ON RELIABILITY STRESS-STRENGTH MODEL USES SHOCK-MODEL APPROACH TO FOLLOW MUKHERJEE-ISLAM DISTRIBUTION

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Abstract

A component or system exposed to shocks will result in system or device damage. The strength of a produced product is a variable quantity that requires modelling as a random variable. It is necessary to evaluate the stress conditions of the operating system. Strength and stress are both thought of as random variables. which is made clear by the term "stress-strength model." With the inter-arrival time following the Mukherjee-Islam distribution and strength following an exponential distribution, we attempted to create a model in this study. Graphs are used to support numerical examples that are provided.

Keywords: Shock-Model, CDM, Stress-Strength, exponential distribution, Mukherjee – Islam distribution.

I. Introduction

The concept of shock model introduced by Esary, et al., [2]. In the subject of reliability theory, Ramanarayanan [16] examined the cumulative damage process, which introduces the idea of the system's workers' attentiveness. Cumulative damage process (CDP) is an attractive one, which helps in the interpretation of the behaviours of complex mechanisms. Any component or device exposed to shocks cause damage to the device or system. The device fails when the total of the damages exceeds a threshold level. Assume that the damages $X_1, X_2, \dots, X_n$ caused by successive shocks are mutually independent identically distributed random variable with distribution function *G(.)* independent of the threshold whose distribution function is *H(.)*. Then the probability that the device survives k damages is denoted as

$$P_k = \int_0^\infty g_k(x)[1 - H(x)]dx, \quad k = 1, 2, 3, \dots \tag{1}$$

where, $g_K(X)$ is the k fold convolution of $G(x)$ with itself and $G_0(X) = 1$. The Reliability $R(t)$ of the device is given by

$$P(t) = \sum_{k=0}^{\infty} P_k V_k(t) \tag{2}$$

where is the device's reliability R(t), and is the k-fold convolution of G(x) with itself. Where, $V_k(t)$ is the probability that k damages are caused during (0, t]. In those models H(y) has exponential distribution with parameter μ and G(x) is general. So we get,

$$P_k = \int_0^\infty g_k(x)\, e^{-\mu x} dx, \qquad k = 1, 2, 3, \ldots \ \ and\ g^{*k}(\mu) = \alpha^k \ where\ \alpha = g^{*k}(\mu)$$

II. Stress-Strength Models for Reliability

A produced unit's strength is a variable that ought to be represented as a random variable. The foundation of all reliability modelling is this truth. In addition, when performing the stress conditions of the operational environment must be considered when evaluating the viability of a material or the dependability of equipment. In other words, uncertainty regarding the real environmental stress that will be experienced ought to be modeled as random. According to the stress-strength paradigm, "stress and strength" are both considered to be random variables.

Let X be the amount of stress that an operating environment puts on a unit. In many applications, X is interpreted as the highest value that a crucial type of stress can reach. In the most basic stress-strength paradigm, Y is the unit's strength and X is the stress that the operating environment places on it. If a unit's strength exceeds the force placed on it, it may carry out its intended purpose. Regarding this, the reliability (R) as, R = Probability that the unit performs its task satisfactorily.

R is the likelihood that the unit will complete its duty to a high standard. Reliability, then, is the likelihood that the unit will be robust enough to withstand stress.

Assume that the distribution of stress X is continuous. The distribution of F(X) and strength Y is continuous. If X and Y can be regarded as independent, then G(y), when X and Y can be treated as independent then,

$$R = F(y) dG\,(y) = \int [1 - G(x)]\, d\, F(x) = P[Y > X] \tag{5}$$

This model, first considered by Birnbaum [1]. Has discovered more and more uses in the fields of mechanical, aerospace, and civil engineering.

III. Cumulative Damage Model

Strength can be identified with the Threshold of the system and whenever the total damage exceeds the threshold then the system fails. In manpower planning one of the important aspects of study is to determine the likely or expected time to recruitment which follows necessary as a consequence of the wastage in a given manpower system. There are different methods to estimate the likely time at which recruitment becomes necessary. The expected time to recruitment can be determined by using shock model and cumulative damage process which has been discussed by Sathyamoorthy. R., et.al., [17,18]. In doing so the appropriate probability distribution corresponding to the random variable namely the threshold, the magnitude which in other worlds is the magnitude of depletion and inter arrival time between successive epoch of life time are taken to account. S. Parthasarathy and B. Vijayakumar [10] also Optimum Steps of Accelerated Life Testing for Exponentiated Generalized Pareto Distribution. Optimum Steps of Accelerated Life Testing for Exponentiated Exponential Distribution S. Parthasarathy and B. Vijayakumar [9]. Rajarathinam, A., and Manoharan, M., [15]. Using the Lindley Distribution, a stochastic model is used to estimate the anticipated time to hire, and Manoharan, M., & Rajarathinam, A [6]. A stochastic model for estimation of expected time to elapsed time of recruitment using two-grade manpower systems and Manoharan, M., & Rajarathinam, A [7]. A stochastic model for estimation of expected time to recruitment when the threshold has generalized exponentiated gamma distribution. Several authors contributed to developed the above scenario whereas the threshold follows generic distribution.

The study of shock models and their role in system reliability was pioneered by Esary et al. [2]. Who introduced mathematical frameworks for understanding the effects of random shocks and wear on systems. This foundational work laid the groundwork for modeling cumulative damage processes, a concept later elaborated by Ramanarayanan [16]. who incorporated the aspect of worker alertness in cumulative damage scenarios.

Birnbaum [1] first introduced the stress-strength reliability model, where both stress and strength are treated as independent random variables. His work formed a cornerstone for reliability analysis in engineering applications, especially in mechanical and aerospace contexts. Sathiyamoorthi and Parthasarathy [17, 18] extended shock model approaches to manpower systems, particularly focusing on recruitment timing based on cumulative damage and stochastic modeling. Their approach integrates threshold distributions and inter-arrival times to estimate failure or recruitment epochs effectively.

Mukherjee and Islam [21]. Introduced a novel probability distribution that has since found utility in lifetime data modeling and reliability analysis. Though initially underutilized, Ghitany et. al., [3]. Later demonstrated its strength in outperforming traditional exponential models for specific lifetime data sets. Their work revived interest in the Mukherjee-Islam distribution for reliability modeling.

Parthasarathy and Vijayakumar have contributed extensively to reliability modeling using accelerated life testing methods across various generalized distributions, including Exponentiated Exponential, Lomax, Generalized Pareto, and Mukherjee-Islam distributions [8] and [11,12,13 and 14]. Their studies apply shock model frameworks to these distributions, providing practical methods to estimate the expected time to failure or recruitment under different stress and threshold conditions. Other researchers such as Shanker et al. [19, 20]. and Lindley [5] have contributed alternative distributions such as Lindley and Akash, offering improved fit in certain reliability and lifetime data scenarios.

This cumulative body of work provides a strong theoretical and methodological foundation for modeling stress-strength reliability using shock models, particularly with underutilized yet powerful distributions like the Mukherjee-Islam.

In this paper, the concept of stress is treated as a shock, and the threshold can be treated as strength. We made an attempt to develop a model based on strength, which follows an exponential distribution, and inter-arrival time between successive epochs of stress, which follows the Mukherjee-Islam distribution. The mean and variance are derived from this model.

IV. Mukherjee-Islam Distribution

The Mukherjee-Islamdistribution was introduced by Mukherjee-Islam [21] as a novel distribution that is helpful for lifetime data analysis, particularly for modelling stress-strength reliability. Ghitanyet. al., [22]. Studied application of the Mukherjee-Islam distribution. Additionally, they demonstrated through a numerical example that the Mukherjee-Islam distribution outperforms the exponential distribution-based model in terms of modelling. As point out by Ghitany et al., [4]. The Mukherjee-Islam distribution has not received much attention in the literature because of the exponential distribution's widespread use in statistics and many other applicable fields.

If X~M-I *(x, θ)*, then it's probability density function is

$$f(x,\theta) \;=\; \frac{\alpha}{\theta^{\alpha}} x^{\alpha-1},\;\; 0 \;<\; x <\; \theta, \alpha \;>\; 0, \theta \;>\; 0 \tag{4}$$

and cumulative distribution function

$$F(x,\theta) = \left(\frac{\alpha}{\theta}\right)^{\alpha},\;\; 0 \;<\; x <\; \theta, \alpha \;>\; 0, \theta \;>\; 0 \tag{5}$$

V. Assumptions

The process of depletion is linear and cumulative.

- i.i.d. random variables are the arrival times between consecutive life events.
- The approach will break down or fail if the total stress is greater than the strength level Y, which is a random variable.
- The sequence of stressors, the strength, and the process that creates the exits are all independent of one another.

Notations

X_i : A continuous random variable that indicates the amount of stress the component failure on the i^{th} occasion brought to the system occasion *i = 1, 2, 3, …, k.*

$g(.)$: The probability density functions of X.

$g_k(.)$: The k- fold convolution of $g(.)$ i.e., p.d.f. of $\sum_{i=1}^{k} X_i$

$g^*(.)$: Laplace transform of $g\left(.\right)$.

$g_k^*(.)$: Laplace transform of $g^*(.)$.

T : A continuous random variable denoting time to breakdown of the system

$h(.)$: The p.d.f. of random threshold level which has exponential distribution and H(.) is the corresponding c.d.f.

W : A continuous random variable denoting the inter-arrival times between decision epochs

$f(.)$: p.d.f. of random variable W with corresponding c.d.f. $F(.)$.

$F_k(.)$: The k-fold convolution functions of F(.)

$S(.)$: The survivor function i.e. P[T>t].

L(t) : $1 - \mathrm{S}(\mathrm{t})$.

$V_k(t)$: Probability that there is exactly 'k' plan decisions in (0, 1]

VI. Result

Let Y be a random variable which follows exponential distribution and
$P(x_1 + x_2 + \cdots + x_k < Y) = P$ [The system does not fail, after k epochs of exits].
Now,

$$P\left(\sum X_i < Y\right) = \int_0^{\infty} g_K(x)\overline{H(x)}dx \;\Rightarrow \int_0^{\infty} g_K(x)e^{-\alpha x}dx = \;\alpha^k$$

$S(t) = P(T > t)$ = Probability that the system survives beyond t

$$\sum_{k=0}^{\infty} P\{There\ are\ exactly\ k\ instants\ of\ stresses\ in\ (0,1) * P(The\ system\ does\ not\ fail\ in\ (0,t])\}$$

It is also known from renewal theory that

$$P\left(Exactly\ K\ Probability\ deceisions\ in\ (0,1] = F_k(t) - F_{k+1}(t)\right)\ with\ F_0(t) = 1$$

$$S(t) = \sum_{k=0}^{\infty} V_k(t).P\left[\sum_{i=1}^{k} X_i < Y\right]$$

$$= \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)].[g^*(\alpha)]^k$$

Now,

$$S^*(s) = \sum_{k=0}^{\infty} \frac{[f^k(s) - f^{k+1}(s)]}{s}.\alpha^k$$

$$S^*(s) = \frac{1}{s}[1 - f^*(s)]\frac{1}{1-\alpha.f^*(s)}$$

Now,

$L(t) = 1 - S(t)$

$$L^*(s) = \frac{1}{s} - S^*(s) = \frac{[1-g^*(\alpha)]\, f^*(.)}{[1-f^*(.)g^*(\alpha)]} \tag{6}$$

If W is the inter-occurrence time of stress is not a single random variable but it's a mixture of exp (θ) and M-I (2, θ) so,

$$W = \begin{cases} U\ with\ Probability & p = \frac{\theta}{\theta+1} \\ V\ with\ Probability & q = \frac{\theta}{\theta+1} \end{cases} \tag{7}$$

Where, U~exp (θ) and V~M-I(2,θ)

where $f^*(s)$ is Laplace transform of (t) since the inter arrival times are i.i.d. So

$$f^*(s) = p = \frac{\theta}{\theta+1} + q\left(\frac{\theta}{\theta+s}\right)^2$$

Substitute in the above equation (7),

$$S^*(s) = \frac{1}{s}\left[1 - \left(p.\frac{\theta}{\theta+1} + q\left(\frac{\theta}{\theta+s}\right)^2\right)\right].\frac{1}{1-\alpha.\left[p.\frac{\theta}{\theta+1} + q\left(\frac{\theta}{\theta+s}\right)^2\right]}$$

Computing,

$$E(T) = -\frac{d}{dx}S^*(s), given\ s = 0$$

$$E(T) = -\frac{m_1}{1-\alpha}$$

where, $m_1$is mean of W and $m_1 = \frac{1}{\theta}.\frac{\theta+2}{\theta+1}$

We get,

$$E(T) = \frac{1}{(1-\alpha)}.\frac{1}{\theta}.\frac{\theta+2}{\theta+1} \tag{8}$$

And, $E(T^2) = \frac{d^2S^*(s)}{ds^2}\ given\ s = 0$

$$= \frac{m_2}{2}\left(\frac{1}{1-\alpha}\right) + m_1^2\frac{\alpha}{(1-\alpha)^2}$$

Where, $m_2 = \frac{2}{\theta^2}.\frac{\theta+3}{\theta+1}$

$$Var\ (T) = E(T^2) - [E(T)]$$

Now, substitute m_1 and m_2 in the above equation we get

$$Var\ (T) = \frac{2}{\theta^2}.\frac{\theta+3}{\theta+1}.\frac{1}{1-\alpha} + \frac{1}{\theta^2}.\frac{(\theta+2)^2}{(\theta+1^2)}.\frac{1}{(1-\alpha)^2}.[2\alpha-1]$$

$$= \frac{1}{\theta^2}.\frac{1}{\theta+1}.\frac{1}{(1-\alpha)}\left[2(\theta+3) + \frac{(\theta+2)^2}{(\theta+1)(\theta+\alpha)}.(2\alpha-1)\right] on\ simplification \quad (10)$$

VII. Numerical Illustrations

To illustrate the theoretical findings, we compute the expected time to failure E(T) and variance V(T) for various values of the model parameters λ and θ. Tables 1 and 2 present the computed values for a range of λ (from 0.1 to 0.9) and θ (from 0.2 to 2.0). The results show that as θ increases, the expected time and variance generally decrease, indicating that higher shock frequency leads to faster system degradation. This trend is further visualized in Figures 1 and 2, which graphically show how E(T) and V(T) change with parameter variations.

Table 1: *Expected Time* E(T) *for Various Values of* θ

Mean E(T)					
α	θ=0.2	θ=0.4	θ=0.8	θ=1.0	θ =2.0
0.1	10.185	4.762	2.160	1.667	0.741
0.2	11.458	5.357	2.431	1.875	0.833
0.3	13.095	6.122	2.778	2.143	0.952
0.4	15.278	7.143	3.241	2.500	1.111
0.5	18.333	8.571	3.889	3.000	1.333
0.6	22.917	10.714	4.861	3.750	1.666
0.7	30.556	14.286	6.481	5.000	2.222
0.8	45.833	21.427	9.722	7.500	3.333
0.9	91.667	42.857	19.444	15.000	6.667

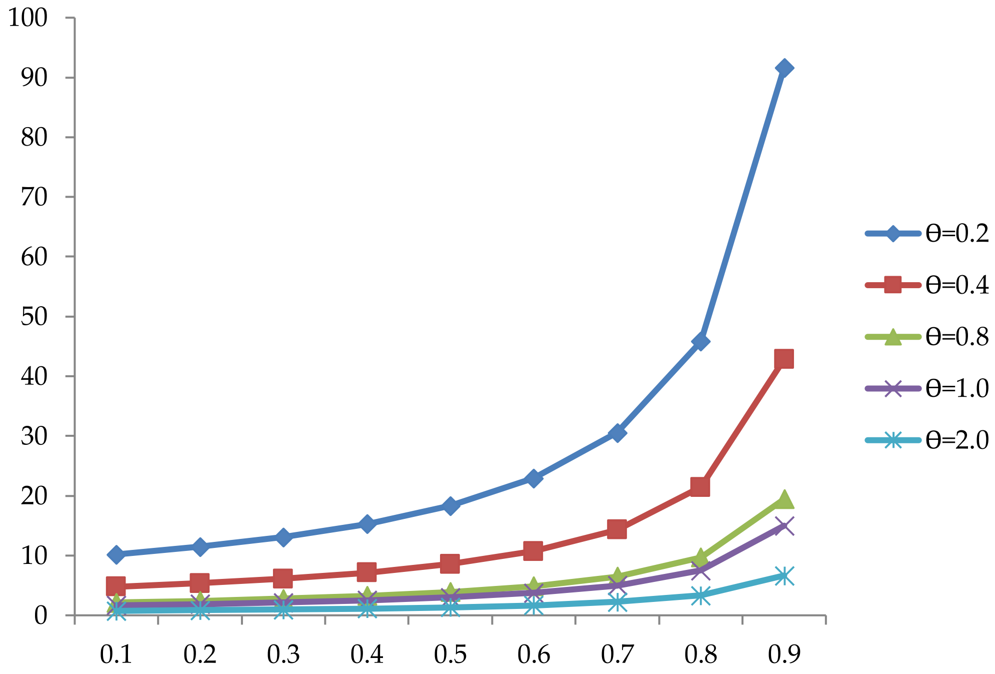

Figure 2: *Plot of Expected Time E(T) versus Parameter θ*

Table 2: *Variance V(T) for Various Values of θ*

	Variance				
α	θ=0.2	θ=0.4	θ=0.8	θ=1.0	θ=2.0
0.1	80.926	19.037	4.306	2.644	0.570
0.2	116.250	26.926	5.978	3.650	0.775
0.3	156.865	36.020	7.912	4.814	1.013
0.4	205.417	46.922	10.239	6.217	1.300
0.5	266.667	60.714	13.194	8.000	1.667
0.6	350.139	79.566	17.249	10.450	2.172
0.7	478.056	108.537	23.503	14.233	2.956
0.8	717.083	162.806	35.255	21.350	4.433
0.9	1400.556	318.265	68.997	41.800	8.689

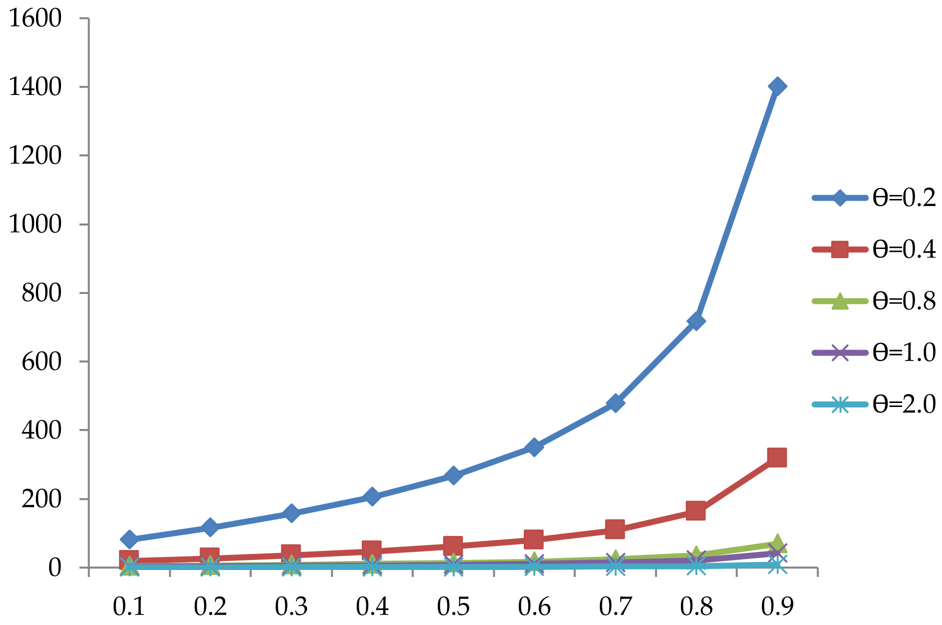

Figure2: *Plot of Variance V(T) versus Parameter θ*

The model assumes that component strength follows an exponential distribution with parameter α, and the shock arrival times follow the Mukherjee–Islam distribution, governed by a parameter θ. Key quantities calculated are the mean time to failure E[T] and its variance Var(T), based on formulas involving α, θ, and the first two moments of the inter-arrival time.

Table 3: *Summary of Two Examples:*

Mean Time to Failure			
Example	Parameters (α,θ)	E[T]	Variance Var (T)
1	(0.3, 0.8)	2.78	0.02
2	(0.7, 0.4)	14.29	6.24

Higher α (weaker components) leads to longer lifetimes only if shocks are infrequent (lower θ). The balance between component strength and shock frequency is critical in determining system reliability.

VIII. Conclusion

In this paper, we developed a reliability model for stress-strength analysis using a shock model framework. The model considers inter-arrival times of stress following the Mukherjee-Islam distribution and strength following an exponential distribution. We derived expressions for the expected time to failure and its variance. Numerical examples, supported by tables and graphical representations, demonstrate how the model behaves under different parameter settings. This approach highlights the usefulness of incorporating less-common distributions like Mukherjee-Islam in reliability theory. Future work may extend this model to include multiple types of shocks, other threshold distributions, or real-world data applications.

The shock-model stress–strength framework with the Mukherjee–Islam inter-arrival distribution provides a flexible way to capture both the randomness in shock timing and heterogeneity in component strength. Designers can leverage the closed-form expressions for E[T] and Var(T) to perform sensitivity analyses, optimizing either the component's strength (α) or the expected shock clustering (θ) to meet reliability targets.

Future work might explore non-exponential strength distributions or extend the model to allow dependency between shock magnitudes and inter-arrival times. Incorporating covariate effects (e.g., environmental factors) into the Mukherjee–Islam parameters could further enhance practical applicability.

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