ATTRIBUTE CONTROL CHART BASED ON RAYLEIGH DISTRIBUTION: A BAYESIAN APPROACH

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Abstract

Statistical Process Control is a quality control approach that uses statistical tools to assess, monitor, and optimize processes. A control chart plays a crucial role in this method by tracking the performance of a process and providing a visual signal to detect any irregular deviations caused by specific assignable factors. It compares the values of a quality attribute against predetermined control limits. In typical quality control procedures, control charts are often created without considering parameter uncertainty. However, it is essential to identify any shifts in the parameters of the probability distribution linked to one or more variables associated with the process to ensure effective monitoring. Evaluating these parameters is vital, as it can influence the long-term functionality of the control chart in both controlled and uncontrolled environments. This article introduces an innovative attribute control chart that utilizes a Bayesian methodology based on the Rayleigh lifetime distribution, specifically applied under accelerated life testing with a hybrid censoring technique. The parameters for the control chart will be established through various combinations of values using a Bayesian approach, while the effectiveness of the control chart will be evaluated through the Average Run Length (ARL). Numerical examples will illustrate the proposed control chart, and simulated data will showcase its potential application. The effectiveness of the proposed control chart will be quantified through ARLs for different shift constants, sample sizes, and additional parameters such as p, Lower Control Limit, and Upper Control Limit.

Keywords: Rayleigh Distribution, Predictive Distribution, Attribute Control Chart, Average Run Length

1. Introduction

The manufacturing industry is currently facing significant challenges due to variations in processes, which lead to defects and increased costs. In the realm of continuous manufacturing, it is crucial to determine whether a process is in control and to identify any existing variations. These variations may stem from random factors or identifiable causes. Quality control involves the processes of monitoring, measuring, and correcting a product's performance in relation to established criteria or specifications. One of the most common and effective methods for ensuring quality is the control chart. This tool provides a graphical representation of data gathered from the manufacturing sector. Each control chart includes a central line (CL) that indicates the average, as well as an upper line that marks the upper control limit (UCL) and a lower line that represents the lower control limit (LCL).

To effectively track the production process in various scenarios, several new control charting techniques have been developed. These control charts come in different types, each specifically designed to address the unique characteristics of the quality attribute being examined. They can be categorized into two main types based on the nature of the data: 'variable' and 'attribute'. Variable control charts focus on accurately measured data, while attribute control charts are used to identify items as either compliant or non-compliant with established standards.

The lifetime of a product is considered a crucial quality attribute for certain goods. Life tests are conducted to monitor the manufacturing process of these products. Depending on the results of the life test, a product may be classified as either conforming or non-conforming. However, this testing method can be quite time-consuming due to the lengthy nature of the tests. In this context, it is important to acknowledge the use of censoring techniques. Common censoring methods used in life testing include type-I censoring, type-II censoring, and hybrid censoring. In hybrid censoring, the life test is concluded at either the designated test time (t) or upon detecting the $(UCL+1)^{th}$ failure, whichever comes first. If the failures observed during the life test remain within the upper control limit (UCL) and lower control limit (LCL) at time t, the production process is in-control; otherwise, it is considered out-of-control. If a product fails the life test before reaching the specified termination time t, it is classified as a non-conforming product.

Accelerated life testing (ALT) is an evaluative technique designed to forecast the probability of future product failures. This approach is well-regarded for its effectiveness in 'accelerating time,' making it a preferred method for testing. ALT is particularly advantageous when it is impractical to wait for failures to occur at their normal rates, thereby offering insights into anticipated future failures. During ALT, products are subjected to heightened conditions such as increased stress, strain, temperature, and pressure, resulting in a faster failure rate than traditional life testing. As a result, employing ALT with hybrid censoring can lead to a significant reduction in both the overall time to failure and the costs associated with inspections.

Attribute control charts, including the np control chart, are based on the assumption that the quality characteristic follows a normal distribution. However, the actual distribution of these quality characteristics may differ from this assumption. Such discrepancies can mislead industrial engineers when using the existing control chart, potentially leading to an increase in non-conforming items. A significant body of literature examines the development of attribute control charts utilizing various lifetime distributions, including but not limited to: Weibull[1–3], Pareto[4], Birnbaum Saunder[5], Inverse Rayleigh[6], Burr X & XII, Inverse Gaussian & Exponential[7], Lognormal[8], Exponentiated Half Logistic[9], Dagum[10], Pareto[11], Weibull Pareto[12], Gamma[13], Log–Logistic[14], Generalized Log Logistic[15], Generalized Exponential[16], Inverse Weibull[17], Length-Biased Weighted Lomax[18], Rayleigh[19], Half Normal & Half Exponential Power[20], Exponentiated Inverse Kumaraswamy[21], Lindley[22], Generalized Rayleigh[23] and Exponentiated Exponential[24].

The attribute control charts referenced in earlier literature are predicated on the assumption that the mean number of defective items remains unchanged over time when the process is in control. Control charts that depend entirely on sample variance, like the np-chart, can produce numerous false alarms if the actual number of defectives is inconsistent during an in-control state. These false alarms arise because the observed count of nonconforming items is influenced by both process variability and sampling variation, while the control limits are calculated based solely on sampling variance. Larger sample sizes tend to reveal any underlying process variations more clearly, resulting in the actual number of nonconforming items varying from one sample to another based on an underlying distribution. The Bayesian approach is widely recognized for its effectiveness in addressing uncertainty related to the parameter of interest. A pioneering Bayesian process control chart for attributes was introduced by [25]. Subsequently, numerous researchers, as noted in references [26–35] have expanded the literature by developing a variable control chart for various lifetime distributions using the Bayesian approach.

The Rayleigh distribution is frequently applied as a lifetime distribution in the context of research related to electro-vacuum devices and communication engineering. Its hazard function, characterized by a linear increase, suggests that it is well-suited for items that deteriorate rapidly. This distribution is a particular case of the Weibull distribution, which is a statistical model frequently utilized in reliability engineering and life testing. Initially proposed as the Rayleigh power (and amplitude) distribution in reference [36], it was later renamed to Rayleigh distribution in subsequent studies [37, 38], which examined its properties, estimation methods, and applications. The significance of this skewed distribution has led to extensive research by various authors in engineering-related domains. Additionally, the Rayleigh distribution can be used to model the lifespan of products resulting from manufacturing processes. A review of the literature indicates that there has been no research conducted on control charts that incorporate life tests for non-normal distributions, such as the Rayleigh distribution, particularly in the context of accelerated life testing with hybrid censoring using a Bayesian approach.

In this paper, the design of an attribute control chart based on a predictive distribution of the lifetime is considered under the assumption that the lifetime follows a Rayleigh distribution and the process parameter is modelled by an Inverse-Rayleigh distribution under accelerated life test with hybrid censoring. The control chart coefficient was determined, and the performance of the proposed control chart was discussed with the ARL. When the process parameter changes, the proposed control chart is designed. With the use of simulated data, the proposed control chart's applicability is described. Simulated data is used to illustrate the proposed control chart. Section 2 presents designing the Bayesian control chart and provides a procedure for obtaining the control chart for given strength. The real-life application of the proposed control chart through simulation study is described in Section 3. The behaviour of the ARL is discussed in Section 4. Section 5 presents the conclusion of the study.

2. Designing a Bayesian Attribute control chart

Let "T" be the lifetime of the products produced in a production process. Assume that "T" follows a Rayleigh distribution with parameter θ given for $t > 0$ by its density function $g_{T|\{\theta\}}$:

$$g_{T|\{\theta\}}(t) = \frac{te^{-\frac{t^2}{2\theta^2}}}{\theta^2}; t > 0, \theta > 0 \tag{1}$$

The distribution function of $G_{T|\{\theta\}}$ is given by:

$$G_{T|\{\theta\}}(t) = 1 - e^{-\frac{t^2}{2\theta^2}}; t > 0, \theta > 0 \tag{2}$$

According to [39], Consider the process parameter θ with $\theta > 0$ as random variable Θ with prior distribution given by the inverse Rayleigh distribution with density function for $\theta > 0$:

$$h_{\Theta|\{\lambda\}}(\theta) = \frac{1}{\theta^3\lambda^2}e^{-\frac{1}{2\lambda^2\theta^2}}; \theta > 0, \lambda > 0 \tag{3}$$

The updated lifetime of the product is described by the predictive distribution of "T" whose density function can be obtained from the prior distribution (3) and the sampling distribution (1) as:

$$f_{T|\{\lambda\}}(t) = \frac{2\lambda^2 t}{[1+\lambda^2 t^2]^2}; t > 0 \tag{4}$$

Hereafter, the above predictive distribution referred as "Rayleigh-Inverse-Rayleigh" distribution and the distribution function of $T|\{\lambda\}$ is obtained as:

$$F_{T|\{\lambda\}}(t) = 1 - \frac{1}{[1+\lambda^2 t^2]}; t > 0 \tag{5}$$

The product's mean lifetime under Rayleigh-Inverse-Rayleigh distribution is given by

$$\mu = \frac{\pi}{2\lambda} \tag{6}$$

The product's expected lifetime under standard (usual) conditions is indicated by the symbol t_U and is modelled by an Rayleigh-Inverse-Rayleigh distribution with a particular parameter (λ_U). Thus, the subsequent equation can be applied to evaluate the product's lifetime in these standard conditions

$$F_U(t_U|\lambda_U) = 1 - \frac{1}{[1 + \lambda_U^2 t_U^2]} \tag{7}$$

Similarly, the product's lifetime when subjected to accelerated conditions is denoted as t_A, with the assumption that it follows an Rayleigh-Inverse-Rayleigh distribution defined by a particular parameter (λ_A). Thus, the product's lifetime under such accelerated conditions is presented as follows

$$F_A(t_A|\lambda_A) = 1 - \frac{1}{[1 + \lambda_A^2 t_A^2]} \tag{8}$$

The mean lifetime of the product under accelerated conditions is provided by

$$\mu_A = \frac{\pi}{2\lambda_A}$$

By employing the definition of the Acceleration Factor (AF), we can formulate it in the following way

$$\lambda_A = \frac{\lambda_U}{AF}$$

Therefore, Equation (8) becomes

$$F_A(t_A|\lambda_A) = 1 - \frac{1}{\left[1 + \left(\frac{\lambda_U}{AF}\right)^2 t_A^2\right]}$$

$$F_A(t_A|\lambda_A) = 1 - \frac{AF^2}{[AF^2 + \lambda_U^2 t_A^2]}$$

The probability of product failure represented as p_0 when the process is maintained under control, is defined by the likelihood that the product will fail before the censoring time t_A.

$$p_0 = 1 - \frac{AF^2}{[AF^2 + \lambda_U^2 t_A^2]} \tag{9}$$

The mean life of the product under accelerated conditions is represented by μ_A when the process is in control. The experimental duration t_A can be expressed with the mean of the in-control process, specifically as $t_A = a * \mu_A$, where "a" signifies the experiment termination ratio. Consequently, Equation (9) can be revised by incorporating the values for t_A and μ_A.

$$p_0 = 1 - \frac{AF^2}{\left[AF^2 + \left(\frac{a\pi}{2}\right)^2 \left(\frac{\lambda_U}{\lambda_A}\right)^2\right]}$$

It is presumed that the parameter remains constant when the process is under control, specifically $\lambda_A = \lambda_U$. Let p_0 as the failure probability of an item when the process is maintained under controlled accelerated conditions, which can be derived from the following:

$$p_0 = 1 - \frac{AF^2}{\left[AF^2 + \left(\frac{a\pi}{2}\right)^2\right]} \tag{10}$$

Reference [40] pointed out that "realistically, however, when a special cause influences a process, it may cause a shift in more than one parameter (location, scale, skewness, etc.) of the process distribution". It means that, due to the occurrence of an assignable cause, the process may be affected and subsequently the process mean or shape parameter of the distribution will be shifted from the specified target level. In addition, this parameter shift doesn't greatly affect the distribution so it reduces from one to another. But in such a situation, the process will be declared as out-of-control and stopped until the cause of process variations is removed. Also, it is assumed that the process parameter is shifted to $\lambda_A = f\lambda_U$, where f is a shift constant. Then the failure probability of an item when the process is out of control, denoted by p_1, is given as

$$p_1 = 1 - \frac{AF^2}{\left[AF^2 + \left(\frac{a\pi}{2f}\right)^2\right]} \tag{11}$$

Usually, for the given values of a & f, the failure probabilities are calculated by using (10) and (11).

We develop a new np control chart for the lifetime of the product distributed according to the Rayleigh-Inverse-Rayleigh distribution based on the number of products in each subgroup. The following is the designed np control chart's operational procedure:

Step 1: Select a sample of n items randomly from the subgroup of the production process and put the sample items for the life test of specified termination time t_0.
Step 2: Observe the number of failed items and denotes it as "d".
Step 3: Terminate the life test either after reached at time t_0 or $d > UCL$ before reaching time t_0, whichever is earlier.
Step 4: If $d < LCL$ or $d > UCL$, then declare the production process as out-of-control. Otherwise, declare the production process as in control.

The proposed control chart is considered to be an np control chart since the number of failed items in a subgroup of fixed sample size is plotted against the subgroup. The number of failed items in the life test follows a binomial (n, p_0) distribution when the production process is in control. The probability of an item failing before test termination time t_A, is denoted as p_0. As a result, the following equations are used to determine the control limits for the proposed control chart:

$$UCL = np_0 + k\sqrt{np_0(1 - p_0)} \tag{12}$$

$$LCL = max[0, np_0 - k\sqrt{np_0(1 - p_0)}] \tag{13}$$

Where "k" is the control limits coefficient and the failure probability p_0 of the product before the time t_A is calculated using (10).

The probability that the process is declared to be in control is denoted by P_{in}^0 and given as

$$P_{in}^0 = P(LCL \leq d \leq UCL|p_0)$$

$$P_{in}^0 = \sum_{d=[LCL]+1}^{[UCL]} \binom{n}{d} p_0^d (1 - p_0)^{n-d} \tag{14}$$

Generally, the average run length is used to evaluate the performance of the control chart. The in-control ARL of the proposed control chart denoted by ARL_0, is determined by

$$ARL_0 = \frac{1}{1 - [\sum_{d=[LCL]+1}^{[UCL]} \binom{n}{d} p_0^d (1 - p_0)^{n-d}]} \tag{15}$$

The probability that, the process is declared to be in control after shifting is denoted by P_{in}^1 and given as

$$P_{in}^{1} = P(LCL \leq d \leq UCL|p_1)$$

$$P_{in}^{1} = \sum_{d=[LCL]+1}^{[UCL]} \binom{n}{d} p_1^d (1-p_1)^{n-d} \tag{16}$$

The ARL for the shifted process (out-of-control) is denoted by ARL_1 and is obtained by:

$$ARL_1 = \frac{1}{1 - [\sum_{d=[LCL]+1}^{[UCL]} \binom{n}{d} p_1^d (1-p_1)^{n-d}]} \tag{17}$$

To evaluate the effectiveness of the proposed control chart in identifying process shifts, we establish the optimal parameters LCL, UCL, a, and k based on predetermined in-control ARL values. Specifically, we examined four scenarios of in-control ARL, namely r_0 = 200, 300, 400, and 500, while maintaining a fixed sample size of n = 25 and 30, AF = 1 and 2 for a one-sided control chart. Additionally, we analysed four sample sizes, 20, 25, 30, and 35, while keeping the in-control ARL (r_0) constant at 250 and 370, AF = 1 and 2 for the same type of control chart. Furthermore, we investigated four instances of in-control ARL across two values of r_0 and two sample sizes, specifically r_0 = 350 and 450, with n = 20 and 30, AF = 1, 2 for a two-sided control chart. The selection of optimal parameters aims to ensure that the in-control ARL is as close as possible to the specified ARL values, while the out-of-control ARLs are reported for varying shift constant values. The optimal parameters, along with their corresponding out-of-control ARLs, are presented in Tables 1 through 5.

Table 1: *ARL when the process parameter one-sided shift*

	n = 25							
	AF = 1				AF = 2			
Parameter	r_0 = 200	r_0 = 300	r_0 = 400	r_0 = 500	r_0 = 200	r_0 = 300	r_0 = 400	r_0 = 500
LCL	6	6	5	5	0	0	0	0
UCL	20	21	20	20	12	12	13	13
a	0.6633	0.6758	0.6365	0.6583	0.6220	0.6578	0.6672	0.6860
k	2.7950	2.9070	2.8010	2.9691	3.3892	3.1791	3.4608	3.2932
$Shift(f)$	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
1.00	200.099	300.145	400.257	500.060	200.155	300.473	400.137	500.338
0.90	190.252	224.537	353.284	236.532	184.222	255.520	279.701	377.636
0.80	50.550	142.095	81.976	55.257	130.979	66.746	161.980	112.626
0.70	13.676	32.275	19.721	14.590	27.656	15.636	31.304	23.157
0.60	4.610	8.929	5.921	4.816	6.647	4.437	7.160	5.776
0.50	2.061	3.226	2.379	2.113	2.206	1.768	2.296	2.037
0.40	1.268	1.609	1.345	1.280	1.186	1.110	1.205	1.160
0.30	1.040	1.123	1.053	1.042	1.008	1.004	1.010	1.007
0.20	1.002	1.010	1.002	1.002	1.000	1.000	1.000	1.000
0.10	1.000	1.000	1.000	1.000	1.0000	1.000	1.000	1.000

Table 2: *ARL when the process parameter one-sided shift*

$n = 30$								
	$AF = 1$				$AF = 2$			
Parameter	$r_0 = 200$	$r_0 = 300$	$r_0 = 400$	$r_0 = 500$	$r_0 = 200$	$r_0 = 300$	$r_0 = 400$	$r_0 = 500$
LCL	8	7	7	9	0	0	0	0
UCL	24	23	24	26	12	13	13	14
a	0.6597	0.6366	0.6413	0.7469	0.6476	0.6838	0.6088	0.7142
k	2.9354	3.1035	3.1439	3.2825	2.8618	2.9305	3.2453	2.8584
$Shift(f)$	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
1.00	200.253	300.496	400.392	499.811	199.990	300.080	400.339	499.720
1.10	53.495	87.743	98.041	114.768	155.086	265.998	177.470	465.747
1.20	18.797	28.973	31.604	35.044	137.855	232.026	83.065	366.882
1.30	8.538	12.374	13.291	14.072	71.471	112.628	44.985	167.343
1.40	4.729	6.482	6.872	7.013	41.213	61.450	27.382	87.021
1.50	3.050	3.975	4.167	4.139	26.178	37.278	18.233	50.737
1.60	2.208	2.751	2.857	2.786	17.958	24.601	13.017	32.383
1.70	1.746	2.090	2.153	2.077	13.092	17.356	9.817	22.205
1.80	1.474	1.704	1.745	1.674	10.020	12.915	7.734	16.126
1.90	1.306	1.467	1.494	1.432	7.975	10.034	6.311	12.269
2.00	1.200	1.314	1.333	1.281	6.5539	8.0750	5.2989	9.6950

Table 3: *ARL when the process parameter one-sided shift*

$r_0 = 250$								
	$AF = 1$				$AF = 2$			
Parameter	$n = 20$	$n = 25$	$n = 30$	$n = 35$	$n = 20$	$n = 25$	$n = 30$	$n = 35$
LCL	4	6	8	9	0	0	1	0
UCL	17	21	24	26	12	12	15	13
a	0.6612	0.6664	0.6711	0.6355	0.7205	0.7327	0.8003	0.6186
k	3.0784	3.0315	3.0314	3.0321	3.4687	2.8626	2.8349	2.9044
$Shift(f)$	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
1.00	250.283	250.028	250.191	250.218	250.234	249.949	250.236	250.184
1.10	84.552	71.127	67.231	68.351	109.185	219.511	241.302	225.011
1.20	33.746	26.070	22.737	21.614	55.458	171.212	152.969	196.977
1.30	16.277	11.996	9.973	9.115	32.146	87.324	67.999	96.822
1.40	9.142	6.594	5.360	4.813	20.626	49.513	35.027	53.498
1.50	5.788	4.163	3.369	3.032	14.314	30.957	20.444	32.841
1.60	4.029	2.926	2.387	2.150	10.561	20.931	13.164	21.911
1.70	3.022	2.236	1.854	1.688	8.180	15.061	9.157	15.611
1.80	2.405	1.823	1.543	1.424	6.586	11.392	6.774	11.720
1.90	2.007	1.563	1.352	1.266	5.473	8.972	5.264	9.178
2.00	1.738	1.391	1.231	1.168	4.666	7.305	4.258	7.439

Table 4: *ARL when the process parameter one-sided shift*

	$r_0 = 370$							
	$AF = 1$				$AF = 2$			
Parameter	$n = 20$	$n = 25$	$n = 30$	$n = 35$	$n = 20$	$n = 25$	$n = 30$	$n = 35$
LCL	4	5	8	14	0	0	1	1
UCL	17	20	24	31	12	13	15	16
a	0.6909	0.6316	0.7012	0.8638	0.7549	0.7813	0.7418	0.7346
k	2.8343	2.8397	2.9146	2.8954	3.2127	2.8439	2.9169	2.9080
$Shift(f)$	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
1.00	370.047	370.356	370.096	370.073	370.283	370.294	370.086	370.182
0.90	295.564	282.919	171.096	240.076	267.304	96.125	173.819	84.293
0.80	86.130	90.027	38.110	95.016	143.458	24.019	36.435	17.879
0.70	24.891	21.201	10.069	21.970	32.352	6.951	8.596	4.782
0.60	8.337	6.225	3.498	6.520	8.300	2.580	2.738	1.847
0.50	3.403	2.449	1.689	2.603	2.770	1.359	1.341	1.133
0.40	1.769	1.362	1.148	1.440	1.367	1.044	1.032	1.006
0.30	1.199	1.056	1.016	1.085	1.037	1.001	1.000	1.000
0.20	1.027	1.003	1.000	1.006	1.001	1.000	1.000	1.000
0.10	1.001	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 5: *ARL when the process parameter two-sided shift*

	$AF = 1$				$AF = 2$			
	$r_0 = 350$		$r_0 = 450$		$r_0 = 350$		$r_0 = 450$	
Parameter	$n = 20$	$n = 30$	$n = 20$	$n = 30$	$n = 20$	$n = 30$	$n = 20$	$n = 30$
LCL	3	6	5	9	0	0	0	0
UCL	16	22	18	26	12	14	12	14
a	0.6154	0.6046	0.7758	0.7417	0.7495	0.7338	0.7773	0.7202
k	3.0601	3.0250	2.9858	3.2412	3.4640	2.9457	3.1778	2.8852
$Shift(f)$	ARL	ARL	ARL	ARL	ARL	ARL	ARL	ARL
0.10	1.000	1.000	1.008	1.000	1.000	1.000	1.000	1.000
0.20	1.008	1.000	1.106	1.010	1.001	1.000	1.000	1.000
0.30	1.102	1.004	1.466	1.128	1.040	1.000	1.028	1.000
0.40	1.501	1.077	2.454	1.643	1.386	1.015	1.297	1.019
0.50	2.743	1.468	5.143	3.418	2.864	1.201	2.436	1.241
0.60	6.631	2.910	13.132	10.073	8.763	2.066	6.713	2.258
0.70	20.121	8.392	39.439	29.949	34.813	5.337	24.220	6.206
0.80	72.012	32.915	133.904	99.772	156.206	18.685	101.617	23.014
0.90	259.127	156.136	431.665	421.938	283.359	79.060	379.733	102.129
1.00	350.171	350.162	450.390	449.755	350.087	350.051	450.461	449.933
1.10	139.299	122.038	156.674	103.577	152.461	312.114	210.600	391.229
1.20	54.342	39.414	57.032	32.142	74.566	279.782	99.623	298.978
1.30	25.305	16.324	25.108	13.099	41.685	216.640	53.811	181.052
1.40	13.727	8.296	13.008	6.614	35.942	109.495	32.511	93.319
1.50	8.405	4.940	7.684	3.950	17.546	62.222	21.444	53.984
1.60	5.664	3.324	5.043	2.683	12.666	38.853	15.155	34.226
1.70	4.119	2.459	3.603	2.016	9.629	26.148	11.313	23.336
1.80	3.183	1.956	2.755	1.636	7.631	18.687	8.825	16.865
1.90	2.582	1.646	2.225	1.407	6.253	14.021	7.135	12.776
2.00	2.179	1.446	1.877	1.264	5.268	10.947	5.940	10.059

From the examination of Tables 1 and 4, it is evident that the out-of-control ARL (ARL_1) is reduced when the shift constant f is lowered. Similarly, Tables 2 and 3 demonstrate that an increase in the shift constant f leads to a decrease in the out-of-control ARL (ARL_1). Additionally, Table 5 shows that deviations of the shift constant from the in-control ARL result in a decrease in the out-of-control ARL. This finding highlights the proposed control chart's proficiency in promptly identifying process shifts. An algorithm is proposed to determine the optimal parameters for the design of the control chart, accommodating various combinations of specified in-control ARL and sample size.

1. Specify the values of ARL say r_0, sample size n and acceleration factor (AF).
2. Set a = 0.0001 and k = 0.0001.
3. Determine the failure probability of the product (i.e., p_0) when the process is in control by using equation (10) and also determine the control chart parameters (LCL and UCL) by substituting the values of n, a and k in equation (12) and (13).
4. Calculate for in-control ARL by using equation (15) and compare it with r_0. Continue this process, for different combinations of (a and k) until getting an in-control ARL is very close to the specified ARL, r_0.
5. Whenever such ARL exist, then the corresponding parameters (LCL, UCL, a, k) are the required optimal parameters. Then calculate the out-of-control ARL (i.e., ARL_1) using these optimal parameters in equation (17).

3. Application of Proposed Control Chart

3.1. Real-Life Application

In this section, a design example is given to demonstrate a real-life application of the proposed control chart under the Bayesian approach. Assume that the product lifetime follows a Rayleigh-Inverse-Rayleigh distribution with parameter λ_A. Considering the following product values: μ_A is 1000 hours, r_0 = 200, n is 20 and AF = 2. The control chart parameters provided in Table 1 include: k at 3.3892, and a at 0.6220, with a lower control limit (LCL) of 0 and an upper control limit (UCL) of 12. Therefore, the control chart to be designed as follows:

Step 1: Select a sample of 20 products from each subgroup and conduct a life test for 622 hours. Record the number of failed products during the life test and denote it as "d".
Step 2: Terminate the life test either after reaching 622 hours or the number of failed products greater than twenty before reaching 622 hours whichever is earlier.
Step 3: If d is between 0 and 12, the process is considered in control; otherwise, it is classified as out of control.

3.2. Simulation Study

This section provides a demonstration of the proposed control chart's application using simulated data. The data is produced from a Rayleigh distribution while the process is in a controlled state, with parameters that follow an inverse Rayleigh distribution under standard conditions (i.e., AF = 1) and a target average lifetime of 1000 hours. A random sample of size n = 20 is utilized for each batch, where the first fifteen samples are derived from the in-control process, and the subsequent fifteen samples are from a shifted process with f = 0.40, resulting in a total of 30 sample batches. From Table V, with n = 20 and ARL_0 = 350, the constants are established as k = 3.0601, a = 0.6154, UCL = 16, and LCL = 3. The number of products with a lifetime below 615 hours is considered as nonconforming items and denoted it as 'D', with the following values: 12, 10, 15, 11, 14, 10, 13, 9, 11, 10, 14, 9, 13, 14, 12, 16, 18, 15, 19, 18, 17, 16, 19, 17, 18, 20, 17, 19, 16, 18. The values of nonconforming items are plotted alongside two control limits (LCL = 3 and UCL = 16) shown in Figure 1.

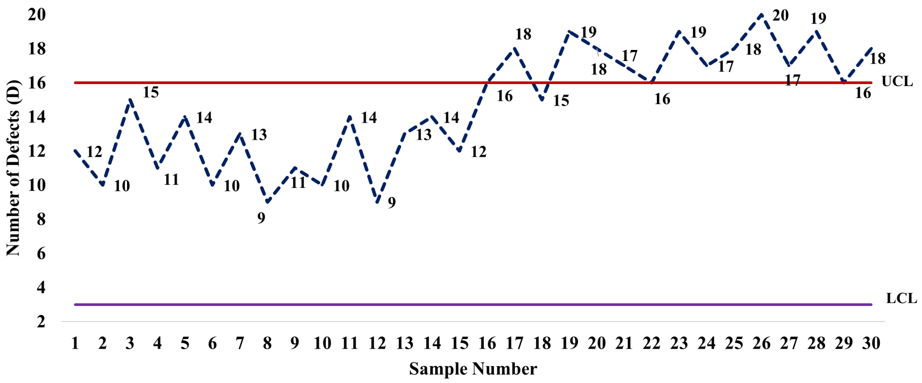

Figure 1: *Control Chart for Simulation Data*

As illustrated in Figure 1, the proposed control chart identifies a shift occurring at the 17^{th} sample, specifically the second observation following the shift, with a calculated Average Run Length (ARL) of 1.501. Consequently, this control chart effectively recognises shifts in the manufacturing process under standard conditions. Therefore, it can be concluded that the proposed control chart is capable of detecting process shifts more rapidly.

4. Behaviour of the Average Run Length

Empirical analysis can investigate the behavior of the ARL under standard conditions (i.e., AF = 1). The characteristics of the ARL curves can be examined concerning the parameters p, n, f, LCL, and UCL while keeping the other parameters constant. To begin with, the effect of "p" on the ARL is assessed. Figure 2 illustrates a set of four ARL curves that correspond to various sets of control chart parameters: (n = 25, k = 3.0, UCL = 20, LCL = 5), (n = 30, k = 3.0, UCL = 20, LCL = 5), (n = 35, k = 3.0, UCL = 20, LCL = 5), and (n = 40, k = 3.0, UCL = 20, LCL = 5). The shapes of these curves reveal an upward trend as the values of "p" increase until reaching the ARL_0, after which the ARL begins to decline with further increases in "p". For instance, with the control chart parameters set at (n = 35, k = 3.0, UCL = 20, LCL = 5), the values of "p" are taken as 0.30, 0.32, 0.34, 0.36, 0.38, and 0.40. The corresponding ARL values for these p values are 36.871, 64.288, 105.883, 135.762, 113.262, and 70.379, respectively. The analysis of the behavior

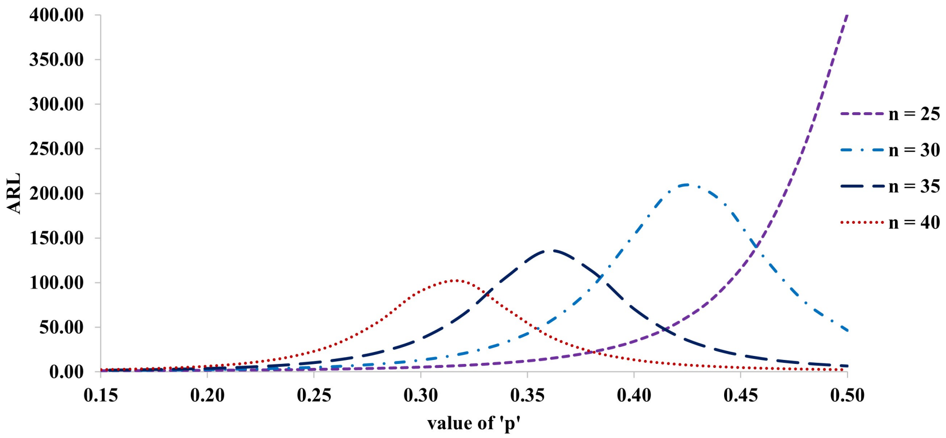

Figure 2: *Behaviour of ARL under "p" and "n"*

of ARL_0 and ARL_1 concerning sample size is presented in Figure 2. This figure indicates that the ARL_0 value rises as the sample size diminishes. Specifically, for sample sizes of n = 25, 30, 35, and 40, the corresponding ARL_0 values are 400.98, 207.29, 135.76, and 101.39, respectively. Conversely, the ARL_1 value exhibits an increase with larger sample sizes while maintaining a constant "p" value. For instance, the ARL_1 values for sample sizes n = 25, 30, 35, and 40 are 7.11, 20.48, 64.29, and 101.39, respectively, with a fixed p-value of 0.32. Thus, it can be concluded that in these control charts, an increase in sample size leads to a decrease in the in-control ARL, while the out-of-control ARL value increases with larger sample sizes. The subsequent analysis

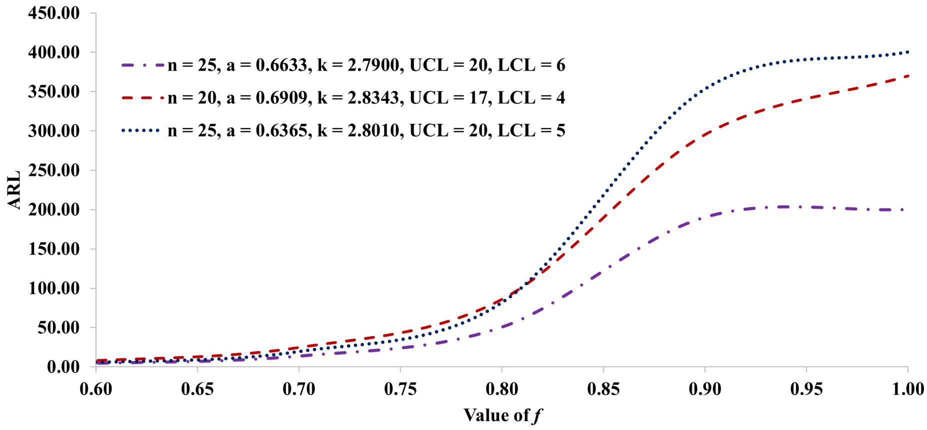

Figure 3: *Behaviour of ARL under shift constant "f"*

focuses on the behaviour of ARL_1 concerning the shift constant "f" as illustrated in Figure 3. This figure presents three distinct ARL curves, each corresponding to varying sets of control chart parameters: (n = 25, a = 0.6633, k = 2.7900, UCL = 20, LCL = 6), (n = 20, a = 0.6909, k = 2.8343, UCL = 17, LCL = 4), and (n = 25, a = 0.6365, k = 2.8010, UCL = 20, LCL = 5). A total of ten different values for the shift constant "f" are examined, specifically f = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0, while maintaining a fixed ARL_0 of 370. The data depicted in Figure 3 indicates that an increase in the shift constant correlates with a rise in the ARL_1 value at a constant ARL_0. For instance, with the control chart parameters (n = 20, a = 0.6909, k = 2.8343, UCL = 17, LCL = 4), the values of "f" are assessed as 0.1 through 1.0, yielding corresponding ARL_1 values of 1.00, 1.03, 1.20, 1.77, 3.40, 8.34, 24.89, 86.13, 295.56, and 370.05. This demonstrates that an increase in the shift constant results in a corresponding increase in the ARL_1 value. The

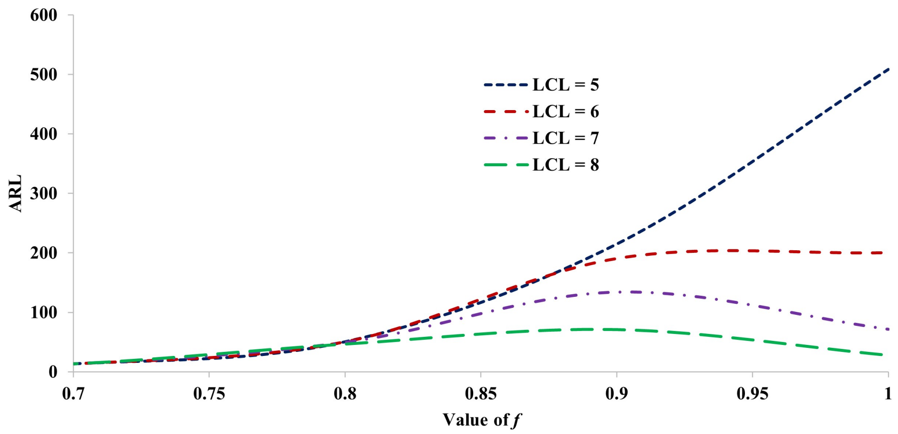

Figure 4: *Behaviour of ARL under "LCL"*

subsequent analysis focuses on the behaviour of ARL_0 concerning the LCL, with the findings presented in the form of ARL curves as illustrated in Figure 4. This figure showcases four distinct

ARL curves corresponding to LCL values of 5, 6, 7, and 8, while maintaining fixed control chart parameters of n = 25, a = 0.6633, k = 2.7900, UCL = 20, and ARL_0 = 200. It is observed from Figure 4 that an increase in the LCL results in a decrease in the value of ARL_0. For instance, with the parameters n = 25, a = 0.6633, k = 2.7900, UCL = 20, and ARL_0 = 200, the ARL_0 values for the LCL values of 5, 6, 7, and 8 are recorded as 508.415, 200.099, 71.801, and 27.981, respectively.

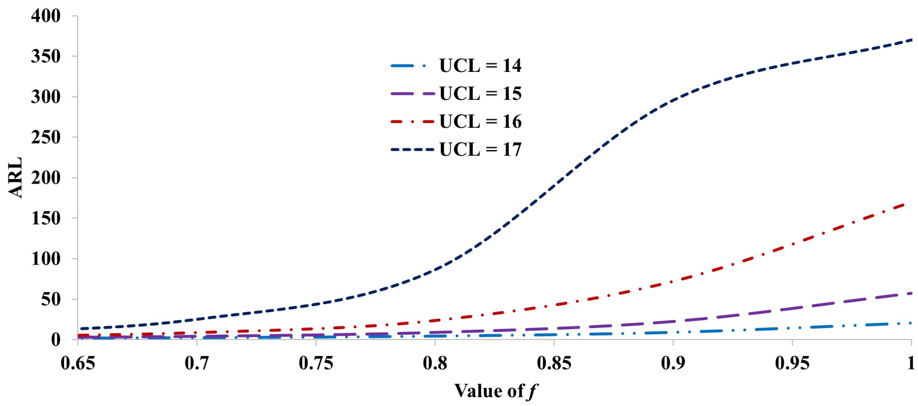

Figure 5: *Behaviour of ARL under "UCL"*

The analysis of the behaviour of ARL_0 concerning the UCL has been conducted, with the findings presented in the form of ARL curves as illustrated in Figure 5. This figure showcases four distinct ARL curves corresponding to UCL values of 14, 15, 16, and 17 while maintaining fixed control chart parameters of n = 20, a = 0.6909, k = 2.8343, LCL = 4, and ARL_0 = 370. It is observed from Figure 5 that an increase in the UCL results in a corresponding increase in the value of ARL_0. Specifically, for the parameters n = 20, a = 0.6909, k = 2.8343, LCL = 4, and ARL_0 = 370, the ARL_0 values for the UCLs of 14, 15, 16, and 17 are 20.5004, 57.3531, 169.508, and 370.047, respectively.

5. Discussion

In this paper, we have developed a methodology for a new Bayesian attribute control chart for the lifetime of products. The life tests are performed by accelerated life test with hybrid censoring assuming a Rayleigh distribution for the lifetime and an inverse Rayleigh distribution as a prior. The proposed control chart is to ensure the mean lifetime of the product as the quality criterion. The newly established control chart is highly adaptable and may be used to monitor the lifetime of quality products. The tables are provided for industrial usage and are explicated with the use of simulated data. It is generated using R software from a Rayleigh-Inverse-Rayleigh distribution. The performance of the proposed control chart is expressed in terms of ARLs for various shift constants (f), sample size (n), other parameters p, LCL and UCL. It should be noted that if the hybrid censoring scheme is used to carry out the life test, executing the sampling inspection will reduce the time and cost of conducting the life test. The developed attribute control chart has the potential to be extended for use with various other statistical distributions as part of a continuing research study.

References

[1] Muhammad Aslam and Chi-Hyuck Jun (2015). Attribute Control Chart for the Weibull Distribution under Truncated Life Tests. *Quality Engineering*, 27(3): 283 – 288.

[2] Muhammad Aslam, Osama H. Arif and Chi-Hyuck Jun (2017). An Attribute Control Chart for a Weibull distribution under Accelerated Hybrid Censoring. *PLoS ONE*, 12(3): e0173406: 1 – 11.

[3] Ali Hussein Al-Marshadi, Ambreen Shafqat, Muhammad Aslam and Abdullah Alharbey (2021). Performance of a New Time-Truncated Control Chart for Weibull Distribution under Uncertainty. *International Journal of Computational Intelligence Systems*, 14(1): 1256 – 1262.

[4] Aslam M, Nasrullah Khan and Chi – Hyuck (2016a). A control chart for time truncated life tests using Pareto distribution of the second kind. *Journal of Statistical Computation and Simulation*, 86(11): 2113 – 2122.

[5] Aslam M, Arif, O. H and Jun, C (2016b). An Attribute Control Chart based on the Birnbaum-Saunders Distribution using Repetitive Sampling. *IEEE Access*, 5: 9350 – 9360.

[6] Nanthakumar C and T. Kavitha (2017). Design of Attribute Control Chart based on Inverse Rayleigh Distribution under Type – I Censoring. *International Journal of Statistical Sciences*, 4(6): 17 – 22.

[7] Ambreen Shafqat, Jaffer Hussain, Amjad D. Al-Nasser and Muhammad Aslam (2018). Attribute Control Chart for some popular distributions. *Communication in Statistics – Theory and Methods*. 47(8): 1978 – 1988.

[8] Arif O. H, Muhammad Aslam and K.A. Butt (2018). Attribute Control Chart for a Lognormal distribution under Accelerated Time Censoring. *Journal of Computational and Theoretical Nanoscience*, 15(3): 919 – 923.

[9] Srinivasa Rao G (2018). A Control Chart for Time Truncated Life Tests Using Exponentiated Half Logistic Distribution. *Applied Mathematics & Information Sciences: An International Journal*, 12(1): 125 – 131.

[10] Srinivasa Rao Gadde, A.K. Fulment and P.K Josephat (2019). Attribute Control Charts for Dagum distribution under truncated life tests. *Life cycle Reliability and Safety Engineering*, 8: 329 – 335.

[11] Balamurali S and P. Jeyadurga (2019). An Attribute np Control Chart for Monitoring Mean Life using Multiple Deferred State Sampling based on Truncated Life Tests. *International Journal of Reliability, Quality and Safety Engineering*, 26(1): 1950004-1 – 1950004-18.

[12] Enayat M. Abd Elrazik (2020). Attribute control charts for the New Weibull Pareto Distribution under truncated life tests. *Journal of Statistics Applications & Probability*, 9(1): 43 – 49.

[13] Ahmad Ibrahim Shawky, Muhammad Aslam and Khushnoor Khan (2020). Multiple dependent state sampling-based charts using Belief Statistic under Neutrosophic Statistics. *Journal of Mathematics*, 2020(7680286): 1- 14.

[14] Srinivasa Rao Gadde, Edwin Paul (2020). Time Truncated Control Chart using Log- Logistic Distribution. *Biometrics & Biostatistics International Journal*, 9: 76 - 82.

[15] Rosaiagh, K., G. Srinivasa Rao and S. V. S. V. S. V. Prasad (2021). A Control Chart for Time Truncated Life Tests using type-II Generalized Log Logistic Distribution. *Biometrics & Biostatistics International Journal*, 10(4): 138 – 143.

[16] Adeoti O. A. and P. Ogundipe (2021). A Control Chart for the generalized exponential distribution under time truncated life test. *Life Cycle Reliability and Safety Engineering*, 10: 53 – 59.

[17] Ayman Baklizi and Sawsan Abu Ghannam (2022). An attribute control chart for the inverse Weibull distribution under truncated life tests. *Heliyon*, 8(e11976): 1 - 5.

[18] Srinivasa Rao Gadde and Amer Ibrahim Al-Omari (2022). Attribute control charts based on TLT for Length-Biased Weighted Lomax Distribution. *Journal of Mathematics*, 2022(3091850): 1- 15.

[19] Olatunde Adebayo Adeoti and Gadde Srinivasa Rao (2022). Attribute control chart for Rayleigh distribution using Repetitive sampling under truncated life test. *Journal of Probability and Statistics*, 2022(8763091): 1 - 11.

[20] Muhammad Naveed, Muhammad Azam, Nasrullah Khan, Muhammad Aslam, Muhammad Saleem and Muhammad Saeed (2023). Control Chart for Half Normal and half Exponential power Distributed Process. *Scientific Reports*, 13(8632): 1- 20.

[21] Srinivasa Rao B, M. Rami Reddy and K. Rosaiah (2023). An attribute Control Chart for Time

Truncated Life Tests using Exponentiated Inverse Kumaraswamy Distribution. *Reliability: Theory & Applications*, 3:74(18): 206 – 217.

[22] Sri Ramachandran G V., Srinivasa Rao Gadde and Muhammed Aslam (2024). Utilizing Repetitive Sampling in the Construction of a Control Chart for Lindley Distribution with Time Truncation. *Journal of Statistical Theory and Applications*, 23: 224 – 239.

[23] Saha M., Pareek P, Tripathi H, and A Devi (2024). Time Truncated Attribute Control Chart for Generalized Rayleigh Distributed Quality Characteristics and Beyond *International Journal of Quality & Reliability Management*, 41(5): 1400 – 1416.

[24] Gunasekaran Munian (2024). A New Attribute Control Chart based on Exponentiated Exponential Distribution under Accelerated Life Test with Hybrid Censoring. *Reliability: Theory & Applications*, 19(4): 850 – 860.

[25] Joel M. Calabrese (1995). Bayesian Process Control for Attributes. *Management Science*, 41(4): 637 – 645.

[26] Ulrich Menzefricke (2002). On the Evaluation of Control Chart Limits based on Predictive Distributions. *Communication in Statistics – Theory and Methods*, 31(8): 1423 – 1440.

[27] Apley D. W (2012). Posterior distribution charts: A Bayesian approach for graphically exploring a process mean. *Technometrics*, 54(3): 279 – 310.

[28] Amin Shaka Aunali and D. Venkatesan (2017). Comparison of Bayesian Method and Classical Charts in Detection of Small Shifts in the Control Charts. *International Journal of Operations Research and Optimization*, 6(2): 101 – 114.

[29] Kumar N and S. Chakraborti (2017). Bayesian Monitoring of times between events: The Shewhart tr - chart. *Journal of Quality Technology*, 49(2): 136 – 154.

[30] Amin Shaka Aunali and D. Venkatesan (2019). Bayesian Approach in Control Chart Techniques. *International Journal of Scientific Research in Mathematical and Statistical Sciences*, 6(2): 217 – 221.

[31] Amin Shaka Aunali, Venkatesan. D. and Michele Gallo (2019). Bayesian Control Chart using Gamma Prior. *International Journal of Scientific & Technology Research*, 8(12): 1567 - 1570.

[32] Ali S (2020). A Predictive Bayesian Approach to sequential time-between-events monitoring. *Quality and Reliability Engineering International*, 36(1): 365 – 387.

[33] Kadoishi S and H. Kawamura (2020). Control Charts based on Hierarchical Bayesian Modeling. *Total Quality Science*, 5(2): 72 – 80.

[34] Konstantinos Bourazas, Dimitrios Kiagias and Panagiotis Tsiamyrtzis (2022). Predictive Control Charts (PCC): A Bayesian Approach in Online monitoring of short runs. *Journal of Quality Technology*, 54(4): 367 – 391.

[35] Fatimah Alshahrani, Ibrahim M. Almanjahie, Majid Khan, Syed M. Anwar, Zahid Rasheed and Ammara N. Cheema (2023). On Designing of Bayesian Shewhart-Type Control Charts for Maxwell Distributed Processes with Application of Boring Machine. *Mathematics*, 11(1126): 1 - 20.

[36] Lord Rayleigh, O. M. F. R. S (1919). On the problem of random vibrations, and of random flights in one, two, or three dimensions. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Series 6, 37(220): 321 – 347.

[37] Siddiqui M. M (1962). Some Problems Connected with Rayleigh Distributions. *Journal of Research of the National Bureau of Standards – D. Radio Propagation*, 66D(2): 167 – 174.

[38] Siddiqui M. M (1964). Statistical Inference for Rayleigh Distribution. *Radio Science Journal of Research NBS/USNC-URSI*, 68D(9): 1005 – 1010.

[39] Kalaiselvi S., A. Loganathan and R. Vijayaraghavan (2014). Bayesian Reliability Sampling Plans under the Conditions of Rayleigh-Inverse-Rayleigh Distribution. *Economic Quality Control*, 29(1): 29 – 38.

[40] Saad T and Bakir (2012). A nonparametric Shewhart-type quality control chart for monitoring broad changes in a process distribution. *International Journal of Quality, Statistics, and Reliability*, 2012(147520): 1 – 10.