NEUTROSOPHIC TOPOLOGIZED BIPARTITE GRAPH

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Abstract

Neutrosophic topologized bipartite graph analyses unreliability in bipartite structured graphs by incorporating neutrosophic concepts with topologic possessions. This is a novel approach by providing theorems that integrate truth, indeterminacy and falsity values in topology axioms of neutrosophic graphs. The study introduces a topological space where each singleton set is either open or closed, and the boundary of each element is limited to two or fewer connections. And, it extends the topologized graph into various graph structures like a neutrosophic star graph, a neutrosophic bistar, a neutrosophic bipartite graph, a neutrosophic tree and a neutrosophic complete bipartite graph through theorems and examples.

Keywords: Neutrosophic star graph, Neutrosophic labeling, Neutrosophic complete bipartite graph, neutrosophic cycle.
AMS Subject Classification: 03B52, 03E72, 05C72, 54A40

1. Introduction

Generalizations of fuzzy sets and intuitionistic fuzzy sets are given by Smarandache and Atanassov, respectively. A neutrosophic set involves truth, indeterminacy and falsity values/degrees to indulge the effectiveness of an uncertain environment. Many researchers have submitted and researched various aspects of neutrosophic sets in different fields. Antonia Vella [1] used to refer to the basic conceptualization of a topologized graph. As we go through the path, pre-path and classical topology [2], compact space, pre-cycle, bond space and cycle space we refer to [3]. By the hypergraph $\mathscr{H}$ local connectedness and ferns are defined in [4] as $\mathscr{V}_{\mathscr{H}} \cup \mathscr{E}_{\mathscr{H}}$ a collection of all elements in $\mathscr{U}$ such that if it $\mathscr{U}$ has a vertex v, then it contains all hyperedges incident with v. The fact these topologies were either defined in the union of the vertex set or in the vertex set and edge set. The application of topological graph theory involves printing/coding theory and decoding the codewords from secret or journalistic departments. In [5] the authors discussed the Tanner graph, which is also a bipartite graph, where one subvertex is a codeword and the other subvertex set represents the digit combination without codeword errors. we referred to [6], [7] and [8] for basic graph theory and topology concepts to implement in neutrosophic sets.

Smarandache [9] introduced neutrosophic sets as a generalization of intuitionistic fuzzy sets and explained the basic concept of neutrosophic sets for further research. In [10] applied neutrosophic logic in artificial intelligence, decision making and control systems and provided the basic foundation for neutrosophic graphs in uncertain environments. Akram [11] developed a graph theory framework involving neutrosophic logic by introducing adjacency matrices, shortest paths, and spanning trees in neutrosophic to handle imprecise information in uncertainty. Kanchana M and Kavitha K [12] proposed a heuristic algorithm to solve shortest-path problems in transportation networks modeled with interval-valued neutrosophic sets, effectively handling

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uncertainty and negative edge weights. The method improves decision-making in complex and indeterminate environments. Kanchana M and Kavitha K [13] studied a bipolar neutrosophic framework to solve transportation problems on specific graphs such as symmetric graphs, capturing dual uncertainties through positive and negative membership degrees. This contributes to robust logistics modeling in uncertain and bidirectional network settings. Kanchana M and Kavitha K [14] have proposed the Modified Incident Edge Path Algorithm for computing shortest paths in networks with negative weights, avoiding negative cycles. Also, they further introduced a sensitivity analysis [15] for neutrosophic transportation problems to assess route robustness. Their methods enhance decision-making in logistics and urban planning under uncertainty. Both approaches address fuzziness and indeterminacy using neutrosophic logic. Smarandache et al. [16] introduced the foundational conceptualizations of neutrosophic graphs and applied them in complex systems to model uncertainty and fuzzy decision frameworks.

Gross et al. [17] explored the interaction between topology and graph theory, including the analysis of embedding graphs on surfaces and their topological properties. These are applicable in network visualization, biological modeling, and computational topology. Amir [18] discussed the embedding of neutrosophic graphs in spheres and m-toruses. He concluded that neutrosophic sets and crisp graphs can be used to construct neutrosophic graphs that are embeddable on various topological surfaces. Vetrivel et al.In [19] explored the forgotten topological index and its properties in neutrosophic graphs through theoretical formulations and applications. Hamid [20] surveyed box complexes and simplicial complexes related to graphs, offering topological insights by providing lower bounds through chromatic numbers. Max et al. [21] introduced TOGL, which investigates the global topological data of graphs using homology. TOGL can be integrated into graph neural networks, enhancing their functionality compared to traditional models. Vimala [22] introduced foundational concepts of topologized bipartite graphs for stars, trees, and cycles under specific topological conditions. These concepts lead to the implementation of topology in neutrosophic bipartite graphs that satisfy topological axioms, termed as neutrosophic topological bipartite graphs. The considered neutrosophic graphs are finite, undirected, simple, and planar, without multiple edges. The associated topological space is also finite.

This article explores neutrosophic topological graphs and neutrosophic bipartite graphs satisfying topological axioms via theorems and examples. Section 2 presents the fundamental definitions used to demonstrate these concepts. Section 3 contains theorems and solved examples for neutrosophic star, complete bipartite, tree, and cycle graphs. Section 4 concludes the study.

2. Preliminaries

Let [9] the universal set be $\mathcal{U}$ and the neutrosophic set $\mathscr{A}$ in $\mathcal{U}$ is defined as,
$\mathscr{A} = \{(x, T_{\mathscr{A}}, I_{\mathscr{A}}, F_{\mathscr{A}})|x \in \mathcal{U}\}$, $T_{\mathscr{A}}$ is truth membership, $I_{\mathscr{A}}$ is indeterminacy membership and $F_{\mathscr{A}}$ is falsity membership functions of x; where $T_{\mathscr{A}}$, $I_{\mathscr{A}}$, $F_{\mathscr{A}} \in [0,1]$ satisfying
$0 \leq T_{\mathscr{A}}(x) + I_{\mathscr{A}}(x) + F_{\mathscr{A}}(x) \leq 3$.

Let $\mathscr{G}(\mathscr{V}, \mathscr{E}, \mathcal{T}, \mathscr{N})$ be a neutrosophic topological graph in a topological space, consisting of the following [14], [15]:

- $\mathscr{V}$ a vertex set
- $\mathscr{E}$ edge set, $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{E}$
- $\mathcal{T}$ a topology defined on $\mathscr{V} \cup \mathscr{E}$, forms a neutrosophic topological space $(\overline{X}, \mathcal{T})$ where $\overline{X} = \mathscr{V} \cup \mathscr{E}$
- $\mathscr{N} : \overline{X} \to [0,1] \times [0,1] \times [0,1]$ assigning truth $(T(x))$, indeterminacy $(I(x))$ and falsity $(F(x))$ membership values $\forall x \in \overline{X}$ satisfies $0 \leq T(x) + I(x) + F(x) \leq 3$.

Property of Neutrosophic Topology:

- A neutrosophic subset $\mathscr{U} \subseteq \overline{X}$ is neutrosophically open if $\forall a \in \mathscr{U}$ there is a neutrosophic neighborhood of $x \in \mathscr{U}$.
- Neutrosophic boundary vertex/edge edge x is denoted as $\partial(x)$ satisfying $|\partial| \leq 2$.

These are the separation axioms for neutrosophic topologized graphs.

Let $\mathscr{G}(\mathscr{V},\mathscr{W},\mathscr{E},\mathscr{N})$ be a neutrosophic bipartite graph consisting of $\mathscr{V},\mathscr{W}$ two disjoint vertex subsets such that $\mathscr{V}\cap\mathscr{W}=\emptyset$, $\mathscr{E}$ edge set, $\forall e\in\mathscr{E}$ linking $\mathscr{V}$ to $\mathscr{W}$, $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{E}$. $\mathscr{N}$ a neutrosophic membership function defined as $\mathscr{N}:\mathscr{V}\cup\mathscr{U}\cup\mathscr{W}\rightarrow[0,1]\times[0,1]\times[0,1]$ having truth (T), indeterminacy (I) and falsity (F) memberships. Bipartition condition for neutrosophic graph [15]:

- The neutrosophic graph does not contain odd-length cycles
- Each and every partition $\mathscr{V}$ and $\mathscr{W}$ are neutrosophically distinct; that is, $\forall u\in\mathscr{V},w\in\mathscr{W},\mathscr{N}(u)\neq\mathscr{N}(w)$.

We refer the following theorems from [12], [14] and [15] for resulting theorems.

Theorem 1. A neutrosophic graph $\mathscr{G}$ is a neutrosophically bipartite graph iff it contains only cycles of even length, where the truth, indeterminacy and falsity memberships of adjacent vertices satisfy bipartition conditions.

Theorem 2. Every neutrosophic tree is neutrosophically bipartite.

3. Results on Neutrosophic topologized bipartite graph

The following theorems and their examples are proved using the definitions and theorems mentioned in section 2 and refer to some articles from the references mentioned in section 1.

Theorem 3. Let $\bar{\mathcal{K}}_{1,m(T,I,F)}$ be a neutrosophic star graph with m pendant vertices. The neutrosophic graph $\bar{\mathcal{K}}_{1,m(T,I,F)}$ is a neutrosophic topologized graph for $0<m\leq 2$.

Proof. Let $\bar{\mathcal{K}}_{1,m(T,I,F)}$ be a neutrosophic star graph, and consider $(\bar{X},\mathcal{T})$ a topological space where the topology $\mathcal{T}$ is defined by $\mathscr{V}\cup\mathscr{E}$. For all $\mathscr{V},\mathscr{W}\in\mathscr{V}$, $\bar{\mathcal{K}}_{1,m(T,I,F)}$ has a central vertex $\mathscr{V}$ linked to m pendant vertices $w_1,w_2,\ldots,w_m\in\bar{\mathcal{K}}_{1,m(T,I,F)}$, where $m\in\mathbb{N}$ and $0<m\leq 2$. The vertex $\mathscr{V}$ is bipartite with every vertex of $w_1,w_2,\ldots,w_m$. Each vertex and edge of the neutrosophic graph is associated with truth, indeterminacy, and falsity values.

Case (i): For $m=1$, the neutrosophic graph consists of two pendant vertices and one central vertex w_1,w_2, and $\mathscr{V}$, respectively. Let $e\in\mathscr{E}$ be the edge connecting $\mathscr{V}$ to w_1. Thus, $|\bar{X}|=3$. Clearly, $\forall x\in\bar{X},|\partial(x)|\leq 2$ (since the boundary of each x is at most 2). Moreover, for every $x\in\bar{X}$, x is open if its truth value is large and closed if its falsity value is large. Since $\{x\}$ is closed, for any $y\in\bar{X}$, $\bar{X}\setminus\{x\}$ is open. The neutrosophic edge set $\mathscr{E}$'s points, which are not closed, are open, and its complement, the neutrosophic vertex set $\mathscr{V}$, is closed. Thus, $\bar{\mathcal{K}}_{1,1(T,I,F)}$ is a neutrosophic topologized graph.

Case (ii): For $m=2$, $|\bar{X}|=5$, consisting of three pendant vertices w_1,w_2,w_3 and a central vertex $\mathscr{V}$. There exist two edges $e_1=(\mathscr{V},w_1)$ and $e_2=(\mathscr{V},w_2)$ connecting $\mathscr{V}$ to w_1,w_2,w_3. Each singleton vertex is either open or closed based on its truth and falsity values. Each vertex has at most two neighboring vertices, so $|\partial(x)|\leq 2$ is satisfied.

Thus, $\bar{\mathcal{K}}_{1,2(T,I,F)}$ is a neutrosophic topologized graph.

Hence, $\bar{\mathcal{K}}_{1,m(T,I,F)}$ is a neutrosophic topologized graph for $0<m\leq 2$. ■

Example 3.1:

Consider $\bar{\mathcal{K}}_{1,1(T,I,F)}$, a neutrosophic star graph. Let $\forall\mathscr{V},\mathscr{W}\in\mathscr{V}$, $\mathscr{V}=\{v_1\}$, $\mathscr{W}=\{w_1\}$, and $\mathscr{E}=\{e_1=(v_1,w_1)\}$. The topological space is $\bar{X}=\{v_1,e_1,w_1\}$, and the neutrosophic topology is defined as: $\mathcal{T}=\{\emptyset,\{v_1\},\{w_1\},\{v_1,w_1\},\{v_1,e_1\},\{v_1,e_1,w_1\}\}$. For every $\{x\}\in\bar{X}$, it is open or closed. Here, $\mathcal{T}$ consists of open sets, and its neutrosophic values are: $\mathscr{N}(v_1)=(T_{v_1},I_{v_1},F_{v_1})$, $\mathscr{N}(e_1)=(T_{e_1},I_{e_1},F_{e_1})$, $\mathscr{N}(w_1)=(T_{w_1},I_{w_1},F_{w_1})$. The boundary value of each vertex and edge is at most 2 neighboring elements, satisfying $|\partial(x)|\leq 2$. $\partial(v_1)=\{w_1\}\implies|\partial(v_1)|=1$, $\partial(e_1)=\{v_1,w_1\}\implies|\partial(e_1)|=2$,
$\partial(w_1)=\{v_1\}\implies|\partial(w_1)|=1$.

Hence, $\bar{\mathcal{K}}_{1,1(T,I,F)}$ is a neutrosophic topological graph and given in the figure 1

Theorem 4. A bistar graph $\mathcal{B}(n,n)$ is not a neutrosophic topologized graph.

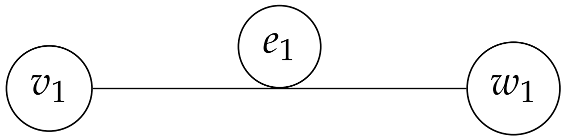


Figure 1: *$\bar{\mathcal{K}}_{1,1(T,I,F)}$ is a neutrosophic topological graph.*

Proof. Let $(\bar{X}, \mathcal{T})$ be a topological space where the topology $\mathcal{T}$ is defined by $\mathscr{V} \cup \mathscr{E}$. The space contains a bistar graph $\mathcal{B}(n,n)$, which consists of two star graphs $\mathcal{S}(n)$ connected by a central vertex v_c. Each star graph $\mathcal{S}(n)$ has n leaf vertices, resulting in a total of $2n$ leaf vertices in $\mathcal{B}(n,n)$. The central vertex v_c connects to all $2n$ leaf vertices. Let v_l denote a leaf vertex connected to v_c. The boundary values of the vertices are as follows:

$$|\partial(v_l)| = 1 \quad \text{and} \quad |\partial(v_c)| = 2n.$$

Each vertex and edge in $\mathcal{B}(n,n)$ is associated with truth (T), indeterminacy (I), and falsity (F) neutrosophic values. For the central vertex v_c, the boundary leads to $2n$ connections, which overlap and violate the neutrosophic values T, I, and F. Since the sum of T, I, and F must satisfy:

$$0 \leq T + I + F \leq 3.$$

It is not possible for v_c to lie on the boundary while satisfying this condition, as the boundary size $2n$ exceeds the limit for valid neutrosophic values. Furthermore, every singleton v_c must be either open or closed neutrosophically. However, v_c becomes indeterminate due to its large boundary, violating the neutrosophic topology.
Hence, $\mathcal{B}(n,n)$ is not a neutrosophic topologized graph. ■
Example 3.2
Let $\mathcal{B}(3,3)$ be a bistar graph, where the central vertex v_c connects to the leaf vertices $v_1, v_2, v_3, v_4, v_5, v_6$ of $\mathcal{S}(3)$. The boundary of v_c is $|\partial(v_c)| = 6 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. For a leaf vertex v_1, we have $|\partial(v_1)| = 1$.

Let the neutrosophic values for $v_1, v_2, v_3, v_4, v_5, v_6$ be $T = 0.9$, $I = 0.1$, and $F = 0$. Then for v_c, we calculate:

$$T_{(v_c)} = \sum_{i=1}^{6} T_{(v_i)} = 6 \times 0.9 = 5.4, \quad I_{(v_c)} = \sum_{i=1}^{6} I_{(v_i)} = 0.6, \quad F_{(v_c)} = \sum_{i=1}^{6} F_{(v_i)} = 0.0.$$

These values are not valid in neutrosophic logic since $T + I + F \geq 3$ is not allowed. Hence, $\mathcal{B}(3,3)$ is not a neutrosophic topologized graph.

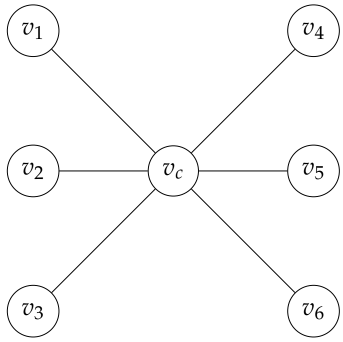


Figure 2: *Bistar graph $\mathcal{B}(3,3)$ with central vertex v_c and leaf vertices $v_1, v_2, v_3, v_4, v_5, v_6$.*

Theorem 5. Let $\mathscr{G}(\mathscr{V}, \mathscr{E}, T, I, F)$ be a neutrosophic bipartite graph. $\mathscr{G}_n$ is topologized if every vertex has a degree at most 2.

Proof. Consider the neutrosophic bipartite graph $\mathscr{G}$ whose vertices have degree at most 2. Let the topological space $(\overline{X}, \mathcal{T})$, where the boundary of any vertex or edge is determined by its adjacent or incident vertices. Since the degree of every vertex is at most 2, the boundary of any vertex or edge is less than 3. Each singleton vertex or edge is either open or closed, as their boundaries are limited. Thus, $\mathscr{G}$ is a topologized neutrosophic graph. Moreover, the neutrosophic topology on $\mathscr{G}$ satisfies the required separation axioms $\mathscr{T}_0$ and $\mathscr{T}_1$:

- $\mathscr{T}_0$: Distinct vertices can be separated by neutrosophic open sets.
- $\mathscr{T}_1$: There exists a neutrosophic open set that separates each vertex with distinct membership values.

Hence, $\mathscr{G}$ is a neutrosophic topologized bipartite graph. ■

Example 3.3:

Consider the neutrosophic bipartite graph $\mathscr{G}(\mathscr{V}, \mathscr{E}, T, I, F)$, where:
$\mathscr{V} = \mathscr{U} \cup \mathscr{W}$, $\mathscr{U} = \{u_1, u_2\}$, $\mathscr{W} = \{w_1, w_2\}$, and
$\mathscr{E} = \{e_1, e_2\}$, $e_1 = (u_1, w_1)$, $e_2 = (u_2, w_2)$.
let $\overline{X} = \{u_1, u_2, w_1, w_2, e_1, e_2\}$ be the topological space defined by:
$\mathcal{T} = \{\overline{X}, \varnothing, \{e_1\}, \{e_2\}, \{u_1, e_1, w_2\}, \{u_2, e_2, w_1\}, \{e_1, e_2\}, \{u_1, e_1, w_2, e_2\}, \{u_2, e_2, w_1, e_1\},$
$\{u_1, e_1\}, \{u_1, e_1, e_2\}, \{u_1, e_1, u_2, e_2, w_1\}\}$.
Assigning the membership values as:

$$T_{(u_1)} = \{w_2\} = 1, \quad T_{(u_2)} = \{w_1\} = 1, \quad T_{(w_1)} = \{u_2\} = 1, \quad T_{(w_2)} = \{u_1\} = 1,$$

$$T_{(e_1)} = \{u_1, w_1\} = 2, \quad T_{(e_2)} = \{u_2, w_2\} = 2.$$

The degree of each vertex is at most two and satisfies the boundary condition:
$|\partial(u_1)| = |\partial(w_2)| = |\partial(u_2) = |\partial(w_1)| = 1$; $|\partial(e_1)| = |\partial(e_2)| = 2$ therefore,

$$|\partial(x)| \leq 2 \quad \text{for all } x \in \overline{X}.$$

Thus, the neutrosophic topology defined on the neutrosophic bipartite graph $\mathscr{G}$ satisfies all the required conditions, making it a neutrosophic topologized graph.

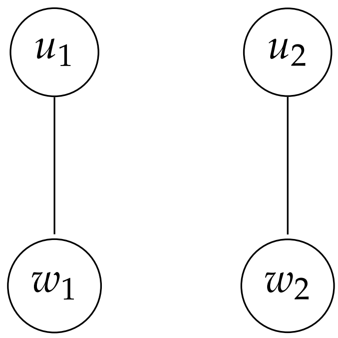


Figure 3: *Neutrosophic bipartite graph $\mathscr{G}_n$*

Corollary 3.1: Let $\mathscr{G}$ be a tree. If $\mathscr{G}$ is a neutrosophic bipartite graph, then it is a neutrosophic topologized graph.

Proof: Let $\mathscr{G}$ be a tree and also a neutrosophic bipartite graph, which is acyclic by definition, and its vertex set $\mathscr{V}$ can be decomposed into $\mathscr{V}_1$ and $\mathscr{V}_2$, two vertex sets, where $\mathscr{V}_1$ is connected to $\mathscr{V}_2$ by edges of $\mathscr{E}$ of $\mathscr{G}$. Assuming the boundary conditions under a neutrosophic setting, For any edge $e = u_1, v_1 \in \mathscr{E}$, e connects adjacent vertices, each vertex has a degree at most 2 determining neutrosophic boundary values, which implies that the boundary of each vertex is at most 2. since $\mathscr{G}$ is a tree, every vertex has a well-defined neutrosophic neighborhood. Thus, every singleton vertex is either open or closed under truth membership values and is connected through a finite

number of edges satisfying the separation properties of neutrosophic bipartite nature. Hence, $\mathscr{G}$ is a neutrosophic topologized graph.
Example 3.4:

Defining the neutrosophic topological space $\overline{X} = \{v_1, v_2, w_1, w_2, e_1, e_2, e_3\}$, and its topology $\mathcal{T}$ as:
$\mathcal{T} = \{\overline{X}, \varnothing, \{v_1\}, \{v_2\}, \{w_1, e_1, e_2, v_2\}, \{v_2, e_3, w_2\}, \{v_2, w_2\}, \{v_1, v_2\}, \{v_1, w_1, e_1, e_2, v_2\}, \{v_1, v_2, e_3, w_2\}, \{v_1, v_2, w_2\}, \{v_2, e_3, w_2, w_1, e_1, e_2\}, \{w_1, e_1, e_2, v_2, w_2\}, \{v_1, v_2, w_2\}, \{v_1, w_1, e_1, e_2, v_2, w_2\}, \{w_1, w_2, e_1, e_2, v_2, e_3\}, \{v_1, w_1, e_1, e_2, v_2, w_2\}, \{w_1, e_1, e_2, v_2\}\}$.

For every singleton $x \in \overline{X}$, it is open or closed such that $|\partial(x)| \leq 2$. satisfying boundary conditions for each vertex are:

$$|\partial(v_1)| = \{w_1\} = 1, \quad |\partial(v_2)| = \{w_1, w_2\} = 2,$$
$$|\partial(w_1)| = \{v_1, v_2\} = 2, \quad |\partial(w_2)| = \{v_2\} = 1,$$
$$|\partial(e_1)| = \{v_1, w_1\} = 2, \quad |\partial(e_2)| = \{v_2, w_1\} = 2,$$
$$|\partial(e_3)| = \{v_2, w_2\} = 2.$$

The boundary of each vertex and edge is $|\partial(x)| \leq 2$, and the neutrosophic topology separation axioms are;
$\mathscr{T}_0$: Every singleton vertex is open or closed satisfying $|\partial(x)| \leq 2$.
$\mathscr{T}_1$: For any two distinct vertices there exists an open set. Thus, $\mathscr{G}$ is a neutrosophic topologized graph.

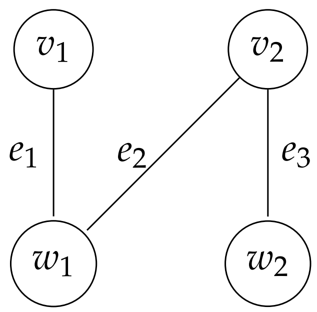


Figure 4: *$\mathscr{G}$ is a neutrosophic topologized graph where $\mathscr{G}$ is a tree*

Theorem 6. All neutrosophic cycles in a neutrosophic bipartite graph are neutrosophically topologized.

Proof. Let $\mathscr{G}(\mathscr{V}, \mathscr{E})$ be a neutrosophic bipartite graph having topological space $(\overline{X}, \mathcal{T})$, where its neutrosophic topology $\mathcal{T}$ is defined by its vertex and Edge set, $\mathscr{V}$ and $\mathscr{E}$ respectively.

Every neutrosophic bipartite graph contains only even cycles, and where these cycles are regular. For each vertex $v \in \mathscr{V}$ and edge $e \in \mathscr{E}$, the boundary $\partial(x)$ satisfies:

$$|\partial(x)| \leq 2.$$

By the neutrosophic topology separation axiom, every singleton vertex or edge in a cycle is either open or closed. Since every cycle in the neutrosophic bipartite graph is created from $\mathscr{V}$ and $\mathscr{E}$, each having a boundary of at most 2, then all neutrosophic cycles are neutrosophically topologized. Hence, every neutrosophic cycle in a neutrosophic bipartite graph is neutrosophically topologized. ■
Example 3.5:

Let $\mathcal{B}(3,4)$ be a neutrosophic bipartite graph.

The neutrosophic bipartite graph $\mathcal{B}(3,4)$ defined by the following vertex sets and edge sets: $\mathscr{V} = \{v_1, v_2, v_3\}$, $\mathscr{W} = \{w_1, w_2, w_3, w_4\}$ and $\mathscr{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ where,

$$(e_1) = \{v_1, w_1\}, \quad (e_2) = \{v_1, w_2\}, \quad (e_3) = \{v_1, w_3\}$$

$$(e_4) = \{v_2, w_1\}, \quad (e_5) = \{v_2, w_4\}, \quad (e_6) = \{v_3, w_2\}$$
$$(e_7) = \{v_3, w_3\}, \quad (e_8) = \{v_3, w_4\}$$

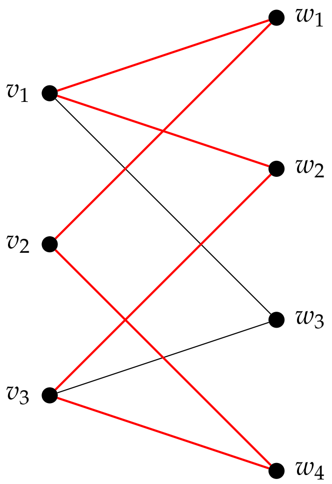
the neutrosophic topological space is $\overline{X} = \{v_1, v_2, v_3, w_1, w_2, w_3, w_4, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$
Let the neutrosophic cycle $\mathcal{C}_1$ is $\mathcal{C}_1 = \{v_1, e_1, w_1, e_4, v_2, e_5, w_4, e_8, v_3, e_6, w_2, e_2, v_1\}$
The neutrosophic values (T, I, F) for each v_i, w_j and e_k are:

$$\mathscr{N}(v_i) = (T_{v_i}, I_{v_i}, F_{v_i}), \quad \mathscr{N}(w_j) = (T_{w_j}, I_{w_j}, F_{w_j})$$

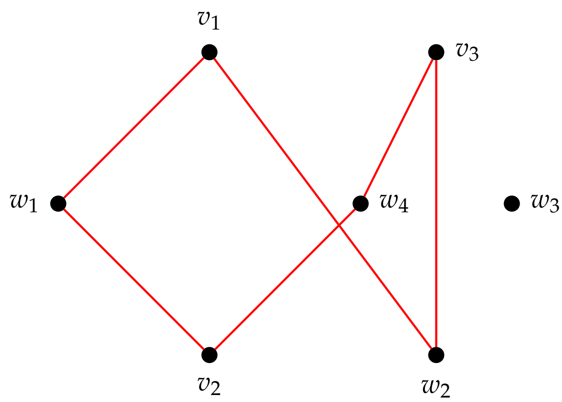
$$\mathscr{N}(e_k) = (T_{e_k}, I_{e_k}, F_{e_k}), \quad \forall v_i \in \mathscr{V}, \quad \forall w_j \in \mathscr{W}, \quad \forall e_k \in \mathscr{E}.$$

$$|\partial(v_1)| = \{w_1, w_2\} = 2, \quad |\partial(v_2)| = \{w_1, w_4\} = 2, \quad |\partial(v_3)| = \{w_2, w_4\} = 2$$
$$|\partial(w_1)| = \{v_1, v_2\} = 2, \quad |\partial(w_2)| = \{v_1, v_3\} = 2, \quad |\partial(w_4)| = \{v_2, v_3\} = 2$$
$$|\partial(e_1)| = \{v_1, w_1\} = 2, \quad |\partial(e_2)| = \{v_1, w_2\} = 2, \quad |\partial(e_4)| = \{v_2, w_1\} = 2$$
$$|\partial(e_5)| = \{v_2, w_4\} = 2, \quad |\partial(e_6)| = \{v_3, w_2\} = 2, \quad |\partial(e_8)| = \{v_3, w_4\} = 2$$

Since every $|\partial(x)| \leq 2$ is satisfied, hence, $\mathcal{B}(3,4)$ is a neutrosophic topologized graph and similarly, $\mathcal{C}_2, \mathcal{C}_3, ..., \mathcal{C}_{11}$ neutrosophic cycles exist.



(a) *$\mathcal{B}(3,4)$ neutrosophic bipartite graph*



(b) *Neutrosophic cycle $\mathcal{C}_1$ in $\mathcal{B}(3,4)$*

Figure 5: *Graph $\mathcal{B}(3,4)$ and the neutrosophic cycle $\mathcal{C}_1$*

As shown in Figure 5, the cycle $\mathcal{C}_1$ forms a closed path

Theorem 7. Let $\overline{\mathcal{K}}_{m,n}$ be a bipartite graph with $(m, n \geq 3)$. Then $\overline{\mathcal{K}}_{m,n}$ is not a neutrosophic topologized bipartite graph.

Proof. A neutrosophic bipartite graph $\overline{\mathcal{K}}_{m,n}$ consists of two disjoint sets $\mathscr{V}_1$ and $\mathscr{V}_2$, where: $|\mathscr{V}_1| = m$ and $|\mathscr{V}_2| = n$. Each vertex in $\mathscr{V}_1$ is connected to all vertices in $\mathscr{V}_2$ and respectively. For a neutrosophic topologized graph, any vertex x must satisfy the boundary condition $|\partial(x)| \leq 2$. In $\overline{\mathcal{K}}_{m,n}$, the degree of any vertex $v \in \mathscr{V}_1$ is n, and the degree of any vertex $w \in \mathscr{V}_2$ is m. Since $(m, n \geq 3)$, the degree of each vertex in $\overline{\mathcal{K}}_{m,n}$ is:

$$|\partial(v)| = n \geq 3 \quad \text{and} \quad |\partial(w)| = m \geq 3.$$

which violates $|\partial(x)| \leq 2$. Thus, $\overline{\mathcal{K}}_{m,n}$ is not a neutrosophic topologized bipartite graph. ■
Example 3.6:
$\overline{\mathcal{K}}_{4,5}$ is not a neutrosophic topologized graph.

Consider the structure of $\overline{\mathcal{K}}_{4,5}$ as;

$$\mathscr{V} = \{v_1, v_2, v_3, v_4\}, \quad \mathscr{W} = \{w_1, w_2, w_3, w_4, w_5\}$$

$$\mathscr{E} = \{e_1, e_2, \ldots, e_{20}\}, \quad (e_{ij}) = \{v_i, w_j\}, \quad \forall e_{ij} \in \mathscr{E}$$

Degree of vertices are,
$|\partial(v_i)| = 5, \quad \forall v_i \in \mathscr{V}, |\partial(w_j)| = 4, \quad \forall w_j \in \mathscr{W}$
the neutrosophic graph is neutrosophically topologized if $|\partial(x)| \leq 2, \quad \forall x \in \mathscr{V} \cup \mathscr{W}$
Violation of degree:

$$|\partial(v_i)| = 5 > 2, \quad \forall v_i \in \mathscr{V}, \quad |\partial(w_j)| = 4 > 2, \quad \forall w_j \in \mathscr{W}$$

thus, $|\partial(x)| \leq 2$ is violated since: $\exists x \in \mathscr{V} \cup \mathscr{W}$ such that $|\partial(x)| > 2$.
Hence, $\overline{\mathcal{K}}_{4,5}$ is not a neutrosophic topologized graph and shown in Fig. 6 .

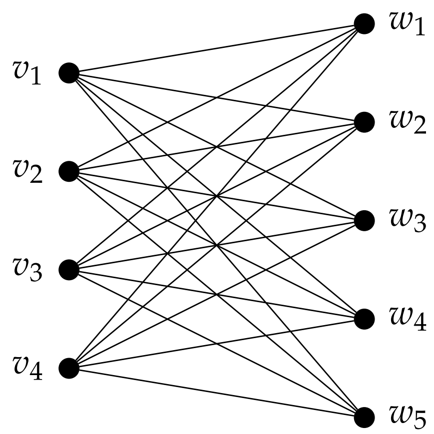


Figure 6: $\overline{\mathcal{K}}_{4,5}$ *not a neutrosophic topologized graph*

4. Discussion and Conclusion

The neutrosophic topologically bipartite graph was introduced and analyzed by incorporating the ideals of the neutrosophic idea with graph topology. By integrating the neutrosophic values of truth, indeterminacy and falsity in topological properties into graph structures. The neutrosophic star graph, Bistar graph, neutrosophic bipartite graph, neutrosophic tree, and neutrosophic complete bipartite graphs satisfying the boundary condition $0 < m \leq 2$ were verified using theorem proofs with suitable examples. Also, the non-topological graph under neutrosophic topological axioms was demonstrated in the theorem. These theorems provided a new approach to model a topology graph structure into a neutrosophic topological graph structure by defining truth, indeterminacy and falsity values.

4.1. Future work

Expanding these theorems into high-dimensional topology by defining its structure in neutrosophic values providing neutrosophic graphs, and investigating the influence of dynamic transpose in neutrosophic values. Expanding this into an algorithmic approach for application in artificial intelligence, such as optimization and modeling for uncertain environments.

Conflicts of interest : “The authors declare there is no conflict of interest.”

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