x,\zeta))^{\alpha}}, \quad x > 0, \tag{4}$$

and pdf can be defined as follows,

$$f(x) = \frac{2\alpha g(x,\zeta)\,(G(x,\zeta))^{\alpha-1}}{\left(1+(G(x,\zeta)^{\alpha}\right)^2}, \quad x > 0, \alpha > 0. \tag{5}$$

Amal et al. ([6]) proposed type II half logistic Weibull (TIIHLW) by taking Weibull as the baseline distribution in (4). Recently more flexible extension of TIIHLG is proposed by Altun et al. [7], referred to as New Type II Half Logistic-G (NTIIHLG) family of distributions. The cdf of NTIIHL G family can be expressed as follows

$$F(x;\alpha,\lambda,\zeta) = 1 - \int_0^{-log\left(\frac{G(x,\zeta)^{\lambda}}{G(x,\zeta)^{\lambda}+(1-G(x,\zeta))^{\lambda}}\right)} \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})^2}\, dt, \tag{6}$$

where, $x > 0$, $\alpha > 0, \lambda > 0$ and $G(x,\zeta)$ is a baseline cdf, which depends on a parameter vector ζ. The distribution function (6) provides New type II half logistic generated distributions ([7]).

After simplification the cdf of NTIIHLG can written as follows,

$$F(x) = \frac{2G(x,\zeta)^{\alpha\lambda}}{G(x,\zeta)^{\alpha\lambda} + \left(G(x,\zeta)^{\alpha} + \bar{G}(x,\zeta)^{\alpha}\right)^{\lambda}}, \quad x > 0, \tag{7}$$

where $\alpha > 0, \lambda > 0$.

The pdf of NTIIHLG is given by,

$$f(x) = \frac{2\alpha\lambda g(x,\zeta)G(x,\zeta)^{\alpha\lambda-1}\bar{G}(x,\zeta)^{\alpha-1}\left(G(x,\zeta)^{\alpha}+\bar{G}(x,\zeta)^{\alpha}\right)^{\lambda-1}}{\left[G(x,\zeta)^{\alpha\lambda}+\left(G(x,\zeta)^{\alpha}+\bar{G}(x,\zeta)^{\alpha}\right)^{\lambda}\right]^{2}}, \quad x>0,\ \alpha>0. \tag{8}$$

In this study we derive new extension of one parameter inverse Weibull (IW) using New Type II Half Logistic-G family of distributions. The Weibull distribution was frequently utilized in survival and reliability analysis due to straightforward mathematical form. An Inverted version of the Weibull distribution called the Inverse Weibull distribution has been used to represent real-world occurrences since it has an inverted bathtub failure rate. Inverted bathtub failure rate distributions are frequently observed in medical and biological domains, such as in patient data related to lung cancer, bladder cancer, and leukemia ([1], [5]). Detailed study of Inverse Weibull distribution were found in Drapella [8], Mudholkar and Kollia [9] and Jiang et al. [10].

A random variable X is said to have a one parameter Inverse Weibull distribution with shape parameter β, if its probability density function (pdf) is given by

$$g(x;\beta) = \beta x^{-(\beta+1)}\mathrm{e}^{-x^{-\beta}}, \tag{9}$$

and the cumulative distribution function (cdf) is

$$G(x) = \mathrm{e}^{-x^{-\beta}}, \tag{10}$$

for all $x > 0$ and $\beta > 0$. Extension of IW distribution called generalized inverse Weibull (GIW) is discussed in de Gusmão et al. [11].

A random variable X is said to have two parameter Generalized Inverse Weibull distribution with parameters $\alpha,\ \beta$, if its probability density function (pdf) is given by

$$g(x;\alpha,\beta) = \alpha\beta x^{-(\beta+1)}\mathrm{e}^{-\alpha x^{-\beta}}, \tag{11}$$

and the cumulative distribution function (cdf) is

$$G(x) = \mathrm{e}^{-\alpha x^{-\beta}}, \tag{12}$$

for all $x > 0$ and $\alpha,\ \beta > 0$. In this paper we study a new extension of inverse Weibull distribution called New Type II Half Logistic- Generalized inverse Weibull (NTIIHLGIW) family of distribution to model lifetime data or survival data. Motivation behind introducing this new family with three parameters is that it provide more flexibility to handle complex data sets arising from variety of fields.

The paper is organized as follows: Section 2 describes the NTIIHLGIW distribution, whereas section 3 illustrate various statistical and mathematical properties, which include the hazard function, quantile function, skewness, kurtosis, moments, moment generating function, characteristics function, etc. We derive the distribution of the Order Statistics in section 4. Section 5 investigate the different methods of estimation including maximum likelihood estimation, weighted least square, Andarson-darling and Cramer Von approach. Section 6 describe the simulation study to validate the proposed model estimation procedures and two real data sets are analysed to demonstrate the efficiency of the proposed model in section 7. The concluding remarks were provided in Section 8.

2. New Type II Half Logistic Generalized Inverse Weibull Distribution

A random variable X is said to have New Type II Half Logistic- Generalized Inverse Weibull (NTIIHLGIW) with parameters $(\alpha,\beta,\lambda,\gamma)$ denoted by $X \sim NTIIHLGIW(\alpha,\beta,\lambda,\gamma)$ if its pdf is given by,

$$f(x) = \frac{2\alpha\beta\lambda\gamma x^{-(\beta+1)}(\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha\lambda}(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha-1}\left((\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}+(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda-1}}{\left[(\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha\lambda}+\left((\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}+(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda}\right]^{2}}, \quad x>0, \tag{13}$$

and its cdf is,

$$F(x) = \frac{2(e^{-\gamma x^{-\beta}})^{\alpha\lambda}}{(e^{-\gamma x^{-\beta}})^{\alpha\lambda} + \left((e^{-\gamma x^{-\beta}})^{\alpha} + (1 - e^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda}}, \quad x > 0, \tag{14}$$

for all $x > 0$, α, β λ, $\gamma > 0$.

The plots of the pdf and cdf of the NTIIHLGIW (α, β λ, γ) distribution for particular values of its parameters are presented in Fig.1 and Fig. 2 respectively.

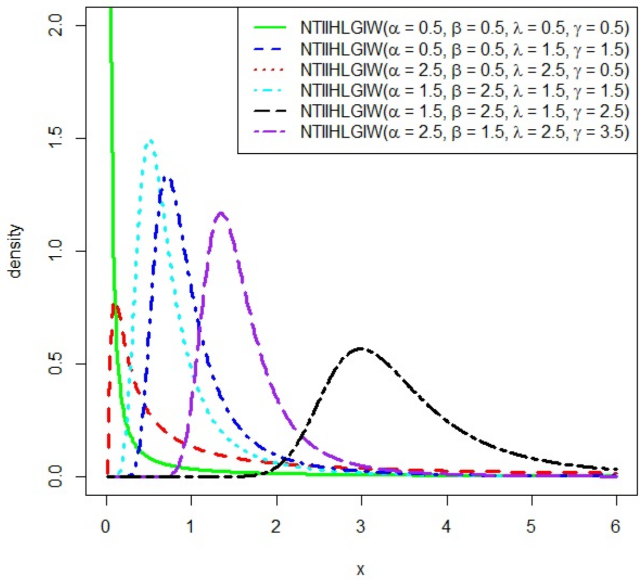

Figure 1: *Plot of pdf of NTIIHLGIW for different parameter values*

Theorem 1. Given that X follows the NTIIHLGIW $(\alpha, \beta, \lambda, \gamma)$ distribution with pdf $f(x)$ and cdf $F(x)$ as given in (13 and (14) respectively, then: 1. $\lim_{x \to \pm\infty} f(x) = 0$, and 2. $\lim_{x \to \pm\infty} F(x) = 1$.

Proof. Trivial and hence it is omitted. ■

3. Mathematical and reliability properties

It is important to understand the mathematical properties of probability distributions for a number of reasons. First off, these qualities make it possible to create a number of statistical measurements that characterize a random variable's behavior. We outline some mathematical and reliability properties of the NTIIHLGIW distribution.

3.1. Survival Function

The survival function describes the probability that an item or individual will survive past a certain time. A survival function can be expressed as $S(x) = 1 - F(x)$,

$$S(x) = \frac{\left((e^{-\gamma x^{-\beta}})^{\alpha} + (1 - e^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda} - (e^{-\gamma x^{-\beta}})^{\alpha\lambda}}{\left((e^{-\gamma x^{-\beta}})^{\alpha} + (1 - e^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda} + (e^{-\gamma x^{-\beta}})^{\alpha\lambda}}, \quad x > 0. \tag{15}$$

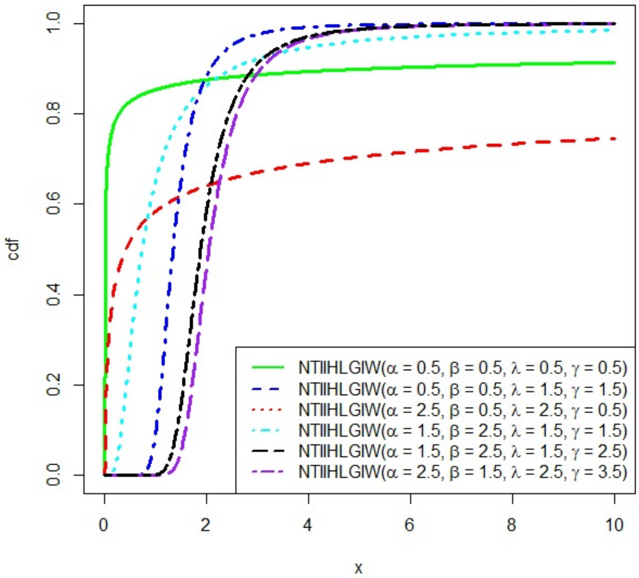

Figure 2: *Plot of cdf of NTIIHLGIW for different parameter values*

The plots of the sf and hf of the NTIIHLIW $(\alpha,\ \beta\ \lambda,\ \gamma)$ distribution for particular values of its parameters are presented in Fig.3 and Fig. 4 respectively. Fig. 4 demonstrates that the hf of NTIIHLGIW has reversed-J shape as well as an upside-down bathtub (UBT) shape.

3.2. Hazard Function

The hazard function (hf) is given by $h(x) = \frac{f(x)}{1-F(x)}$, is the instantaneous rate at which a component will fail given that it has already survived a length of time x. For the cdf and pdf given in Eq.(14) and (13), respectively, $h(x)$ takes the form:

$$h(x) = \frac{2\alpha\beta\lambda\gamma x^{-(\beta+1)}(\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha-1}\left((\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}+(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda-1}}{\left((1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}+(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{2\lambda}-(1-\mathrm{e}^{-\gamma x^{-\beta}})^{2\alpha\lambda}},\quad x>0. \tag{16}$$

Theorem 2. The limit of the hazard rate function of the NTIIHLGIW$(\alpha,\beta,\lambda,\gamma)$ distribution as $t \to \pm\infty$ is zero:

$$\lim_{t\to\pm\infty} h(t) = 0.$$

Proof. Trivial and hence it is omitted. ■

3.3. Reverse Hazard Function

The Revised Hazard Function $r(x) = \frac{f(x)}{F(x)}$ is given by,

$$r(x) = \frac{\alpha\beta\lambda\gamma x^{-(\beta+1)}(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha-1}\left((\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}+(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda-1}}{\left[(\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha\lambda}+\left((\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}+(1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda}\right]},\quad x>0. \tag{17}$$

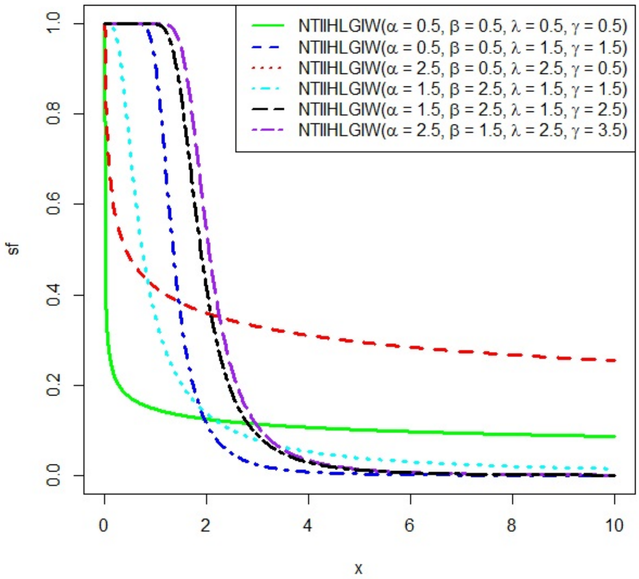

Figure 3: *Survival plot of NTIIHLGIW distribution for different parameter values*

The plots of the rhf and of of the NTIIHLIW $(\alpha,\ \beta\ \lambda,\ \gamma)$ distribution for particular values of its parameters are presented in Fig.5 and Fig. 6 respectively.

3.4. Cumulative Hazard Function

The Cumulative Hazard Function, $H(x) = -log(S(x))$ is obtained using the relation,

$$H(x) = -log\left[\frac{\left((\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha} + (1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda} - (\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha\lambda}}{\left((1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha} + (1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda} + (\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha\lambda}}\right], \tag{18}$$

where log refers to natural logarithm.

3.5. The Odd Function

The Odd Function is obtained using the relation $Q(x) = \frac{F(x)}{S(x)}$ and is given by

$$O(x) = \frac{2\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha\lambda}}{\left((\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha} + (1-\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha}\right)^{\lambda} - (\mathrm{e}^{-\gamma x^{-\beta}})^{\alpha\lambda}}. \tag{19}$$

3.6. Quantile Function, Median, Skewness and Kurtosis

The q^{th} quantile function of the NTIIHLG distribution is given by

$$\xi_q = Q_G\left(\frac{q^{\frac{1}{\alpha\lambda}}}{q^{\frac{1}{\alpha\lambda}} + \left(1 + (1-q)^{\frac{1}{\lambda}} - q^{\frac{1}{\lambda}}\right)^{\frac{1}{\alpha}}}\right). \tag{20}$$

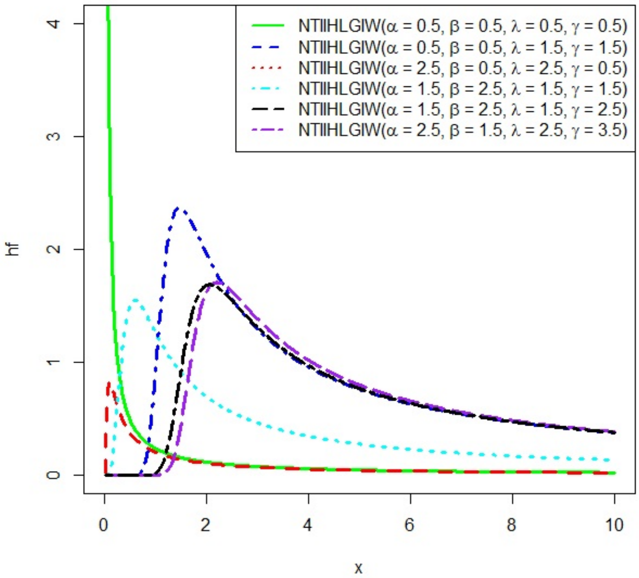

Figure 4: *Hazard plot of NTIIHLGIW distribution for different parameter values*

where, where $Q_G(q)$ is the baseline quantile function, q is a uniform distribution on the interval (0,1). Then the q^{th} quantile function of the NTIIHLGIW distribution is given by

$$X_q = \alpha \left(-\frac{1}{\gamma} \log \left(\frac{q^{\frac{1}{\alpha\lambda}}}{q^{\frac{1}{\alpha\lambda}} + \left(1 + (1-q)^{\frac{1}{\lambda}} - q^{\frac{1}{\lambda}}\right)^{\frac{1}{\alpha}}} \right) \right)^{-\frac{1}{\beta}}. \tag{21}$$

By utilizing equation (21), we can compute the first and third quartiles by substituting $q = 0.25$ and $q = 0.75$, respectively.

The median can be obtained as,

$$x_{0.5} = \alpha \left[-\frac{1}{\gamma} \log \left(\frac{0.5^{\frac{1}{\alpha\lambda}}}{0.5^{\frac{1}{\alpha\lambda}} + 1} \right) \right]^{-\frac{1}{\beta}}. \tag{22}$$

By utilizing equation (21), we can compute Galton's [22] skewness (S_k) and Moor's[23] Kurtosis (K). The measure of skewness (S_k) is given by

$$S_k = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)}, \tag{23}$$

and the measure of Kurtosis,

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{24}$$

3.7. Moments

Now we derive the r^{th} raw moment of NTIIHLGIW using the power series representation of NTIIHLG family derived by Altun et al. [7]. If the pdf given by a random variable X in equation

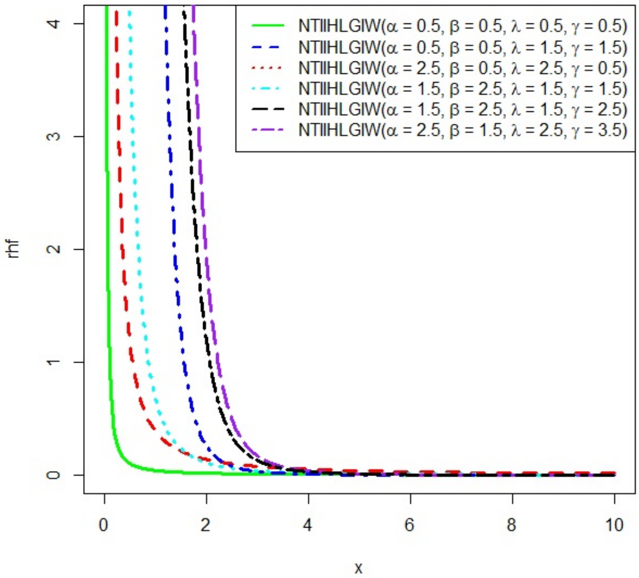

Figure 5: *Plot of rhf of NTIIHLGIW distribution for different parameter values*

13, then the corresponding r^{th}moment is given by:

$$\begin{aligned}\mu'_r &= E(x^r)\\ &= \int_0^\infty x^r f(x)dx\\ &= \sum_{r,h=0}^{\infty} c_{k+1}\frac{(k+1)(-1)^h\gamma^{\frac{r}{\beta}}\alpha^r\binom{k}{h}\Gamma\left(1-\frac{r}{\beta}\right)}{(h+1)^{\left(1-\frac{r}{\beta}\right)}}, r<\beta,\end{aligned}$$

where c_k ia as in Altun et al. [7] and is given by,
$c_{k+1} = \frac{1}{b_0}\left[a_{k+1} - \frac{1}{b_0}\sum_{r=1}^{k} b_r c_{k-r}\right]$, $c_0 = \frac{a_0}{b_0}$, $a_k = 2\sum_{j=k}^{\infty}(-1)^{k+j}\binom{\alpha\lambda}{j}\binom{j}{k}$, $b_k = \frac{a_k}{2} + h_k(\lambda)$ and h(x) is the pdf of exponentiated generalized inverse Weibull distribution.

3.8. Moment Generating Functions

Let X be a random variable having the pdf defined in equation (13) then its Moment Generating Function (mgf) is given by

$$\begin{aligned}M_X(t) &= E[e^{tX}]\\ &= \int_0^\infty e^{tx} f(x)\,dx.\end{aligned}$$

The mgf of NTIIHLGIW distribution is given by

$$M_x(t) = \sum_{r,h=0}^{\infty} c_{k+1}\frac{(k+1)(-1)^h t^r\gamma^{\frac{r}{\beta}}\alpha^r\binom{k}{h}\Gamma\left(1-\frac{r}{\beta}\right)}{(h+1)^{\left(1-\frac{r}{\beta}\right)}}, r<\beta. \tag{25}$$

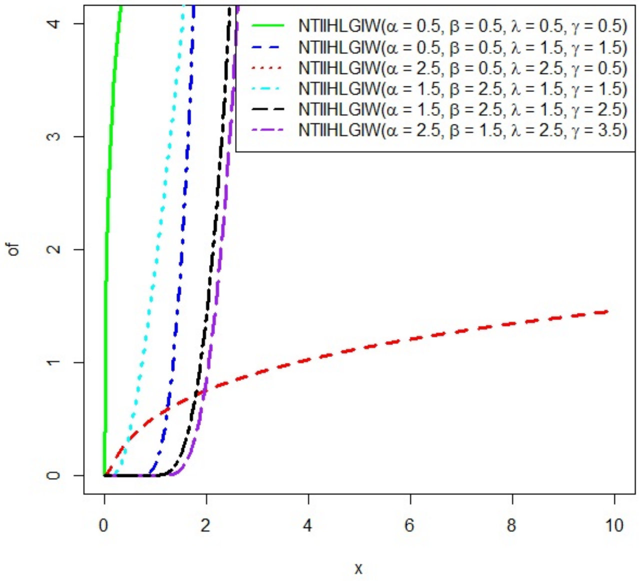

Figure 6: *Plot of Odd function of NTIIHLGIW distribution for different parameter values*

4. Order Statistics

In this section, we derive the pdf of the r^{th} order statistic of the NTIIHLGIW distribution. Order statistics have applications in system reliability. Let $X_1, X_2, X_3, ..., X_n$ be a simple random sample from the NTIIHLGIW distribution with cdf and pdf given by equation (14) and (13), respectively, and let $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq ... \leq X_{n:n}$, denote the order statistics obtained from this sample. We now give the pdf of $X_{r:n}$ say $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = B_{r:n} \left[F(x;\alpha,\beta,\lambda,\gamma)\right]^{r-1} \left[1 - F(x;\alpha,\beta,\lambda,\gamma)\right]^{n-r} f(x;\alpha,\beta,\lambda,\gamma). \tag{26}$$

For all $x > 0$, where F and f are given by equation (14) and equation (13) respectively, and $B_{r:n} = \frac{n!}{(n-1)!(n-r)!}$. Thus, using binomial series expansion,

$$(1-x)^{\alpha-1} = \sum_{j=0}^{\infty}(-1)^j \binom{\alpha-1}{j} x^j.$$

We obtain,

$$f_{r:n}(x) = C_{r:n} \sum_{s=0}^{\infty}(-1)^s \binom{n-r}{s} \left[F(x;\alpha,\beta,\lambda,\gamma)\right]^{r+s-1} f(x;\alpha,\beta,\lambda,\gamma). \tag{27}$$

The cdf of the r^{th} order statistics is given by

$$\begin{aligned} F_{r:n}(x) &= \sum_{i=r}^{n} \binom{n}{i} (F(x))^i (1-F(x))^{n-i} \\ &= \sum_{i=r}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n}{i} \binom{n-i}{j} (F(x))^{i+j}. \end{aligned} \tag{28}$$

Inserting cdf Equation (14) in Equation (28), we get the cdf of the r^{th} order statistics of NTIIHLGIW distribution. The pdf of the smallest order statistics $X_{1:n}$ is obtained from Equation (27), by putting

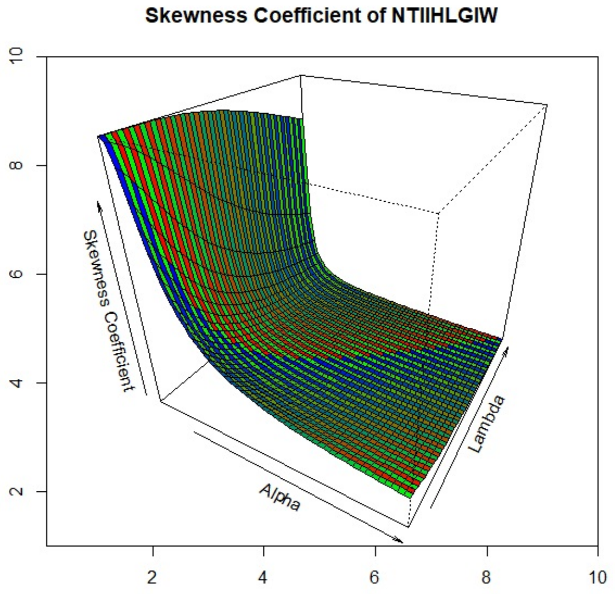

Figure 7: *Plot of skewness NTIIHLGIW distribution for* $\beta = 0.5$ *and* $\gamma = 0.5$

$r = 1$. Also, the pdf of the largest order statistics $X_{n:n}$ can be obtained from Equation (27), by putting $r = n$.

The pdf of joint distribution of r^{th} and s^{th} order statistics are given by,

$$f_{r:s:n}(x) = B_{r:s:n}\,[F(x_r)]^{r-1}\,[F(x_r s) - F(x_r)]^{s-r-1}\,[1 - F(x_s)]^{n-r}\,f(x_r)f(x_s), \tag{29}$$

where $B_{r:s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

5. Methods of Estimation

5.1. Maximum likelihood Estimation

Let X be a random variable with the pdf of the NTIIHLGIW distribution defined in equation (13) then its log-likelihood function is defined by:

$$\begin{aligned} L(x;\alpha,\beta,\lambda,\gamma) = {} & n\log 2 + n\log\alpha + n\log\beta + n\log\lambda + n\log\gamma - (\beta+1)\sum_{i=1}^{n}\log x_i + \alpha\lambda\sum_{i=1}^{n}\log g(x_i) \\ & + (\alpha-1)\sum_{i=1}^{n}\log(1 - G(x_i)) + (\lambda-1)\sum_{i=1}^{n}\log\left(g(x_i)^{\alpha} + (1-g(x_i))^{\alpha}\right) \\ & - 2\sum_{i=0}^{\infty}\log\left(g(x_i)^{\alpha\lambda} + (g(x_i)^{\alpha} + (1-g(x_i))^{\alpha})^{\lambda}\right), \end{aligned} \tag{30}$$

where $g(x_i) = e^{-\gamma x^{-\beta}}$. The MLE's $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$, $\hat{\lambda}_{MLE}$ and $\hat{\gamma}_{MLE}$ were obtained by maximizing the likelihood function in Equation (30) numerically by using packages like "Adequacy" and "MaxLik" packages available in the R statistical software. We applied PSO and the Optim function of BFGS algorithm of Nash [14] in R [15].

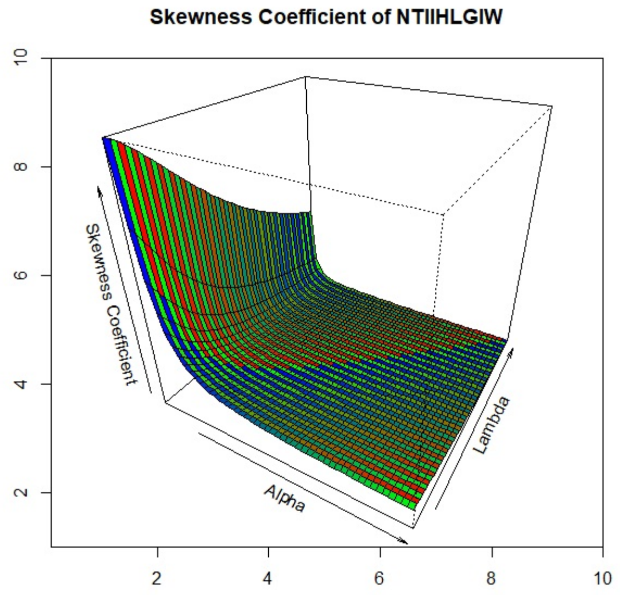

Figure 8: *Plot of skewness NTIIHLGIW distribution for* $\beta = 2.5$ *and* $\gamma = 0.5$

5.2. Least square estimation

The method of Least square estimation was proposed by Adrien-Marie Legendre [12] to determine the unknown parameters. Let $X_1, X_2, X_3, ..., X_n$, as the random sample of size n, taken from a continuous distribution function, then let $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq ... \leq X_{(n)}$ be its corresponding order statistics. Then the expected value and variance of the empirical cumulative distribution function (ecdf) are defined as follows,

$$E\big(F(X_{(i)}\big) = \frac{i}{n+1}, \qquad i = 1, 2, ..., n \tag{31}$$

$$Var\big(F(X_{(i)}\big) = \frac{i(n-i+1)}{(n+1)^2(n+2)}, \qquad i = 1, 2, ..., n. \tag{32}$$

The least square estimators (LSE) of the unknown parameters of NTIIHLGIW can be obtained by minimizing $S = \sum_{i=1}^{n} \left(F(X_{(i)}) - \frac{i}{n+1}) \right)^2$ with respect to the unknown parameters of NTIIHLGIW.

5.3. Weighted Least Square

The weighted Least Square estimators (WLSE) for estimating the unknown parameters of the NTIIHLGIW can be obtained by minimising the following equation

$$S = \sum_{i=1}^{n} W_i \left(F(X_{(i)}) - \frac{i}{n+1}) \right)^2,$$

where $W_i = 1$ for all $i = 1, 2, ..., n$ in case of LSE method and $W_i = \frac{1}{Var\left(F(X_{(i)}\right)}$ in Equation (32).

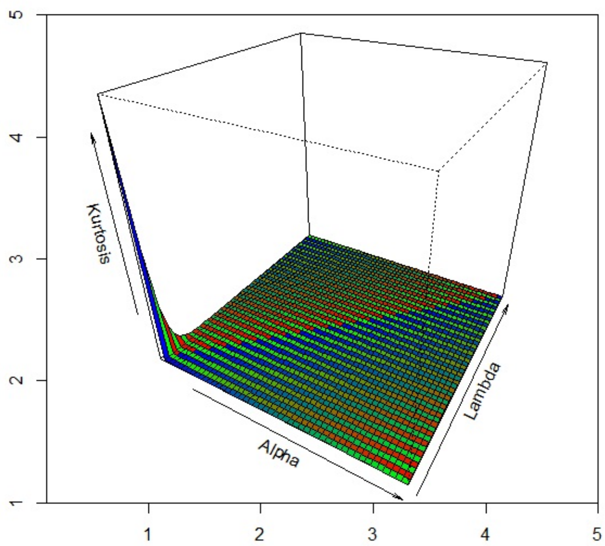

Figure 9: *Plot of Kurtosis NTIIHLGIW distribution for $\beta = 0.5$ and $\gamma = 0.5$*

6. Simulation Study and Data Analysis

6.1. Simulation Study

In this section, simulation study is conducted to check the performance of the estimation algorithms. For this, we consider $\alpha = 1$, $\beta = 0.5$, $\lambda = 1$ and $\gamma = 1.5$. We simulate data from NTIIHLGIW model for different sample sizes $n = 50, 100$ and 250, calculate for MLE, LSE and WLSE estimates. The result is listed in Table 1. Here, we can see that as the n increases, mean squared error (MSE) decreases also bias is decreases. From Table 1 we can see that the MLE method performs good for estimating the parameters of NTIIHLGIW distribution compared to LSE and WLSE methods.

6.2. Data Analysis

In this section, we demonstrate the usefulnesss of the proposed $NTIIHLGIW(\alpha, \beta, \lambda, \gamma)$ distribution. We fit this distribution to two real life data set and compare the results with some recent efficient models those corresponding to the Generalized inverse Weibull distribution, Transmuted Inverse Weibull distribution, TIIHLIW and TIIHLGIW. The corresponding pdf are presented below:

- Inverse Generalized Weibull Distributions:

$$f(x, \alpha, \beta, \lambda) = \alpha\beta\lambda^{\beta} e^{-\left(\frac{\lambda}{x}\right)^{\beta}} x^{-(\beta+1)} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^{\beta}}\right)^{\alpha-1}.$$

- Generalized Inverse Generalized Weibul Distribution:

$$f(x, \alpha, \beta, \lambda, \gamma) = \alpha\beta\gamma\lambda^{\beta} e^{-\gamma\left(\frac{\lambda}{x}\right)^{\beta}} x^{-(\beta+1)} \left(1 - e^{-\gamma\left(\frac{\lambda}{x}\right)^{\beta}}\right)^{\alpha-1},.$$

Data Set 1: This data set has been taken from ([21]).The data on survival of 40 patients suffering from leukemia, from the Ministry of Health Hospitals in Saudi Arabia taken from Abouammoh

Table 1: *Simulation study for $\alpha = 1$, $\beta = 0.5$, $\lambda = 1.5$ and $\gamma = 0.5$ for sample sizes 50, 100 and 250.*

n	Parameter	Method	Bias	MSE
50	α	MLE	0.101	0.069
		LSE	-0.114	0.086
		WLSE	-0.108	0.071
	β	MLE	0.078	0.041
		LSE	-0.091	0.083
		WLSE	-0.088	0.062
	λ	MLE	0.094	0.088
		LSE	0.098	0.101
		WLSE	0.098	0.097
	γ	MLE	0.038	0.025
		LSE	-0.046	0.062
		WLSE	-0.041	0.059
100	α	MLE	0.027	0.015
		LSE	-0.036	0.024
		WLSE	-0.027	0.023
	β	MLE	0.012	0.004
		LSE	0.021	0.014
		WLSE	0.018	0.012
	λ	MLE	0.021	0.015
		LSE	-0.028	0.024
		WLSE	-0.024	0.016
	γ	MLE	0.013	0.009
		LSE	0.021	0.026
		WLSE	0.016	0.019
250	α	MLE	0.003	0.002
		LSE	0.008	0.003
		WLSE	0.005	0.003
	β	MLE	0.002	0.003
		LSE	0.003	0.004
		WLSE	0.003	0.004
	λ	MLE	0.005	0.006
		LSE	0.008	0.011
		WLSE	0.005	0.008
	γ	MLE	0.002	0.003
		LSE	0.004	0.003
		WLSE	0.003	0.003

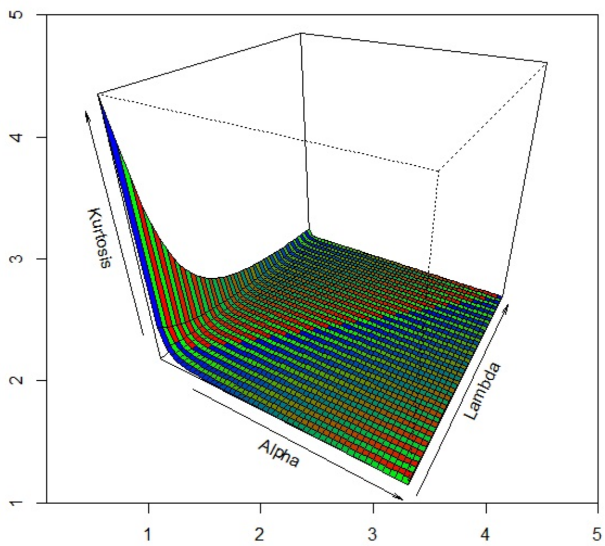

Figure 10: *Plot of Kurtosis NTIIHLGIW distribution for* $\beta = 2.5$ *and* $\gamma = 0.5$

et al. (1994):

115	181	255	418	441	461	516	739	743	789	807	865	924	983
1024	1062	1063	1165	1191	1222	1222	1251	1277	1290	1357	1369	1408	1455
1478	1549	1578	1578	1599	1603	1605	1696	1735	1799	1815	1852.		

Table 2: *Estimates and Goodness-of-fit measures based on AIC,BIC,AICC and CAIC for Data Set 1*

Distribution	**Estimates**	**Log-Likelihood**	**AIC**	**BIC**	**AICC**	**CAIC**
NTIIHLGIW	$\alpha = 64.3676$	-326.9782	629.9564	655.023	660.623	665.023
	$\beta = 10.69195$					
	$\theta = 740.11404$					
TIIHLIW	$\alpha = 0.042609$	-346.17	698.3402	703.4068	699.006	703.4068
	$\beta = 11.354601$					
	$\theta = 123.703578$					
TGIW	$\alpha = 0.03236$	-367.4545	742.9098	749.664	744.0518	749.6644
	$\beta = 8.30675$					
	$\theta = 14.26147$					
	$c = 47.2107282$					
GIW	$\alpha = 0.03236$	-367.4545	742.9098	749.664	744.0518	749.6644
	$\beta = 8.30675$					
	$\theta = 14.26147$					
	$c = 47.2107282$					

From Table 2, it shows that the proposed NTIIHLGIW distribution model has the lowest AIC, BIC, AICC, and CAIC values among the other distributions, it suggesting that it provides the best fit to the dataset.

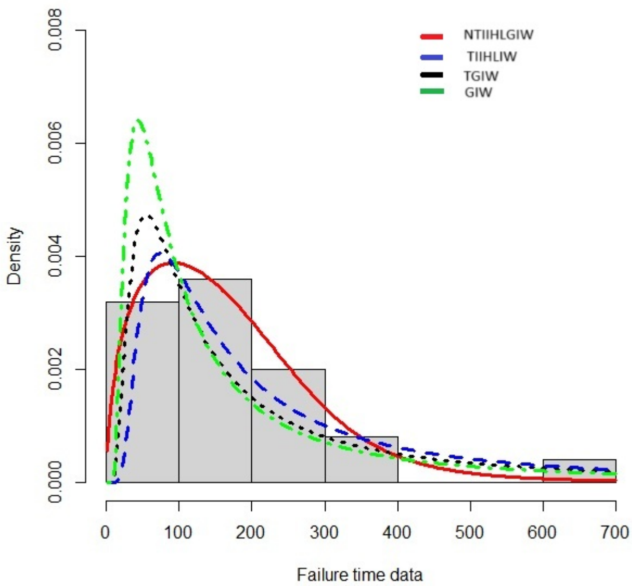

Figure 11: *Plot of fitted pdf of NTIIHLGIW distribution for data set 1*

Data set 2: This data set represents survival times in Days, from a Two-Arm Clinical Trial considered by [25] and [24] . The survival time in days for the 31 patients from Arm B are :

37	84	92	94	110	112	119	127	130	133	140	146	155	159	173	179
194	195	209	249	281	319	339	432	469	519	633	725	817	1557	1776	.

Table 3: *Estimates and Goodness-of-fit measures based on AIC,BIC,AICC and CAIC for Data Set 2*

Distribution	Estimates	Log-Likelihood	AIC	BIC	AICC	CAIC
NTIIHLGIW	$\alpha = 23.0618$	-206.6578	419.3155	423.6175	420.2044	423.6175
	$\beta = -0.3304$					
	$\lambda = 303.1340$					
	$\gamma = 303.1340$					
TIIHLIW	$\alpha = 0.0596$	-217.3668	440.7335	445.0355	441.6224	445.03
	$\beta = 9.8165$					
	$\lambda = 40.5081$					
TGIW	$\alpha = 0.0596$	-217.3668	440.7335	445.0355	441.6224	445.03
	$\beta = 9.8165$					
	$\lambda = 40.5081$					
GIW	$\alpha = 0.6665$	-217.0158	442.0317	447.7676	443.5701	447.7676
	$\beta = 8.8546$					
	$\theta = 40.9719$					
	$c = 1.0867$					

From Table 3, We can see that our model NTIIHLGIW has Minimum AIC, BIC, AICC, and CAIC values than TIIHLIW, TGW and GIW distributions. So we can conclude that the newly proposed model is a better fit for the given data compared to other models.

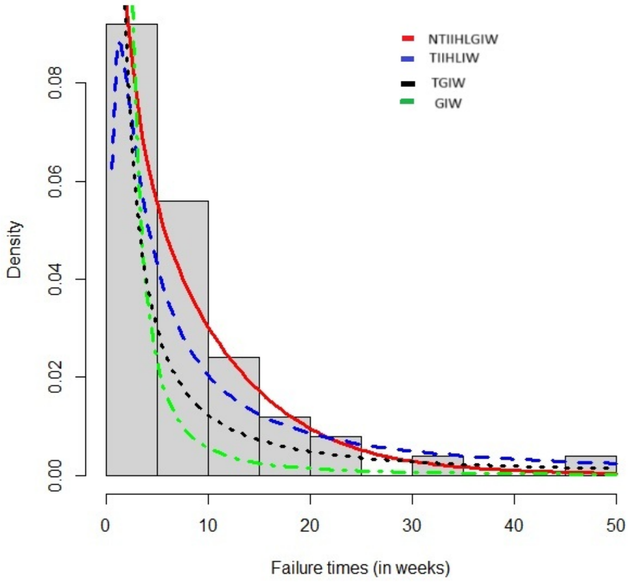

Figure 12: *Plot of fitted pdf of NTIIHLGIW distribution for data set 2*

7. Conclusions

The present research work introduced new flexible extension of Inverse Weibull distribution called new type II half logistic generalized Inverse Weibull distribution. The NTIIHLGIW distribution's structural features and reliability measures have been derived and examined. To estimate the distribution's parameters, a maximum likelihood estimation, lease square and weighted least square estimation methods were used. For the simulated data set, the results are shown in table 2. We can see that the estimated values obtained are near to the predefined parameters and that as n increases, MSE decreases, confirming the law of large numbers. We to illustrate the usefulness of NTIIHLGIW we used two real-life data sets and it shows that the NTIIHLGIW distribution has better fit than other competing models GIW, TGIW and TIIHLIW distribution by using Goodness of fit based on AIC,BIC,AICC and CAIC.

Conflict of Interest
The authors declare that they have no conflict of interest.

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