TWO STAGE FLOW SHOP SCHEDULING PROBLEM INCLUDING PROBABILITY UNDER LR FUZZY NUMBER

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Abstract

Often known as the hybrid flow shop (HFS), the scheduling of flow shops with several parallel machines per stage is a challenging integrated problem that arises in many real-world scenarios. This study addresses bi stage flow shop scheduling problem involving parallel equipotential machines at each stage, where processing times are characterized as LR-type fuzzy numbers to encapsulate inherent uncertainties. Recognizing the criticality of minimizing the makespan under imprecise conditions, we employ four distinct reference functions to compute fuzzy makespan, thereby offering a diversified perspective on solution robustness. Four distinct fuzzy ranking (reference) functions are employed for defuzzification and comparison. The scheduling problem is solved using a Branch and Bound (B&B) algorithm tailored to accommodate fuzzy parameters and parallel machine constraints. Comparative analyses with an adapted version of branch and bound method are conducted to benchmark performance under varying degrees of fuzziness. Results underscore the efficacy and adaptability of the proposed approach, revealing nuanced improvements in makespan estimation and scheduling performance across varying fuzziness levels. This investigation not only enriches the theoretical landscape of fuzzy scheduling but also provides practical insights for complex manufacturing systems operating under uncertainty. This work not only extends classical scheduling methodologies into the fuzzy domain but also highlights the potential of integrating multiple reference functions within exact solution frameworks. The objective is to minimize the makespan under fuzzy conditions. To handle the fuzziness in processing times, A comparative analysis of the results obtained using each reference function is conducted to evaluate their effectiveness in producing optimal or near-optimal schedules. The study highlights the impact of the choice of reference function on the computational efficiency and solution quality, offering insights for better decision-making in uncertain manufacturing scenarios. These functions serve to transform fuzzy numbers into comparable crisp values, thereby enabling traditional scheduling algorithms to process the data effectively. The focus of this research is to evaluate the impact of each reference function on the scheduling performance and to analyze the variation in results with respect to makespan minimization. The findings offer significant theoretical contributions to fuzzy scheduling literature and practical implications for industries where uncertainty in processing times is a critical factor.

Keywords: flow time (makespan), LR fuzzy number, flow shop scheduling, transportation technique, B&B, scheduling

I. Introduction

Scheduling problems in manufacturing systems continue to attract considerable research attention due to their critical impact on operational efficiency and resource utilization. Among these, the two-stage FSSP, wherein jobs must sequentially pass through two stages equipped with parallel and equipotential machines, represents a class of problems that is both practically relevant and theoretically challenging. While classical approaches often presume deterministic processing times, real-world manufacturing environments are inherently uncertain. To address this, we model processing times as LR-type Fuzzy Numbers, allowing a more accurate representation of vagueness and imprecision intrinsic to production processes.

The foundations of modern scheduling optimization under uncertainty began with classical algorithmic approaches. Brown and Lomnicki [1] applied the B&B algorithm to scheduling problems, introducing an effective exact method for exploring complex solution spaces. Dubois and Prade [2] advanced this framework by detailing arithmetic operations on fuzzy numbers, which would later become critical in modeling uncertain scheduling times and costs. In the same year, Zadeh [3] introduced the groundbreaking concept of fuzzy sets, enabling partial membership functions to model imprecise data, a method far more aligned with real-world complexity than traditional binary logic. McCahon and Lee [4] made a key contribution by applying fuzzy set theory to job sequencing with fuzzy processing times. Their approach captured real-life uncertainty in job durations and demonstrated the practical benefits of using fuzzy modeling in production scheduling. Ishibuchi and Lee [5] further pushed these concepts into practice by formulating a fuzzy FSSP using fuzzy processing times. Giachetti and Young [6] complemented this with a parametric representation of fuzzy numbers, improving computational efficiency and clarity in fuzzy arithmetic operations.

In the early 2000s, Yao and Lin [7] introduced a model that constructed a fuzzy FSS framework grounded in statistical data, linking theoretical fuzzy models to practical industrial observations. Wang and Kuo [8] later proposed a three-parameter fuzzy arithmetic approximation method suitable for applications in fuzzy neural networks, marking the integration of fuzzy logic into AI techniques. Saneeifard [9] focused on ranking LR fuzzy numbers using weighted averaging at various confidence levels, a method useful in job prioritization within fuzzy scheduling. Kaur and Kumar [10] proposed Mehar's method for solving fully fuzzy linear programming problems, tailored for systems with LR fuzzy parameters. Thorani and Shankar [11] developed fuzzy transportation models that applied fuzzy logic to supply chains and logistics. Gupta, Sharma, and Aggarwal [12] tackled two-machine FSSP under a fuzzy environment, incorporating setup time and a single transport facility. Their work demonstrated the real-world feasibility of fuzzy models in production scheduling.

Zhou et al. [13] expanded on fuzzy arithmetic operations for LR fuzzy numbers and applied them to fuzzy programming, supporting the solution of fuzzy optimization problems with improved numerical performance. Gupta, Goel, and Mangla [14] focused on optimizing scheduling in two-stage flow shops with equipotential machines in the first stage, under deterministic assumptions. Gupta and Goel [15] then extended the model by applying the B&B technique under uncertain (fuzzy) conditions at both stages. These studies provided strong evidence for the viability of B&B methods in complex fuzzy scheduling environments. Kaushik, Gupta, and Goel [16] conducted a comparative analysis of B&B with NEH and CDS heuristics for bi-stage fuzzy FSSP, illustrating the balance between solution quality and computational efficiency. Another study by Kaushik, Sonia, Gupta, and Goel [17] compared B&B, genetic algorithms (GA), and the Palmar heuristic in minimizing processing times under fuzzy settings, concluding that hybrid techniques often outperform exact methods in scalability.

II.Preliminaries

I. Fuzzy Set

A fuzzy set, a mathematical representation of a set that allows for vagueness and uncertainty, is defined in equation as χ = {(y, μχ(y)) ∣y∈Y}

where χ is the fuzzy set, Y is the set of all possible elements, μχ(y)is the membership function, which assigns a degree of membership to each element y in Y.

II. Fuzzy Number

A fuzzy number is a mathematical representation of a numerical value that is uncertain or imprecise, is defined as χ = {(y, μχ(y)) ∣y∈R}

where, χ is the fuzzy number, R is the set of real numbers, μχ(y)is membership function, which assigns a degree of membership to each real number y.

III. LR Fuzzy Number

A number F =$(a, s, d, g)_{LR}$ is an LR fuzzy number if$\mu_F(x)=\begin{cases} L(\frac{a-x}{d}) & , x \le a, d > 0 \\ R(\frac{x-s}{g}) & , x \ge s, g > 0 \\ 1 & , otherwise \end{cases}$

If a = s then F = $(a, s, d, g)_{LR}$will be converted into F = $(a, d, g)_{LR}$, triangular LR-fuzzy number. Reference functions (L&R) are continuous, non-increasing and defines the left and right shapes of $\mu_F(x)$ respectively and L(0)=R(0) = 1.

Reference functions and their inverses are given in table

Name of Function	Reference Function	Inverse
Linear	$R_F(x) = \max\{0, 1 - x\}$	R_F^{-1}(x)=1-β
Power	$R_F(x) = \max\{0, 1 - x\}$	R_F^{-1}(x)=-$\sqrt[q]{1-\beta}$
Exponential	$R_F(x) = e^{-qx}, q \ge 1$	R_F^{-1}(x)=$-\frac{(ln\beta)}{q}$
Exponent Power	$R_F(x) = e^{-x^q}, q \ge 1$	R_F^{-1}(x)=$\sqrt[q]{-ln\beta}$
Rational	$R_F(x) = \dfrac{1}{1 + x^q}, q \ge 1$	R_F^{-1}(x)=$\sqrt[q]{\frac{1-\beta}{\beta}}$

IV. $\boldsymbol{\lambda}$ −cut for LR-Fuzzy Number

If F = $(a, s, d, g)_{LR}$is a LR -fuzzy number and $\boldsymbol{\lambda}$∈R, $\boldsymbol{\lambda}$∈ [0,1]. Then
$F_k = \{x \in X: \mu_F(x) \ge \lambda\}$=$a - dL^{-1}(\boldsymbol{\lambda})$ +$gR^{-1}(\boldsymbol{\lambda})$ is said to the $\boldsymbol{\lambda}$ −cut of F.

V. Yager's Ranking Formula

If F = $(a, s, d, g)_{LR}$is a LR -fuzzy number then R(F)=$\frac{1}{2}[\int_0^1((a - d\,L^{-1}(\boldsymbol{\lambda}))d\boldsymbol{\lambda}+\int_0^1(s + gR^{-1}(\boldsymbol{\lambda}))d\boldsymbol{\lambda}]$
Let F and K be two fuzzy numbers then
1. F>K if R(F)>R(K); 2.F=K if R(F)=R(K); 3. F<K if R(F)<R(K)
Case 1: L(x)= R(x) = max $\{0, 1 - |x|\}$, then R(F)=$\frac{1}{2}[a - \frac{d}{2} + s + \frac{g}{2}]$ (1)

Case 2: L(x)= R(x) = e^{-x}, thenR(F)=$\frac{1}{2}[a-d+s+\frac{g}{2}]$ (2)

Case 3: L(x)= max $\{0,1-|x|\}$, R(x) = e^{-x}, then R(F)=$\frac{1}{2}[a-\frac{d}{2}+s+g]$ (3)

Case 4: L(x) = e^{-x}, R(x)= max $\{0,1-|x|\}$, then R(F)=$\frac{1}{2}[a-d+s+\frac{g}{2}]$ (4)

VI. Arithmetic Operation

Let $(a_1, s_1, d_1, g_1)_{LR}$ and $(a_2, s_2, d_2, g_2)_{LR}$ be two LR-fuzzy numbers. Then

- Addition: F+H=$(a_1+a_2, s_1+s_2, d_1+d_2, g_1+g_2)_{LR}$
- Subtraction: F-H= $(a_1-s_2, s_1-a_2, d_1+g_2, g_1+d_2)$
- Scalar multiplication: If x>0, x∈R then x × F=$(xa_1, xs_1, xd_1, xg_1)_{LR}$

If x<0, x∈R then x × F=$(xa_1, xs_1, -xd_1, -xg_1)_{LR}$

III. Practical Situation

In pharmaceutical manufacturing, producing tablets involves highly regulated and structured steps. Two of the most critical sequential processes are:

- Stage 1: Granulation and Blending
  This involves preparing the raw materials (active pharmaceutical ingredients and excipients), which are mixed and granulated to form a uniform composition.
- Stage 2: Tablet Compression
  The blended material is compressed into tablets using tablet press machines.

In a large-scale facility, there are multiple granulators/blenders (Stage 1) and multiple tablet presses (Stage 2), all of which are equipotential—that is, they can process any formulation batch with identical performance specifications. These machines are designed to handle a range of products within the same capacity and accuracy levels. The production begins when different batches (jobs), each representing a specific formulation (e.g., a painkiller, an antibiotic, or a vitamin tablet), are queued for processing. Each batch must first go through Stage 1 (granulation and blending). A job scheduler allocates each batch to any available granulator or blender, since all machines are considered equal in capability and speed. As the batches vary in volume or sensitivity, they may require different processing times, which are typically known in advance but may also be estimated using fuzzy logic if there is uncertainty due to raw material variability.

Once a batch is assigned and begins processing in Stage 1, the system keeps track of its progress. When processing is complete, the job transitions to Stage 2 (tablet compression). At this point, the batch enters a waiting buffer if no tablet press is currently available. As soon as one of the presses becomes free, the next available job from the buffer (usually the one that finished earliest or based on a rule like earliest due date) is assigned to it. The compression process then turns the blend into final tablets, completing the production cycle.

In such a tightly regulated environment, traceability and scheduling efficiency are crucial. Delays or bottlenecks at any stage can halt the entire batch processing, affecting production output and compliance with delivery commitments. Additionally, equipment usage must be optimized not just for speed, but also for compliance with cleaning protocols and cross-contamination prevention, which can be incorporated into the model by adding setup times or blocking constraints.

Figure 1: *Tablet production in pharmaceutical manufacturing unit*

IV. Mathematical Formulation

Assume that n jobs are being considered for processing on two different types of parallel equipotential machines, $\hat{S}$ and $\hat{C}$. The utilization time of n^{th} jobs on $\hat{S}$ and $\hat{C}$ machines is shown by $(\hat{k}_n,\hat{l}_n,\hat{m}_n,\hat{n}_n)_{\boldsymbol{LR}}$ and $(\hat{r}_n,\hat{h}_n,\hat{k}_n,\hat{d}_n)_{\boldsymbol{LR}}$, respectively. The processing time is taken as LR-fuzzy number. Each job does not have to be processed on every one ofmachines of type $\hat{S}$ and $\hat{C}$. The model's matrix form is shown as follows:

Table 1: *Tabular form of problem*

Job				Processor $\hat{S}$						Processor $\hat{C}$		
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_n$	Processing time Of$\hat{S}$	Probability p_i	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_m$	Processing time of $\hat{C}$	Probability q_i
1	s_{11}	s_{12}	s_{13}	s_{1n}	$(\hat{k}_1,\hat{l}_1,\hat{m}_1,\hat{n}_1)_{LR}$	p_1	c_{11}	c_{12}	c_{13}	c_{1n}	$(\hat{r}_1,\hat{h}_1,\hat{k}_1,\hat{d}_1)_{LR}$	q_1
2	s_{21}	s_{22}	s_{23}	s_{2n}	$(\hat{k}_2,\hat{l}_2,\hat{m}_2,\hat{n}_2)_{LR}$	p_2	c_{21}	c_{22}	c_{23}	c_{2n}	$(\hat{r}_2,\hat{h}_2,\hat{k}_2,\hat{d}_{2,})_{LR}$	q_2
3	s_{31}	s_{32}	s_{33}	s_{3n}	$(\hat{k}_3,\hat{l}_3,\hat{m}_3,\hat{n}_3)_{LR}$	p_3	c_{31}	c_{32}	c_{33}	c_{3n}	$(\hat{r}_3,\hat{h}_3,\hat{k}_3,d_3)_{LR}$	q_3
4	s_{41}	s_{42}	s_{43}	s_{4n}	$(\hat{k}_4,\hat{l}_4,\hat{m}_4,\hat{n}_4)_{\boldsymbol{LR}}$	p_4	c_{41}	c_{42}	c_{43}	c_{4n}	$(\hat{r}_4,\hat{h}_4,\hat{k}_4,\hat{d}_4)_{\boldsymbol{LR}}$	q_4
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
n	s_{n1}	s_{n2}	s_{n3}	s_{nn}	$(\hat{k}_n,\hat{l}_n,\hat{m}_n,\hat{n}_n)_{\boldsymbol{L}}$	p_n	c_{n1}	c_{n1}	c_{n1}	c_{nm}	$(\hat{r}_n,\hat{h}_n,\hat{k}_n,\hat{d}_n)_{\boldsymbol{L}}$	q_n
t_{pj}	t_{11}	t_{12}	t_{13}	t_{1n}			t_{21}	t_{22}	t_{23}	t_{2m}		

I. Notations

i: job index
j: machine index
$(\hat{k}_i,\hat{l}_i,\hat{m}_i,\hat{n}_i)_{\boldsymbol{LR}}$: processing time of i^{th}job on machine $\hat{S}$.
$(\hat{r}_i,\hat{h}_i,\hat{k}_i,\hat{d}_i)_{\boldsymbol{LR}}$: processing time ofi^{th} job on machine $\hat{C}$.
s_{n3}: unit operational cost of job i (1,2,3,.....,n) on $\hat{S}_j$ like machine j =1,2,3.
c_{n1}: unit operational cost of job i (1,2,3,.....,n) on $\hat{C}_j$ like machine j =1,2,3.
p_i, q_i: Probability of machines $\hat{S}_j$ and $\hat{C}_j$ respectively.

II. Assumptions

- Every task is processed separately.
- Processing all jobs on all parallel machines is not required. One task can be finished by working exclusively on the first, second, or third machine.
- Costs of operation vary depending on the job.
- Machines are available at all times.
- Pre-emption is not taken into account.

III. Algorithm

The following course of action can be used to solve the FSSP given in table 1. Calculate expected processing time by multiplying probability with processing time and then using the ranking formula, we convert the LR fuzzy utilisation time into a crisp value in both phases. Next, in order to determine the best allocation, we use transportation techniques such as VAM& MODI. The best sequence is then determined using B&B.

Step 1: Calculate the expected processing time as $\check{S}_i = \hat{S}_i * p_i, \check{C}_i = \hat{C}_i * q_i$

Step 2:Defuzzify the processing time taken as LR fuzzy number into crisp values using ranking rule given in equation (1) in Case 1, equation (2) in Case 2, equation (3) in Case 3 and using equation (4) in Case 4 respectively.

Step 3: Apply transportation technique like VAM and MODI. To apply any transportation technique, we examine the constraint

$$\sum_{j=1}^{3} t_{1j} = \sum_{i=1}^{n} \check{S}_i \text{ and } \sum_{j=1}^{3} t_{2j} = \sum_{i=1}^{n} \check{C}_i \tag{5}$$

If equation (5) is satisfied, problem is balanced otherwise add dummy job to balance it.

Step 4: Use B&B method by applying the rule

$$b_1 = \max \left(\sum_{i=1}^{n} S_{ij}\right)_{j=1,2,..,m} + \min_{i \in J_k'}\{\max_{j=1,2,......p} C_{ij}\} \;\&\; b_2 = \{\max_{i \in J_k} S_{ij}\}_{j=1,2....,m} + \max\left(\sum_{i=1}^{n} C_{ij}\right)_{j=1,2,..,p}$$

Step 5:Calculate LB=max$\{b_1, b_2\}$ for all the jobs and label the appropriate nodes of the branch from the calculated values.Determine the least of all values of LB and begin at that node. Continue the aforementioned procedure until we reach the branch termination point.In-out table for the obtained sequence gives minimum elapsed time.

Step 6:Repeat step 2 to 5 for all reference functions.

V. Numerical Illustration

Three machines $\hat{S}_1$, $\hat{S}_2 and \hat{S}_3$ of type $\hat{S}$ and three machines $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ of type $\hat{C}$ are accessible for various amounts of time and have differing running expenses with probabilities associated with the machines at both stages as presented in table 2. Processing time is taken as LR fuzzy number to account for the uncertainty.

Table 2: *Mathematical model*

Job	Processor $\hat{S}$					Processor $\hat{C}$				
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	Processing time of $\check{S}_i$	Probability p_i	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	Processing time of $\hat{C}_i$	Probability q_i
1	2	6	5	$(7,8,2,4)_{LR}$	0.2	8	3	11	$(5,7,3,4)_{LR}$	0.1
2	8	2	10	$(6,7,2,4)_{LR}$	0.1	9	4	5	$(6,7,4,5)_{LR}$	0.2
3	7	8	4	$(4,5,2,3)_{LR}$	0.3	6	2	7	$(8,9,4,6)_{LR}$	0.1
4	5	9	6	$(3,5,2,4)_{LR}$	0.2	4	8	3	$(6,7,3,5)_{LR}$	0.2
5	11	3	9	$(7,9,1,6)_{LR}$	0.1	2	7	9	$(8,10,4,6)_{LR}$	0.2
Available time	1.8	2.1	1.66		2.1	1.7	2.38	1.8		

Solution:
After multiplication of processing time of the machines with their probabilities, obtained values are shown in table 3.

Table 3: *Problem after multiplying probability with processing time*

Job	Processor $\hat{S}$				Processor $\hat{C}$			
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	Processing time of $\check{C}_i$
1	2	6	5	$(1.4,1.6,0.4,0.8)_{LR}$	8	3	11	$(0.5,0.7,0.3,0.4)_{LR}$
2	8	2	10	$(0.6,0.7,0.2,0.4)_{LR}$	9	4	5	$(1.2,1.4,0.8,1.0)_{LR}$
3	7	8	4	$(1.2,1.5,0.6,0.9)_{LR}$	6	2	7	$(0.8,0.9,0.4,0.6)_{LR}$
4	5	9	6	$(0.6,1.0,0.4,0.8)_{LR}$	4	8	3	$(1.2,1.4,0.6,1.0)_{LR}$
5	11	3	9	$(0.7,0.9,0.1,0.6)_{LR}$	2	7	9	$(1.6,2.0,0.8,1.2)_{LR}$
Available time	1.8	2.1	1.66		2.1	1.7	2.38	

Now we defuzzify the LR processing time into crisp value using below given reference function and equation (1) which is given in table 4.

Case 1: If L(x)= R(x) = max $\{0, 1 - |x|\}$

Table 4: *Reduced problem after defuzzifying LR fuzzy number into crisp value*

Job	Processor $\hat{S}$				Processor $\hat{C}$			
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	Processing time of $\check{C}_i$
1	2	6	5	1.60	8	3	11	0.63
2	8	2	10	0.70	9	4	5	1.35
3	7	8	4	1.43	6	2	7	0.9
4	5	9	6	0.90	4	8	3	1.4
5	11	3	9	0.93	2	7	9	1.9
Available time	1.8	2.1	1.66		2.1	1.7	2.38	

Since the equation (5) is satisfied therefore, the problem is balanced. Now, we will find the optimal allocations of processing time and unit cost by applying VAM and MODI method to optimize the problem which is given in table 5.

Table 5: *Problem after optimal allocation*

Job	Processor $\hat{S}$				Processor $\hat{C}$			
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	Processing time of $\check{C}_i$
1	1.6	0	0	1.60	0	0.63	0	0.63
2	0	0.7	0	0.70	0	0.17	1.18	1.35
3	0	0	1.43	1.43	0	0.9	0	0.90
4	0.2	0.47	0.23	0.90	0.2	0	1.2	1.40
5	0	0.93	0	0.93	1.9	0	0	1.90

Now B&B algorithm is applied to further optimize the allocation obtained after the application of VAM and MODI given in table 6.

Table 6: *Applying branch and bound*

(i)	b_1 =max $(\sum_{i=1}^{n} S_{ij})_{j=1,2,..,m}$ + $\min_{i\in J_k'}\{\max_{j=1,2,......p} C_{ij}\}$	$b_2 = \{\max_{i\in J_k} S_{ij}\}_{j=1,2.....,m}$+max $(\sum_{i=1}^{n} C_{ij})_{j=1,2,..,p}$	LB=max$\{b_1, b_2\}$
1	2.1+0.18=2.28	1.6+2.38=3.98	3.98
2	2.1+0.63=2.73	0.7+2.38=3.08	3.08
3	2.1+0.18=2.28	1.43+2.38=3.81	3.81
4	2.1+0.18=2.28	0.47+2.38=2.85	2.85
5	2.1+0.18=2.28	0.93+2.38=3.31	3.31

Here minimum of all the LB's is 2.85, which corresponds to task 4. So, position of job 4 in the optimum sequence will be at first place. Now repeat the above process for the subsequences {4,1}, {4,2}, {4,3}and {4,5} given in table 7.

Table 7: *Applying B&B technique for 1st node*

(i)	b_1 =max$(\sum_{i=1}^{n} S_{ij})_{j=1,2,..,m}$+ $\min_{i\in J_k'}\{\max_{j=1,2,......p} C_{ij}\}$	$b_2 = \{\max_{i\in J_k} S_{ij}\}_{j=1,2.....,m}$+max $(\sum_{i=1}^{n} C_{ij})_{j=1,2,..,p}$	LB=max$\{b_1, b_2\}$
41	2.1+0.18=2.28	2.43+2.38=4.81	4.81
42	2.1+0.63=2.73	2.35+2.38=4.73	4.73
43	2.1+0.18=2.28	2.56+2.38=4.94	4.94
45	2.1+0.18=2.28	3.3+2.38=5.68	5.68

Now, the lowest lower bound is 4.73, which corresponds to sub sequence {4,2}. Therefore, fixing job 2 at 2ndposition in the optimal schedule. Now repeat the above process for the subsequences {4,2,1}, {4,2,3} and {4,2,5} given in table 8.

Table 8: *Applying B&B technique for 2nd node*

(i)	b_1 =max$(\sum_{i=1}^{n} S_{ij})_{j=1,2,..,m}$+ min$\{\max_{i\in J_k'} C_{ij}\}_{j=1,2,.....p}$	b_2 = $\{\max_{i\in J_k} S_{ij}\}_{j=1,2....,m}$+max $(\sum_{i=1}^{n} C_{ij})_{j=1,2,..,p}$	LB=max$\{b_1, b_2\}$
421	2.1+0.9=3	2.23+2.38=4.61	4.61
423	2.1+0.63=2.73	2.33+2.38=4.71	4.71
425	2.1+0.63=2.73	4+2.38=6.38	6.38

Clearly, minimum of LB's is 4.61which corresponds to sub sequence {4,2,1}. So, position of job 1 in the optimum sequence will be at third place. Now repeat the above process for the subsequences {4,2,1,3}and {4,2,1,5} given in table 9.

Table 9: *Applying B&B technique for 3rd node*

(i)	b_1 =max$(\sum_{i=1}^{n} S_{ij})_{j=1,2,..,m}$+ min$\{\max_{i\in J_k'} C_{ij}\}_{j=1,2,.....p}$	b_2 = $\{\max_{i\in J_k} S_{ij}\}_{j=1,2....,m}$+max $(\sum_{i=1}^{n} C_{ij})_{j=1,2,..,p}$	LB=max$\{b_1, b_2\}$
4213	2.1+1.9=4	3.13+2.38=5.51	5.51
4215	2.1+0.9=3	2.83+2.38=5.21	5.21

Here minimum lower bound corresponds to subsequence {4,2,1,5}. So, job 3 will be at last position of optimal schedule. Hence, {4,2,1,5,3} is the required optimal sequence and table 10represents corresponding in-out table.

Table 10: *In-out table*

Job	Processor $\hat{S}$			Processor $\hat{C}$		
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$
4	0-0.2	0-0.47	0-0.23	0.47-0.67	-	0.47-1.67
2	-	0.47-1.17	-	-	1.17-1.34	1.67-2.85
1	0.2-1.8	-	-	-	1.8-2.43	-
5	-	1.17-2.1	-	2.1-4	-	-
3	-	-	0.23-1.66	-	2.43-3.33	-

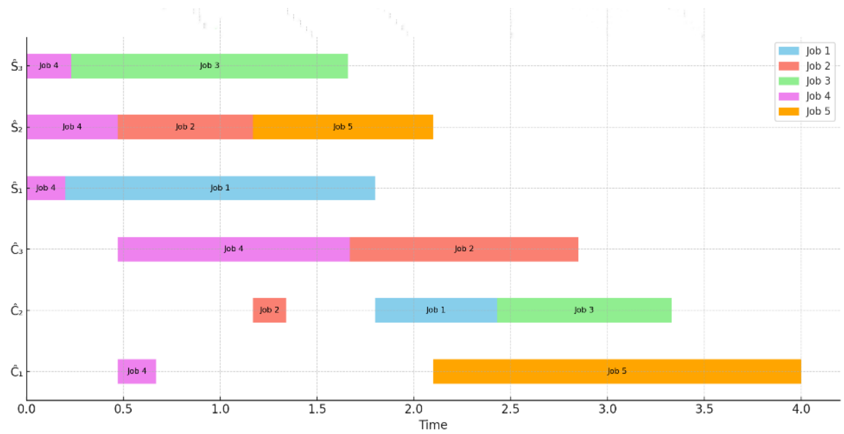

Figure 2: *Gantt chart for case 1.*

Case 2: If L(x)= R(x) = e^{-x}

Now we change the LR processing time into crisp value using above reference function and equation (2). Defuzzified values are given in table 11.

Table 11: *Reduced problem after defuzzifying LR fuzzy number into crisp value*

Job	Processor $\hat{S}$					Processor $\hat{C}$				
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_4$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$	Processing time of $\check{C}_i$
1	2	6	5	0	1.70	8	3	11	0	0.65
2	8	2	10	0	0.75	9	4	5	0	1.40
3	7	8	4	0	1.50	6	2	7	0	0.95
4	5	9	6	0	1.00	4	8	3	0	1.50
5	11	3	9	0	1.05	2	7	9	0	2.00
Available time	1.8	2.1	1.66	0.44		2.1	1.7	2.38	0.32	

Since the equation (5) is not satisfied therefore, the problem is unbalanced. Therefore, we add dummy machine$\hat{S}_4$at 1st stage and $\widehat{C}_4$at 2nd stage to balance the problem. Now, we find the optimal allocations of processing time and optimal allotment of unit cost for each job by applying VAM and MODI method to optimize the problem which is given in table 12.

Table 12: *Problem after application of VAM& MODI method*

Job	Processor $\hat{S}$					Processor $\hat{C}$				
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_4$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$	Processing time of $\check{C}_i$
1	1.7	0	0	0	1.70	0	0.65	0	0	0.65
2	0	0.75	0	0	0.75	0	0.1	0.98	0.32	1.40
3	0	0	1.5	0	1.50	0	0.95	0	0	0.95
4	0.1	0.3	0.16	0.44	1.00	0.1	0	1.4	0	1.50
5	0	1.05	0	0	1.05	2.0	0	0	0	2.00

Now we apply B&B method to obtain the sequence of jobs for optimal solution. The optimal sequence obtained is 42153 and table 13 represents corresponding in-out table.

Table 13: *In-out table.*

Job	Processor $\hat{S}$				Processor $\hat{C}$			
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_4$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$
4	0-0.1	0-0.3	0-0.16	0-0.44	0.44-0.54	-	0.44-1.84	-
2	-	0.3-1.05	-	-	-	1.05-1.15	1.84-2.82	1.05-1.37
1	0.1-1.8	-	-	-	-	1.8-2.45	-	-
5	-	1.05-2.1	-	-	2.1-4.1	-	-	-
3	-	-	0.16-1.66	-	-	2.45-3.4	-	-

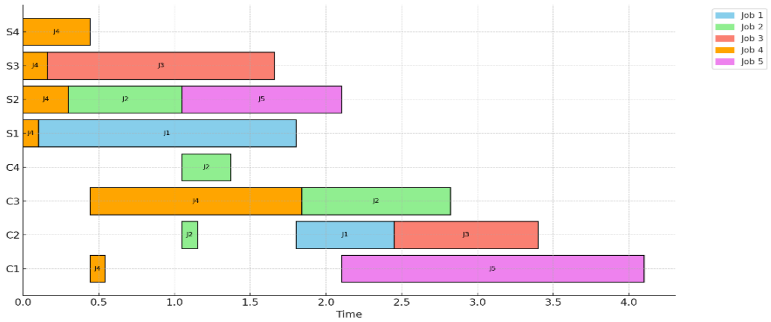

Figure 3: *Gantt chart for case 2.*

Case 3: If L(x)= max $\{0, 1 - |x|\}$, R(x) = e^{-x}

Now we defuzzify the LR processing time into crisp value using above reference function and equation (3). Defuzzified values are given in table 14.

Table 14: *Defuzzifying LR fuzzy processing time*

Job	Processor $\hat{S}$					Processor $\hat{C}$				
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_4$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$	Processing time of $\check{C}_i$
1	2	6	5	0	1.8	8	3	11	0	0.73
2	8	2	10	0	0.8	9	4	5	0	1.6
3	7	8	4	0	1.65	6	2	7	0	1.05
4	5	9	6	0	1.1	4	8	3	0	1.65
5	11	3	9	0	1.08	2	7	9	0	2.2
Available time	1.8	2.1	1.66	0.87		2.1	1.7	2.38	1.05	

Since the equation (5) is not satisfied therefore, the problem is unbalanced. Therefore, we add dummy machine $\hat{S}_4$ at 1st stage and $\hat{C}_4$ at 2nd stage to balance the problem. Now, we find the optimal allocations of processing time and optimal allotment of unit cost for each job by applying VAM and MODI method to optimize the problem which is given in table 15.

Table 15: *Problem after optimal allocation*

Job	Processor $\hat{S}$					Processor $\hat{C}$				
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_4$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$	Processing time of $\check{C}_i$
1	1.58	0.22	0	0	1.8	0	0.65	0	0.08	0.73
2	0	0.8	0	0	0.8	0	0	0.73	0.87	1.6
3	0	0	0.65	0	1.65	0	1.05	0	0	1.05
4	0.22	0	0.01	0.8	1.1	0	0	1.65	0	1.65
5	0	1.08	0	0	1.08	2.1	0	0	0.1	2.2

Now we apply B&B algorithm and calculate the lower bound of jobs to obtain the sequence of jobs for optimal solution as in case 1&2. Required optimal sequence comes out to be {2,1,3,4,5}. Table 16 represents in-out table.

Table 16: *In-out table.*

Job	Processor $\hat{S}$				Processor $\hat{C}$			
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{S}_4$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$
2	-	0-0.8	-	-	-	-	0.8-1.53	0.8-1.67
1	0-1.58	0.8-1.02	-	-	-	1.58-2.23	-	1.58-1.66
3	-	-	0-0.65	-	-	0.65-1.7	-	-
4	1.58-1.8	-	0.65-0.66	0-0.8	-	-	1.8-3.45	-
5	-	1.02-2.1	-	-	2.1-4.2	-	-	2.1-2.2

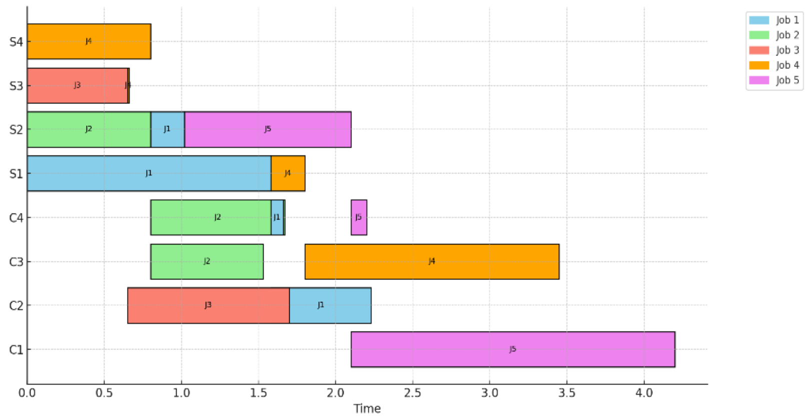

Figure 4: *Gantt chart for case 3.*

Case 4: If L(x) = e^{-x}, R(x)= max $\{0, 1 - |x|\}$

Now we again defuzzify the LR processing time into crisp value using above reference function and equation (4) shown in table 17.

Table 17: *Problem after defuzzification of LR fuzzy numbers*

Job	Processor $\hat{S}$Processor $\hat{C}$							
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	Processing time of $\check{C}_i$
1	2	6	5	1.60	8	3	11	0.63
2	8	2	10	0.70	9	4	5	1.35
3	7	8	4	1.43	6	2	7	0.9
4	5	9	6	0.90	4	8	3	1.4
5	11	3	9	0.93	2	7	9	1.9
6	0	0	0	0.43	0	0	0	0.73
Available time	1.8	2.1	1.66		2.1	1.7	2.38	

Since the equation (5) is not satisfied therefore, the problem is unbalanced. Therefore, we add dummy job$\hat{S}_4$ at 1st stage and $\widehat{C}_4$at 2nd stage to balance the problem. Now, we find the optimal allocations of processing time and optimal allotment of unit cost for each job by applying VAM and MODI method to optimize the problem which is given in table 18.

Table 18: *Reduced problem after optimal allocation*

Job	Processor $\hat{S}$				Processor $\hat{C}$			
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	Processing time of $\check{S}_i$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	Processing time of $\check{C}_i$
1	1.5	0	0	1.60	0	0.55	0	0.63
2	0	0.65	0	0.70	0	0.35	0.8	1.35
3	0	0	1.28	1.43	0	0.8	0	0.9
4	0.3	0.12	0.38	0.90	0	0	1.25	1.4
5	0	0.9	0	0.93	1.7	0	0	1.9
6	0	0.43	0	0.43	0.4	0	0.33	0.73

Now by applying B&B algorithm to further optimize the allocation obtained after the application of VAM and MODI, optimal sequence obtained is 462351. Table 19 represents in-out table.

Table 19: *In-out table.*

Job	Processor $\hat{S}$			Processor $\hat{C}$		
i	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$
4	0-0.3	0-0.12	0-0.38	-	-	0.38-1.63
6	-	0.12-0.55	-	0.55-0.95	-	1.63-1.96
2	-	0.55-1.2	-	-	1.2-1.55	1.96-2.76
3	-	-	0.38-1.66	-	1.66-2.46	-
5	-	1.2-2.1	-	2.1-3.8	-	-
1	0.3-1.8	-	-	-	2.46-3.01	-

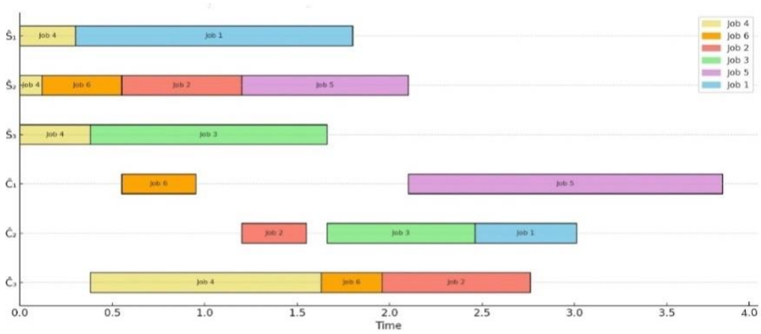

Figure 5: *Gantt chart for case 4.*

VI. Comparison

Values calculated for the other reference function is shown in the table 20.

Table 20: *Comparison table*

S.No.	Reference Function	Optimal Sequence	Total Flow Time(in hrs)
1	L(x)= R(x) = max $\{0,1-\lvert x\rvert\}$	{4,2,1,5,3}	4
2	L(x)= R(x) = e^{-x}	{4,2,1,5,3}	4.1
3	L(x)= max $\{0,1-\lvert x\rvert\}$, R(x) = e^{-x}	{2,1,3,4,5}	4.2
4	L(x) = e^{-x}, R(x)= max $\{0,1-\lvert x\rvert\}$	{4,2,3,5,1}	3.8

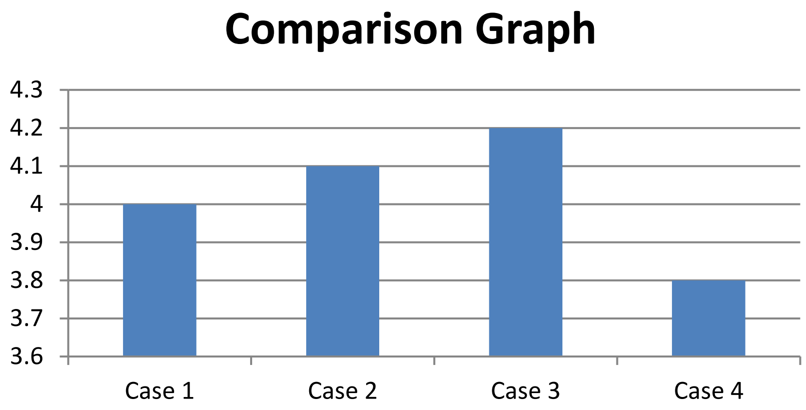

Figure 6: *Comparison of total elapsed time corresponding to reference functions*

VII. Conclusion

An approach to scheduling in the LR-fuzzy environment is presented in this work. In FSSPs where processing time is taken as LR-Fuzzy numbers, B&B technique has been utilized to determine the best order for two stage FSSP with parallel equipotential machines at both stages including probabilities. The example shows that the best results are obtained using either a linear or exponential function on the LR-fuzzy number.

VIII. Future Scope

The present study on the two-stage FSSP with parallel machines at both stages, incorporating processing times as LR fuzzy number and probabilities, solved through the Branch and Bound (B&B) method with linear and exponential reference functions, offers several directions for future research. Multi-stage or hybrid flow shop systems are common in complex industrial settings, and extending the framework would enhance its applicability in real-world manufacturing scenarios with varied processing routes and interdependencies. Incorporating dynamic scheduling elements such as stochastic job arrivals, machine failures, and other real-time disruptions presents another promising direction. The LR fuzzy modeling approach is well-suited for dynamic systems, as it supports continuous updates of uncertainty parameters and belief levels based on real-time data inputs.

While the B&B method guarantees optimality, its computational burden increases significantly with problem size. Future work could investigate the integration of metaheuristic algorithms which may yield near-optimal solutions with reduced computational effort, particularly for large-scale instances. Further, a multi-objective optimization approach may be employed to address additional performance criteria, including total flow time, transportation time, job block, energy consumption and machine utilization.

Another important area for development is the adaptive selection of reference functions. Instead of relying on fixed linear or exponential forms, future research may explore learning-based or heuristic methods to dynamically identify the most appropriate reference structure based on job features or operational context. Also, the development of a software-based decision-support system integrating these features would support practical adoption by manufacturing decision-makers.

Conflicts of interests: The authors declare that there are no conflicts of interest.

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