ANALYSIS OF $M, MAP/PH/2$ NON- PREEMPTIVE PRIORITY QUEUEING INVENTORY SYSTEM WITH BREAKDOWN, REPAIR AND (s, S) POLICY

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Abstract

We analyze a non-preemptive priority queueing model in this research that has two heterogeneous servers, each of which has its own queue. Queue 1 possesses a low priority status with infinite capacity and queue 2 possesses a high priority status with finite capacity. We made the assumption that arrival follows M, MAP and service time follows Phase-type distribution. Server 1 is always available in the system; Server 2 is an unreliable server. With a (s, S) policy, the inventory is filled up and an exponential distribution is scheduled for the replenishment time. Using the matrix analytical approach, a stationary probability vector was assessed, and a stability criterion was created for our system. Performance metrics are also studied using this technique. Both two and three-dimensional displays are used to show the numerical examples.

Keywords: Marked Markovian arrival process, phase distribution, multiple vacation, non-preemptive priority, breakdown, repair, (s, S) policy.

1. Introduction

The Markovian Arrival Process (MAP) is highly suited for matrix analytical solutions and has one of the most important modeling tools in its basic structure. The formulation of the MAP as a flexible Markovian point process, a specific instance of a Batch Markovian Arrival Process $(BMAP)$, was initially presented by Neuts [16]. In general, a queueing system may have an input process with many arrival kinds. We suggest marking all the intriguing arrivals and looking at the lines that the marked arrivals experience in order to study such queueing systems. To excite the marked point process $(MMAP)$, we use the Markov arrival process with marked transitions. To obtain a comprehensive and thorough analysis of $MMAP$, readers are directed to Ayyappan and Archana [2], Kim et al. [7], Raj and Jain [17].

The attached inventory has an impact on the service, which makes the queueing-inventory system distinct from the standard queueing system. Service will be disrupted if inventory is not available. Furthermore, the inventory is used at the serving rate rather than the customer arrival rate even while there are customers waiting for service, which is another way that it differs from standard inventory management.

Jeganathan et al [12], examined a Markovian (s, Q) inventory queueing system with finite buffer queueing model with two heterogeneous servers and mixed priority service. Performance analysis of queueing inventory systems and networked queueing systems in stochastic modeling. With dual service stations, it offers two different kinds of service facilities, Jeganathan et al [13], handled commodity sales and non-commodity services with a single server and a multi-server, respectively. The queueing inventory system was researched by Rasmi et al. [18], using items with PH-distributed common life times and clients of different classes arriving based on $MMAP$. Customers are given priority access to inventories based on their class number and the items' freshness. Positive lead time and the (s, S) policy are used to control the inventory. A queueing inventory system has been extensively examined by Indhumathi and Karthikeyan [11] because of its many uses in the contemporary world and capacity to improve accuracy in this area. Adequate knowledge regarding queueing inventory systems is also provided.

Priority services in queuing systems discriminate against units based on their class; this kind of discrimination happens in everyday setting in engineering systems, particularly in communication networks channel access protocols and computer operating systems. Preemptive and non-preemptive priority queues are the two varieties. In preemptive priority, the arrival of priority units interrupts the operation of an ordinary unit. However, under non-preemptive priority, even if the priority unit enters the system, the service of an ordinary unit continues uninterrupted. According to Krishnamoorthy et al [14], a priority-generated client can only wait once, and a priority-generated customer must exit the system for emergency assistance elsewhere. The authors of the study, Haghighi [1] examined a new model of a conveyor-like priority queueing system. In other words, priority is intended to reduce system idleness. They also examined a general parallel finite buffer multi-server priority queueing system with reneging and balking, as well as a two-station scenario. Ayyappan and Meena [3], studied a preemptive priority queueing model for single vacation, repair, and impatient consumers. They found that high priority customers lose priority and move to low priority queues for feedback.

Halfin and Segal [9], developed a queueing model for a system that serves two types of clients. Principal clients are randomly assigned to servers with negative exponential service, while secondary customers are served by servers not occupied by primary clients. Servers join a queue if all of them are busy, with primary customers always getting service first. Srinivas [19], assumed a single server non-preemptive priority queueing approach. Since there are very few servers that are completely dependable, many authors have already examined queuing models with unreliable servers. Research on the aforementioned topic is known as queuing systems with server failures. The unreliable server queueing model was introduced by White and Lee [10]. Every consumer using the system makes this choice independently of the others, as was examined by Dudin et al. [6].

2. Mathematical Model

Examine a non-preemptive priority queueing model with two different arrival kinds for two heterogeneous servers. They are both low priority (LPC) and high priority (HPC) clients and the Marked Markovian Arrival Process $(MMAP)$ is used to track their arrival. For no arrival in the system, the square matrix D_0 is used; for an LPC arrival, the square matrix D_1 is used; and for an HPC arrival, the square matrix D_2 is used. The LPC has an infinite capacity, while the HPC is limited to a finite capacity of size N. The normal arrival rate is $\lambda_i = \pi D_i e_m$, i=1, 2 for HPC and LPC, respectively.

Both servers collectively adhere to a shared service rate characterized by a PH-distribution (α, W) of order m. The vector W^0 is explicitly defined as $W^0 = -We$. Server 2 is susceptible to breakdowns, with an exponential distribution characterized by the parameter ψ. To ensure continuity of service during a server 2 breakdown, server 1 temporarily takes over the treatment of the current customer from queue 2 who was being served by server 2, following a non-preemptive priority approach. The remaining customers in queue 2 remain in the queue, waiting until server 2 becomes available again. As fast as a breakdown is detected, server 2 enters a repair process immediately. This repair process is also subject to an exponential distribution with a parameter denoted as τ. Once the repair is successfully completed, server 2 resumes its regular work, ready to serve the incoming customers once again.

In cases, when there are either no HPC in the system or atmost one inventory in the system and both exists, server 2 takes multiple vacation and the occurrence of vacation is denoted as η. But, when there are either no customers in the system or atmost one inventory in the system and both exists, server 1 remains idle. When both servers are busy implies there exists atleast two inventory in the system. The ordering policy of the inventory is (s, S), where S is the ordering quantity and s is the reorder point. It is assumed that the order's lead time follows an exponential distribution with a rate of β. The schematic picture of this model is provided in Figure 1.

$\otimes$ - Kronecker two matrices of different sizes are multiplied to create a block matrix. $\oplus$ is the Kronecker result of two matrices of various sizes, resulting in the block matrix. e denotes an appropriate order column vector with each of its entries is one. O - It indicates zero matrices in the appropriate sequence.

3. The Generator Matrix

Let $I(t) \in \{0, 1, 2, \ldots, S\}$, $Y(t) \in \{0, 1, 2, 3, 4, 5\}$, $N_1(t) \in \{0, 1, 2, \ldots, \infty\}$ and $N_2(t) \in \{0, 1, 2, \ldots, N\}$ suggest, respectively, the inventory level, server status, the number of LPC in the queue and the number of HPC queue. The system's server status can be described as follows:

$$Y(T) = \begin{cases} 0, & \text{if the server 1 is idle and server 2 is on vacation,} \\ 1, & \text{if the server 1 is idle and server 2 is busy with HPC,} \\ 2, & \text{if the server 1 is idle and server 2 is on repair,} \\ 3, & \text{if the server 1 is busy and server 2 is on vacation,} \\ 4, & \text{if the server 1 is busy with LPC and server 2 is busy with HPC,} \\ 5, & \text{if the server 1 is busy and server 2 is on repair.} \end{cases}$$

It is obvious that {$N_1(t)$, $N_2(t)$, Y(t), I(t), S(t), L(t): $t \geq 0$}is a CTMC with the State Space.

$$\omega = \ell(0) \bigcup_{i=1}^{\infty} \ell(i),$$

Figure 1: *Schematic representation*

$$\begin{aligned} l(0) = &\{(0, j, 0, k, l_1) : 0 \leq j \leq N, 0 \leq k \leq S, 1 \leq l_1 \leq n\} \\ &\cup \{(0, j, 1, k, l_0, l_1) : 1 \leq j \leq N, 1 \leq k \leq S, 1 \leq l_0 \leq m, 1 \leq l_1 \leq n\} \\ &\cup \{(0, j, 2, k, l_1) : 0 \leq j \leq N, 0 \leq k \leq S, 1 \leq l_1 \leq n\}, \end{aligned}$$

and for $i \geq 1$,

$$\begin{aligned} l(i) = &\{(0, j, 0, k, l_1) : 0 \leq j \leq N, k = 0, 1 \leq l_1 \leq n,\} \\ &\cup \{(0, j, 1, k, l_0, l_1) : 1 \leq j \leq N, k = 1, 1 \leq l_0 \leq m, 1 \leq l_1 \leq n,\} \\ &\cup \{(0, j, 2, k, l_1) : 0 \leq j \leq N, k = 0, 1 \leq l_1 \leq n\} \\ &\cup \{(0, j, 3, k, l_0, l_1) : 1 \leq j \leq N, 1 \leq k \leq S, 1 \leq l_0 \leq m, 1 \leq l_1 \leq n,\} \\ &\cup \{(0, j, 4, k, l_0, l_1) : 1 \leq j \leq N, 2 \leq k \leq S, 1 \leq l_0 \leq m, 1 \leq l_1 \leq n\} \\ &\cup \{(0, j, 5, k, l_0, l_1) : 1 \leq j \leq N, 1 \leq k \leq S, 1 \leq l_0 \leq m, 1 \leq l_1 \leq n\}. \end{aligned}$$

4. The infinitesimal generation matrix of the QBD procedure

Our methodology for generating QBD(Quasi Birth-Death) processes is explained in this section using the following notations and presumptions.

$$Q = \begin{bmatrix} B_{00} & B_{01} & & & & & \\ B_{10} & A_1 & A_0 & & & & \\ & A_2 & A_1 & A_0 & & & \\ & & A_2 & A_1 & A_0 & & \\ & & & A_2 & A_1 & A_0 & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The entries in the block matrices of Q are defined as follows:
$B_{00{(i_2,j_2,k_2,l_2) \atop (i_1,j_1,k_1,l_1)}}$ as transition sub matrices that contain transitions of the form $(0, i_1, j_1, k_1, l_1) \to (0, i_2, j_2, k_2, l_2)$ and $B_{01{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}}$ as transition sub matrices which contain transitions of the form $(0, i_1, j_1, k_1, p_1, l_1) \to (1, i_2, j_2, k_2, p_2, l_2)$. Let $(i_1, j_1, k_1, p_1, l_1)$ denotes the number of LPC, server status, number of items, service phases and arrival phases in the row vector and $(i_2, j_2, k_2, p_2, l_2)$ denotes the number of HPC, server status, number of items, service phases and arrival phases in the column vector.

$$B_{00} = \begin{bmatrix} B_{0011} & B_{0012} & 0 \\ B_{0021} & B_{0022} & B_{0023} \\ B_{0031} & 0 & B_{0033} \end{bmatrix}, \quad B_{01} = \begin{bmatrix} B_{0111} & 0 & 0 & B_{0114} & 0 & 0 \\ 0 & B_{0122} & 0 & 0 & B_{0125} & B_{0126} \\ 0 & 0 & B_{0133} & 0 & 0 & B_{0136} \end{bmatrix},$$

$$B_{0012{(i_2,j_2,k_2,l_2) \atop (i_1,j_1,k_1,l_1)}} = \begin{cases} I_S \otimes \alpha \otimes \eta I_n, & i_1 = 0, 1 \le i_2 \le N, j_1 = j_2 = 0, \\ & 1 \le k_1, k_2 \le S, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0021{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} I_S \otimes W^0 \otimes I_n, & i_1 = 0, i_2 = 1, j_1 = 1, j_2 = 0, 1 \le k_1, k_2 \le S, \\ & 1 \le p_1, p_2 \le m, 1 \le l_1, l_2 \le n, \\ W^0 \otimes I_n, & i_1 = 0, 2 \le i_2 \le N, j_1 = 1, j_2 = 0, k_1, k_2 = 1, \\ & 1 \le p_1, p_2 \le m, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0023{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} I_S \otimes e_m \otimes \psi I_n, & \text{if } i_1 = i_2 = 0,\ j_1 = 1,\ j_2 = 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0031{(i_2,j_2,k_2,l_2) \atop (i_1,j_1,k_1,l_1)}} = \begin{cases} I_{S+1} \otimes \tau I_n, & \text{if } i_1 = 0,\ 0 \le i_2 \le N,\ j_1 = 2,\ j_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0011}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} I_{s+1} \otimes D_0 - \beta I_n, & \\ e_{s+1} \otimes \beta I_n, D_0, & i_1 = i_2 = 0, j_1 = j_2 = 0, \\ & 1 \le l_1, l_2 \le n, \\ I_{S+1} \otimes I_n \otimes D_2, & i_1 = 1, 1 \le i_2 \le N, j_1 = j_2 = 0, \\ & 0 \le k_1, k_2 \le S, 1 \le l_1, l_2 \le n, \\ D_0 - \beta I_n, & i_1 = 0, i_2 = 1, j_1 = j_2 = 0, k_1, k_2 = 0, \\ & 1 \le l_1, l_2 \le n, \\ I_s \otimes D_0 - (\eta + \beta) I_n, & \\ e_s \otimes \beta I_n, & i_1 = 0, 1 \le i_2 \le N, j_1 = j_2 = 0, \\ & 1 \le k_1, k_2 \le S, 1 \le l_1, l_2 \le n, \\ I_{S-s} \otimes D_0 - \eta I_n, & i_1 = 0, 1 \le i_2 \le N, j_1 = j_2 = 0, \\ & s+1 \le k_1, k_2 \le S, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0022}{}^{(i_2,j_2,k_2,p_2,l_2)}_{(i_1,j_1,k_1,p_1,l_1)} = \begin{cases} I_{s+1} \otimes W \oplus D_0 - (\psi + \beta) I_{mn}, e_{s+1} \otimes \beta I_{mn}, & \\ I_{S-s} \otimes W \oplus D_0 - \psi I_{mn}, & i_1 = 0, 1 \le i_2 \le N, j_1 = j_2 = 1, \\ & 1 \le k_1, k_2 \le S, 1 \le p_1, p_2 \le m, 1 \le l_1, l_2 \le n, \\ I_S \otimes I_m \otimes D_2, & i_1 = 0, 1 \le i_2 \le N, j_1 = 1, j_2 = 1, \\ & 1 \le k_1, k_2 \le S, 1 \le p_1, p_2 \le m, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0033}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} I_{s+1} \otimes D_0 - (\tau + \beta) I_n, & \\ e_{s+1} \otimes \beta I_n, & \\ I_{S-s} \otimes D_0 - \tau I_n, & i_1 = 0, 0 \le i_2 \le N, j_1 = j_2 = 2, \\ & 0 \le k_1, k_2 \le S, 1 \le l_1, l_2 \le n, \\ I_{S+1} \otimes D_2, & i_1 = 0, i_2 = 1, j_1 = j_2 = 2, \\ & 1 \le k_1, k_2 \le S, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0111}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} D_1, & i_1 = i_1 + 1, 0 \le i_2 \le N, j_1 = j_2 = 0, \\ & k_1, k_2 = 0, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0114}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} I_S \otimes \alpha \otimes D_1, & i_1 = i_1 + 1, 0 \le i_2 \le N, j_1 = 0, j_2 = 3, \\ & 1 \le k_1, k_2 \le S, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0122}{}^{(i_2,j_2,k_2,p_2,l_2)}_{(i_1,j_1,k_1,p_1,l_1)} = \begin{cases} I_m \otimes D_1, & i_1 = i_1 + 1, 1 \le i_2 \le N, j_1 = j_2 = 1, \\ & k_1, k_2 = 1, 1 \le p_1, p_2 \le m, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0125}{}^{(i_2,j_2,k_2,p_2,l_2)}_{(i_1,j_1,k_1,p_1,l_1)} = \begin{cases} I_{S-1} \otimes I_m \otimes D_1, & i_1 = i_1 + 1, 1 \le i_2 \le N, j_1 = 1, j_2 = 4, \\ & 2 \le k_1, k_2 \le S, 1 \le p_1, p_2 \le m, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0126}{}^{(i_2,j_2,k_2,p_2,l_2)}_{(i_1,j_1,k_1,p_1,l_1)} = \begin{cases} I_S \otimes \psi \otimes I_{mn}, & i_1 = i_1 + 1, 1 \le i_2 \le N, j_1 = 1, j_2 = 5, \\ & 2 \le k_1, k_2 \le S, 1 \le p_1, p_2 \le m, 1 \le l_1, l_2 \le n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0133_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}} = \begin{cases} D_1, & i_1 = i_1+1, 0 \leq i_2 \leq N, j_1 = j_2 = 2, \\ & k_1,k_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{0136_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_S \otimes \alpha \otimes D_1, & i_1 = i_1+1, 0 \leq i_2 \leq N, j_1 = 2, j_2 = 5, \\ & 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$B_{10_{(i_1,j_1,k_1,p_1 l_1)}^{(i_2,j_2,k_2,p_2,l_2)}}$'s entries can be defined as transition sub matrices that contain transitions of the form $(1, i_1, j_1, k_1, p_1, l_1) \rightarrow (0, i_2, j_2, k_2, p_2, l_2)$ and $A_{1_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)}}$ serve as transition sub matrices that make up transitions with the following structure:
$(h, i_1, j_1, k_1, p_1, l_1) \rightarrow (h, i_2, j_2, k_2, p_2, l_2)$, where $h \geq 1$. In the row vector, let $(i_1, j_1, k_1, p_1, l_1)$ represent the number of LPC, server status, number of items, service phases, and arrival phases; in the column vector, let $(i_2, j_2, k_2, p_2, l_2)$ represent the number of HPC, server status, number of items, service phases, and arrival phases.

$$B_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{1041} & 0 & 0 \\ 0 & B_{1052} & 0 \\ 0 & 0 & B_{1063} \end{bmatrix}, \quad A_1 = \begin{bmatrix} A_{111} & 0 & 0 & A_{114} & 0 & 0 \\ A_{121} & A_{122} & 0 & 0 & A_{125} & A_{126} \\ A_{131} & 0 & A_{133} & 0 & 0 & A_{136} \\ 0 & 0 & 0 & A_{144} & A_{145} & 0 \\ 0 & 0 & 0 & A_{154} & A_{155} & A_{156} \\ 0 & 0 & 0 & A_{164} & A_{165} & A_{166} \end{bmatrix},$$

$$B_{1041_{(i_1,j_1,k_1,p_1 l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_S \otimes W^0 \otimes I_n, & i_1 = i_1-1, 0 \leq i_2 \leq N, j_1 = 3, j_2 = 0, 1 \leq k_1 \leq S, \\ & 0 \leq k_2 \leq S, 1 \leq p_1 \leq m, p_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{1052_{(i_1,j_1,k_1,p_1 l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_{S-1} \otimes W^0 \otimes \alpha \otimes I_n, & i_1 = i_1-1, 0 \leq i_2 \leq N, j_1 = 4, j_2 = 1, \\ & 2 \leq k_1 \leq S, 1 \leq k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{1063_{(i_1,j_1,k_1,p_1 l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_S \otimes W^0 \otimes I_n, & i_1 = i_1-1, 0 \leq i_2 \leq N, j_1 = 5, j_2 = 2, 1 \leq k_1 \leq S, \\ & 0 \leq k_2 \leq S, 1 \leq p_1 \leq m, p_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{111_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}} = \begin{cases} D_0 - \beta I_n, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = j_2 = 0, \\ & k_1, k_2 = 0, l_1, l_2 = n, \\ D_2, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = j_2 = 0, \\ & k_1, k_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{114_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}} = \begin{cases} \alpha \otimes \beta I_n, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = j_2 = 0, \\ & k_1, k_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{121_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} W_0 \otimes I_n, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = 1, j_2 = 0, \\ & k_1, k_2 = 1, 1 \leq l_1, l_2 \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{125{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} \beta I_{nm}, & i_1 = i_2 = 1, j_1 = 1, j_2 = 4, \\ & k_1, k_2 = 1, 1 \leq l_1, l_2 \leq n, \\ 0, & otherwise. \end{cases}$$

$$A_{122{(i_2,j_2,k_2,l_2) \atop (i_1,j_1,k_1,l_1)}} = \begin{cases} W \oplus D_0 - (\psi + \beta) I_{nm}, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = j_2 = 1, \\ & k_1, k_2 = 1, 1 \leq l_1, l_2 \leq n, \\ I_m \otimes D_2, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = j_2 = 1, \\ & k_1, k_2 = 1, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{126{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} I_m \otimes \psi I_n, & i_1 = i_2 = 1, j_1 = 1, j_2 = 5, \\ & k_1 = k_2 = 1, 1 \leq l_1 \leq m, 1 \leq l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{131{(i_2,j_2,k_2,l_2) \atop (i_1,j_1,k_1,l_1)}} = \begin{cases} \tau I_n, & i_1 = 1, 1 \leq i_2 \leq n, j_1 = 2, j_2 = 0, \\ & k_1, k_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{133{(i_2,j_2,k_2,l_2) \atop (i_1,j_1,k_1,l_1)}} = \begin{cases} D_0 - (\tau + \beta) I_n, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = j_2 = 2, \\ & k_1, k_2 = 0, 1 \leq l_1, l_2 \leq n, \\ D_2, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = j_2 = 2, \\ & k_1, k_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{136{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} \alpha \otimes \beta I_n, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = 2, j_2 = 4, k_1, k_2 = 0, \\ & 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{145{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} I_{S-1} \otimes \eta I_{mn}, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = j_2 = 3, \\ & 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{154{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} I_{S-1} \otimes W^0 \alpha \otimes I_n, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = 1, j_2 = 0, \\ & 2 \leq k_1, k_2 \leq S, 1 \leq l_1, l_2 \leq n, \\ W^0 \alpha \otimes I_n, & i_1 = 1, 2 \leq i_2 \leq N, j_1 = 4, j_2 = 3, \\ & k_1, k_2 = 2, 1 \leq p_1 \leq m, p_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{156{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} I_{S-1} \otimes \psi I_{mn}, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = 4, j_2 = 5, \\ & 2 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & otherwise. \end{cases}$$

$$A_{164{(i_2,j_2,k_2,p_2,l_2) \atop (i_1,j_1,k_1,p_1,l_1)}} = \begin{cases} I_S \otimes \tau I_{mn}, & i_1 = 1, i_2 = 0, j_1 = 5, j_2 = 3, \\ & 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ \tau I_{mn}, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = 5, j_2 = 3, \\ & k_1, k_2 = 1, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{144_{(i_1,j_1,k_1,p_1 l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_s \otimes W \oplus D_0 - \beta I_{mn}, e_s \otimes \beta I_{mn}, \\ I_{S-s} \otimes W \oplus D_0, & i_1 = 1, i_2 = 0, j_1 = j_2 = 3, \\ \quad 1 \leq k_1, k_2 \leq s, 1 \leq l_1, l_2 \leq n, 1 \leq p_1, p_2 \leq m, \\ I_{S-s} \otimes W \oplus D_0, & i_1 = 1, i_2 = 0, j_1 = j_2 = 3, \\ \quad s+1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ I_S \otimes D_2 \otimes I_m, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = j_2 = 3, \\ \quad 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ I_s \otimes W + D_0 - (\eta + \beta) I_{mn}, \\ e_s \otimes \beta I_{mn}, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = j_2 = 3, \\ \quad 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ I_{S-s} \otimes W + D_0 - \eta I_{mn}, \\ & i_1 = 1, 1 \leq i_2 \leq N, j_1 = j_2 = 3, \\ \quad 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1 l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{155_{(i_1,j_1,k_1,p_2,l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_{s-1} \otimes W^0 \oplus D_0 - \psi + \beta I_{mn}, e_{s-1} \otimes \beta I_{mn}, \\ I_{((S-1)-(s-1))} \otimes W \oplus D0 - \psi I_{mn}, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = j_2 = 4, \\ \quad 2 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ I_{S-1} \otimes I_m \otimes D_2, & i_1 = 1, 2 \leq i_2 \leq N, j_1 = j_2 = 4, \\ \quad 2 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ I_{S-2} \otimes W^0 \alpha \otimes I_n, & i_1 = 1, 2 \leq i_2 \leq N, j_1 = j_2 = 4, \\ \quad 3 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{165_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_{S-1} \otimes \tau I_{mn}, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = 5, j_2 = 4, \\ \quad 1 \leq k_1 \leq S, 2 \leq k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{166_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)}} = \begin{cases} I_s \otimes W^0 \oplus D_0 - \tau + \beta I_{mn}, e_s \otimes \beta I_{mn}, \\ I_{(S-s)} \otimes W \oplus D0 - \tau I_{nm}, & i_1 = 1, 0 \leq i_2 \leq N, j_1 = j_2 = 5, \\ \quad 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ I_S \otimes I_m \otimes D_2, & i_1 = 1, 1 \leq i_2 \leq N, j_1 = j_2 = 5, \\ \quad 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$A_{2_{(i_1,j_1,k_1,p_1 l_1)}^{(i_2,j_2,k_2,p_2,l_2)}}$'s entries should be defined as transition submatrices that contain transitions of the form $(h, i_1, j_1, k_1, p_1, l_1) \rightarrow (h-1, i_2, j_2, k_2, p_2, l_2)$, where $h > 1$ and $A_{0_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)}}$ function as transition submatrices that comprise transitions of the form $(h, i_1, j_1, k_1, p_1, l_1)$ $\rightarrow (h+1, i_2, j_2, k_2, p_2, l_2)$, where $h \geq 1$. In the row vector, let $(i_1, j_1, k_1, p_1, l_1)$ represent the number of LPC, server status, number of items, service phases, and arrival phases; in the column vector, let $(i_2, j_2, k_2, p_2, l_2)$ represent the number of HPC, server status, number of items, service phases, and arrival phases.

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{241} & 0 & 0 & A_{244} & 0 & 0 \\ 0 & A_{252} & 0 & 0 & A_{255} & 0 \\ 0 & 0 & A_{263} & 0 & 0 & A_{266} \end{bmatrix}, A_0 = \begin{bmatrix} A_{011} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{022} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{033} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{044} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{055} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{066} \end{bmatrix},$$

$$A_{241}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} W^0 \otimes I_n, & i_1 = i_1 - 1, i_2 = 1, j_1 = 3, j_2 = 0, 1 \leq k_1, \leq S, k_2 = 0 \\ & 1 \leq p_1 \leq m, p_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{244}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} I_{S-1} \otimes W^0\alpha \otimes I_n, & i_1 = i_1 - 1, i_2 = 1, j_1 = j_2 = 3, 1 \leq k_1, k_2 \leq S, \\ & 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{252}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} W^0\alpha \otimes I_n, & i_1 = i_1 - 1, i_2 = 1, j_1 = 4, j_2 = 1, 2 \leq k_1 \leq S, \\ & 1 \leq k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{255}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} W^0\alpha \otimes I_n, & i_1 = i_1 - 1, i_2 = 1, j_1 = j_2 = 4, 2 \leq k_1, k_2 \leq S, \\ & 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{263}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} W^0 \otimes I_n, & i_1 = i_1 - 1, i_2 = 1, j_1 = 5, j_2 = 0, 1 \leq k_1 \leq S, \\ & k_2 = 0, 1 \leq p_1 \leq m, p_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & otherwise. \end{cases}$$

$$A_{266}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} I_{S-1} \otimes W^0\alpha \otimes I_n, & i_1 = i_1 - 1, i_2 = 1, j_1 = j_2 = 5, \\ & 1 \leq k_1, k_2 \leq S, 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & otherwise. \end{cases}$$

$$A_{011}{}_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)} = \begin{cases} D_1, & i_1 = i_1 + 1, i_2 = 2, j_1 = j_2 = 0, 1 \leq k_1 \leq S, k_2 = 0, \\ & 1 \leq p_1 \leq m, p_2 = 0, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{022}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} I_m \otimes D_1, & i_1 = i_1 + 1, i_2 = 2, j_1 = j_2 = 1, k_1, k_2 = 1, \\ & 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{033}{}_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)} = \begin{cases} D_1, & i_1 = i_1 + 1, i_2 = 2, j_1 = j_2 = 1, \\ & k_1, k_2 = 1, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{044}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} I_S \otimes I_m \otimes D_1, & i_1 = i_1 + 1, i_2 = 2, j_1 = j_2 = 3, 1 \leq k_1, k_2 \leq S \\ & 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{055}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} I_{S-1} \otimes I_m \otimes D_1, & i_1 = i_1 + 1, i_2 = 2, j_1 = j_2 = 4, 2 \leq k_1, k_2 \leq S \\ & 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{066}{}_{(i_1,j_1,k_1,p_1,l_1)}^{(i_2,j_2,k_2,p_2,l_2)} = \begin{cases} I_S \otimes I_m \otimes D_1, & i_1 = i_1 + 1, i_2 = 2, j_1 = j_2 = 5, 1 \leq k_1, k_2 \leq S \\ & 1 \leq p_1, p_2 \leq m, 1 \leq l_1, l_2 \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

5. Steady State Analysis

5.1. Stability Condition

A generator matrix is indicated by the equation $A = A_0 + A_1 + A_2$. In the steady state, let π be the probability vector of A that satisfies $\pi e = 1$ and $\pi A = 0$. The vector π is divided by

$$\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = \big(\pi_{000}, \pi_{001}, \ldots, \pi_{00S}, \pi_{010}, \pi_{011}, \ldots, \pi_{01S}, \pi_{0N0}, \pi_{0N1}, \ldots, \pi_{0NS}, \pi_{111}, \pi_{121}, \ldots, \pi_{1N1}, \pi_{200}, \pi_{210}, \ldots, \pi_{2N0}, \pi_{301}, \pi_{302}, \ldots, \pi_{30S}, \pi_{311}, \pi_{312}, \ldots, \pi_{31S}, \pi_{3N1}, \pi_{3N2}, \ldots, \pi_{3NS}, \ldots, \pi_{412}, \ldots, \pi_{42S}, \pi_{422}, \ldots, \pi_{42S}\big)$$

Where π_0 has a dimension of (N+1)(S+1)n, π_1 has a dimension of nmN, π_2 has a dimension of (N+1)n, π_3 has a dimension of nm(N+1)S, π_4 has a dimension of N(S-1)mn and π_5 has a dimension of (N+1)Smn. subject to normalizing condition

$$\pi_0 e_n + \pi_1 e_{mn} + \pi_2 e_n + \pi_3 e_{mn} + \pi_4 e_{mn} + \pi_5 e_{mn} = 1.$$

Within the framework of QBD, our model's stability should meet the necessary and sufficient conditions of $\pi A_0 e < \pi A_2 e$,
Which is found to be of the below form,

$$\sum_{j_1=0}^{N} \pi_{0j_1}[D_1 e_n] + \sum_{j_1=1}^{N} \pi_{1j_1}[I_m \otimes D_1 e_{mn}] + \sum_{j_1=0}^{N} \pi_{2j_1}[D_1 e_n]$$
$$+ \sum_{j_1=0}^{N}\sum_{j_2=1}^{S} \pi_{2j_1j_2}[I_m \otimes D_1 e_{mn}] + \sum_{j_1=1}^{N}\sum_{j_2=2}^{S} \pi_{3j_1j_2}[I_m \otimes D_1 e_{mn}] + \sum_{j_1=0}^{N}\sum_{j_2=1}^{S} \pi_{4j_1j_2}[I_m \otimes D_1 e_{mn}]$$
$$< \sum_{j_1=0}^{N}\sum_{j_2=2}^{S} \pi_{3j_1j_2}[W^0 \otimes I_n e_{mn}(1+\alpha)]$$
$$+ \sum_{j_1=1}^{N}\sum_{j_2=2}^{S} \pi_{4j_1j_2}[W^0\alpha \otimes I_n e_{mn}] + \sum_{j_1=0}^{N}\sum_{j_2=2}^{S} \pi_{5j_1j_2}[W^0 \otimes I_n e_{mn}(1+\alpha)].$$

5.2. The Transition Probability Vector

Examine the following partitioning of the invariant probability vector of Q, represented by the symbol z.: $z = (z_0, z_1, z_2, \ldots)$. z_0 is $2(N+1)(S+1)n + NSmn$ in dimension. Whereas the dimensions of $z_1, z_2, \ldots$ are $2(N+1)n + Nmn + 2(N+1)Smn + N(S-1)mn$. Then, z satisfies the conditions $zQ = 0$ and $ze = 1$. The rate matrix R can be evaluated using the logarithmic reduction procedure. The steady-state probability vector z might also be found using the following equation.

$$z_i = z_1 R^{i-1}, i \geq 2,$$

where R represents the quadratic equation's non-negative solution.

$$R^2 A_2 + R A_1 + A_0 = 0,$$

as well as satisfying the relation $RA_2 e = A_0 e$. The vectors z_0 and z_1 are derived using the following equations.

$$z_0 B_{00} + z_1 B_{10} = 0,$$
$$z_0 B_{01} + z_1 [A_1 + RA_2] = 0,$$

based on the Normalizing condition,

$$z_0 e_{2(N+1)(S+1)n+NSmn} + z_1 [1-R]^{-1} e_{2(N+1)n+Nmn+2(N+1)Smn+N(S-1)mn} = 1.$$

[15] state that by employing crucial steps in the logarithmic reduction process, the R matrix can be generated analytically.

6. Measures of performance

In this section, we present some key performance measures to highlight the qualitative aspects of the model under study. The measures and their calculation formulas are listed below.

- Probability that (server 1) S1 is idle while (Server 2) S2 is on vacation:
  $P_{S1IS2V} = \sum_{u_2=0}^{N} \sum_{u_3=0}^{S} \sum_{u_5=1}^{n} X_{0u_2u_3u_50}$.
- Probability that S1 is idle while S2 is busy:
  $P_{S1IS2B} = \sum_{u_2=1}^{N} \sum_{u_3=1}^{S} \sum_{u_4=1}^{m} \sum_{u_5=1}^{n} X_{0u_2u_3u_4u_51}$.
- Probability that S1 is idle while S2 is under repair process:
  $P_{S1IS2R} = \sum_{u_2=0}^{N} \sum_{u_3=0}^{S} \sum_{u_5=1}^{n} X_{0u_2u_3u_52}$.
- Probability that S1 is busy while S2 is on vacation:
  $P_{S1BS2V} = \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{u_3=1}^{S} \sum_{u_4=1}^{m} \sum_{u_5=1}^{n} X_{u_1u_2u_3u_4u_53}$.
- Probability that both S1 and S2 is busy:
  $P_{S1BS2B} = \sum_{u_1=1}^{\infty} \sum_{u_2=1}^{N} \sum_{u_3=1}^{S} \sum_{u_4=1}^{m} \sum_{u_5=1}^{n} X_{u_1u_2u_3u_4u_54}$.
- Probability that S1 is busy while S2 is under repair process:
  $P_{S1BS2R} = \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{u_3=1}^{S} \sum_{u_4=1}^{m} \sum_{u_5=1}^{n} X_{u_1u_2u_3u_4u_55}$.

7. Numerical Implementation

In this section, we examine the output of our system using both graphical and numerical representations. The variance and correlation patterns given below identify the five distinct MAP representations, each with a mean value of 1. Since ERA, EXA, and HYA represen the renewal process, there is no correlation between the first three arrival processes.

- **Erlang arrival of order 2 (ERA):**

$$D_0 = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- **Exponential (EXA) Arrival:**

$$D_0 = [-1], \quad D_1 = [0.6], \quad D_2 = [0.4].$$

- **Arrival in Hyper exponential (HYA):**

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.026 & 0.114 \\ 0.1026 & 0.0114 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.684 & 0.076 \\ 0.0684 & 0.0076 \end{bmatrix}.$$

- **MAP-Negative Correlation (MNCA) arrival:**

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.006012 & 0 & 0.595446 \\ 134.1234 & 0 & 1.3548 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.004008 & 0 & 0.396964 \\ 89.4156 & 0 & 0.9032 \end{bmatrix}.$$

- **MAP-Positive Correlation (MPCA) arrival:**

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.595446 & 0 & 0.006012 \\ 1.3548 & 0 & 134.1234 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.396964 & 0 & 0.004008 \\ 0.9032 & 0 & 89.4156 \end{bmatrix}.$$

Let us consider PH-distributions for the service as follows:

- **Erlang service of order 2 (ERS):** $\alpha = [1, 0]$, $W = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$.
- **Exponential Service (EXS):** $\alpha = [1]$, $W = [-1]$
- **Hyper exponential service (HYS):** $\lambda = [0.3, 0.7]$, $W = \begin{bmatrix} -12 & 6 \\ 5 & -10 \end{bmatrix}$.

7.1. Illustration 1

We have examined how the service rate (μ) affects the E_{system} in the following Tables 1 to 3. Fix $N = 5$, $s = 3$, $S = 6$, $\lambda = 2$, $\beta = 2$, $\eta = 2$, $\tau = 1$, $\psi = 1$, we observe that from the following Tables 1 to 3 : As the service rate (μ) increases, the corresponding E_{system} decreases proportionately to the range of arrival and service time arrangements. Observe the arrival timings, in comparison to all other arrival times, E_{system} declines significantly in HYA and considerably more slowly in ERA.

Table 1: *E_{system} vs Service rate (μ) - EXS*

μ	ERA	EXA	HYA	$MNCA$	$MPCA$
11	0.05593823	0.06346217	0.07735655	0.07877835	3.51783032
12	0.05135673	0.05839620	0.07166483	0.07258573	3.37347062
13	0.04748172	0.05414040	0.06691189	0.06736384	3.25215178
14	0.04416124	0.05051471	0.06288302	0.06290067	3.14867954
15	0.04128409	0.04738877	0.05942438	0.05904185	3.05931843
16	0.03876697	0.04466588	0.05642283	0.05567228	2.98131269

Table 2: *E_{system} vs Service rate (μ) - ERS*

μ	ERA	EXA	HYA	$MNCA$	$MPNA$
11	0.05605629	0.06423748	0.07868989	0.07863574	3.51697311
12	0.05144156	0.05904635	0.07278352	0.07244778	3.37176961
13	0.04754427	0.05469343	0.06786383	0.06723433	3.24987784
14	0.04420844	0.05099088	0.06370287	0.06278133	3.14602334
15	0.04132044	0.04780305	0.06013782	0.05893328	3.05641556
16	0.03879549	0.04502960	0.05704930	0.05557447	2.97826031

Table 3: *E_{system} vs Service rate (μ) - HYS*

μ	ERA	EXA	HYA	$MNCA$	$MPNA$
11	0.01121492	0.01559324	0.02501944	0.01925209	2.11066096
12	0.01036636	0.01471293	0.02408268	0.01814938	2.08193365
13	0.00964837	0.01396924	0.02329224	0.01721792	2.05738093
14	0.00903297	0.01333265	0.02261633	0.01642087	2.03614293
15	0.00849963	0.01278159	0.02203174	0.01573125	2.01758391
16	0.00803298	0.01229990	0.02152113	0.01512886	2.00122295

7.2. Illustration 2

In the figures 2 to 6, the effect of the vacation rate μ on the E_{system} has been examined. Adjust $\lambda = 2$, $\beta = 2$, $\mu = 15$, $\tau = 1$, $\psi = 1$, $N = 5$, $s = 3$, $S = 6$, we observe that from the following figures 2 to 6: The variety of arrival and service time arrangements rises with the vacation rate (η) and the associated E_{system} increases as well. An rise in the vacation rate suggests that the server could have to spend more time traveling to the service station and serving the customers, which would raise E_{system}. Notice how E_{system} rises significantly in HYA and MNCA and considerably more slowly in MPCA compared to all other arrival times.

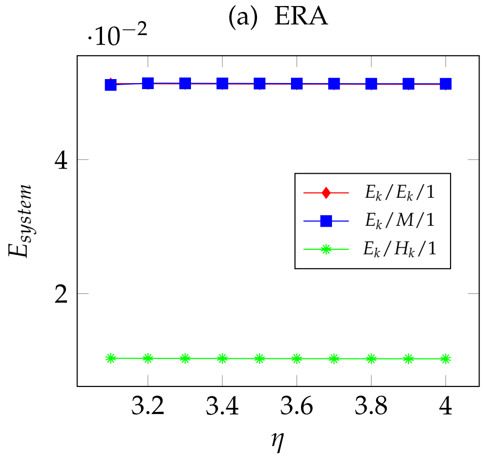

Figure 2: *Vacation rate vs. E_{system}*

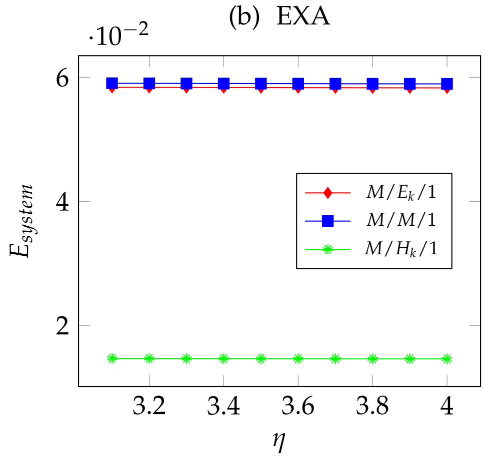

Figure 3: *Vacation rate vs. E_{system}*

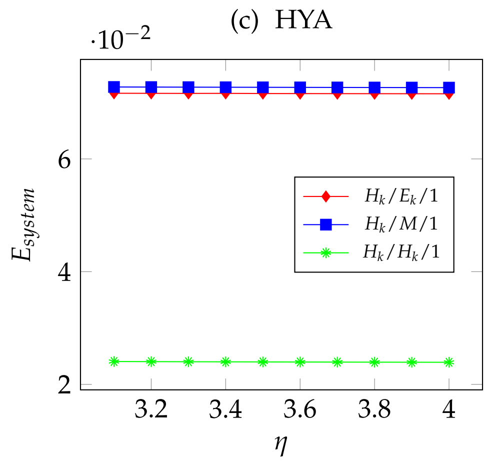

Figure 4: *Vacation rate vs. E_{system}*

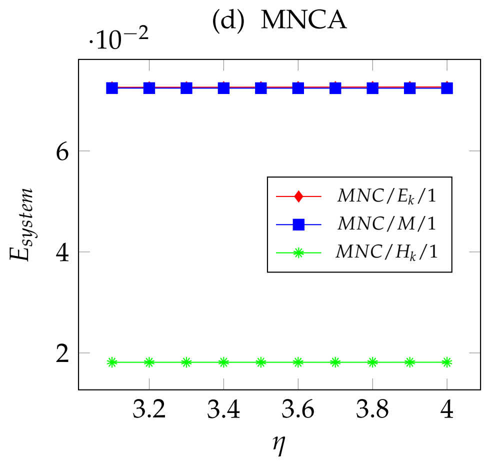

Figure 5: *Vacation rate vs. E_{system}*

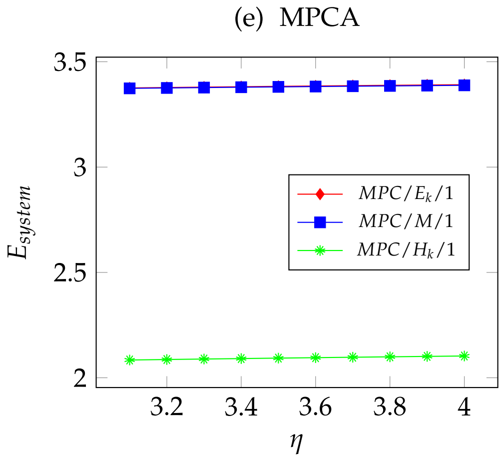

Figure 6: *Vacation rate vs.* E_{system}

8. Conclusion

In this study, we considered customers arriving according to a Markovian Modulated Arrival Process (MMAP) and a phase-type service distribution provided by two servers operating under a non-preemptive priority queueing inventory system with breakdowns, repairs and an (s, S) policy. We analyzed visual representations using 2D graphs and obtained tabular numerical results by examining the effects of various model parameters on key system performance metrics. To this end, we employed the matrix-analytic approach to study the MMAP/PH/2 queueing inventory system with breakdown and repair mechanisms. For future work, we plan to extend the current model by incorporating a Batch Markovian Arrival Process (BMAP) for arrivals, which is particularly suitable for modeling systems with batch arrivals.

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