SENSITIVITY ANALYSIS OF TRIANGULAR AND SYMMETRIC SPLITTING METHODS FOR POSITIVE DEFINITE LINEAR SYSTEMS - BLOCK STOCHASTIC MATRICES

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Abstract

In this paper, triangular and symmetric(TS) splitting method is applied to regularized linear system of block stochastic coefficient matrix for finding the steady state probability vector and discussed the sensitivity analysis. The homogeneous system is transformed into regularized non-homogeneous linear system by using preconditioned matrix with the small perturbation. The sensitivity analysis depends on the perturbation parameter. From the numerical results, it is concluded that, the sensitivity analysis and convergence analysis of TS method changes rapidly when a small change in the perturbation parameter. Moreover, the numerical value of spectral radius gives the convergence analysis and the bounds of TS method.

Keywords: Block Stochastic rate matrix, TS splitting method, Condition number, Spectral radius, Perturbation parameter, Error analysis.

1. Introduction

Many problems in engineering and science are mathematically modelled as the following system of homogenous equations:

$$\pi\mathbb{Q} = 0 \ \ and \ \ \pi e = 1, \tag{1}$$

where, $\mathbb{Q}$ is block stochastic rate matrix and π is the steady state probability vector. The performance measures of any real time system pertaing to the homogenous system depends on the unique solution. Performance measure of finite queue was triggered by Rykov **et.al.** [1]. Shao **et.al.** and Rajaiah **et.al.** suggested a new method for finding the performance measures of internet router with self-similar input process [2, 3]. The solution of above system gives one dimensional null space. But, for a unique non-zero solution, the above homogenous system is converted into the non-homogenous regularized linear system as follows:

DOI: https://doi.org/10.24412/1932-2321-2025-489-262-276

Taking transpose on both sides of Eq. (1),

$$\begin{aligned} \pi\mathbb{Q} &= 0 \\ \Rightarrow (\pi\mathbb{Q})^T &= 0 \\ \Rightarrow \mathbb{Q}^T\pi^T &= 0 \\ \Rightarrow \overline{\mathbb{A}}x &= 0 \end{aligned} \tag{2}$$

where, $\overline{\mathbb{A}}$ and x are the transposes of the stochastic rate matrix $\mathbb{Q}$ and steady state vector π respectively. In particular, the coefficient matrix $\overline{\mathbb{A}}$ has zero column sum, positive diagonal entries and non-positive off-diagonal entries.
Since Eq. (2) does not give a unique solution, a non-negative constant $\epsilon > 0$ is added such that it is converted into the preconditioned regularized linear system [7, 8],

$$\mathbb{A}x = (\mathbb{Q}^T + \epsilon I_{n^2})x = e_{n^2} \tag{3}$$

where, $e_{n^2} = [0,\ 0,\ \ldots,\ 0,\ 1]^T$ is the unit vector. The steady-state probability vector π is then obtained by normalizing the vector x.
The steady state probability vector π of regularized linear system is solved by many researchers using different splitting methods and its convergence criteria is also discussed for general and stochastic rate matrices [4, 5, 6, 7, 8, 9, 10, 11, 12]. The significant improvements in convergence rates is achieved from Krylov subspace methods [5, 13], preconditioning techniques by [11, 12, 13], and two splitting and multi splitting iterative methods by [4, 9, 14, 15]. Two alternative methods Hermitian and Skew-Hermitian (HSS) and Positive definite Skew-Symmetric (PSS) methods were proposed which converge unconditionally to a unique solution for the system of equations [4, 16]. Triangular and skew-symmetric splitting method is used to find the steady state vector of regularized linear system [7]. Triangular and symmetric splitting method is used to find the steady state vector of regularized linear system of circulant matrices [8]. Hence, in this chapter, an improved convergence solution for the regularized positive definite linear systems is developed by using block triangular and symmetric (BTS) iteration method.

On the other hand, the perturbation theory was introduced as a new branch in linear algebra and attracted many researchers to focus their attention towards the sensitivity analysis of perturbation linear systems [17, 18, 19, 20]. Sensitivity analysis of perturbation linear system was taken into consideration while calculating the errors using condition number [20]. The condition number plays an important role in the numerical linear algebra. The condition number measures the sensitivity analysis of regularized linear system for small perturbation $\epsilon > 0$. The sensitivity analysis of iterative solution for the regularized linear system and its convergence analysis of a unique non-zero solution depends on the regularized matrix $\mathbb{A}$. If matrix $\mathbb{A}$ is non-singular, then condition number of the matrix decides the convergence of the iterative solution of the regularized linear system $\mathbb{A}x = b$. The system $\mathbb{A}x = b$ posses a unique non-zero solution for non-singular matrix. If the condition number of matrix $\mathbb{A}$ moves to an indefinitely large value for different values of the small perturbation ϵ, then the matrix $\mathbb{A}$ tends to a singular matrix. If matrix $\mathbb{A}$ is singular, then the solution of the system Eq. (3) does not exist. If the matrix is non-singular, then the condition number of $\mathbb{A}$ of the system Eq. (3) is finite. The ill-posed and well-posed solution of the regularized system depends on the perturbation ϵ. The regularized linear system Eq. (3) is well-posed for small condition number and ill-posed for large condition number. In this chapter, the sensitivity analysis of BTS iterative solution using the condition number and spectral radius of regularized stochastic matrix is discussed.

The rest of the chapter is organized as follows: In section 2, condition number and convergence analysis of regularized linear system is discussed. In section 3, the BTS iteration method and its convergence is studied. In section 4, the choice of the contraction factor α is analyzed. Finally, the numerical results are presented in section 5, and the conclusion in section 6

2. Regularized Linear System

Here, first the norm of a matrix, condition number of regularized matrix for sensitivity analysis of regularized linear system are defined. For convergence analysis of a unique non-zero solution of regularized linear system, some basic definitions, and which are used to prove regularized matrix is positive definite are presented. Some theorems for a unique convergence solution of the regularized linear system are proved.

Definition 1. If $\mathbb{A}$ is $m \times n$ matrix then 1-norm, ∞-norm, 2-norm are defined as follows:

- $\|\mathbb{A}\|_1 = Max \sum_{i=1}^{m} \mid a_{ij} \mid$ for $j = 1,\ 2,\ ...,\ n$
- $\|\mathbb{A}\|_\infty = Max \sum_{j=1}^{n} \mid a_{ij} \mid$ for $i = 1,\ 2,\ ...,\ n$
- $\|\mathbb{A}\|_2 = \sqrt{\lambda_{max}(\mathbb{A}^T\mathbb{A})}$

Definition 2. Condition number: The Condition number of a matrix $\mathbb{A}$ is denoted by $\kappa(\mathbb{A})$ and defined as $\kappa(\mathbb{A}) = \|\mathbb{A}\|\|\mathbb{A}^{-1}\|$.
Definition 3. The matrix $\mathbb{A}$ is said to be ill-conditioned, if $\mathbb{A}$ is near to singularity and it is said to well-conditioned, if $\mathbb{A}$ is non-singular matrix.
Definition 4. Any matrix $\mathbb{A} \in R^{n\times n}$ of the form $\mathbb{A} = sI - B, s > 0, B \geq 0$ is called an M-matrix, if $s \geq \rho(B)$.
If $s > \rho(B)$, then $\mathbb{A}$ is well-conditioned M-matrix, otherwise $\mathbb{A}$ is ill-conditioned M-matrix.

Lemma 1. *Let $\mathbb{P}$ be a doubly stochastic block transition probability matrix, then $\mathbb{Q}$ is a doubly stochastic block transition rate matrix.*
Proof. Let

$$\mathbb{P} = \frac{1}{n}\begin{bmatrix} \mathbb{P}_{11} & \mathbb{P}_{12} & \mathbb{P}_{13} & \dots & \mathbb{P}_{1n} \\ \mathbb{P}_{21} & \mathbb{P}_{22} & \mathbb{P}_{23} & \dots & \mathbb{P}_{2n} \\ \mathbb{P}_{31} & \mathbb{P}_{32} & \mathbb{P}_{33} & \dots & \mathbb{P}_{3n} \\ \dots & \dots & \dots & \ddots & \vdots \\ \mathbb{P}_{n1} & \mathbb{P}_{n2} & \mathbb{P}_{n3} & \dots & \mathbb{P}_{nn} \end{bmatrix} \tag{4}$$

is a doubly stochastic block transition probability matrix, where each transition block matrix of order $n \times n$ is defined as

$$\mathbb{P}_{ij} = \begin{bmatrix} p_{11}^{ij} & p_{12}^{ij} & p_{13}^{ij} & \dots & p_{1j}^{ij} & \dots & p_{1n}^{ij} \\ p_{21}^{ij} & p_{22}^{ij} & p_{23}^{ij} & \dots & p_{2j}^{ij} & \dots & p_{2n}^{ij} \\ p_{31}^{ij} & p_{32}^{ij} & p_{33}^{ij} & \dots & p_{3j}^{ij} & \dots & p_{3n}^{ij} \\ \dots & \dots & \dots & \dots & \ddots & \dots & \vdots \\ p_{n1}^{ij} & p_{n2}^{ij} & p_{n3}^{ij} & \dots & p_{nj}^{ij} & \dots & p_{nn}^{ij} \end{bmatrix}$$

Let π be the steady state vector satisfying,

$$\begin{aligned} \pi &= \pi\mathbb{P} \\ \Rightarrow \pi - \pi\mathbb{P} &= 0 \\ \Rightarrow \pi(I_{n^2} - \mathbb{P}) &= 0 \\ \Rightarrow \pi\mathbb{Q} &= 0 \end{aligned}$$

where,

$$\mathbb{Q} = I_{n^2 \times n^2} - \mathbb{P} = \begin{bmatrix} I_n - \frac{1}{n}\mathbb{P}_{11} & -\frac{1}{n}\mathbb{P}_{12} & -\frac{1}{n}\mathbb{P}_{13} & \dots & -\frac{1}{n}\mathbb{P}_{1n} \\ -\frac{1}{n}\mathbb{P}_{21} & I_n - \frac{1}{n}\mathbb{P}_{22} & -\frac{1}{n}\mathbb{P}_{23} & \dots & -\frac{1}{n}\mathbb{P}_{2n} \\ -\frac{1}{n}\mathbb{P}_{31} & -\frac{1}{n}\mathbb{P}_{32} & I_n - \frac{1}{n}\mathbb{P}_{33} & \dots & -\frac{1}{n}\mathbb{P}_{3n} \\ \dots & \dots & \dots & \ddots & \vdots \\ -\frac{1}{n}\mathbb{P}_{n1} & -\frac{1}{n}\mathbb{P}_{n2} & -\frac{1}{n}\mathbb{P}_{n3} & \dots & I_n - \frac{1}{n}\mathbb{P}_{nn} \end{bmatrix} \tag{5}$$

From the above matrix, we have, $\frac{1}{n}\sum_{j=1}^{n} \mathbb{P}_{ij} = I_n$ for each $1 \le i \le n$.
In this case, the sum of each row

$$\begin{aligned} I_n - \frac{1}{n}\mathbb{P}_{i1} - \frac{1}{n}\mathbb{P}_{i2} - \frac{1}{n}\mathbb{P}_{i3} - \dots - \frac{1}{n}\mathbb{P}_{in} &= I_n - \frac{1}{n}(\mathbb{P}_{i1} + \mathbb{P}_{i2} + \dots + \mathbb{P}_{in}) \\ &= 0. \end{aligned}$$

Similarly, it is proved that the sum of each column is zero.
Therefore, the matrix $\mathbb{Q}$ is doubly stochastic rate matrix.

■

Theorem 1. *Let $\mathbb{P}$ be a stochastic transition probability matrix of order $n^2 \times n^2$ with block transition probability matrices, then the block stochastic rate matrix $\mathbb{Q} \in R^{n^2 \times n^2}$ is M-Matrix.*
Proof. From Eq. (4), $\mathbb{P}$ is a block stochastic transition probability matrix of order $n^2 \times n^2$ with block transition probability matrices.
From Eq. (5), and Lemma 5.1, $\mathbb{Q}$ is a transition rate matrix with block matrices.
From, Eq. (5), we have,

$$\mathbb{Q}^T = \begin{bmatrix} I_n - \frac{1}{n}\mathbb{P}_{11} & -\frac{1}{n}\mathbb{P}_{21} & -\frac{1}{n}\mathbb{P}_{31} & \dots & -\frac{1}{n}\mathbb{P}_{n1} \\ -\frac{1}{n}\mathbb{P}_{12} & I_n - \frac{1}{n}\mathbb{P}_{22} & -\frac{1}{n}\mathbb{P}_{32} & \dots & -\frac{1}{n}\mathbb{P}_{n2} \\ -\frac{1}{n}\mathbb{P}_{13} & -\frac{1}{n}\mathbb{P}_{23} & I_n - \frac{1}{n}\mathbb{P}_{33} & \dots & -\frac{1}{n}\mathbb{P}_{n3} \\ \dots & \dots & \dots & \ddots & \vdots \\ -\frac{1}{n}\mathbb{P}_{1n} & -\frac{1}{n}\mathbb{P}_{2n} & -\frac{1}{n}\mathbb{P}_{3n} & \dots & I_n - \frac{1}{n}\mathbb{P}_{nn} \end{bmatrix}$$

In order prove the matrix $\mathbb{Q}$ is M-matrix, it is sufiicient to prove that $\frac{\mathbb{Q} + \mathbb{Q}^T}{2}$ is M-matrix.
Now,

$$\frac{\mathbb{Q}+\mathbb{Q}^T}{2} = \begin{bmatrix} I_n - \frac{1}{n}\mathbb{P}_{11} & \frac{-\mathbb{P}_{12}-\mathbb{P}_{21}}{2n} & \frac{-\mathbb{P}_{13}-\mathbb{P}_{31}}{2n} & \dots & \frac{-\mathbb{P}_{1n}-\mathbb{P}_{n1}}{2n} \\ \frac{-\mathbb{P}_{12}-\mathbb{P}_{21}}{2n} & I_n - \frac{1}{n}\mathbb{P}_{22} & \frac{-\mathbb{P}_{23}-\mathbb{P}_{32}}{2n} & \dots & \frac{-\mathbb{P}_{2n}-\mathbb{P}_{n2}}{2n} \\ \frac{-\mathbb{P}_{13}-\mathbb{P}_{31}}{2n} & \frac{-\mathbb{P}_{23}-\mathbb{P}_{32}}{2n} & I_n - \frac{1}{n}\mathbb{P}_{33} & \dots & \frac{-\mathbb{P}_{3n}-\mathbb{P}_{n3}}{2n} \\ \dots & \dots & \dots & \ddots & \vdots \\ \frac{-\mathbb{P}_{1n}-\mathbb{P}_{n1}}{2n} & \frac{-\mathbb{P}_{2n}-\mathbb{P}_{n2}}{2n} & \frac{-\mathbb{P}_{3n}-\mathbb{P}_{n3}}{2n} & \dots & I_n - \frac{1}{n}\mathbb{P}_{nn} \end{bmatrix}$$

$= I_{n^2\times n^2} - R$
where,

$$R = \begin{bmatrix} \frac{1}{n}\mathbb{P}_{11} & \frac{1}{2n}(\mathbb{P}_{12}+\mathbb{P}_{21}) & \frac{1}{2n}(\mathbb{P}_{13}+\mathbb{P}_{31}) & \dots & \frac{1}{2n}(\mathbb{P}_{1n}+\mathbb{P}_{n1}) \\ \frac{1}{2n}(\mathbb{P}_{12}+\mathbb{P}_{21}) & \frac{1}{n}\mathbb{P}_{22} & \frac{1}{2n}(\mathbb{P}_{23}+\mathbb{P}_{32}) & \dots & \frac{1}{2n}(\mathbb{P}_{2n}+\mathbb{P}_{n2}) \\ \frac{1}{2n}(\mathbb{P}_{13}+\mathbb{P}_{31}) & \frac{1}{2n}(\mathbb{P}_{23}+\mathbb{P}_{32}) & \frac{1}{n}\mathbb{P}_{33} & \dots & \frac{1}{2n}(\mathbb{P}_{3n}+\mathbb{P}_{n3}) \\ \dots & \dots & \dots & \ddots & \vdots \\ \frac{1}{2n}(\mathbb{P}_{1n}+\mathbb{P}_{n1}) & \frac{1}{2n}(\mathbb{P}_{2n}+\mathbb{P}_{n2}) & \frac{1}{2n}(\mathbb{P}_{3n}+\mathbb{P}_{n3}) & \dots & \frac{1}{n}\mathbb{P}_{nn} \end{bmatrix} \geq 0 \quad (6)$$

Now, we have to find spectral radius of R.
If r is the largest eigen value of R, and $r_1, r_2, \dots, r_n$ are non zero row sums of the matrix R, then from the paper **[?]**, we have

$$\min_j \frac{1}{r_j}\frac{1}{n}\sum_{j=1}^{n} p_{jn} r_j = \rho(R) \leq r \leq \rho(R) = \max_j \frac{1}{r_j}\frac{1}{n}\sum_{j=1}^{n} p_{jn} r_j$$

Since R is the non- negative matrix with non zero row sums $r_1, r_2, \dots, r_n$ and p_{ij} $(1 \leq i,\ j \leq n)$ are transition probablity matrices, then the sum of each row and each column is unity. That is,

$$r_1 = r_2 = \dots\dots = r_n = 1$$
$$\Rightarrow 1 = \rho(R) \leq r \leq \rho(R) = 1$$
$$\Rightarrow \rho(R) = 1$$

Since $\frac{\mathbb{Q}+\mathbb{Q}^T}{2} = I_{n^2\times n^2} - R$ and $\rho(R) = 1$ then by the definition, the stochastic rate matrix $\mathbb{Q}$ with block matrices is M-matrix.

■

Theorem 2. *Let $\mathbb{P}$ be a transition probability matrix with block transition matrices and $\mathbb{Q} \in R^{n^2\times n^2}$ is a stochastic rate matrix with block matrices, then there exist $\epsilon > 0$ such that, $\mathbb{A} = (\mathbb{Q}^T + \epsilon I_{n^2})$ is positive definite.*

Proof. In order prove the matrix $\mathbb{A}$ is positive definite, it is sufficient to prove that its symmetric part *i.e.*, $\frac{\mathbb{A}+\mathbb{A}^T}{2}$ is positive definite.

From Eq. (3), we have,

$$\frac{\mathbb{A}+\mathbb{A}^T}{2} = \frac{\mathbb{Q}+\mathbb{Q}^T}{2} + \epsilon I_{n^2}$$
$$\Rightarrow R = (1+\epsilon)I_{n^2} - \frac{\mathbb{A}+\mathbb{A}^T}{2}$$
$$\Rightarrow R \text{ is the non-negative matrix}$$

Let r be the maximal eigen value of R, such that $\rho(R) = r$,

$$\Rightarrow \mid R - rI_{n^2} \mid = 0$$
$$\Rightarrow (1+\epsilon-r) \text{ is the eigen value of } \frac{\mathbb{A}+\mathbb{A}^T}{2}$$
$$\Rightarrow (1+\epsilon) > \rho(R)$$

Hence, the matrix $\mathbb{A}$ is a positive definite matrix.
Next section involves the iterative solution using the triangular and symmetric splitting method of the non-homogeneous positive-definite regularized linear system $\mathbb{A}x = b$. ■

3. Model description

In this section, the steady state probability vector π of Eq. (1) obtained using regularized linear system Eq. (3) and then analyze its convergence criteria. Here, TS splitting method for finding the solution of linear system Eq. (1) with coefficient matrix is a block stochastic matrix is considered. Let the coefficient matrix $\mathbb{A}$ of the regularized linear system Eq. (3) be splitted in the form, $\mathbb{A} = (\mathbb{L} + \mathbb{D} - \mathbb{U}^T) + (\mathbb{U} + \mathbb{U}^T)$,
where,

$$\mathbb{D} = \begin{bmatrix} I_n - \frac{1}{n}\mathbb{P}_{11} & 0 & 0 & \dots & 0 \\ 0 & I_n - \frac{1}{n}\mathbb{P}_{22} & 0 & \dots & 0 \\ 0 & 0 & I_n - \frac{1}{n}\mathbb{P}_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_n - \frac{1}{n}\mathbb{P}_{nn} \end{bmatrix}$$

$$\mathbb{L} = \begin{bmatrix} \frac{\epsilon}{2}I_n & 0 & 0 & \dots & 0 \\ \frac{1}{n}\mathbb{P}_{21} & \frac{\epsilon}{2}I_n & 0 & \dots & 0 \\ \frac{1}{n}\mathbb{P}_{31} & \frac{1}{n}\mathbb{P}_{32} & \frac{\epsilon}{2}I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n}\mathbb{P}_{n1} & \frac{1}{n}\mathbb{P}_{n2} & \frac{1}{n}\mathbb{P}_{n3} & \dots & \frac{\epsilon}{2}I_n \end{bmatrix}, \text{ and } \mathbb{U} = \begin{bmatrix} \frac{\epsilon}{2}I_n & \frac{1}{n}\mathbb{P}_{12} & \frac{1}{n}\mathbb{P}_{13} & \dots & \frac{1}{n}\mathbb{P}_{1n} \\ 0 & \frac{\epsilon}{2}I_n & \frac{1}{n}\mathbb{P}_{23} & \dots & \frac{1}{n}\mathbb{P}_{2n} \\ 0 & 0 & \frac{\epsilon}{2}I_n & \dots & \frac{1}{n}\mathbb{P}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\epsilon}{2}I_n \end{bmatrix}.$$

Let,

$$\mathbb{A} = (\mathbb{L} + \mathbb{D} - \mathbb{U}^T) + (\mathbb{U} + \mathbb{U}^T) = \mathbb{T} + \mathbb{S} \tag{7}$$

where, $\mathbb{T} \in R^{n^2 \times n^2}$ is a triangular block stochastic matrix and its diagonal elements are positive, and $\mathbb{S} \in R^{n^2 \times n^2}$ is symmetric block stochastic matrix.

3.1. Iterative Method

Given an initial guess $x^{(0)}$, compute the next approximations using the following scheme [4, 7, 8, 16]:

$$\begin{aligned} (\alpha I_{n^2} + \mathbb{T})x^{(k+1/2)} &= (\alpha I_{n^2} - \mathbb{S})x^{(k)} + b \\ (\alpha I_{n^2} + \mathbb{S})x^{(k+1)} &= (\alpha I_{n^2} - \mathbb{T})x^{(k+1/2)} + b \end{aligned} \tag{8}$$

for $k = 0, 1, 2, \dots$, until $x^{(k)}$ converges for any positive contraction factor α.
The above iterative scheme could be written as

$$x^{(k+1)} = \mathbb{M}(\alpha)x^{(k)} + \mathbb{N}(\alpha)b \tag{9}$$

for $k = 0, 1, 2, \dots$, where,

$$\mathbb{M}(\alpha) = (\alpha I_{n^2} + \mathbb{S})^{-1}(\alpha I_{n^2} - \mathbb{T})(\alpha I_{n^2} + \mathbb{T})^{-1}(\alpha I_{n^2} - \mathbb{S}) \tag{10}$$

and,

$$\mathbb{N}(\alpha) = 2\alpha(\alpha I_{n^2} + \mathbb{S})^{-1}(\alpha I_{n^2} + \mathbb{T})^{-1}$$

Here, $\mathbb{M}(\alpha)$ is called the iteration matrix of BTS iteration method. The BTS iteration method converges to a unique solution, if $\rho(\mathbb{M}(\alpha)) < 1$.

Lemma 2. If $\mathbb{S} = \mathbb{U} + \mathbb{U}^T$ is a symmetric matrix and $\epsilon < 1$ then $\|\mathbb{S}\| < 1$
Proof. By the definition,

$$\begin{aligned} \|\mathbb{S}\| &= \|\mathbb{U} + \mathbb{U}^T\| \\ &\leq \|\mathbb{U}\| + \|\mathbb{U}^T\| \\ &\leq Max(\|\mathbb{U}\| + \|\mathbb{U}^T\|) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ \|\mathbb{S}\| &\leq \epsilon \end{aligned}$$

If $\epsilon < 1$ then $\|\mathbb{S}\| < 1$. ■

Lemma 3. If $\|\mathbb{S}\| < 1$ and $\|(\alpha I_n + \mathbb{S})^{-1}\|$ exists then $\|(I_n + \mathbb{S})^{-1}\| \leq \frac{1}{1-\|\mathbb{S}\|}$

Proof. The proof of the theorem is straight forward as in paper [15] ■

Lemma 4. If $\|\frac{\mathbb{S}}{\alpha}\| < 1$ and $\|(\alpha I_{n^2} + \mathbb{S})^{-1}\|$ exists then $\|(\alpha I_{n^2} + \mathbb{S})^{-1}\| \leq \frac{1}{\alpha - \|\mathbb{S}\|}$

Proof. Let $\alpha > 0$ and $\|\frac{\mathbb{S}}{\alpha}\| < 1$

Let $\mathbb{S}$ is symmetric matrix and $\|(\alpha I_{n^2} + \mathbb{S})^{-1}\|$ exists

Now,

$$\begin{aligned}
\|(\alpha I_{n^2} + \mathbb{S})^{-1}\| &= \|[\alpha(I_{n^2} + \frac{\mathbb{S}}{\alpha})]^{-1}\| \\
&= \|(\alpha)^{-1}(I_{n^2} + \frac{\mathbb{S}}{\alpha})^{-1}\| \\
&\leq \frac{1}{\alpha}\left[\frac{1}{1-\|\frac{\mathbb{S}}{\alpha}\|}\right] \\
&\leq \frac{1}{\alpha}\left[\frac{\alpha}{\alpha-\|\mathbb{S}\|}\right] \\
&\leq \frac{1}{\alpha-\|\mathbb{S}\|}
\end{aligned}$$

■

Theorem 3. If $\|\frac{\mathbb{S}}{\alpha}\| < 1$, $\|(\alpha I_n^2 - \mathbb{S})\| < \alpha - \|\mathbb{S}\|$, and $\mathbb{Q}(\alpha) = (\alpha I_n^2 - \mathbb{S})(\alpha I_n^2 + \mathbb{S})^{-1}$ then $\|\mathbb{Q}(\alpha)\| < 1$.

Proof. Let $\mathbb{S}$ is symmetric matrix and $\|(\alpha I_{n^2} + \mathbb{S})^{-1}\|$ exists

From above Lemma, we have $\|(\alpha I_{n^2} + \mathbb{S})^{-1}\| \leq \frac{1}{\alpha-\|\mathbb{S}\|}$

Now,

$$\begin{aligned}
\|\mathbb{Q}(\alpha)\| &= \|(\alpha I_n^2 - \mathbb{S})(\alpha I_n^2 + \mathbb{S})^{-1}\| \\
&\leq \|(\alpha I_n^2 - \mathbb{S})\| \, \|(\alpha I_n^2 + \mathbb{S})^{-1}\| \\
&\leq \|(\alpha I_n^2 - \mathbb{S})\| \left[\frac{1}{\alpha-\|\mathbb{S}\|}\right] \\
\|\mathbb{Q}(\alpha)\| &< 1
\end{aligned}$$

■

Theorem 4. *Let $\mathbb{A} \in R^{n^2 \times n^2}$ be a block stochastic matrix of regularized linear system Eq. (3) and $\mathbb{M}(\alpha)$ is iteration matrix of BTS iteration method, then the spectral radius of $\mathbb{M}(\alpha) < 1$*

Proof. From Eq. (8), we have,

$$\begin{aligned}
\mathbb{M}(\alpha) &= (\alpha I_{n^2} + \mathbb{S})^{-1}(\alpha I_{n^2} - \mathbb{T})(\alpha I_{n^2} + \mathbb{T})^{-1}(\alpha I_{n^2} - \mathbb{S}) \\
\rho(\mathbb{M}(\alpha)) &= \|\mathbb{M}(\alpha)\|_2 = \|(\alpha I_{n^2} + \mathbb{S})^{-1}(\alpha I_{n^2} - \mathbb{T})(\alpha I_{n^2} + \mathbb{T})^{-1}(\alpha I_{n^2} - \mathbb{S})\|_2 \\
&\leq \|(\alpha I_{n^2} - \mathbb{T})(\alpha I_{n^2} + \mathbb{T})^{-1}\|_2 \|(\alpha I_{n^2} - \mathbb{S})(\alpha I_{n^2} + \mathbb{S})^{-1}\|_2 \qquad (11)
\end{aligned}$$

Let $\mathbb{Q}(\alpha) = (\alpha I_{n^2} - \mathbb{S})(\alpha I_{n^2} + \mathbb{S})^{-1}$. Since $\mathbb{S}$ is a symmetric matrix, then $\mathbb{S}^T = \mathbb{S}$.

$$\begin{aligned}
\mathbb{Q}(\alpha)^T\mathbb{Q}(\alpha) &= ((\alpha I_{n^2} - \mathbb{S})(\alpha I_{n^2} + \mathbb{S})^{-1})^T(\alpha I_{n^2} - \mathbb{S})^{-1}(\alpha I_{n^2} + \mathbb{S}) \\
&= ((\alpha I_{n^2} + \mathbb{S})^{-1})^T((\alpha I_{n^2} - \mathbb{S}))^T(\alpha I_{n^2} - \mathbb{S})^{-1}(\alpha I_{n^2} + \mathbb{S}) \\
&= I_{n^2} \\
&\Rightarrow \mathbb{Q}(\alpha) \text{ is a orthogonal matrix} \\
&\Rightarrow \|\mathbb{Q}(\alpha)\|_2 = 1. \qquad (12)
\end{aligned}$$

Put, $V(\alpha) = (\alpha I_{n^2} - \mathbb{T})(\alpha I_{n^2} + \mathbb{T})^{-1}$.
Since $V(\alpha)$ is a triangular matrix which is positive definite [7], then

$$\|V(\alpha)\|_2 < 1, \forall \alpha > 0 \qquad (13)$$

From Eqs. (11)-(13), we have,

$$\begin{aligned}
&\rho(\mathbb{M}(\alpha)) = \|\mathbb{M}(\alpha)\|_2 \leq \|\mathbb{Q}(\alpha)\|_2\|V(\alpha)\|_2 < 1 \\
\Rightarrow &\rho(\mathbb{M}(\alpha)) < 1, \forall \alpha > 0
\end{aligned}$$

Therefore, the TS iterative solution of the regularized linear system converges to a unique solution. ■

4. Estimation of the Contraction Factor

In this section, the choice of contraction factor and convergence criteria of the iterative solution of regularized linear system is discussed. The following investigation describes formulae in approximating the contraction factor α for BTS iteration method. Contraction factor α plays an important role in convergence solution of the regularized linear system. The contraction factor α minimizes the upper bound of $\rho(\mathbb{M}(\alpha))$. The block coefficient matrix $\mathbb{A}$ of the regularized linear system Eq. (3)) can be splitted into block triangular matrix and block symmetric matrix [4, 10] as follows

$$\begin{aligned}
\mathbb{A} = (\mathbb{L} + \mathbb{D} - \mathbb{U}^T) + (\mathbb{U} + \mathbb{U}^T) = \mathbb{T}_1 + \mathbb{S}_1 \\
\mathbb{A} = (\mathbb{U} + \mathbb{D} - \mathbb{L}^T) + (\mathbb{L} + \mathbb{L}^T) = \mathbb{T}_2 + \mathbb{S}_2
\end{aligned}$$

where, $\mathbb{D}$, $\mathbb{L}$ and $\mathbb{U}$ are block diagonal, lower and upper triangular matrices of block coefficient matrix $\mathbb{A}$ of regularized linear system Eq. (3).
Let, $\mathbb{H}_1 = \mathbb{L} - \mathbb{U}^T$ and $\mathbb{H}_2 = \mathbb{L}^T - \mathbb{U}$ be lower and upper block triangular matrices.
Now,

$$\begin{aligned}
(\alpha I_{n^2} + \mathbb{T}_i)^{-1} &= (\alpha I_{n^2} + \mathbb{D} + \mathbb{H}_i)^{-1}, \quad \text{for} i = 1, 2, \ldots, \\
&= (\alpha I_{n^2} + \mathbb{D})^{-1}(I_{n^2} + \mathbb{H}_i(\alpha I_{n^2} + \mathbb{D})^{-1})^{-1}
\end{aligned}$$

Using Binomial expansion,

$$(\alpha I_{n^2} + \mathbb{T}_i)^{-1} = (\alpha I_{n^2} + \mathbb{D})^{-1}(I_{n^2} - \mathbb{H}_i(\alpha I_{n^2} + \mathbb{D})^{-1} + \mathbb{H}_i^2((\alpha I_{n^2} + \mathbb{D})^{-1})^2 - \ldots)$$

Consider the first order approximation,

$$\begin{aligned}(\alpha I_{n^2}+\mathbb{T}_i)^{-1} &= (\alpha I_{n^2}+\mathbb{D})^{-1}(I_{n^2}-\mathbb{H}_i(\alpha I_{n^2}+\mathbb{D})^{-1})\\ (\alpha I_{n^2}-\mathbb{T}_i)(\alpha I_{n^2}+\mathbb{T}_i)^{-1} &= (\alpha I_{n^2}-\mathbb{D}-\mathbb{H}_i)(\alpha I_{n^2}+\mathbb{D})^{-1}(I_{n^2}-\mathbb{H}_i(\alpha I_{n^2}+\mathbb{D})^{-1})\\ \|(\alpha I_{n^2}-\mathbb{T}_i)(\alpha I_{n^2}+\mathbb{T}_i)^{-1}\|_2 &\approx \|(\alpha I_{n^2}-\mathbb{D})(\alpha I_{n^2}+\mathbb{D})^{-1}\|_2\end{aligned}$$

The contraction factor can be obtained using the procedure [4, 7, 8] as follows:

$$\begin{aligned}\|(\alpha I_{n^2}-\mathbb{T}_i)(\alpha I_{n^2}+\mathbb{T}_i)^{-1}\|_2 &\approx max\|(\alpha I_n-((1+\epsilon)I_n-\frac{1}{n}\mathbb{P}_1))(\alpha I_n-((1+\epsilon)I_n+\frac{1}{n}\mathbb{P}_1))^{-1}\|_2\\ \alpha^\star &\approx arg\min_{\alpha>0} max\|(\alpha I_n-((1+\epsilon)I_n-\frac{1}{n}\mathbb{P}_1))(\alpha I_n-((1+\epsilon)I_n+\frac{1}{n}\mathbb{P}_1))^{-1}\|_2\\ \alpha^\star &\approx arg\min_{\alpha>0} max\|\frac{\alpha-\lambda_{min}}{\alpha+\lambda_{max}}\|\\ \alpha^\star &\approx \sqrt{\lambda_{min}\lambda_{max}}\end{aligned} \tag{14}$$

Therefore, the contraction factor $\alpha=\alpha^\star$ minimizes the upper bound of $\rho(\mathbb{M}(\alpha))$. The computation of optimal parameter α is a hard task, that needs further in-depth study. The two half steps of each step of the BTS iteration method require in finding the coefficient matrices $(\alpha I_{n^2}+\mathbb{T}_i)$ and $(\alpha I_{n^2}+\mathbb{S}_i)$ for $i=1,2\ldots$. To improve computing efficiency of the BTS iteration, ITS iteration method is employed, that is used to solve the two sub systems iteratively[4, 7, 8, 10, 16]. The first subsystem coefficient matrix $(\alpha I_{n^2}+\mathbb{T}_i)$ and second sub system coefficient matrix $(\alpha I_{n^2}+\mathbb{S}_i)$ are solved by Krylov subspace methods. The solution of the first sub system can be obtained by straight forward method, where as the solution of second sub system is obtained by *CGS* method.

5. Validation and Numerical Results

In this section, the sensitive analysis, spectral radius and error analysis of Jacobi, BTS, and BTSS methods of the regularized linear system with the coefficient matrix is a block stochastic matrix is depicted. For illustration purpose, the following block transition probability matrix is considered,

$$\mathbb{P}=\frac{1}{3}\begin{bmatrix}\begin{bmatrix}0.95&0.02&0.03\\0.02&0.9&0.08\\0.03&0.08&0.89\end{bmatrix}&\begin{bmatrix}0.8&0.05&0.15\\0.05&0.7&0.25\\0.15&0.25&0.6\end{bmatrix}&\begin{bmatrix}0.6&0.25&0.15\\0.3&0.65&0.05\\0.1&0.1&0.8\end{bmatrix}\\ \begin{bmatrix}0.8&0.15&0.05\\0.1&0.7&0.2\\0.1&0.15&0.75\end{bmatrix}&\begin{bmatrix}0.9&0.04&0.06\\0.03&0.93&0.04\\0.07&0.03&0.9\end{bmatrix}&\begin{bmatrix}0.65&0.25&0.1\\0.15&0.55&0.3\\0.2&0.2&0.6\end{bmatrix}\\ \begin{bmatrix}0.55&0.35&0.1\\0.25&0.6&0.15\\0.2&0.05&0.75\end{bmatrix}&\begin{bmatrix}0.8&0.05&0.15\\0.1&0.6&0.3\\0.1&0.35&0.55\end{bmatrix}&\begin{bmatrix}0.94&0.01&0.05\\0.02&0.92&0.06\\0.04&0.07&0.89\end{bmatrix}\end{bmatrix}.$$

Let the initial vector $x^{(0)}$ is $\begin{bmatrix}0&0&0&0&0&0&0&1\end{bmatrix}^T$. We compute steady state vector of regularized linear system and corresponding relative error based on the values of ϵ and contraction factor α. The stopping criteria is set as the relative error $< 10^{-15}$. Moreover, the sensitivity analysis and convergence analysis of iterative solution of regularized linear system is obtained by using the condition number and spectral radius of the regularized matrix $\mathbb{A}$ and the results are depicted in Figs. 1-6. Fig. 1 depicts the convergence rate of iterative solution of regularized linear system with absolute error and relative error for the particular values of contraction factor $\alpha=0.76$ and $\epsilon=0.1$. From this figure, it is concluded that, BTS iterative solution converges faster than that of

BTSS and Jacobi methods. Fig. 2 depicts the relative error of BTS iterative method for different values of contraction factor $\alpha = 0.76$ and ϵ. From this figure, one can conclude that BTS iterative solution converges rapidly as ϵ increase. The sensitivity analysis of BTS iterative solution depicted in Figs. 3 and 4. From, Fig. 3, it is concluded that the condition number of the regularized matrix $\mathbb{A}$ is low when the contraction factor moves near to diagonal element. Figs. 4, it is concluded that the condition number of regularized matrix $\mathbb{A}$ is low while comparing with BTSS method. From these two figures, it is concluded that the BTS iterative solution is well conditioned. The spectral radius of the matrix $\mathbb{A}$ depicted in the figure 5. From this, it is clear that the iterative solution converges to unique non-zero solution. The condition number and spectral radius of BTS iterative method for different values of ϵ over contraction factor are depicted in figure 6. From this figure, the sensitivity analysis and convergence analysis of BTS iterative method is effective and efficient while comparing with theoretical results and other existing methods. From this, it is concluded that, the spectral radius of iterative matrix is less than one and which is evident to the theocratical results.

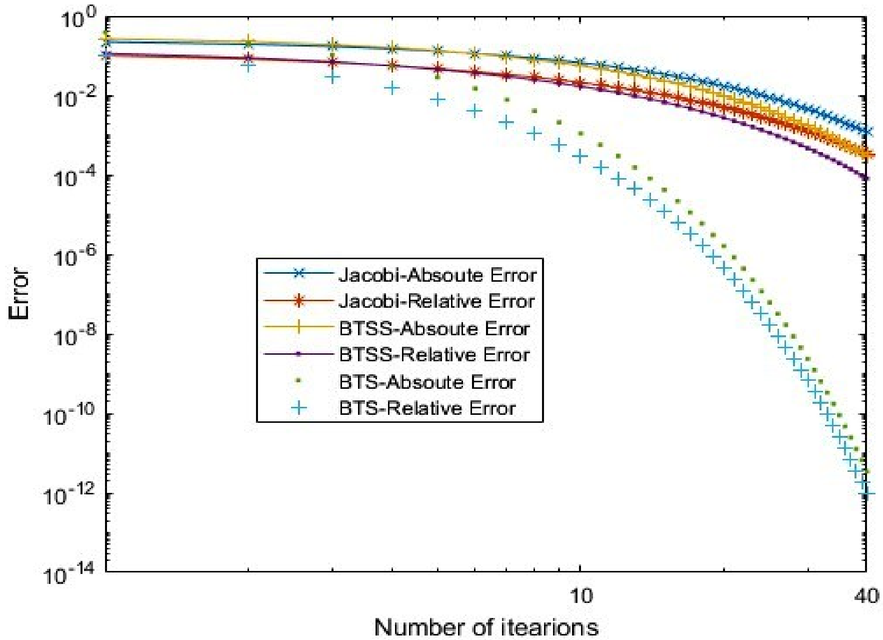

Figure 1: Absolute and Relative error of Jacobi, BTSS, and BTS methods for the contraction factor $\alpha = 0.76$, and $\epsilon = 0.1$.

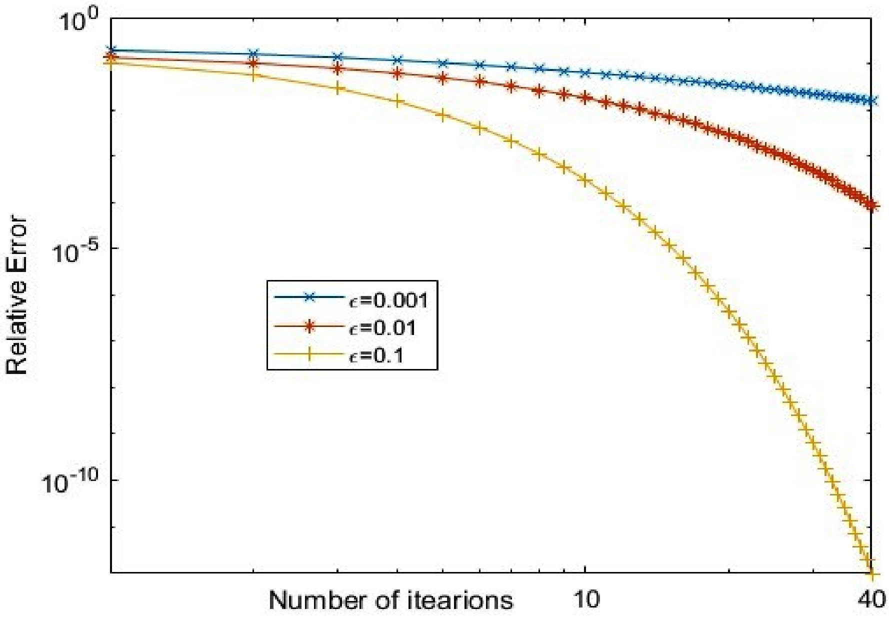

Figure 2: Relative error of the BTS method for the contraction factor $\alpha = 0.76$.

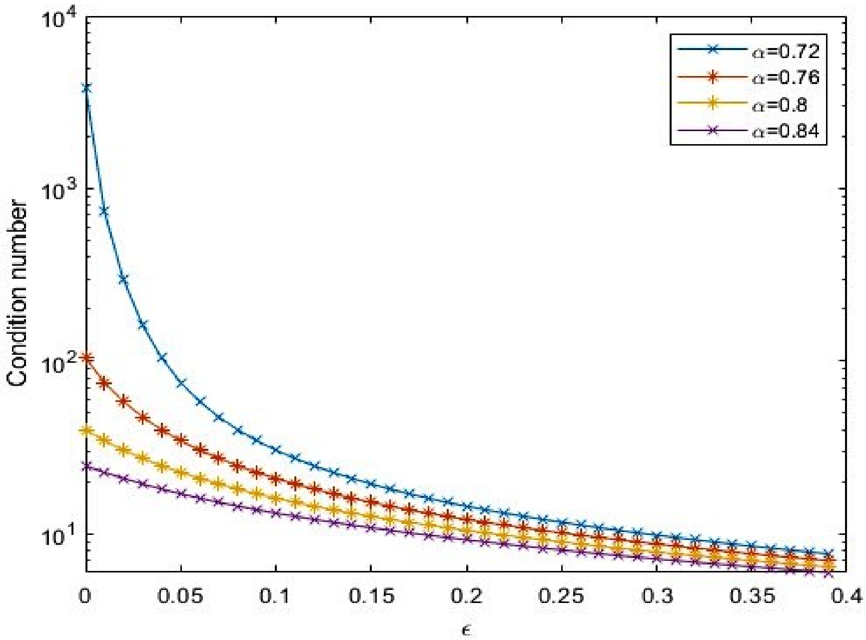

Figure 3: Condition number of the BTS method for the contraction factor $\alpha = 0.76$.

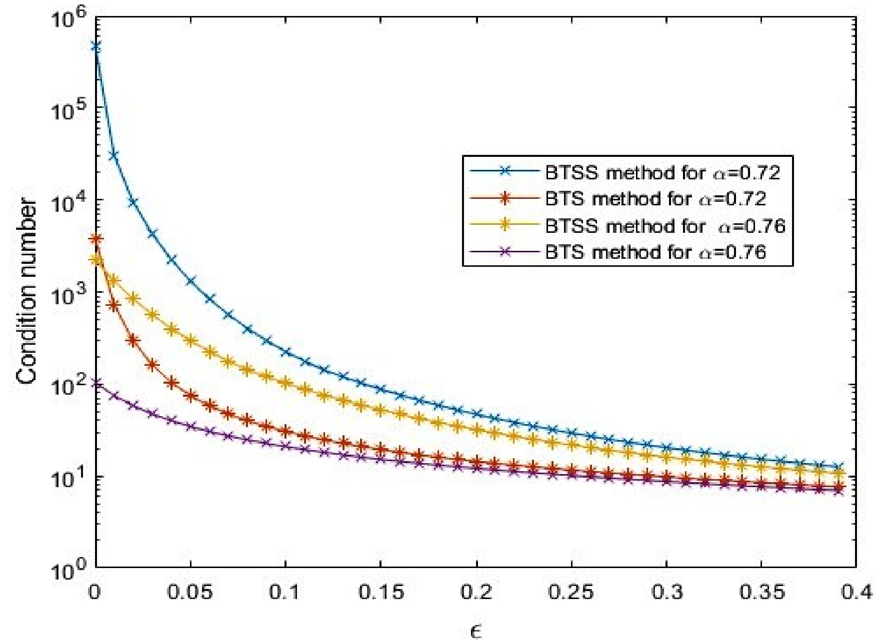

Figure 4: Condition number of the BTSS and BTS methods.

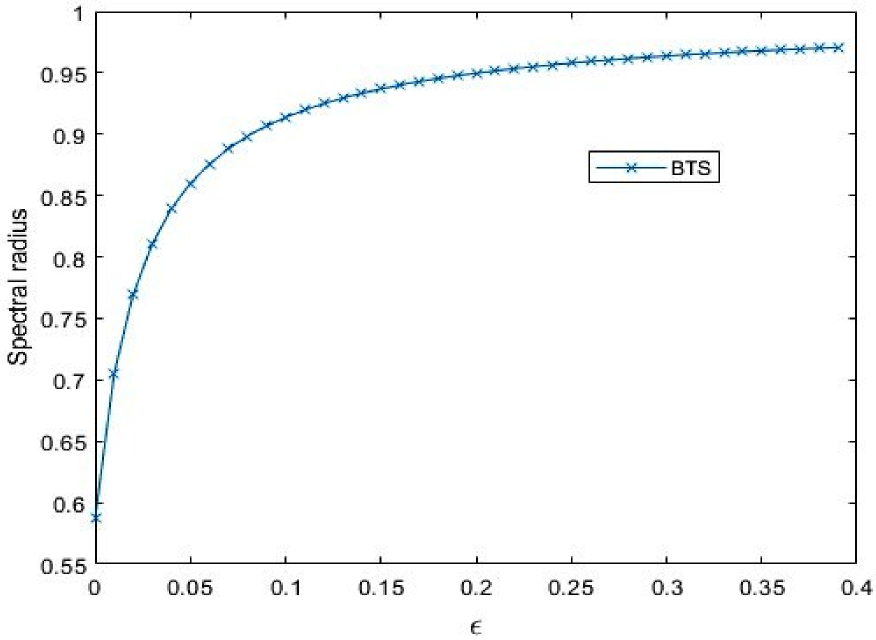

Figure 5: Spectral radius of BTS method

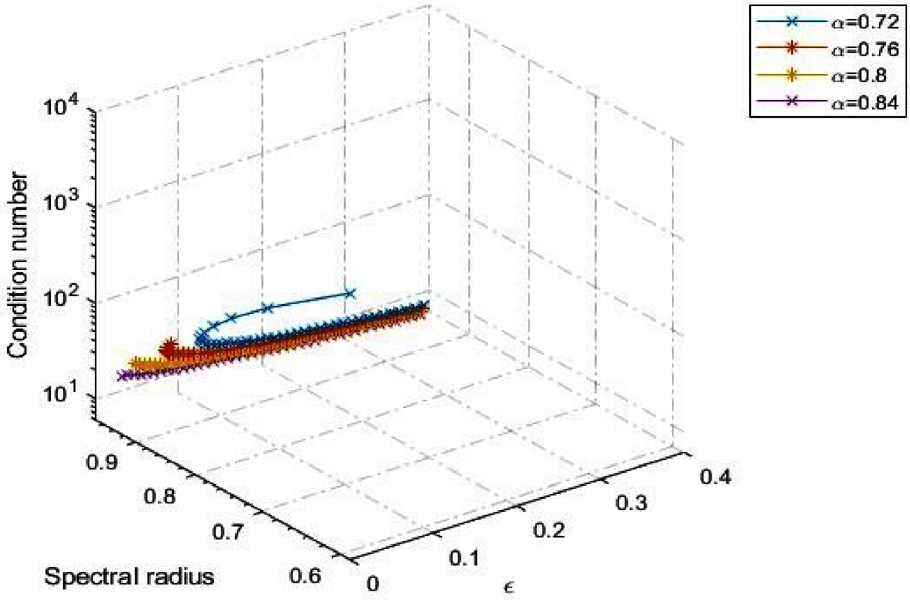

Figure 6: Condition number and spectral radius of BTS method.

6. Conclusion

In this chapter, the steady state vector of regularized linear system of block stochastic matrices using BTS method is estimated. It is proved that the coefficient matrix of regularized linear system Eq. (3) is positive definite. Theoretical analysis shows that the iterative solution of BTS method converges to a unique solution for a wide range of contraction factor α and ϵ. Also, a bound for the' spectral radius of the iteration matrix is derived that gives the contraction factor $\alpha^{\star}$, which minimizes the upper bound. The sensitivity analysis of BTS iteration method is obtained by the condition number and spectral radius of the regularized matrix and concluded that the iterative solution of BTS method is well conditioned.

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