TRANSMUTED X-RAMA DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

In this paper, we introduce a new generalized distribution called Transmuted X-Rama distribution (TXRD) by using the quadratic rank transmutation map (QRTM). The probability density function (PDF), cumulative distribution function (CDF), and key moments of the transmuted distribution are meticulously derived to demonstrate how the transmutation parameter influences the shape of the distribution. Additionally, we analyze its reliability function, hazard rate, odds function, entropy and order statistics all of which are crucial in applications involving uncertainty and longevity data. We propose the method of maximum likelihood estimation (MLE) and method of least squares estimation (LSE) for the estimation of parameters of the distribution. We also show how the transmuted distribution may be fitted to real datasets.

Keywords: Transmuted X-Rama distribution, moments, reliability analysis, order statistics, entropy, simulation.

1. Introduction

Distributions are fundamental concepts in probability and statistics that describe how data values are distributed across possible outcomes. It is not always possible to represent complex data sets using standard distributions due to their rigid architecture and assumptions. Because of these limitations, it is necessary to use generalized distributions, which are extensions of conventional ones that offer more parameters and flexibility.

Recently, several researchers have suggested new families of distributions for better modeling of real-world events by generalizing existing families. Numerous methods exist for creating a new generalized distribution in the literature. One such method is the quadratic rank transmutation map (QRTM) presented by Shaw and Buckley [16]. This method relies on a baseline distribution to generate a fresh distribution called the transmuted distribution. Although many of the characteristics of the baseline distributions are retained, transmuted distributions often offer increased flexibility in modeling data, especially in terms of skewness and kurtosis, due to the additional parameter. The significance of QRTM is demonstrated by a number of published studies in the literature. To name a few, Aryal and Tsokos [4] introduced the transmuted Weibull distribution, Khan, King and Hudson [9] studied the transmuted Kumaraswamy distribution, Demirci [5] proposed the transmuted Xgamma distribution, Mohiuddin et al [10] introduced the transmuted Garima distribution, Onyekwer et al [13] studied the transmuted Shanker distribution. Alzaatreh et al [3] developed a method to generate a family of continuous distributions called *T*-*X* family of distributions. Girish and Jayakumar[11] introduced a combined family of *T*-*X* and transmuted distributions called the T-transmuted X family of distributions.

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According to Shaw and Buckley [16], the quadratic rank transmutation map (QRTM) technique states that a random variable X has a transmuted distribution if its cdf is provided by

$$G(x) = (1+\lambda)F(x) - \lambda\left(F(x)\right)^2; \quad |\lambda| \leq 1 \tag{1}$$

where F(x) is the cdf of the base distribution and G(x) is the cdf of the transmuted distribution. The pdf of Transmuted distribution will be;

$$g(x) = f(x)\left(1+\lambda-2\lambda F(x)\right) \tag{2}$$

where f(x) and g(x) are the corresponding probability density functions of F (x) and G (x), respectively.

The Rama distribution is a one-parameter distribution introduced by Sankar R [15] as a model suitable for skewed lifetime data with a non-constant hazard rate. This distribution is created by combining the Gamma$(4,\theta)$ and Exponential(θ) distributions in a certain proportion. In order to provide the Rama distribution with further flexibility, several generalized variations of it have been documented in the literature. Some of them include, Edith et al [6] proposed two-parameter Rama distribution, Abebe et al [1] introduced two-parameter power Rama distribution, Eyob and Shanker R [8] studied two-parameter weighted Rama distribution, Onyekwere et al [14] proposed the inverted power Rama distribution. Recently, a novel variation of the Rama distribution by Omoruyi et al [12] and the X-Rama distribution by Etaga et al [7] were introduced. The X-Rama distribution combines the Rama distribution and the exponential distribution with a mixing proportion $\frac{\theta^3}{\theta^3+6}$. In this article, we propose a new variation of the X-Rama distribution, called the transmuted X-Rama distribution (TXRD) by applying the technique of quadratic rank transmutation map (QRTM).

2. Transmuted X-Rama distribution

In this section, we provide the formulation of the transmuted X-Rama distribution (TXRD) by applying the QRTM of equation (1) to the X-Rama distribution. Consider the X-Rama distribution proposed by Etaga et al [7] with the probability density function (pdf) as

$$f(x) = \frac{\theta^4}{\left(\theta^3+6\right)^2}\left(\theta^3+6x^3+12\right)e^{-\theta x}, \quad x>0,\ \theta>0 \tag{3}$$

and cumulative distribution function (cdf) as

$$F(x) = 1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{\left(\theta^3+6\right)^2}\right]e^{-\theta x} \tag{4}$$

Using (4) in (1), we get the cdf of the TXRD,

$$\begin{aligned} G(x) = (1+\lambda)&\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\} \\ &-\lambda\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\}^2 \end{aligned} \tag{5}$$

Therefore, a continuous random variable X is said to follow TXRD if its probability density function is given by,

$$g(x;\theta,\lambda) = \theta e^{-\theta x}\left(1+\frac{6(\theta^3x^3-6)}{(\theta^3+6)^2}\right)\times\left(1-\lambda+2\lambda e^{-\theta x}\left(1+\frac{6\theta(\theta^2x^3+3\theta x^2+6x)}{(\theta^3+6)^2}\right)\right);$$

$$x>0,\ |\lambda|\leq 1, \theta>0 \tag{6}$$

The pdf and cdf plots of TXRD for different combinations of parameter values are displayed in Figure (1) and Figure (2) to demonstrate how it behaves in terms of distribution.

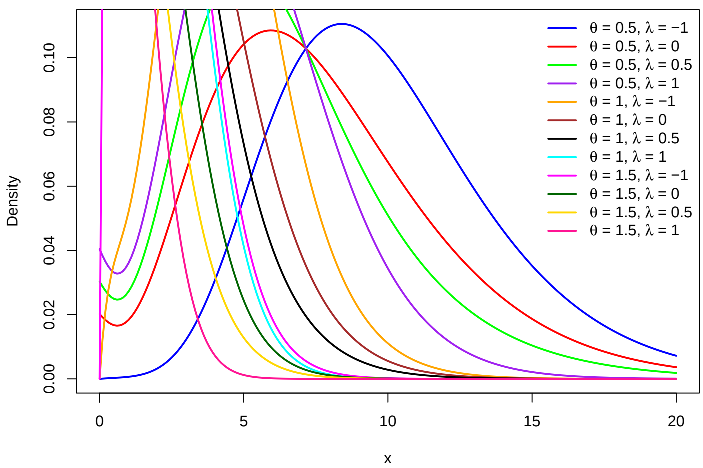

Figure 1: *PDF of TXRD for different values of parameters*

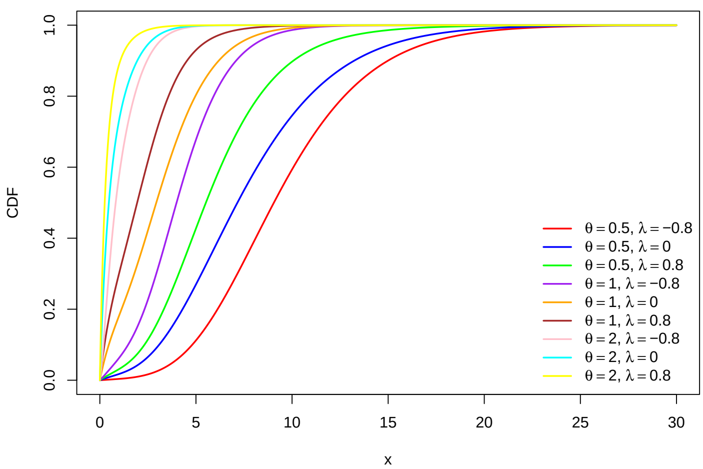

Figure 2: *CDF of TXRD for different values of parameters*

The plots show how the form, location of the peak and spread of the TXRD are mutually controlled by the parameters θ and λ. As θ decreases, the distribution becomes more dispersed with heavier tails, while as θ increases, it shows sharper peaks and faster decay. The transmutation parameter λ modulates the shape by introducing skewness and altering the behavior of the tail, where positive values of λ tend to stretch the right tail and increase asymmetry, and negative values enhance the peakedness around the mode.

3. Properties of transmuted X- Rama distribution

In this section, some mathematical properties such as moments, variance, moment generating function, and characteristic function of TXRD are derived.

3.1. Moments

Moments are quantitative measurements that are associated with the shape of a distribution. They help describe traits such as peakedness, skewness, spread, and location.

Let X be a TXRD random variable, then the r-th order moment will be

$$\begin{aligned}\mu_r' = &\int_0^\infty x^r\,\theta e^{-\theta x}\left(1+\frac{6(\theta^3x^3-6)}{(\theta^3+6)^2}\right)\\ &\times\left[1-\lambda+2\lambda e^{-\theta x}\left(1+\frac{6\theta}{(\theta^3+6)^2}(\theta^2x^3+3\theta x^2+6x)\right)\right]dx\end{aligned}$$

$$\begin{aligned}= &\frac{\lambda}{16(2\theta)^r(\theta^3+6)^2}\Big[9\Gamma(r+7)+54\Gamma(r+6)+136\Gamma(r+5)\\ &-408\Gamma(r+4)-2520\left(\Gamma(r+3)+\Gamma(r+2)\right)\\ &-560\Gamma(r+1)\Big]\\ &+\frac{\theta(1-\lambda)}{\theta^{r+1}}\Gamma(r+1)\\ &+\frac{6\theta(1-\lambda)}{(\theta^3+6)^2}\left(\frac{\Gamma(r+4)-\Gamma(r+1)}{\theta^r}\right)\end{aligned} \tag{7}$$

From equation (7) we can generate the first four raw moments.

$$\mu_1' = \frac{2189\lambda+(1-\lambda)\left[(\theta^3+6)^2+138\theta\right]}{\theta(\theta^3+6)^2} \tag{8}$$

$$\mu_2' = \frac{37955\lambda+(1-\lambda)\left[8(\theta^3+6)^2+2832\theta\right]}{4\theta^2(\theta^3+6)^2} \tag{9}$$

$$\mu_3' = \frac{170895\lambda+(1-\lambda)\left[24(\theta^3+6)^2+17136\theta\right]}{4\theta^3(\theta^3+6)^2} \tag{10}$$

$$\mu_4' = \frac{3349485\lambda+(1-\lambda)\left[384(\theta^3+6)^2+4815536\theta\right]}{16\theta^4(\theta^3+6)^2} \tag{11}$$

3.2. Variance

The variance of the TXRD random variable X is

$$\begin{aligned}V(X) = &\frac{37955\lambda+(1-\lambda)\left[8(\theta^3+6)^2+2832\theta\right]}{4\theta^2(\theta^3+6)^2}\\ &-\left(\frac{2189\lambda+(1-\lambda)\left[(\theta^3+6)^2+138\theta\right]}{\theta(\theta^3+6)^2}\right)^2\end{aligned} \tag{12}$$

3.3. Moment generating function

The moment generating function of the TXRD random variable X is obtained as

$$M_X(t) = \int_0^\infty\left[1+tx+\frac{(tx)^2}{2!}+\cdots\right]f(x,\theta,\lambda)\,dx$$

$$= \int_0^\infty \sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f(x,\theta,\lambda)\, dx$$
$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu'_j$$

substituting the pdf of TXRD (6) we get

$$\begin{aligned} M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \Bigg[& \frac{\theta(1-\lambda)\Gamma(j+1)}{\theta^{j+1}} + \frac{6\theta(1-\lambda)}{(\theta^3+6)^2} \left(\frac{\Gamma(j+4)-\Gamma(j+1)}{\theta^j} \right) \\ & + \frac{\lambda}{16(2\theta)^j(\theta^3+6)^2} \Big(9\Gamma(j+7) + 54\Gamma(j+6) + 136\Gamma(j+5) \\ & - 408\Gamma(j+4) - 2520\Gamma(j+3) - 2520\Gamma(j+2) - 560\Gamma(j+1) \Big) \Bigg] \end{aligned} \quad (13)$$

3.4. Characteristic function

The characteristic function of TXRD is

$$\begin{aligned} \Phi_X(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \Bigg[& \frac{\theta(1-\lambda)\Gamma(j+1)}{\theta^{j+1}} + \frac{6\theta(1-\lambda)}{(\theta^3+6)^2} \left(\frac{\Gamma(j+4)-\Gamma(j+1)}{\theta^j} \right) \\ & + \frac{\lambda}{16(2\theta)^j(\theta^3+6)^2} \Big(9\Gamma(j+7) + 54\Gamma(j+6) + 136\Gamma(j+5) \\ & - 408\Gamma(j+4) - 2520\Gamma(j+3) - 2520\Gamma(j+2) - 560\Gamma(j+1) \Big) \Bigg] \end{aligned} \quad (14)$$

4. Order statistics

Let $X_1, X_2, \ldots, X_n$ be the random sample taken from TXRD with probability density function $g_X(x)$ and cumulative distribution function $G_X(x)$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be its order statistics. The probability density function of the k-th order statistic $X_{(k)}$ is given by,

$$g_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} g_X(x) \left[G_X(x)\right]^{k-1} \left[1 - G_X(x)\right]^{n-k}$$

Using Equations (5) and (6),

$$\begin{aligned} g_{X_{(k)}}(x) = & \frac{n!}{(k-1)!(n-k)!}\, \theta e^{-\theta x} \left[1 + \frac{6(\theta^3 x^3 - 6)}{(\theta^3+6)^2} \right] \\ & \times \left\{ (1-\lambda) + 2\lambda e^{-\theta x} \left[1 + \frac{6\theta}{(\theta^3+6)^2} \left(\theta^2 x^3 + 3\theta x^2 + 6x \right) \right] \right\} \\ & \times B^{k-1} (1-B)^{n-k} \end{aligned} \quad (15)$$

where B is the cdf of the TXRD.

Therefore, the pdf of the higher-order statistic $X_{(n)}$ of TXRD can be obtained as

$$\begin{aligned} g_{X_{(n)}}(x) = & n\theta e^{-\theta x} \left[1 + \frac{6(\theta^3 x^3 - 6)}{(\theta^3+6)^2} \right] \\ & \times \left\{ (1-\lambda) + 2\lambda e^{-\theta x} \left[1 + \frac{6\theta}{(\theta^3+6)^2} \left(\theta^2 x^3 + 3\theta x^2 + 6x \right) \right] \right\} \\ & \times B^{n-1} \end{aligned} \quad (16)$$

The pdf of the first-order statistic $X_{(1)}$can be obtained as

$$g_{X_{(1)}}(x) = n\theta e^{-\theta x}\left[1+\frac{6(\theta^3x^3-6)}{(\theta^3+6)^2}\right] \times\left\{(1-\lambda)+2\lambda e^{-\theta x}\left[1+\frac{6\theta}{(\theta^3+6)^2}\left(\theta^2x^3+3\theta x^2+6x\right)\right]\right\} \times(1-B)^{n-1} \tag{17}$$

5. Renyi entropy

Entropy is a metric used in probability and statistics to measure the degree of randomness in a probability distribution. It describes how unpredictable a system is; the higher entropy means more randomness, while the lower entropy means more certainty.
The Renyi entropy of the TXRD random variable is given by,

$$R_E(\alpha) = \frac{1}{1-\alpha}\log\int_0^\infty f^\alpha(x)\,dx$$

$$= \frac{1}{1-\alpha}\log\left(\int_0^\infty\left[\theta e^{-\theta x}\left(1+\frac{6(\theta^3x^3-6)}{(\theta^3+6)^2}\right) \times\left(1-\lambda+2\lambda e^{-\theta x}\left(1+\frac{6\theta}{(\theta^3+6)^2}(\theta^2x^3+3\theta x^2+6x)\right)\right)\right]^\alpha dx\right)$$

$$= \frac{1}{1-\alpha}\log\left\{\theta^\alpha\sum_{j=0}^\infty\binom{\alpha}{j}\left(\frac{6}{(\theta^3+6)^2}\right)^j\sum_{k=0}^\infty\binom{\alpha}{k}\left(\frac{2\lambda}{1-\lambda}\right)^k \times\sum_{l=0}^\infty\binom{k}{l}\left(\frac{6\theta}{(\theta^3+6)^2}\right)^l\int_0^\infty(\theta^3x^3-6)^j(\theta^2x^3+3\theta x^2+6x)^l e^{-\theta x(\alpha+k)}\,dx\right\}$$

by applying series expansions,

$$R_E(\alpha) = \frac{1}{1-\alpha}\log\theta^\alpha\sum_{j=0}^\infty\binom{\alpha}{j}\left[\frac{6}{(\theta^3+6)^2}\right]^j\sum_{k=0}^\infty\binom{\alpha}{k}\left(\frac{2\lambda}{1-\lambda}\right)^k \times\sum_{l=0}^\infty\binom{k}{l}\left[\frac{6\theta}{(\theta^3+6)^2}\right]^l\sum_{m=0}^\infty\binom{j}{m}\theta^{3m}(-6)^{j-m} \times\sum_{n_1+n_2+n_3=l}\binom{l}{n_1,n_2,n_3}\theta^{2n_1+n_2}3^{n_2}6^{n_3} \times\int_0^\infty x^{3(m+n_1)+2n_2+n_3}e^{-\theta x(\alpha+k)}dx$$

using gamma integral,

$$R_E(\alpha) = \frac{1}{1-\alpha}\log\theta^\alpha\sum_{j=0}^\infty\binom{\alpha}{j}\left[\frac{6}{(\theta^3+6)^2}\right]^j\sum_{k=0}^\infty\binom{\alpha}{k}\left(\frac{2\lambda}{1-\lambda}\right)^k \times\sum_{l=0}^\infty\binom{k}{l}\left[\frac{6\theta}{(\theta^3+6)^2}\right]^l\sum_{m=0}^\infty\binom{j}{m}\theta^{3m}(-6)^{j-m} \times\sum_{n_1+n_2+n_3=l}\binom{l}{n_1,n_2,n_3}\theta^{2n_1+n_2}3^{n_2}6^{n_3} \times\frac{\Gamma\big(3(m+n_1)+2n_2+n_3+1\big)}{\big[\theta(\alpha+k)\big]^{3(m+n_1)+2n_2+n_3+1}}$$

simplifying further, we obtain the Renyi entropy of TXRD as

$$R_E(\alpha) = \frac{1}{1-\alpha} \log \Bigg[\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n_1+n_2+n_3=l} \binom{\alpha}{j} \binom{\alpha}{k} \binom{k}{l} \binom{j}{m} \binom{l}{n_1, n_2, n_3} \times \frac{3^{n_2} (-1)^{j-m} 6^{2j+l-m+n_3} \theta^{\alpha+3m+l+2n_1+n_2}}{(1-\lambda)^k (\theta^3+6)^{2(j+l)}} \cdot \frac{(3(m+n_1)+2n_2+n_3)!}{[\theta(\alpha+k)]^{3(m+n_1)+2n_2+n_3+1}} \Bigg] \quad (18)$$

6. Reliability analysis

In reliability engineering and statistics, a reliability or survival function is a basic idea that expresses the likelihood that a system will run as intended for a particular amount of time under given circumstances. It is a key component in the evaluation of long-term performance of a system and serves as a measurement of its reliability.

6.1. Reliability function

The reliability function or survival function, denoted by R(x) or S(x), represents the probability that a system will operate successfully without failure up to time x.

R(x) = P(X>x)
The reliability function is related to the cumulative distribution function (CDF) of the failure time as

R(x)=1-G(x)
survival function of TXRD will be obtained as

$$R(x) = 1-(1-\lambda)\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\} + \lambda\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\}^2 \quad (19)$$

6.2. Hazard rate function

The hazard rate function measures the probability that a system will fail in the next instant if it has survived to this far.

$$h(x) = \lim_{\Delta x \to 0} \frac{P(x \le X < x+\Delta x \mid X \ge x)}{\Delta x}$$

The hazard rate function of TXRD is obtained as

$$h(x) = \frac{g(x)}{R(x)}$$

$$h(x) = \frac{\theta e^{-\theta x}\left[1+\frac{6(\theta^3x^3-6)}{(\theta^3+6)^2}\right]\left[1-\lambda+2\lambda e^{-\theta x}\left(1+\frac{6\theta}{(\theta^3+6)^2}(\theta^2x^3+3\theta x^2+6x)\right)\right]}{1-(1-\lambda)\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\}+\lambda\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\}^2} \quad (20)$$

and the reverse hazard rate of TXRD is

$$h_r(x) = \frac{g(x)}{G(x)}$$

$$= \frac{\theta e^{-\theta x}\left[1+\frac{6\theta^3x^3-36}{(\theta^3+6)^2}\right]\left\{(1-\lambda)+2\lambda e^{-\theta x}\left[1+\frac{6\theta}{(\theta^3+6)^2}(\theta^2x^3+3\theta x^2+6x)\right]\right\}}{\left\{(1+\lambda)\left[1-\left(1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right)e^{-\theta x}\right]-\lambda\left[1-\left(1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right)e^{-\theta x}\right]^2\right\}} \tag{21}$$

6.3. Odds function

The odds function, often referred to as the odds of failure, is a concept that is derived from the survival function which is used in reliability theory to characterize the probability of failure over time. It gives the ratio of the probability of failure at time x to the probability of survival beyond time x.
Let X be a TXRD random variable then, the odds function of X is given by,

$$\omega_0(x) = \frac{G(x)}{R(x)}$$

$$= \frac{(1+\lambda)\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\} - \lambda\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\}^2}{1-\left[(1-\lambda)\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\} + \lambda\left\{1-\left[1+\frac{6\theta^3x^3+18\theta^2x^2+36\theta x}{(\theta^3+6)^2}\right]e^{-\theta x}\right\}^2\right]} \tag{22}$$

The plots of the reliability function, the hazard rate function, and the odds function of the TXRD for different combinations of parameter values are shown in Figure (3), Figure (4), and Figure (5).

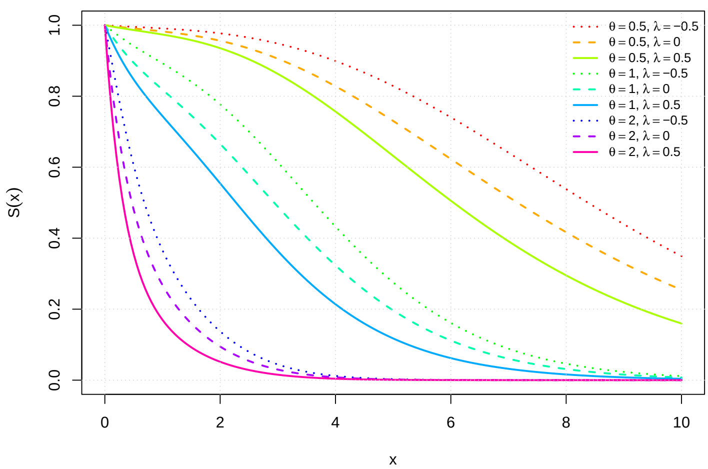

Figure 3: *Survival function of TXRD for different values of parameters*

In the survival function plot (Figure (3)), as θ increases, the survival rate decreases more slowly, indicating longer life spans and slower failure rate and as λ decreases, the survival function decreases more drastically, leading to a greater decrease in survival probabilities. This behavior suggests that higher values of θ improve system durability and prolong survival, while lower values of λ are associated with faster system failures and reduced longevity.

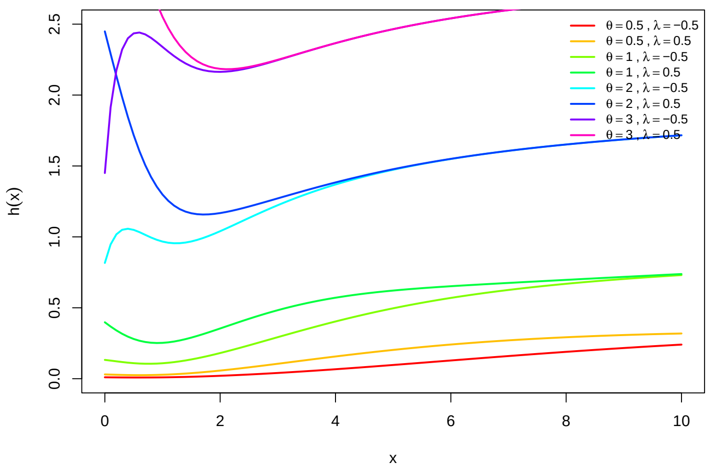

Figure 4: *Hazard rate function of TXRD for different values of parameters*

In the graph of the hazard rate function (Figure(4), the curves exhibit flexible and diverse shapes, reflecting different types of failure mechanisms. As the shape parameter θ increases, the overall hazard rate increases, indicating an increased likelihood of failure at any given time. As the transmutation parameter λ increases, the hazard rate also increases, particularly at early times, reflecting an increased initial risk. However, as λ decreases, the increase in the risk of failure is delayed, often resulting in an initially decreasing hazard rate function. This flexibility makes the TXRD model well suitable for modeling a wide range of real-world reliability scenarios.

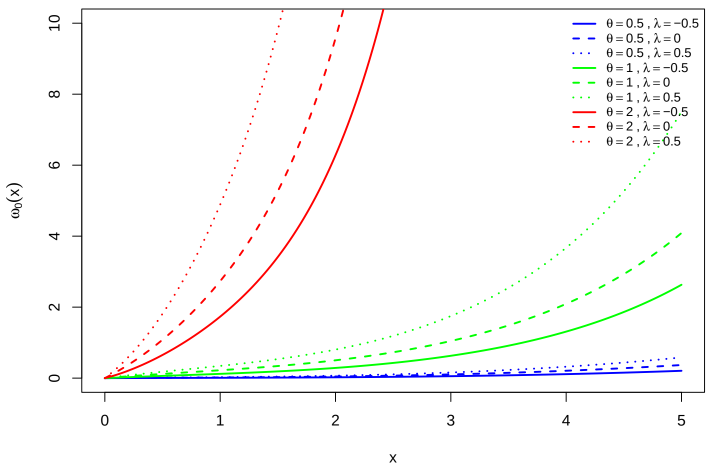

Figure 5: *Odds function of TXRD for different values of parameters*

Figure (5) shows the plot of the odds function $\omega_0(x)$ of TXRD, which illustrates how the parameters θ and λ influence the behavior of the model. As θ increases, the odds function shows a faster increase in the probability of failure, and as θ decreases, the curve becomes flatter, indicating a slower growth in the risk of failure. In contrast, as λ increases, the odds function is further amplified, and as λ decreases, the odds function indicates a lower and more gradual increase. When $\lambda = 0$, the model simplifies to the baseline distribution. The combined effect of θ and λ determines the shape of the odds function, with higher values of both producing the fastest growth in the odds of failure, while lower values of θ and λ show the most gradual increase.

7. Estimation of parameters

In the literature, several parameter estimation techniques have been proposed. Here we discuss the maximum likelihood estimation (MLE) method and the least squares estimation (LSE) method to estimate the parameters of TXRD.

7.1. Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be the random sample selected from TXRD then the likelihood function,

$$L = \prod_{i=1}^{n} \theta e^{-\theta x_i} \left(1 + \frac{6(\theta^3 x_i^3 - 6)}{(\theta^3+6)^2}\right) \left[1 - \lambda + 2\lambda e^{-\theta x_i} \left(1 + \frac{6\theta}{(\theta^3+6)^2}(\theta^2 x_i^3 + 3\theta x_i^2 + 6x_i)\right)\right]$$

$$= \theta^n e^{-n\theta\bar{x}} \prod_{i=1}^{n} \left(1 + \frac{6(\theta^3 x_i^3 - 6)}{(\theta^3+6)^2}\right) \left[1 - \lambda + 2\lambda e^{-\theta x_i} \left(1 + \frac{6\theta}{(\theta^3+6)^2}(\theta^2 x_i^3 + 3\theta x_i^2 + 6x_i)\right)\right]$$

$$\begin{aligned}\log L &= n \log\theta - n\theta\bar{x} \\ &+ \sum_{i=1}^{n} \log\left\{\left[1 + \frac{6(\theta^3 x_i^3 - 6)}{(\theta^3+6)^2}\right]\left[1 - \lambda + 2\lambda e^{-\theta x_i}\left(1 + \frac{6\theta}{(\theta^3+6)^2}\left(\theta^2 x_i^3 + 3\theta x_i^2 + 6x_i\right)\right)\right]\right\}\end{aligned}$$

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} - n\bar{x} + \sum_{i=1}^{n} \frac{1}{A_i} \cdot \frac{\partial A_i}{\partial\theta} + \bar{x} \sum_{i=1}^{n} \frac{1}{B_i} \cdot \frac{\partial B_i}{\partial\theta} = 0 \tag{23}$$

where

$$A_i = \left[1 + \frac{6\left(\theta^3 x_i^3 - 6\right)}{\left(\theta^3+6\right)^2}\right]$$

and

$$B_i = \left[1 - \lambda + 2\lambda e^{-\theta x_i}\left(1 + \frac{6\theta}{\left(\theta^3+6\right)^2}\left(\theta^2 x_i^3 + 3\theta x_i^2 + 6x_i\right)\right)\right]$$

$$\frac{\partial \log L}{\partial\lambda} = \sum_{i=1}^{n} \frac{1}{B_i} \frac{\partial B_i}{\partial\lambda} = 0 \tag{24}$$

Due to the complex structure of the non-linear equations (23) and (24), we employ R software to quantitatively estimate the necessary parameters.

7.2. Least square estimation

Let $G(x;\theta,\lambda)$ be the cumulative distribution function of TXRD (5), the least square estimation minimizes:

$$S(x;\theta,\lambda) = \sum_{i=1}^{n} \left[\hat{G}_n(x_i) - G(x_i;\theta,\lambda)\right]^2$$

where $\hat{G}_n(x_i)$ is the empirical cumulative distribution function (ECDF) obtained by,

$$\hat{G}_n(x_i) = \frac{1}{n}\sum_{j=1}^{n} I(x_j \leq x_i)$$

where $I(x_j \leq x_i)$ is the indicator function, defined as:

$$I(x_j \leq x_i) = \begin{cases} 1, & \text{if } x_j \leq x_i \\ 0, & \text{otherwise} \end{cases}$$

We estimate θ and λ by solving,

$$(\hat{\theta},\hat{\lambda}) = \arg\min_{\theta,\lambda} S(x;\theta,\lambda) \tag{25}$$

We use R software to estimate the necessary parameters numerically.

8. Simulation study

In this section, we simulate the data for TXRD to compare the performance of the maximum likelihood estimates (MLEs) and least squares estimates (LSEs) for the two parameters θ and λ using Monte Carlo simulation. The TXRD samples were generated for different samples of sizes n= 25, 50, 75, 100, 200, 500 for a fixed choice of parameters θ=1 and λ=0.5. Estimates of unknown parameters have been obtained using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method to maximize the log-likelihood function. The estimated values of the parameters θ and λ with their corresponding standard error, bias, and mean square error (MSE) are shown in Table (1).

The conclusions obtained from the simulation study are given as follows.

• MLE has lower bias and mean square error for both parameters θ and λ, indicating more accurate estimates.

• LSE shows significantly higher bias and mean square error, especially for λ, suggesting that it is less reliable for estimating this parameter.

• As the sample size increases, bias and mean square error decrease for both methods, but LSE still maintains a relatively higher bias and MSE compared to the MLE method.

•MLE is the superior method in terms of both, producing more accurate and reliable estimates in the case of the transmuted X-Rama distribution.

Table 1: *Simulation Results for MLE and LSE Estimators*

n	Method	Parameter	Mean	Bias	SE	MSE
10	MLE	θ	1.015088	0.015088	0.195922	0.038536
		λ	0.539817	0.039817	0.503626	0.254718
	LSE	θ	1.051201	0.051201	0.188800	0.038196
		λ	0.650663	0.150663	0.393454	0.177196
25	MLE	θ	1.007458	0.007458	0.155505	0.024189
		λ	0.522935	0.022935	0.467841	0.218963
	LSE	θ	1.016571	0.016571	0.138563	0.019436
		λ	0.595623	0.095623	0.395424	0.165191
50	MLE	θ	1.006305	0.006305	0.122880	0.015109
		λ	0.519831	0.019831	0.393573	0.154983
	LSE	θ	1.011312	0.011312	0.116134	0.013588
		λ	0.570243	0.070243	0.364357	0.137424
75	MLE	θ	1.007654	0.007654	0.109352	0.011993
		λ	0.483514	-0.016490	0.370582	0.137328
	LSE	θ	1.012414	0.012414	0.104396	0.011031
		λ	0.518208	0.018208	0.346516	0.120165
100	MLE	θ	1.004228	0.004228	0.104798	0.010979
		λ	0.489118	-0.010880	0.350514	0.122733
	LSE	θ	1.008898	0.008898	0.102888	0.010644
		λ	0.512336	0.012336	0.337062	0.113536
200	MLE	θ	1.005771	0.005770	0.094230	0.008895
		λ	0.482926	-0.017070	0.329599	0.108709
	LSE	θ	1.006737	0.006737	0.088680	0.007894
		λ	0.503649	0.003649	0.306341	0.093671
500	MLE	θ	1.017436	0.017436	0.071209	0.005365
		λ	0.436896	-0.063104	0.258042	0.070435
	LSE	θ	1.017356	0.017356	0.068564	0.004993
		λ	0.449928	-0.050072	0.253882	0.066834

9. Data analysis

In this section, we demonstrate the adaptability of TXRD to model real-world data sets. To evaluate the model, we compute the goodness-of-fit statistics and estimate the model parameters.

Data Set: 1

The following data represent 40 patients suffering from blood cancer (leukemia) from one of the ministry of health hospitals in Saudi Arabia (Abouammah, Ahmed and Khalique, 2000 [12]).

Table 2: *Data regarding 40 Leukemia Patients from Health Hospital in Saudi Arabia.*

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162	2.211	2.370	2.532	2.693
2.805	2.910	2.912	3.192	3.263	3.348	3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986
4.049	4.244	4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381		

Data Set: 2

The following data represent the strength of the aircraft window glass recorded [2]

Table 3: *Dataset regarding the strength of the aircraft window glass*

18.83	20.80	21.657	23.03	23.23	24.05	24.321	25.50	25.52	25.80
26.69	26.77	26.78	27.05	27.67	29.90	31.11	33.20	33.73	33.76
33.89	34.76	35.75	35.91	36.98	37.08	37.09	39.58	44.045	45.29
45.381									

The Bayesian information criterion (BIC), Akaike information criterion (AIC), Akaike information criterion corrected (AICc), and logL for the data set are used to compare the distribution. The lower values of AIC, BIC, AICc, and -logL are indicative of a better distribution. We also provide the Kolmogorov-Smirnov test (K-S test) to check the goodness-of-fit of the distribution with the considered data sets.

$$\text{AIC} = 2k - 2\log L \quad \text{BIC} = k\log n - 2\log L \quad \text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1}$$

where n is the sample size and k is the number of parameters.

Table 4: *Model comparison based on MLE, log-likelihood, AIC, BIC, and AICc*

Distribution	**MLE**		**logL**	**AIC**	**BIC**	**AICc**	**K-S test**	**P-value**
	$\hat{\theta}$	$\hat{\lambda}$						
TXRD	1.2371	-1.0000	-70.8544	145.7089	149.0867	146.0332	0.1191	0.6222
TSD	0.7026	-0.7625	-78.0412	160.0824	163.4602	160.4068	0.1494	0.3342

Table 5: *Model comparison based on MLE, log-likelihood, AIC, BIC, and AICc*

Distribution	**MLE**		**logL**	**AIC**	**BIC**	**AICc**	**K-S test**	**P-value**
	$\hat{\theta}$	$\hat{\lambda}$						
TXRD	0.1627	-1.0000	-110.1237	224.2475	227.1154	224.6760	0.1696	0.2995
TSD	0.0867	-1.0000	-117.9894	239.9789	242.8468	240.4074	0.2585	0.0258

Table(4) represents the performance distributions of 40 patients suffering from blood cancer (leukemia) modeled using two statistical distributions: the proposed Transmuted X-Rama Distribution (TXRD) and the established Transmuted Shanker Distribution (TSD). The purpose of the analysis was to determine which model best fits the survival data of these patients. Table(5) presents a model comparison between the Transmuted X-Rama Distribution (TXRD) and the transmuted Shanker distribution (TSD) with respect to the strength of the recorded aircraft window glass. It is clear from both tables that TXRD provides a better fit than TSD.

10. Conclusion

In this study, we proposed a new two-parameter lifetime distribution, called the transmuted X-Rama distribution (TXRD) by applying the quadratic rank transmutation map (QRTM). Some distributional properties along with reliability analysis and parameter estimation are also discussed. A simulation study was conducted to evaluate the performance of the Maximum Likelihood Estimation (MLE) and the Least Square Estimation (LSE), and the results showed that the MLE method outperformed the LSE method. We also obtained expressions for order statistics and Renyi entropy. Finally, we illustrated the efficiency of the proposed distribution by applying it to two real-life data sets.

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