DIFFERENT ESTIMATION METHODS AND VALIDATION FOR THE EXTENSION EXPONENTIAL DISTRIBUTION

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Abstract

This study investigates the extended exponential distribution proposed by Nadarajah and Haghighi, which can effectively model data with a mode at zero and accommodate varying hazard rates—whether increasing, decreasing, or constant. Unlike other distributions, it allows for scenarios where the hazard function increases while the probability density function decreases monotonically. The focus is on exploring alternative estimation methods to maximum likelihood estimation for this distribution, including the maximum product of spacing, Cramer-von Mises, Anderson-Darling, Right-tail Anderson-Darling, Left-tail Anderson-Darling, and Kolmogorov-Smirnov tests. A novel approach using initial data is proposed to develop an effective goodness-of-fit criterion tailored for validating this model. Extensive simulations with thousands of samples are conducted to assess the performance of these estimation methods and the practicality of the proposed goodness-of-fit test. Real data applications are also utilized to demonstrate the applicability and effectiveness of the extended exponential distribution in real-world scenarios. This research expands the statistical modelling toolkit by providing robust estimation techniques and validation criteria specifically designed for the characteristics of the extended exponential distribution.

Keywords: Anderson-Darling method, Cramer-von-Mises method, maximum likelihood estimation, maximum product of spacing method, Right-tail Anderson-Darling method, goodness-of-fit test.

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I. Introduction

The recent statistical literature mainly focuses on the introduction of new models capable of describing observations when classical models cannot do so. There are different forms of generalizations either by adding shape parameters, by exponentiation the cumulative distribution or based on generator distributions, one can see [8], [2], [6], [3] and others. Nevertheless, [17] proposed a simple and very interesting model. With two parameters, this one offers a good alternative to gamma, Weibull and the exponentiated exponential distributions for which the hazard functions are only constant or decreasing when the probability density function (pdf) is monotonically decreasing while the rate failure of the new distribution can be increasing. Also, this distribution is able to describe data with mode fixed at zero when Its hazard rates can be increasing, decreasing and constant which is not realized for other models. Making a simple change of variable, the new model is reduced to a truncated Weibull distribution. The proposed distribution has more applications than classical models like the fit of the daily rainfall data (for example) which have a monotonically decreasing probability density (pdf) function when the empirical hazard rate failure (hrf) is increasing. The authors studied mathematical properties such the moments, the Bonferroni and Lorenz curves, the entropy and the order statistics. For estimating the unknown parameters, they proposed the maximum likelihood (MLE) and the moment estimation (MME) methods which cannot give the explicit forms of the estimators, so iterative methods are required and as It well known the results are performed for big sample sizes. It should be noted also, that the validation of this model is not investigate yet till now which motivate us to propose a new technique for testing the fit of this distribution.

In this work, we propose, firstly, different estimation methods which can be used for small and big sample sizes such as the maximum product of spacing (MPS), the Cramer-von-Mises (CM) method, Anderson-Darling (AD) method, Right-tail and Left-tail Anderson-Darling (RAD, LAD) methods and the method of Kolmogorov-Smirnov (KS) to estimate the unknown parameters as alternatives to maximum likelihood estimation method. On the other hand, to have reliable results in any statistical analysis, It must be necessary to verify if the observed data are really described by the model chosen. Using an extension of the chi-square statistic introduced by [18] and [22] for continuous models, we provide a criteria goodness-of-fit test which takes into account the unknown parameters and the initial data for the extension exponential distribution. This new technique recovers all the information given by the observations and can validate effectively the model without comparison with its competitors. Recently, this method was performed for some new distributions, see [23], [25], [24], [19]. An importante simulation study is given to calculate the different estimators, their biases, their mean square errors and their estimated average widths. A comparison study between the method performances is also provided. Beside classical model selection criteria such as AIC, BIC, LL and Kolmogorov-Smirnov statistic, we provided all the elements of the new statistic test which are used to distinguish between this distribution and its competitors. Real data applications are used for illustration purpose.

The objective of this work is to introduce and evaluate alternative estimation methods suitable for both small and large sample sizes, to ensure the chosen model accurately describes the observed data through a new goodness-of-fit test, and to validate the model using comprehensive simulation studies and real data applications.

II.The Extension exponential distribution

Depending on two parameters α and λ (shape and scale), the extension of the exponential distribution introduced by Nadarajah and Haghighi [17] is characterized by the cumulative probability function

$$F(t) = 1 - exp(1 - (1 + \lambda t)^{\alpha}) \tag{1}$$

The probability density function (pdf) is given by

$$f(t) = \alpha\lambda(1 + \lambda t)^{\alpha-1} exp(1 - (1 + \lambda t)^{\alpha}) \qquad \alpha > 0 \text{and} \lambda > 0. \tag{2}$$

The hazard and the cumulative hazard functions are

$$h(t) = \alpha\lambda(1 + \lambda t)^{\alpha-1} \tag{3}$$

$$H(t) = -log(S(t)) = (1 + \lambda t)^{\alpha} - 1 \tag{4}$$

Different shapes of the pdf and the hazard rate functions are represented in Figure 1 and Figure 2.

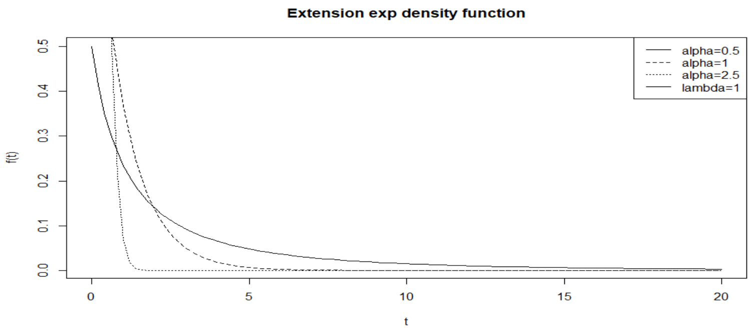

Figure 1: *probability density function of the extension exponential distribution*

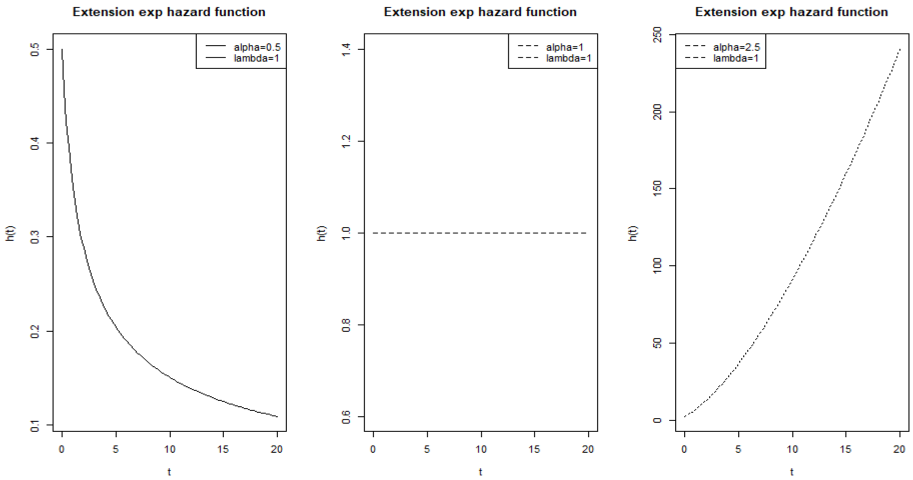

Figure 2: *hazard function of the extension exponential distribution*

The interest of this pdf is to have the mode zero and different hazard shapes (increasing, decreasing and constant), furthermore the extension exponential distribution pdf takes only decreasing shapes whereas the hazard function can be increasing (see Figure 3).

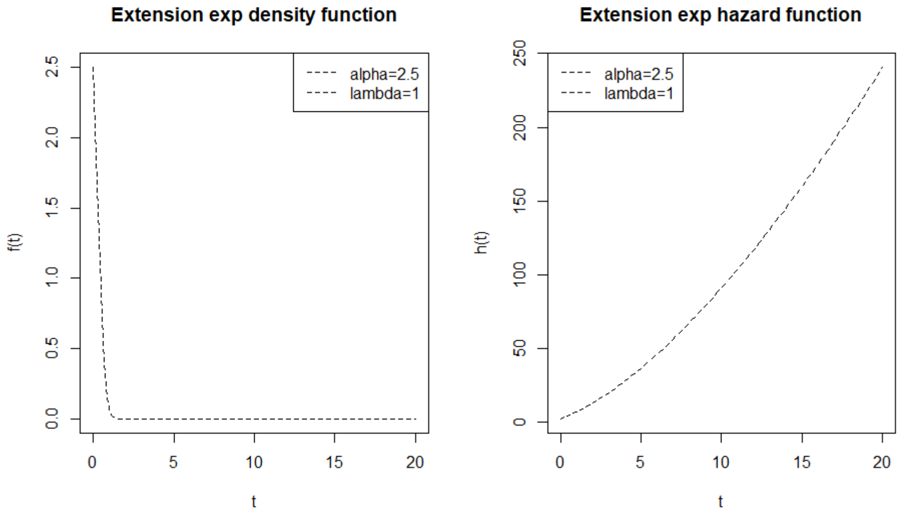

Figure 3: *density and hazard function of the extension exponential distribution with* $\alpha > 1$

The authors studied mathematical properties such as the moments, the Bonferroni and Lorenz curves which have applications in several fields, the entropy and the order statistics. For estimating the unknown parameters, they proposed the maximum likelihood and the moment estimation methods.

III. Maximum likelihood estimation

In this section we resume the maximum likelihood estimators of the unknown parameters in order to compare with those of the proposed methods in this work.

We consider a sample of n independent identically distributed variables $t_1, t_2, \ldots, t_n$ from the extension exponential distribution with the parameter vector $\theta = (\alpha, \lambda)$, the likelihood function is given by

$$L(t_1, t_2, \ldots, t_n; \theta) = \prod_{i=1}^{n} f(t_i; \theta) = \prod_{i=1}^{n} \alpha\lambda(1 + \lambda t_i)^{\alpha-1} exp(1 - (1 + \lambda t_i)^{\alpha}) \tag{5}$$

The log-likelihood function is:

$$l = n + n\ln(\alpha) + n\ln(\lambda) + (\alpha - 1)\sum_{i=1}^{n} \ln(1 + \lambda t_i) - \sum_{i=1}^{n} (1 + \lambda t_i)^{\alpha})]$$

The maximum likelihood estimators MLE are obtained by equaling to zero the following first derivatives of the log-likelihood function with respect to the parameters:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1 + \lambda t_i) - \sum_{i=1}^{n} (1 + \lambda t_i)^{\alpha} \ln(1 + \lambda t_i)] = 0 \tag{6}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + (\alpha - 1)\sum_{i=1}^{n} t_i(1 + \lambda t_i)^{-1} - \alpha \sum_{i=1}^{n} t_i(1 + \lambda t_i)^{\alpha-1} = 0 \tag{7}$$

As the explicit forms of the MLE $(\hat{\alpha}_{MLE}, \hat{\lambda}_{MLE})$ cannot be derived, one can use numerical methods for solving the obtained equations.

IV. Product of spacing estimation method MPS

Introduced by Cheng and Amin (1979, 1983) [7], the product of spacing estimation method (MPS) is an alternative to the maximum likelihood estimation method. This method was also proposed by Ramnaby (1984) [20] as an approximation to the Kullback-Leibler measure of information.

We consider n independent and identically distributed sample drawn from a population with unknown parameters θ. The MPS estimation method is based on the differencies between the successive increasing empirical repartition functions $F(t_i, \theta)$, and where $F(t_0) = 0$ and $F(t_{n+1}) = 1$.

$$P_i(\theta) = F(t_i, \theta) - F(t_{i-1}, \theta) \tag{8}$$

The MPS estimators are obtained by maximizing the quantity $M(\theta)$:

$$M(\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \ln(P_i(\theta)) \tag{9}$$

For the extension exponential distribution, this function is given by:

$$M(\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \ln((1 - exp(1 - (1 + \lambda t_i)^{\alpha})) - (1 - exp(1 - (1 + \lambda t_{i-1})^{\alpha})) \tag{10}$$

$$M(\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \ln(exp(1 - (1 + \lambda t_{i-1})^{\alpha}) - exp(1 - (1 + \lambda t_i)^{\alpha})) \tag{11}$$

So, the MPS estimators are the solutions of the following equations

$$\frac{\partial M(\theta)}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{D1(t_i, \theta) - D1(t_{i-1}, \theta)}{P_i} = 0 \tag{12}$$

$$\frac{\partial M(\theta)}{\partial \lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{D2(t_i, \theta) - D2(t_{i-1}, \theta)}{P_i} = 0 \tag{13}$$

where $D1$ and $D2$ are the first derivatives of the repartition function with respect to the unknown parameters α and λ:

$$D1 = \frac{dF(t)}{d\alpha} = (1+\lambda t)^{\alpha}\ln(1+\lambda t)exp(1-(1+\lambda t)^{\alpha}) \tag{14}$$

$$D2 = \frac{dF(t)}{d\lambda} = \alpha t(1+\lambda t)^{\alpha-1}exp(1-(1+\lambda t)^{\alpha}) \tag{15}$$

Cheng and Amin (1983) [7] showed that the obtained MPS estimators are as efficient and consistent as the maximum likelihood estimators (MLE) under more general conditions.

V. Cramer Von Mises estimation method

Among minimum type estimation methods proposed in the statistical literature, the Cramer-Von-Mises CM method estimation consists in minimizing the distances between theoretical and empirical repartition functions (Cramer-Von-Mises statistic) of the studied model. This method is shown to give the smaller bias estimators than the other minimum distance estimators see [15].

For the extension of the exponential distribution, this statistic is defined by

$$C(a,\lambda) = \frac{1}{12} + \sum_{i=1}^{n}\left((1-exp(1-(1+\lambda t_i)^{a}))) - \frac{2i-1}{2n}\right)^2 \tag{16}$$

To calculate the CM estimators $(\hat{a}_{CV}, \hat{\lambda}_{CV})$ of the unknown parameters, we cancel the first partial derivatives of this statistic $C(a,\lambda)$ with respect to the unknown parameters of the distribution, which amounts to solving the following equations:

$$\sum_{i=1}^{n}\left((1-exp(1-(1+\lambda t_i)^{a})) - \frac{2i-1}{2n}\right)D1 = 0 \tag{17}$$

$$\sum_{i=1}^{n}\left((1-exp(1-(1+\lambda t_i)^{a})) - \frac{2i-1}{2n}\right)D2 = 0 \tag{18}$$

where $D1$ and $D2$ are as given above.

V. Anderson-Darling estimation method

Anderson-Darling method is also a minimum type estimation method, so the estimators $(\hat{a}_{AD}, \hat{\lambda}_{AD})$ are obtained by minimizing the well-known Anderson-Darling statistic A^2 suggested by Anderson and Darling [4] with respect to the unknown parameters. [5] also discussed the properties of the AD estimators and presented its computational form as

$$A^2(a,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\ln(F(t_i)) - \ln(1-F(t_{i^*}))] \tag{19}$$

$$A^2(a,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\ln(1-exp(1-(1+\lambda t_i)^{a}))) - \ln(1-(1-exp(1-(1+\lambda t_{i^*})^{a})))] \tag{20}$$

where $i^* = n-i+1$.

For our model, the estimators $(\hat{a}_{AD}, \hat{\lambda}_{AD})$ are obtained by the resolution of the following equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{D1(t_i)}{(1-exp(1-(1+\lambda t_i)^{a}))} - \frac{D1(t_{i^*})}{(1-(1-exp(1-(1+\lambda t_{i^*})^{a})))}\right] = 0 \tag{21}$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{D2(t_i)}{(1-exp(1-(1+\lambda t_i)^{a}))} - \frac{D2(t_{i^*})}{(1-(1-exp(1-(1+\lambda t_{i^*})^{a})))}\right] = 0 \tag{22}$$

Later, [14] discussed modifications of the standard AD statistics. The most used statistic is the Right-tail AD statistics (see [9], [13] and [21]. The Right tail Anderson-Darling estimators $(\hat{a}_{RAD}, \hat{\lambda}_{RAD})$ are obtained by minimizing the following expression:

$$RAD = \frac{n}{2} - 2\sum_{i=1}^{n} \ln(F(t_i)) - \frac{1}{n}\sum_{i=1}^{n} (2i-1)(1-(F(t_{i^*}))) \tag{23}$$

$$RAD = \frac{n}{2} - 2\sum_{i=1}^{n} \ln((1-exp(1-(1+\lambda t_i)^a))) - \frac{1}{n}\sum_{i=1}^{n} (2i-1)(1-(1-exp(1-(1+\lambda t_{i^*})^a))) \tag{24}$$

which implies the resolution of the following equation system:

$$-2\sum_{i=1}^{n} \frac{D1(t_i)}{(1-exp(1-(1+\lambda t_i)^a)))} + \frac{1}{n}\sum_{i=1}^{n} (2i-1)\frac{D1(t_{i^*})}{(1-((1-exp(1-(1+\lambda t_{i^*})^a)))} = 0 \tag{25}$$

$$-2\sum_{i=1}^{n} \frac{D2(t_i)}{(1-exp(1-(1+\lambda t_i)^a)))} + \frac{1}{n}\sum_{i=1}^{n} (2i-1)\frac{D2(t_{i^*})}{(1-(1-exp(1-(1+\lambda t_{i^*})^a)))} = 0 \tag{26}$$

The Left tail Anderson-Darling estimators $(\hat{a}_{LAD}, \hat{\lambda}_{LAD})$ are obtained by minimizing the following expression

$$LAD = \frac{-3n}{2} + 2\sum_{i=1}^{n} \ln(F(t_i)) - \frac{1}{n}\sum_{i=1}^{n} (2i-1)(1-(F(t_{i^*}))) \tag{27}$$

$$LAD = \frac{-3n}{2} + 2\sum_{i=1}^{n} \ln((1-exp(1-(1+\lambda t_i)^a)))) - \frac{1}{n}\sum_{i=1}^{n} (2i-1)(1-(1-exp(1-(1+\lambda t_{i^*})^a)))) \tag{28}$$

VII. Kolmogorov-Smirnov estimation method

Let us consider an independent and identically distributed sample drawn from a population with unknown vector parameters θ. [27] considered an estimation technique so-called as the minimum Kolmogorov–Smirnov estimation (KS), whose purpose is to minimize the value of the famous Kolmogorov- Smirnov statistic which represents the biggest value of the difference between the theoretical and the corresponding empirical repartition functions of the proposed model:

$$KS = d(F_n, F) = \max_{x\in\Re}|F_n(t) - F(t)| \tag{29}$$

If $F(t)$ is the extension exponential distribution, we have

$$KS = d(F_n, F) = \max_{t\in\Re}|F_n(t) - (1-exp(1-(1+\lambda t)^a)))| \tag{30}$$

where n is the sample size, $F(t)$ is the hypothesized distribution (cdf), and $F_n(t)$ is the empirical cdf, a step-function that increases by $1/n$ at each data value. The KS statistic has been widely used to fit data to continuous populations. The interest of this statistic includes the straightforward computation of the test statistic and Its distribution-free characteristic.

The computational formulas for computing KS are:

$$D^+ = \max_{i=1,2,\ldots,n}(\frac{i}{n} - (1-exp(1-(1+\lambda t_{(i)})^a)) \tag{31}$$

$$D^- = \max_{i=1,2,\ldots,n}((1-exp(1-(1+\lambda t_{(i)})^a) - \frac{i-1}{n}) \tag{32}$$

and

$$KS = \max\{D^+, D^-\} \tag{33}$$

where $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ are the order statistics. The parameter values of that minimize the KS statistic are noted $(\hat{a}_{KS}, \hat{\lambda}_{KS})$.

VIII. Model validation

The validation of the model chosen to describe collected data for the analysis is very important. The most commun used goodness-of-fit tests are those based on empirical distribution functions (EDF) such as the Kolmogorov-Smirnov test and the famous chi-square test based on Pearson statistic X^2. Nevertheless the theory showed that the chi-square goodness-of-fit test cannot be applied when the distribution is not specified and the distribution of the statistic test depends on the method used for the esimation of the unknown parameters. So different modifications were proposed in the statistical literature. In this work, we use the technique introduced by [18] and [22] to construct a criteria test Y^2 to validate the extension exponential distribution.

Let us consider the null hypothesis that a sample $T_1, T_2, \ldots, T_n$ follows the extension exponential distribution $P(T_i < t) = F(t, \theta)$ with the unknown parameter vector $\theta = (\alpha, \lambda)^T$.

The statistic Y^2 is defined as the sum of the Pearson statistic X^2 and a quadratic form Q which depends on the hypothezised model. Based on maximum likelihood estimators on initial data, this statistic follows a chi-square distribution with $r-1$ degrees of freedom where r represents the number of grouped classes.

$$Y^2(\hat{\theta}) = X_n^2(\hat{\theta}) + Q_n(\hat{\theta}) \tag{34}$$

The computational formula of $Q(\hat{\theta})$ is given as (see Van der Vaart, 1998)[26]:

$$Q_n(\hat{\theta}) = \frac{1}{n} L^T(\hat{\theta}_n)\big(I(\hat{\theta}_n) - J(\hat{\theta}_n)\big)^{-1} L(\hat{\theta}_n) \tag{35}$$

where $I(\hat{\theta}_n)$ and $J(\hat{\theta}_n)$ are the estimated information matrices respectively for initial data and grouped data into the r classes I_j chosen:

$$I_j = [a_j, a_{j+1}[; \quad \text{for} \quad j = 1, \ldots, r-1 \tag{36}$$

As the grouped intervals must be equiprobable, so the limits a_j are obtained as

$$a_j = F^{-1}(j/r) \tag{37}$$

$$a_j = F^{-1}(j/r) = \frac{(1-(log(1-\frac{j}{r}))^{1/\alpha}-1}{\lambda} \tag{38}$$

If $v_1, v_2, \ldots, v_r$ are the theoretical frequencies of grouped data in these intervals

$$v_j = card(i, T_i \in I_j, i = 1, .., n) \tag{39}$$

So

$$X_n^2(\hat{\theta}) = X_n^T(\hat{\theta}) X_n(\hat{\theta}) \tag{40}$$

with

$$X_n(\hat{\theta}) = \left(\frac{v_1 - np_1(\theta)}{\sqrt{np_1(\theta)}}, \ldots, \frac{v_r - np_r(\theta)}{\sqrt{np_r(\theta)}}\right) \tag{41}$$

and

$$p_j = F(a_j) - F(a_{j-1}) = (1 - exp(1 - (1 + \lambda a_j)^{\alpha})) - (1 - exp(1 - (1 + \lambda a_{j-1})^{\alpha})) \quad \text{with} \quad j = 1,\dots,r \tag{42}$$

The element of the quadratic form Q for the extension exponential distribution are obtained as follow:

$$L = (L_1 = \sum_{j=1}^{r} \frac{v_j}{p_j(\theta)} \frac{dp_j(\theta)}{d\alpha}, \qquad L_2 = \sum_{j=1}^{r} \frac{v_j}{p_j(\theta)} \frac{dp_j(\theta)}{d\lambda}) \tag{43}$$

$$J(\theta) = B^T(\theta)B(\theta) \text{ with } B(\theta) = (\frac{1}{\sqrt{p_j(\theta)}} \frac{dp_j(\theta)}{d\alpha}, \frac{1}{\sqrt{p_j(\theta)}} \frac{dp_j(\theta)}{d\lambda}) \tag{44}$$

The compenents of Fisher information matrix are

$$I_{11} = (\frac{-1}{n}) \sum_{i=1}^{n} -(\frac{1}{\alpha^2} + (1 + \lambda t_i)^{\alpha} (\ln(1 + \lambda t_i))^2) \tag{45}$$

$$I_{22} = (\frac{-1}{n}) \sum_{i=1}^{n} -(\frac{1}{\lambda^2} + \frac{(\alpha - 1)t_i^2}{(1 + \lambda t_i)^2} + (1 + \lambda t_i)^{\alpha - 2}(\alpha) - 1)\alpha t_i^2)) \tag{46}$$

$$I_{12} = (\frac{-1}{n}) \sum_{i=1}^{n} \frac{t_i}{1 + \lambda t_i} + \frac{t(1 + \lambda t_i)^{\alpha}}{(1 + \lambda t_i)} - \alpha t_i \ln(1 + \lambda t_i)(1 + \lambda t_i)^{\alpha - 1} \tag{47}$$

$$I_{21} = I_{12} = (\frac{-1}{n}) \sum_{i=1}^{n} \frac{t_i}{1 + \lambda t_i} + \frac{t(1 + \lambda t_i)^{\alpha}}{(1 + \lambda t_i)} - \alpha t_i \ln(1 + \lambda t_i)(1 + \lambda t_i)^{\alpha - 1} \tag{48}$$

Thus all the elements of the statistic test Y^2 for extension exponential distribution are provided. The applicability of this test is shown by an intensive simulation study.

IX. Simulation study

I. Estimation:

To evaluate the performances of the different methods proposed in this work, we generated $N = 10{,}000$ samples with different sizes ($n = 15,25,40,80$ and $n = 150$) from the extension exponential distribution. We compute the estimators of the unknown parameters, their biases, their mean square errors and their estimated average widths by different methods with the following values of the parameters $\alpha = 0.2, \lambda = 0.1$. The results are summarized in the following tables (Table1,2,3,4,5,6 and 7).

Table 1: *MLE estimators for parameter values* $\alpha = 0.2, \lambda = 0.1$

n	$(\hat{a}, \hat{\lambda})$	$MSE(\hat{\theta})$	$bias(\hat{\theta})$	AW
15	(0.2252 0.1162)	(0.00368 0.01256)	(0.02527 0.01620)	(0.1188 0.2196)
25	(0.2195 0.1148)	(0.00265 0.01129)	(0.01954 0.01482)	(0.1008 0.2082)
40	(0.2105 0.1105)	(0.00115 0.00540)	(0.01054 0.01055)	(0.0664 0.1440)
80	(0.2054 0.1040)	(0.00049 0.00212)	(0.00542 0.00407)	(0.0433 0.0902)
150	(0.2023 0.1028)	(0.00023 0.00101)	(0.00230 0.00283)	(0.0297 0.0622)

Table 2: *MPS estimators for parameter values* $\alpha = 0.2, \lambda = 0.1$

n	$(\hat{\alpha}, \hat{\lambda})$	$MSE(\hat{\theta})$	$bias(\hat{\theta})$	AW
15	(0.1890 0.2097)	(0.00205 0.05728)	(0.01093 0.10977)	(0.0887 0.4687)
25	(0.1902 0.1785)	(0.00164 0.03929)	(0.00976 0.07854)	(0.0793 0.3885)
40	(0.1918 0.1441)	(0.00089 0.01032)	(0.00811 0.04416)	(0.0584 0.1991)
80	(0.1944 0.1217)	(0.00044 0.00301)	(0.00555 0.02177)	(0.0411 0.1075)
150	(0.1963 0.1116)	(0.00022 0.00121)	(0.00360 0.01165)	(0.0290 0.0681)

Table 3: *CvM estimators for parameter values* $\alpha = 0.2, \lambda = 0.1$

n	$(\hat{\alpha}, \hat{\lambda})$	$MSE(\hat{\theta})$	$bias(\hat{\theta})$	AW
15	(0.2425 0.1425)	(0.02460 0.05036)	(0.04252 0.04250)	(0.3074 0.4398)
25	(0.2193 0.1270)	(0.00393 0.01534)	(0.01931 0.02707)	(0.1228 0.2427)
40	(0.2128 0.1142)	(0.00225 0.00663)	(0.01280 0.01424)	(0.0929 0.1595)
80	(0.2054 0.1074)	(0.00077 0.00270)	(0.00549 0.00747)	(0.0543 0.1018)
150	(0.2032 0.1033)	(0.00038 0.00124)	(0.00322 0.00332)	(0.0382 0.0690)

Table 4: *AD estimators for parameter values* $\alpha = 0.2, \lambda = 0.1$

n	$(\hat{\alpha}, \hat{\lambda})$	$MSE(\hat{\theta})$	$bias(\hat{\theta})$	AW
15	(0.2179 0.1631)	(0.00545 0.03055)	(0.01796 0.06314)	(0.1446 0.3425)
25	(0.2090 0.1329)	(0.00228 0.01512)	(0.00908 0.03295)	(0.0935 0.2410)
40	(0.2055 0.1184)	(0.00122 0.00643)	(0.00557 0.01844)	(0.0684 0.1571)
80	(0.2021 0.1093)	(0.00054 0.00235)	(0.00215 0.00930)	(0.0455 0.0950)
150	(0.2012 0.1049)	(0.00028 0.00117)	(0.00127 0.00499)	(0.0327 0.0670)

Table 5: *RAD estimators for parameter values* $\alpha = 0.2, \lambda = 0.1$

n	$(\hat{\alpha}, \hat{\lambda})$	$MSE(\hat{\theta})$	$bias(\hat{\theta})$	AW
15	(0.2237 0.1801)	(0.00921 0.07814)	(0.02378 0.08012)	(0.1880 0.5478)
25	(0.2115 0.1270)	(0.00232 0.01405)	(0.01152 0.02700)	(0.0944 0.2323)
40	(0.2070 0.1197)	(0.00133 0.00739)	(0.00708 0.01972)	(0.0714 0.1684)
80	(0.2032 0.1092)	(0.00055 0.00280)	(0.00325 0.00922)	(0.0459 0.1037)
150	(0.2018 0.1040)	(0.00028 0.00123)	(0.00186 0.00406)	(0.0327 0.0687)

Table 6: *LAD estimators for parameter values* $\alpha = 0.2, \lambda = 0.1$

n	$(\hat{\alpha}, \hat{\lambda})$	$MSE(\hat{\theta})$	$bias(\hat{\theta})$	AW
15	(0.2227 0.1652)	(0.00519 0.06445)	(0.02277 0.06527)	(0.1412 0.4975)
25	(0.2243 0.1373)	(0.00610 0.02161)	(0.02433 0.03737)	(0.1530 0.2881)
40	(0.2120 0.1213)	(0.00271 0.00849)	(0.01201 0.02133)	(0.1020 0.1805)
80	(0.2057 0.1095)	(0.00097 0.00301)	(0.00576 0.00951)	(0.0610 0.1075)
150	(0.2030 0.1046)	(0.00047 0.00135)	(0.00303 0.00465)	(0.0424 0.0720)

Table 7: *KS estimators for parameter values* $\alpha = 0.2, \lambda = 0.1$

n	$(\hat{\alpha}, \hat{\lambda})$	$MSE(\hat{\theta})$	$bias(\hat{\theta})$	AW
15	(0.2417 0.1501)	(0.01127 0.06134)	(0.04173 0.05015)	(0.2080 0.4854)
25	(0.2229 0.1222)	(0.00441 0.01555)	(0.02292 0.02227)	(0.0411 0.2444)
40	(0.2136 0.1127)	(0.00204 0.00677)	(0.01368 0.01274)	(0.0885 0.1612)
80	(0.2065 0.1066)	(0.00084 0.00282)	(0.00652 0.00665)	(0.0568 0.1040)
150	(0.2036 0.1033)	(0.00040 0.00130)	(0.00366 0.00336)	(0.0392 0.0706)

From numerical experiments, It is observed that the biases and the MSE decrease as the sample size increases and this is for all methods. Regarding tables (1,2,3,4,5,6 and 7), we can see that for little sample sizes, the MPS, the AD and the RAD methods provided very good results but the MPS estimators are the best ones compared to all the other methods. For big sample sizes, all methods gave appromixately the same results, however the AD and the RAD estimators are better than all of them and even that of the maximum likelihood MLE method.

II. criteria test

To show the practibility of the statistic test provided in this work, we test the null hypothesis H_0 that samples are drawn from the extension exponential distribution. The number of rejection of the null hypothesis H_0 for thousands of samples from different sizes ($n = 15,25,40,80$ and 150) are computed and compared with those of their theoretical ones ($\epsilon = 0{,}01; 0{,}05$ and $0{,}1$). The results are presented in Table 8.

Table 8: *critical chi-square and their corresponding empirical values*

n	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.1$
15	(0.03879)	(0.01697)	(0.1230)
25	(0.04918)	(0.01513)	(0.1080)
40	(0.05638)	(0.01503)	(0.1068)
80	(0.05204)	(0.01222)	(0.1022)
150	(0.05073)	(0.01124)	(0.1004)

As expected, the obtained empirical values are very close to their corresponding theoretical ones which implies the feasability of the proposed goodness-of-fit test to validate the extension exponential distribution.

X. Applications on real data sets

In this section, we applied the obtained results to two real data sets.

I. The failure times of the air conditionning system data set

The first one concerns the failure times of the air conditionning system of an air plane studied firstly by [12] and [10] who fitted these data respectively to Weibull and exponentiated exponential distributions. Recently [11] showed that the modified Weibull distribution is more appropriated.
The observed values are given below:

23,261,87,7,120,14,62,47,225,71,246,21,42,20,5,12,120,11,3,14,71,11,14,11,16,90,1,16,52,95.

Using the estimation methods given above, we compute the values of the unknown parameters considering different distributions capable to describe these data namely the extension exponential, Weibull, exponentiated exponential and modified Weibull distributions. The results are given in table 9. The cumulative density function of modified Weibull distribution considered here is:

$$F_{MW} = \frac{2F_W}{1+F_W} = \frac{2(1-e^{-(\frac{t}{\lambda})^{\alpha}})}{2-e^{-(\frac{t}{\lambda})^{\alpha}}}$$

Table 9 : *estimator values of the unknown parameters of competitor distributions of air conditionning data system of air planes*

Distributions	parameters	MLE	CVM	KS	AD	LAD	RAD	MPS
Extensionexp	α	0.5985	0.4866	0.6947	0.5233	0.5531	0.5081	0.5057
	λ	4.3385	6.3226	3.8403	5.4853	5.0388	5.8283	5.7787
Weibull	α	0.8535	0.7854	0.9000	0.7994	0.8775	0.7560	0.7805
	λ	54.612	0.5183	0.4496	0.5324	0.4906	0.5265	0.5577
M.Weibull	α	0.9869	0.8989	1.0260	0.9215	0.9909	0.8829	0.9023
	λ	84.097	0.8633	0.7095	0.8664	0.7808	0.8807	0.9050
Exponentiatedexp	α	0.8090	0.6979	0.8451	0.7082	0.8528	0.6243	0.6990
	λ	0.0145	1.3880	1.9274	1.3489	1.8123	1.1922	1.2200

To differentiated between the competitor distributions, and based on the classical model selection criteria, we calculate the AIC, BIC, LL, KS, CvM and AD statistics, the results are given in Table 10. As It can be seen, for the maximum likelihood estimation method, only the values of $AIC,\ BIC$ and LL of the modified Weibull model are the smallest ones. But for the other estimation methods, all criteria selection models AIC, BIC, LL, KS, CvM and AD of the extension exponential distribution give the smaller values. So, It can be said that the extension exponential distribution might work better than all the other models, and this fact is proved by data plots of the different pdf distributions (see Figure 4).

Table 10 : *AIC, BIC, LL and KS statistic values*

	methods	extension exp	Weibull	Exponentiated	Exp M.Weibull
AIC	MLE	(30.8527)	(31.5635)	(32.0910)	(30.6859)
	CvM	(31.3652)	(31.9047)	(32.5510)	(31.1571)
	KS	(31.6769)	(33.0019)	(33.9210)	(34.2721)
	LAD	(30.9216)	(31.9584)	(33.0875)	(30.8079)
	RAD	(31.1693)	(32.2839)	(33.2811)	(31.3592)
	AD	(31.0697)	(31.7777)	(32.4168)	(30.9512)
	MPS	(31.2078)	(32.0362)	(32.595)	(31.186)
BIC	MLE	(33.6551)	(34.3659)	(34.8934)	(33.4883)
	CvM	(34.1676)	(347071)	(35.3534)	(33.9595)
	KS	(34.4793)	(35.8043)	(36.7234)	(34.2721)
	LAD	(33.7240)	(34.7608)	(35.8899)	(33.6103)
	RAD	(33.9717)	(35.0863)	(36.0835)	(34.1616)
	AD	(33.8721)	(34.5801)	(35.2191)	(33.7536)
	MPS	(34.0101)	(34.8385)	(35.3973)	(33.9883)
LL	MLE	(-13.4263)	(-13.7817)	(-14.0455)	(-13.3429)
	CvM	(-13.6826)	(-13.9523)	(-14.2755)	(-13.5785)
	KS	(-13.8384)	(-14.5009)	(-14.9605)	(-13.7348)
	LAD	(-13.4608)	(-13.9792)	(-14.5437)	(-13.4039)
	RAD	(-13.5846)	(-14.1419)	(-14.6405)	(-13.6796)
	AD	(-13.5348)	(-13.8888)	(-14.2084)	(-13.4756)
	MPS	(-13.6039)	(-14.0181)	(-14.2975)	(-13.5930)
KS	MLE	(0.1318)	(0.1533)	(0.1719)	(0.1427)
	CvM	(0.1206)	(0.1229)	(0.1289)	(0.1202)
	KS	(0.1089)	(0.1120)	(0.1135)	(0.1080)
	LAD	(0.1215)	(0.1312)	(0.1339)	(0.1237)
	RAD	(0.1157)	(0.1303)	(0.1367)	(0.1234)
	AD	(0.1203)	(0.1331)	(0.1411)	(0.1267)
	MPS	(0.1198)	(0.1393)	(0.1595)	(0.5943)

For testing the null hypothesis H_0 that these data follow the extension exponential model, we use the criteria test constructed in this work. We grouped data into $r = 8$ intervals,after calculating the limits of these intervals, we obtain:

$$L = (L_1 = -2.5073, L_2 = -0.6273)$$

$$J(\hat{\theta}) = \begin{bmatrix} 2.1928 & 0.1684 \\ 0.1684 & 0.0145 \end{bmatrix}$$

$$I_n(\hat{\theta}) = \begin{bmatrix} 7.5394 & 0.4651 \\ 0.4651 & 0.0330 \end{bmatrix}$$

we deduce the value of $Y^2 = 9.0753$. For level of significance $\epsilon = 0.05$, the chi-square critical value is $X^2_{r-1} = 14.07$. As $Y^2 = 9.0753 \prec X^2_{r-1} = 14.07$, the null hypothesis H_0 cannot be rejected, therefore the failure times of the air conditionning system of an air plane is effectively modeled by the extension exponential distribution. The pdf plots and the empirical distribution function plots given in Figure 4, show that the extension exponential distribution is best suited to model these data.

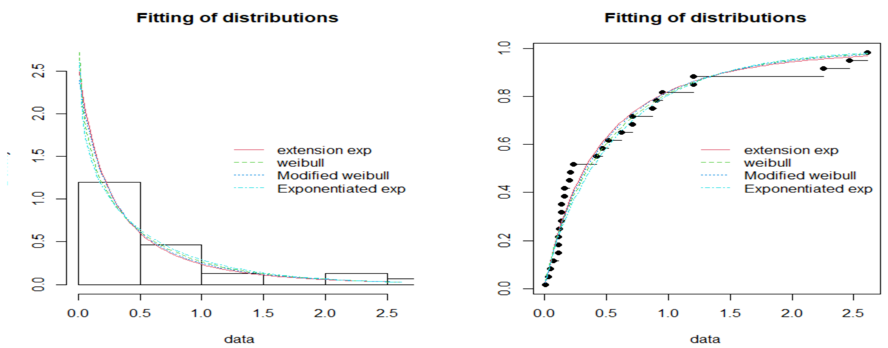

Figure 4: *empirical pdf and cdf versus theorical alternative distributions for the failure times of the air conditionning system of an air plane data*

From these results, we can say that the extension exponential distribution fit these data better than all its alternatives.

II. Floyd River data set

Here we consider a classical flood data of the Floyd River given in Mudholkar and Huston (1996) [16]. In the paper [1], the authors calculated the log-likelihood of these data for different distributions and the obtained values are respectively -376.35 for the exponentiated Weibull distribution, -382.13 for the beta-Weibull distribution and -365.45 for the beta-Pareto model. In this subsection, we propose the extension exponential distribution and its competitors to describe these data. We compute the different estimators of the unkown parameters (Table 11) and the criteria selection models AIC, BIC, LL and KS (Table 12).

Table 11 : *Different estimators of the unkown parameters of competitor distributions for Floyd river data*

Distributions	parameters	MLE	CVM	KS	AD	LAD	RAD	MPS
Extensionexp	α	0.6093	1.2844	0.8757	0.8301	4.0602	0.6793	0.5295
	λ	0.3358	1.1373	0.2053	0.2355	0.0364	0.3358	0.4625
Weibull	α	0.8714	1.1196	1.0977	1.0011	1.2663	0.8886	0.8088
	λ	6.1888	5.0977	5.6478	5.5515	4.8755	5.4194	6.3671
Gamma	α	0.9195	1.2261	1.1642	1.0716	1.4655	0.8615	0.8074
	n	0.1358	0.2498	0.2131	0.1970	0.3194	0.1523	0.1109
Exponentiatedexp	α	0.9688	1.2485	1.1758	1.0996	1.5061	0.8690	0.8357
	λ	0.1445	0.2337	0.2026	0.1963	0.2783	0.1617	0.1213

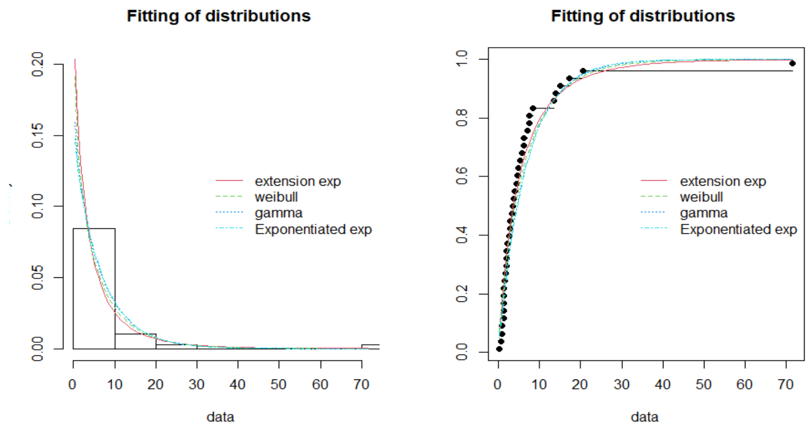

Figure 5: *empirical pdf and cdf versus theorical alternative distributions for a classical flood data of the Floyd River*

Table 12: *AIC, BIC, LL and KS statistic values for Floyd river data*

	methods	Extension exp	Weibull	Gamma	Exponentiated exp
AIC	MLE	(225.9119)	(229.4550)	(231.0047)	(231.167)
	CvM	(246.0783)	(243.3696)	(238.8346)	(237.5375)
	KS	(229.0334)	(237.9247)	(234.7716)	(234.005)
	AD	(228.4938)	(232.8804)	(233.7901)	(233.7134)
	LAD	(571.5404)	(265.8108)	(247.2532)	(243.5087)
	RAD	(226.4296)	(230.1609)	(232.2683)	(232.7168)
	MPS	(226.4262)	(230.0223)	(231.5818)	(231.7741)
BIC	MLE	(229.2391)	(232.7821)	(234.3318)	(234.4941)
	CvM	(249.4054)	(246.6968)	(242.1618)	(240.8647)
	KS	(232.3605)	(241.2518)	(238.0988)	(237.3321)
	AD	(231.8209)	(236.2075)	(237.1172)	(237.0406)
	LAD	(574.8675)	(269.1379)	(250.5803)	(246.8356)
	RAD	(229.7567)	(233.4880)	(235.5954)	(236.0439)
	MPS	(229.7533)	(233.3495)	(234.9089)	(235.1012)
LL	MLE	(-110.956)	(-112.7275)	(-113.5023)	(-113.5835)
	CvM	(-121.0392)	(-119.6848)	(-117.4173)	(-116.7688)
	KS	(-112.5167)	(-116.9624)	(-115.3858)	(-115.0025)
	AD	(-112.2469)	(-114.4402)	(-114.895)	(-114.8567)
	LAD	(-283.7702)	(-130.9054)	(-121.6266)	(-119.7543)
	RAD	(-111.2148)	(-113.0804)	(-114.1341)	(-114.3584)
	MPS	(-111.2131)	(-113.0112)	(-113.7909)	(-113.887)
KS	MLE	(0.1366)	(0.1270)	(0.1471)	(0.1502)
	CvM	(0.1067)	(0.1015)	(0.1003)	(0.0996)
	KS	(0.1034)	(0.0782)	(0.0768)	(0.0764)
	AD	(0.1172)	(0.1059)	(0.0950)	(0.0916)
	LAD	(0.1360)	(0.1264)	(0.1200)	(0.1179)
	RAD	(0.1409)	(0.1425)	(0.1360)	(0.1332)
	MPS	(0.6684)	(0.6715)	(0.7197)	(0.7314)

Compared to the LL values obtained for the different models cited above, we find the value LL=(-110.956) for the extension exponential distribution which proves that this model is better than all the alternative distributions.

NRR statistic On the other hand, we suppose the null hypothesis H_0 that the Floyd river data belong to the extension exponential distribution. If we choose $r = 8$ grouping classes, the intermediate calculation of the criteria test Y^2 are as follow

$$L = (L_1 = 10.2593, L_2 = 5.3251)$$

$$J(\theta)(\hat{\theta}) = \begin{bmatrix} 4.4775 & 4.0355 \\ 4.0355 & 3.9302 \end{bmatrix}$$

$$I_n(\hat{\theta}) = \begin{bmatrix} 6.9144 & 5.1848 \\ 5.1848 & 5.9629 \end{bmatrix}$$

We obtain $Y^2 = 6{,}6724$,, so for the significance level $\epsilon = 0.05$, the critical value $X_{r-1}^2 = 14.07$, therefore we can validate the extension exponential distribution, even the curves of the empirical function distributions and their pdf plots (Figure 5) corroborate the test results.

XI. Conclusion:

As the explicit forms of the maximum likelihood (MLE) estimators cannot be derived, we have used different estimation methods as alternatives to calculate the unknown parameters of the extension exponential distribution. Based on an important simulation study, we have showed that the maximum product spacing (MPS) method gives the best results for little and moderate samples but for big samples the Anderson-Darling method AD and the right tail Anderson-Darling method RAD are better than all the other methods even the MLE. A new statistic which can validate effectively the extension exponential distribution is provided. By using this criteria test, we will not need to compare this model with the possible alternatives in the analysis.

From this study, we conclude that practitionars can use the MPS and the AD estimators respectively for moderate and big samples in the analysis and the obtained statistic test Y^2 permits to fit efficiently observed data to the EE distribution.

As in reliability and survival analyses, data are often censored, we propose in the forthcoming work to construct modified tests to right censored samples to this distribution.

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