ANALYZING SOLUTIONS OF FUZZY NABLA DYNAMIC EQUATIONS ON TIMESCALES: A GENERALIZED HUKUHARA APPROACH

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Abstract

In this paper, we study linear first-order fuzzy nabla dynamic equations on time scales (LFNDET's) under the framework of generalized Hukuhara nabla differentiability and also interpret various results of LFNDET's by employing the Variation of constant formula. A key result concerns the generalized nabla Hukuhara derivative for the product of a crisp function and a fuzzy function on time scales. Furthermore, we derive solutions to these equations using various approaches to generalized Hukuhara nabla differentiability. The behavior of these solutions highlights the importance and applicability of the generalized fuzzy nabla derivative in the context of nabla dynamic equations on time scales.

Keywords: Fuzzy functions; Generalized nabla derivative; Time scales

1. Introduction

S. Hilger introduced the theory of time scales [15] and has shown remarkable potential for a wide range of applications. It unifies continuous and discrete analysis, eliminating the need to prove results separately for differential and difference equations. The framework of dynamic equations on time scales [9] highlights and bridges the differences between these domains. For foundational aspects of calculus on time scales, we refer the reader to [1, 10, 13]. The development of nabla dynamic equations on time scales was initiated in [10], with further contributions by authors such as [16, 19, 24].

Typically, the variables, parameters and the initial conditions in a dynamic model are assumed to be precisely known. However, in practical applications, these components often involve vagueness, imprecision, or incomplete information. To address this uncertainty, fuzzy environments can be incorporated by replacing exact values with fuzzy sets, leading to the formulation of Fuzzy Differential Equations (FDEs). The concept of Hukuhara differentiability, based on the Hukuhara difference (HK-Difference)[18], was introduced by [26], and further used in [20] to study FDEs and establish existence and uniqueness results under the Lipschitz condition.

However, a major limitation of the Hukuhara derivative is that it applies only when the length of the fuzzy support increases, restricting its ability to capture better solutions. To address this drawback, the concept of generalized differentiability for fuzzy-valued functions was introduced in [6, 8]. In [7], the author studied first-order linear fuzzy differential equations (LFDEs) under this generalized framework.

Fard and Bidgoli [12] extended the calculus of fuzzy-valued functions to the setting of time scales. More recently, Vasavi et al. [27] investigated fuzzy dynamic equations on time scales using the generalized delta derivative. In contemporary applications such as economics [3], inventory models [4], and cellular neural networks [14], nabla derivative has often been preferred over the delta derivative due to its fewer restrictions and greater flexibility.

Motivated by these developments and the advantages of the nabla derivative, this paper focuses on the study of linear fuzzy nabla dynamic equations on time scales under generalized Hukuhara differentiability.

The essence of the paper is structured as follows:

Section 2 introduces fuzzy nabla derivatives and integrals on time scales, along with their fundamental properties.

Section 3 discusses the product of two fuzzy functions under different types of generalized Hukuhara differentiability on time scales.

Section 4 is devoted to the analysis of linear fuzzy nabla dynamic equations on time scales (LFNDETs).

2. Preliminaries

It's important to review some basic fuzzy calculus definitions and conclusions. Examine the family of all convex compact nonempty subsets, $R_k(\Re^n)$. The scalar multiplication and set addition are expressed as usual in $R_k(\Re^n)$. The commutative semigroup [20] under addition with cancellation rules is then satisfied by $\Re_k(\Re^n)$. Additionally, if $M, N \in \Re_k(\Re^n)$ and $\alpha_1, \alpha_2 \in \Re$ then

$$\alpha_1(M+N) = \alpha_1 M + \alpha_2 N, \;\; \alpha_1(\alpha_2 M) = (\alpha_1\alpha_2)N, \;\; 1.N = N,$$

and if $\alpha_1, \alpha_2 \geq 0$ then $(\alpha_1+\alpha_2)M = \alpha_1 M + \alpha_2 M$. The distance between M and N using the Poincare-Hausdorff metric, is defined as follows:

$$D(M,N) = \max\{\inf_{n\in N}\sup_{m\in M}\|m-n\|, \sup_{n\in N}\inf_{m\in N}\|m-n\|\}$$

where M and N are any two bounded nonempty subsets of $\Re^n$ $||.||$ is the norm in $\Re^n$. Then $(\Re_k(\Re^n), D)$ becomes a complete seperable metric space [20].

Let $\mathbb{E}_n = \{u : \Re^n \to [0,1]/\mu \text{ satisfies(a)-(d) below}\}$, also define $u : \Re^n \to [0,1]$ be a fuzzy subset of the real line. The function u is called a fuzzy number and fulfills the following conditions:

(a) Normality: There is some $t \in \Re^n$ such that $u(t) = 1$.

(b) convexity: For all $t \in [0,1]$ and $m, n \in \Re^n$ the inequality $u(tm + (1-t)n) \geq \min\{u(m), u(n)\}$ holds.

(c) convexity: The function u is upper semi-continuous on $\Re^n$, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $u(m) - u(m_0) < \epsilon$ when $|m - m_0| < \delta$.

(d) Compact support: The support of u is a compact set.

For $0 < \gamma \leq 1$, denote $[u]^\gamma = \{t \in \Re^n : u(t) \geq \gamma\}$, then based on the above setting we have the γ-level set $[u]^\gamma \in \Re_k(\Re^n)$.

It is well established that, for every continuous function g, we have $[g(s,t)]^\gamma = g([s]^\gamma, [t]^\gamma)$ $\forall s, t \in \mathbb{E}_n$. The scalar addition($\oplus$) and multiplication($\odot$) in $\mathbb{E}_n$ of s, t is defined as

$$[s \oplus t]^\gamma = [s]^\gamma + [t]^\gamma, [k \odot s]^\gamma = k[s]^\gamma, \text{ where } s, t \in \mathbb{E}_n, \; k \in \Re, \; 0 \leq \gamma \leq 1.$$

The theorem presented below extends the features of scalar multiplication and addition of fuzzy number valued functions ($\Re_F = \mathbb{E}_1$) to $\mathbb{E}_n$.

Theorem 1. *[2]*

(a) *If $\tilde{0}$ is the zero element in $\Re_F$, then $\hat{0} = \left(\tilde{0}, \tilde{0}, \ldots, \tilde{0}\right)$ is the zero element in $\mathbb{E}_n$. i.e. $s \oplus \hat{0} = \hat{0} \oplus s = s \ \forall s \in \mathbb{E}_n$;*

(b) *For any $s \in \mathbb{E}_n$ has no inverse with respect to '$\oplus$';*

(c) *For any $\beta_1, \beta_2 \in \Re$ with $\beta_1, \beta_2 \geq 0$ or $\beta_1, \beta_2 \leq 0$ and $m \in \mathbb{E}_n$, then $(\beta_1 + \beta_2) \odot m = (\beta_1 \odot m) \oplus (\beta_1 \odot m)$;*

(d) *For any $\beta_1 \in \Re$ and $m, n \in \mathbb{E}_n$, we have $\beta_1 \odot (m \oplus n) = (\beta_1 \odot m) \oplus (\beta \odot n)$;*

(e) *For any $\beta_1, \beta_2 \in \Re$ and $m \in \mathbb{E}_n$, we have $\beta_1 \odot (\beta_2 \odot m) = (\beta_1 \beta_2) \odot m$.*

If for every $Y, Z \in \mathbb{E}_n$, there is an $X \in \mathbb{E}_n$ such that $Y = Z \oplus X$. Then X is called the Hukuhara difference (HK-Difference) of Y and Z and is denoted by $Y \ominus_h Z$. For any $W, X, Y, Z \in \mathbb{E}_n$ and $\beta \in \Re$, the following holds if the HK-Difference exists.

(a) $D_H(Y, Z) = 0 \Leftrightarrow Y = Z$;

(b) $D_H(\beta Y, \beta Z) = |\beta| D_H(Y, Z)$;

(c) $D_H(X \oplus Z, Y \oplus RZ = D_H(X, Y)$;

(d) $D_H(X \ominus_h Z, Y \ominus_h Z) = D_H(X, Y)$;

(e) $D_H(W \oplus X, Y \oplus Z) \leq D_H(W, Y) + D_H(X, Z)$;

This section introduces essential concepts and properties of the Hukuhara derivatives of fuzzy functions over the interval $I = [x, y]$, $x, y \in \Re$, along with necessary definitions and some results from the theory of time scales.

Definition 1. *[20] Consider a fuzzy-valued function $Q : I \to \mathbb{E}_n$ where $I \subset \Re$ is a compact interval. The function Q is said to be Hukuhara differentiable (or H-differentiable) at a point $s \in I$, if there exist an element $Q^{'}(s)$ in $\mathbb{E}_n$ such that both of the following limits exist and are equal in the metric space $(\mathbb{E}_n, D_H)$*

$$\lim_{h \to 0^+} \frac{Q(s+h) \ominus_h Q(s)}{h}, \lim_{h \to 0^+} \frac{Q(s) \ominus_h Q(s-h)}{h}$$

At the endpoints of the interval I, only the appropriate one-sided limit is considered for the derivative.

Definition 2. *[9]*

(a) *Time scales: A closed nonempty subset of real numbers ($\Re$) which is denoted by $\mathbb{T}$.*

(b) *Backward Jump operator: The operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(s) = \sup \{s_0 \in \mathbb{T} : s_0 < s\}$*

(c) *Graininess function: The operator $\nu : \mathbb{T} \to \mathbb{R}^+$ defined as $\nu(s) = s - \rho(s)$ for $s \in \mathbb{T}$*

(d) *If $\rho(s) = s$, then the operator ρ is called left-dense, otherwise left scattered (in short we use 'ls' for left scattered).*

(e) *If $\mathbb{T}$ is right scattered minimum m then $\mathbb{T} - \{m\} = \mathbb{T}_k$, Otherwise $\mathbb{T} = \mathbb{T}_k$.*

(f) *for each $s \in \mathbb{T}$, the function is $h^\rho : \mathbb{T} \to \Re$ defined by $h^\rho(s) = h(\rho(s))$.*

(g) *The interval of time scale $\mathbb{T}$ is given by*

$$\mathbb{T}^{[x,y]} = \{t \in \mathbb{T} : x \leq t \leq y\} = [x, y] \cap \mathbb{T}$$

and

$$\mathbb{T}_k^{[x,y]} = \begin{cases} \mathbb{T}^{[x,y]}, & \text{if } x \text{ is right dense} \\ \mathbb{T}^{[\sigma(x),y]}, & \text{if } x \text{ is right scattered.} \end{cases}$$

Definition 3. *[9] Let $g : \mathbb{T} \to \mathbb{R}$ be a function and let $s \in \mathbb{T}_k$. Then the* ***nabla derivative*** $g^{\nabla}(s)$ *exists (as a real number)* ***if and only if****, for every $\epsilon > 0$, there exists a neighborhood N_δ of s (i.e., $N_\delta = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that:*

$$|(g^{\rho}(s)) - g(s_0)) - g^{\nabla}(s)(\rho(s) - s_0)| \leq \epsilon|\rho(s) - s_0|, \text{ for all } s_0 \in N_\delta,$$

where, $g^{\nabla}(s)$ is called the Nabla derivative of g at s. A function g is called Nabla (or Hilger) differentiable on $\mathbb{T}_k$, if $g^{\nabla}(s)$ exists for all s in $\mathbb{T}_k$. The function $g^{\nabla} : \mathbb{T}_k \to \Re$ is then called the Nabla derivative of g on $\mathbb{T}_k$.

Definition 4. *[10] Let $g : \mathbb{T} \to \Re$ be a real-valued function defined on the time scales $\mathbb{T}$. The function g is said to be regulated if it admits finite left-sided limits at all left dense points(left-dense) of $\mathbb{T}$, and finite right-sided limits at all right-dense points of $\mathbb{T}$.*

Definition 5. *[10] Let $g : \mathbb{T} \to \Re$ be a real-valued function defined on the time scales $\mathbb{T}$. The function g is said to be left-dense continuous (ld-continuous) if it is continuous at every left-dense point of $\mathbb{T}$, and if the right-hand limit $\lim_{s_0 \to s^+} g(s_0)$ exists and is finite at each right-dense point $s \in \mathbb{T}$.*

Definition 6. *[10] A function α defined from $\mathbb{T}$ to $\mathbb{R}$ is ν-regressive if $1 - \nu(s)\alpha(s) \neg 0$ on the time scales $\mathbb{T}$. The set of all ν-regressive ld-continuous functions defined on $\mathbb{T}_k$ is denoted by*

$$\Re_\nu = \{\alpha : \mathbb{T} \to \mathbb{R} \quad \text{where } \alpha \text{ is } \nu\text{-regressive, ld continuous functions.}\}$$

Theorem 2. *[10] The nabla initial value problem(NIVP) is defined as*

$$\chi^{\nabla} = \alpha(s)\chi(s) + g(s), \chi(s_0) = \chi_0$$

with the initial conditions $u_0 \in mathbbT$ admits unique solution $\chi : \mathbb{T} \to \Re^n$ and is explicitly given by

$$\chi(s) = \hat{e}_\alpha(s, s_0)\chi_0 + \int_{s_0}^{s} \hat{e}_\alpha(s, s_0)g(\tau)\nabla(\tau)$$

where $\alpha \in \Re_\nu$ and $g : \mathbb{T} \to \Re^n$ is ld-continuous.

If we consider the NIVP $\chi^{\nabla} = \alpha(m)\chi, \ \ \chi(m_0) = 1$ on time scales, then the solution is of the form

$$\hat{e}_\alpha(m, m_0) = exp\left\{\int_{m_0}^{m} \hat{\xi}_{\nu(m)}(\alpha(\tau))\nabla\tau\right\}$$

We now list several important properties of the nabla exponential function, which will be further useful.

(1) $\hat{e}_p(m, m) = 1; \hat{e}_0(m, n) = 1;$

(2) $\hat{e}_\alpha(\rho(m), n) = [1 - \nu(m)\alpha(m)]\hat{e}_\alpha(m, n);$

(3) $\hat{e}_\alpha(m, n)\hat{e}_\alpha(n, p) = \hat{e}_\alpha(m, p);$

(4) $\hat{e}_\alpha(m, n) = \dfrac{1}{\hat{e}_\alpha(n, m)}.$

In particular, if $\alpha \in \Re$ is such that $1 - \nu(r)\alpha(r) > 0 \quad \forall r \in \mathbb{T}$,

$$\hat{e}_\alpha(m, 0) = \begin{cases} (1 - rp)^{\frac{m}{p}}, & \text{if } \mathbb{T} = p\mathbb{Z} \text{ with } h > 0, \\ \dfrac{1}{1-\alpha}^{r}, & \text{if } \mathbb{T} = \mathbb{Z}, \\ \hat{e}^{\alpha m}, & \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

If $\mathbb{T} = \mathbb{R}$, the graininess function ν is identically zero and for the discrete case like $\mathbb{T} = \mathbb{Z}$ it remains constant, i.e. $\nu(s) = 1$, for all s $\in \mathbb{Z}$. The other time scales, which include continuous graininess functions, are any closed intervals of real numbers and uniform discrete sets like $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. The following is an example of time scales with nonconstant graininess:

$$\mathbb{T}_k^{[a,b]} = \bigcup_{n=0}^{\infty} [n(a+b), n(a+b)+a]; \quad a, b > 0.$$

Then

$$\rho(t) = \begin{cases} t, \text{ if } t \in \bigcup_{n=0}^{\infty}[n(a+b), n(a+b)+a), \\ t-b, \text{ if } t \in \bigcup_{n=0}^{\infty} \{n(a+b), n(a+b)+a\}. \end{cases}$$

And

$$\nu(t) = \begin{cases} 0, \text{ if } t \in \bigcup_{n=0}^{\infty}[n(a+b), n(a+b)+a), \\ b, \text{ if } t \in \bigcup_{n=0}^{\infty} \{n(a+b), n(a+b)+a\}. \end{cases}$$

3. On the Generalized Nabla Hukuhara Differentiability and Integrability

In this section, we develop the concept of generalized nabla Hukuhara differentiability for fuzzy-valued functions defined on arbitrary time scales. By extending classical differentiation to the fuzzy setting using the nabla calculus framework, we establish conditions under which such functions are differentiable in the generalized Hukuhara sense. Furthermore, we examine the corresponding notions of integrability, providing definitions and results that unify and generalize continuous and discrete cases. These concepts are crucial for analyzing dynamic systems involving uncertainty over nonuniform time domains.

Definition 7. *[23]*
Let $Q : \mathbb{T}^{[x,y]} \to$ be a fuzzy function. We say that Q is generalized nabla Hukuhara differentiable (symbolically ∇_g-differentiable) at $s \in \mathbb{T}_k^{[x,y]}$, if both the left and right generalized Hukuhara derivatives of Q exist at s and coincide, that is,

$$Q_{-}^{\nabla_g}(s) = Q_{+}^{\nabla_g}(s) = Q^{\nabla_g}(s),$$

where $Q_{-}^{\nabla_g}(s)$ or $Q_{+}^{\nabla_g}(s)$ is called ∇_g derivative of Q at $s \in \mathbb{T}_k^{[x,y]}$ and it is denoted by $Q^{\nabla_g}(s)$. Furthermore, if this derivative exists at every point in the domain $\mathbb{T}_k^{[x,y]}$, then the function Q is said to be ∇_g-differentiable on $\mathbb{T}_k^{[x,y]}$.

Remark 1. *[23] Let $Q : \mathbb{T}^{[x,y]} \to \mathbb{E}_n$ be a fuzzy valued function. Then Q is said to be genralized HuKuhara nabla differentiable (∇_g-differentiable) at $s \in \mathbb{T}_k$, If there exists an element $Q^{\nabla_g}(s) \in \mathbb{E}_n$, such that at least one of the following conditions is satisfied:*

$\nabla - GH1$: *There exists $\delta > 0$ such that for all $0 < h < \delta$, the HK-Difference $Q^{\rho}(s) \ominus_h Q(s-h)$ and $Q(s+h) \ominus_h Q^{\rho}(s)$, exist and the following limits are egual:*

$$\mathbb{T} - \lim_{h \to 0} \frac{1}{h - \nu(s)} \odot (Q^{\rho}(s) \ominus_h Q(s-h)) = Q^{\nabla_g}(s) = \mathbb{T} - \lim_{h \to 0} \frac{1}{h + \nu(s)} \odot (Q(s+h) \ominus_h Q^{\rho}(s))$$

or

$\nabla - GH2$ *There exists $\delta > 0$ such that for all $0 < h < \delta$, the HK-Difference $Q(s-h) \ominus_h Q^{\rho}(s)$ and $Q^{\rho}(s) \ominus_h Q(s+h)$, exists and the following limits are egual:*

$$\mathbb{T} - \lim_{h \to 0} \frac{-1}{h - \nu(s)} \odot (Q(s-h) \ominus_h Q^{\rho}(s)) = Q^{\nabla_g}(s) = \mathbb{T} - \lim_{h \to 0} \frac{-1}{h + \nu(s)} \odot (Q^{\rho}(s) \ominus_h Q(s+h)$$

or

$\nabla - GH3:$ *There exists $\delta > 0$ such that for all $0 < h < \delta$, the HK-Difference $Q^{\rho}(s) \ominus_h Q(s-h)$ and $Q^{\rho}(s) \ominus_h Q(s+h)$, the limits exists*

$$\mathbb{T} - \lim_{h \to 0} \frac{1}{h - \nu(s)} \odot (Q^{\rho}(s) \ominus_h Q(s-h)) = Q^{\nabla_g}(s) = \mathbb{T} - \lim_{h \to 0} \frac{-1}{h + \nu(s)} \odot (Q^{\rho}(s)) \ominus_h (s+h)).$$

or

$\nabla - GH4:$ *There exists $\delta > 0$ such that for all $0 < h < \delta$, the HK-Difference $Q(s-h) \ominus_h Q^{\rho}(s)$ and $Q(s+h) \ominus_h Q^{\rho}(s)$, the limits exist*

$$\mathbb{T} - \lim_{h \to 0} \frac{-1}{h - \nu(s)} \odot (Q(s-h) \ominus_h Q(\rho(s))) = Q^{\nabla_g}(s) = \mathbb{T} - \lim_{h \to 0} \frac{1}{h + \nu(s)} \odot (Q(s+h) \ominus_h Q^{\rho}(s)).$$

When such a limit exists at each $s \in \mathbb{T}_k^{[x,y]}$, the function $Q^{\nabla_g} : \mathbb{T}_k^{[x,y]} \to \mathbb{E}_n$ is called the generalized Hukuhara nabla derivative of Q and is said to be ∇_g-differentiable on $\mathbb{T}_k$.

Example 1. *Let $\mathbb{T} = \mathbb{R}$, and consider the fuzzy-valued function*

$$G(s) = \tilde{k} \odot s^2, \quad s \in [0,1],$$

where $\tilde{k} \in \mathbb{E}_n$ is a fixed fuzzy number, and $\odot$ denotes scalar multiplication of a fuzzy number. Assume $\tilde{a}$ is a triangular fuzzy number given by $\tilde{k} = (1,2,3)$.

Since the time scale is $\mathbb{R}$, the graininess function is $\nu(s) = 0$, and the backward jump operator satisfies $\rho(s) = s$. In this continuous setting, the generalized nabla Hukuhara derivative coincides with the standard Hukuhara derivative.

To compute the derivative, we consider:

$$Q(s+h) \ominus_h Q(s) = \tilde{k} \odot \left((s+h)^2 - s^2\right) = \tilde{k} \odot (2sh + h^2).$$

Taking the limit as $h \to 0^+$, we obtain:

$$Q^{\nabla^g}(s) = \lim_{h \to 0^+} \frac{Q(s+h) \ominus_h Q(s)}{h} = \tilde{K} \odot 2s.$$

Thus, the function $Q(s) = \tilde{k} \odot s^2$ is ∇^g-differentiable on $[0,1]$, and its generalized nabla Hukuhara derivative is:

$$Q^{\nabla^g}(s) = \tilde{k} \odot 2s, \quad \forall s \in [0,1].$$

The following figure 1 illustrates the α-cut representation of the fuzzy derivative $G^{\nabla^g}(s)$ for $s \in [0,1]$. Each shaded region corresponds to a different level of $\alpha \in [0,1]$. As α increases, the uncertainty (width of the fuzzy value) decreases, converging to the crisp core at $\alpha = 1$. Let $\mathbb{T} = \mathbb{Z}$, and consider the fuzzy-valued function

$$Q(s) = \tilde{k} \odot s, \quad s \in \mathbb{Z}^{[0,5]},$$

where $\tilde{k} = (1,2,3)$ is a triangular fuzzy number and $\odot$ denotes scalar multiplication.

Since the graininess function on $\mathbb{Z}$ is $\nu(s) = 1$, and the backward jump operator is $\rho(s) = s - 1$, the generalized nabla Hukuhara derivative becomes:

$$Q^{\nabla^g}(s) = \frac{Q(s) \ominus_h Q(\rho(s))}{\nu(s)} = \frac{\tilde{k} \odot s \ominus_h \tilde{k} \odot (s-1)}{1} = \tilde{k}.$$

Thus, $Q^{\nabla^g}(s) = \tilde{k}$ is constant for all $s \in \mathbb{Z}^{[1,5]}$. Note that the derivative is undefined at $s = 0$, since $\rho(0) = -1 \notin [0,5]$.

Theorem 3. *[23] Let $Q : \mathbb{T}^{[x,y]} \to \mathbb{E}_n$ be a fuzzy valued function that is continuous at $s \in \mathbb{T}^{[x,y]}$ and suppose $'s'$ is a ls-points, then the following cases may occur:*

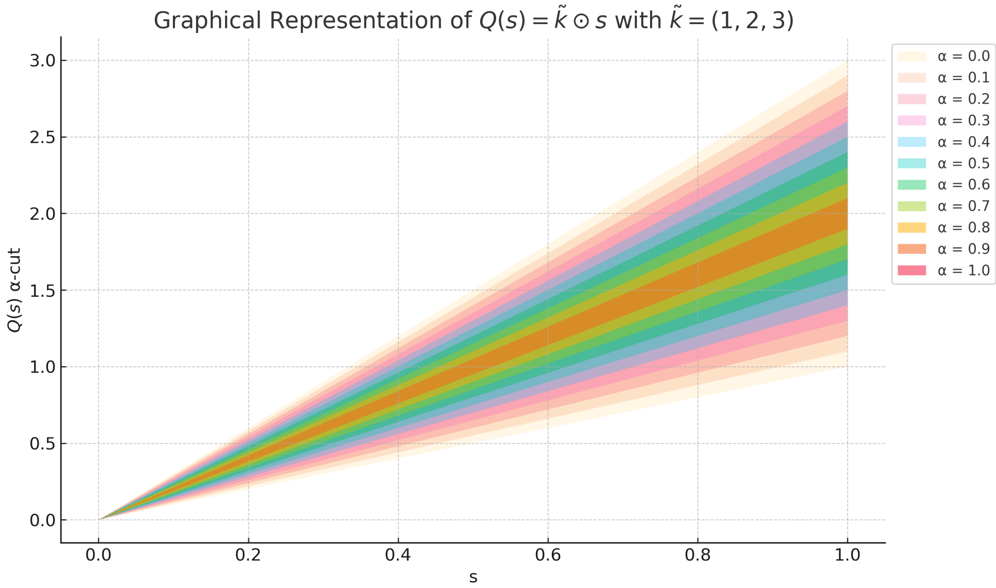

Figure 1: *α-cut representation of the fuzzy derivative $G^{\nabla^g}(s)$ for $s \in [0, 1]$.*

(i) If Q is either $\nabla - GH1$ type differentiable function or $\nabla - GH4$ type differentiable with

$$Q^{\nabla_g}(s) = \frac{1}{\nu(s)} \odot (Q(s) \ominus_h Q^{\rho}(s)) = \frac{-1}{\nu(s)} \odot (Q^{\rho}(s) \ominus_h Q(s)) \in \Re^n.$$

(ii) If Q satisfies $\nabla - GH4$ type differentiable at $'s'$ then

$$Q^{\nabla_g}(s) = \frac{-1}{\nu(s)} \odot (G^{\rho}(s) \ominus_h Q(s)).$$

(iii) If Q is $\nabla - GH4$ type differentiable at $'s'$ then

$$Q^{\nabla_g}(s) = \frac{1}{\nu(s)} \odot (Q(s) \ominus_h Q^{\rho}(s)).$$

Theorem 4. *[23] Let $Q_1, Q_2 : \mathbb{T}^{[x,y]} \to \mathbb{E}_n$ be ∇_g-differentiable at $s \in \mathbb{T}_k^{[x,y]}$. Then:*

(I) If both Q_1 and Q_2 are either ∇-GH1 or ∇-GH4 type differentiable fuzzy functions at left-dense points or at ls-points, then:

(a) $(Q_1 \oplus Q_2)^{\nabla_g}(s) = Q_1^{\nabla_g}(s) \oplus Q_2^{\nabla_g}(s)$

(b) $(Q_1 \ominus_h Q_2)^{\nabla_g}(s) = Q_1^{\nabla_g}(s) \ominus_h Q_2^{\nabla_g}(s)$, provided the HK-Difference exists.

(II) If Q_1 and Q_2 are such that: - one is ∇-GH1 and the other is ∇-GH2 type differentiable at left-dense points, or - one is ∇-GH3 and the other is ∇-GH4 type differentiable at ls-points, then:

(a) $(Q_1 \oplus Q_2)^{\nabla_g}(s) = Q_1^{\nabla_g}(s) \ominus_h [(-1) \odot Q_2^{\nabla_g}(s)]$

(b) $(Q_1 \ominus_h Q_2)^{\nabla_g}(s) = Q_1^{\nabla_g}(s) \oplus [(-1) \odot Q_2^{\nabla_g}(s)]$

(1) If s is left-dense, then the function Q satisfies the following conditions:

(GL1) At $s \in \mathbb{T}^{[x,y]}$, if both $Q^{\rho}(s) \ominus_h Q(s-h)$ and $Q(s+h) \ominus_h Q^{\rho}(s)$ exist for sufficiently small $h > 0$, (GL2) At $s \in \mathbb{T}^{[x,y]}$, if both $Q(s-h) \ominus_h Q^\rho(s)$ and $Q^\rho(s) \ominus_h Q(s+h)$ exist for sufficiently small $h > 0$.

(2) If s is left-scattered, then the function Q satisfies:

(GL3) At $s \in \mathbb{T}_k^{[x,y]}$, if $Q^\rho(s) \ominus_h Q(s)$ exists,

(GL4) At $s \in \mathbb{T}_k^{[x,y]}$, if $Q(s) \ominus_h Q^\rho(s)$ exists.

Theorem 5. *Let $Q : \mathbb{T} \to \mathbb{E}^n$ be a ∇_g-fuzzy differentiable function, and let $h : \mathbb{T} \to \mathbb{R}$ be ∇-differentiable. Then:*

(a) If $h^\rho(s)h^\nabla(s) > 0$ and Q is ∇-GH1-type differentiable function at ld-point $'s'$, or Q is ∇-GH4-type differentiable at ls-point $'s'$, then $h \odot Q$ is ∇_g-differentiable and

$$(h \odot Q)^{\nabla_g}(s) = h(s) \odot Q^{\nabla_g}(s) \oplus h^\nabla(s) \odot Q^\rho(s).$$

(b) If $h(s)h^\nabla(s) < 0$ and Q is ∇-GH4-type differentiable at left-dense or ls-point $'s'$, then

$$(h \odot Q)^{\nabla_g}(s) = h(s) \odot Q^{\nabla_g}(s) \oplus h^\nabla(s) \odot Q^\rho(s).$$

(c) If $h^\rho(s)h^\nabla(s) > 0$, Q is ∇-GH4-type differentiable and $h \odot Q$ satisfies condition (GL1) at left-dense point $'s'$ or condition (GL3) at ls-point $'s'$, then

$$(h \odot Q)^{\nabla_g}(s) = h(s) \odot Q^{\nabla_g}(s) \ominus_h (-h^\nabla(s)) \odot Q^\rho(s).$$

(d) If $h(s)h^\nabla(s) < 0$, Q is ∇-GH1-type differentiable and $h \odot Q$ satisfies (GL1) at left-dense point $'s'$, or Q is ∇-GH4-type differentiable and $h \odot Q$ satisfies (GL3) at a ls-point $'s'$, then

$$(h \odot Q)^{\nabla_g}(s) = h^\nabla(s) \odot Q(s) \ominus_h h^\rho(s) \odot (-Q^{\nabla_g}(s)).$$

(e) If $h^\rho(s)h^\nabla(s) > 0$, Q is ∇-GH4-type differentiable and $h \odot Q$ satisfies (GL2) at left-dense point $'s'$, or (GL4) at ls-point $'s'$, then

$$(h \odot Q)^{\nabla_g}(s) = h^\nabla(s) \odot Q^\rho(s) \ominus_h (-h(s)) \odot Q^{\nabla_g}(s).$$

(f) If $h(s)h^\nabla(s) < 0$, Q is ∇-GH1-type differentiable and $h \odot Q$ satisfies (GL2) at left-dense point $'s'$, or Q is ∇-GH4-type differentiable and $h \odot Q$ satisfies (GL4) at ls-point $'s'$, then

$$(h \odot Q)^{\nabla_g}(s) = Q^{\nabla_g}(s) \odot h^\rho(s) \ominus_h Q(s) \odot (-h^\nabla(s)).$$

Proof. If $'s'$ is left-dense, then $\rho(s) = s$ and $\nu(s) = 0$. The proof of case (a) is analogous to that of Theorem 5 in [6]. We now prove cases $(b) - (f)$.

(b) Assume $h(s) > 0$, $g^\nabla(s) < 0$, and $Q^\rho(s) = Q(s) \oplus u(s)$. Let $h^\rho(s) = h(s) + v(s)$. Then:

$$h^\rho(s) \odot Q^\rho(s) = h(s) \odot Q(s) \oplus h(s) \odot u(s) \oplus v(s) \odot Q(s) \oplus v(s) \odot u(s).$$

So,

$$(h^\rho(s) \odot Q^\rho(s)) \ominus_h (h(s) \odot Q(s)) = h(s) \odot Q^{\nabla_g}(s) \oplus h^\nabla(s) \odot Q^\rho(s).$$

(c) Suppose $h^\rho(s) > 0$, $h^\nabla(s) > 0$, and $Q^\rho(s) = Q(s) \oplus u(s)$. Let $h(s) = h^\rho(s) + v(s)$. Then:

$$(h^\rho(s) \odot Q^\rho(s)) \ominus_h (h(s) \odot Q(s)) = h(s) \odot Q^{\nabla_g}(s) \ominus_h Q^\rho(s) \odot (-h^\nabla(s)).$$

(d) Suppose $h(s) < 0$, $h^\nabla(s) > 0$, and $Q(s) = Q^\rho(s) \oplus u(s)$, so that

$$(h^\rho(s) \odot Q^\rho(s)) \ominus_h (h(s) \odot Q(s)) = h^\nabla(s) \odot Q(s) \ominus_h h^\rho(s) \odot (-Q^{\nabla_g}(s)).$$

(e) Let $h^{\rho}(s) > 0$, $h^{\nabla}(s) > 0$, and $Q^{\rho}(s) = Q(s) \oplus u(s)$. Suppose that

$$Q(s-h) \ominus_h Q^{\rho}(s) \text{ and } Q^{\rho}(s) \ominus_h Q(s+h) \text{ exist.}$$

Since $h \odot Q$ satisfies (GL2), we have:

$$h^{\nabla}(s) \odot Q^{\rho}(s) \ominus_h (-h(s)) \odot Q^{\nabla_g}(s)$$

exists, which proves the formula for case (e).

(f) Suppose $h(s) < 0$, $g^{\nabla}(s) < 0$, and Q is ∇-$GH4$-type differentiable. Let $Q(s) = Q^{\rho}(s) \oplus u(s)$, and assume $h \odot Q$ satisfies condition (GL4), i.e., $Q(s) \ominus_h Q^{\rho}(s)$ exists. Then:

$$(h \odot Q)^{\nabla_g}(s) = Q^{\nabla_g}(s) \odot h^{\rho}(s) \ominus_h Q(s) \odot (-h^{\nabla}(s)).$$

■

Definition 8 ([22]). *Let $Q : \mathbb{T}^{[x,y]} \to \mathbb{E}_n$ be a fuzzy-valued function. If Q admits a left-dense continuous (ld-continuous) measurable nabla-sector on $\mathbb{T}^{[x,y]}$, then Q is said to be ∇-integrable on $\mathbb{T}^{[x,y]}$.*

*The ∇-integral of Q over $\mathbb{T}^{[x,y]}$ is defined **level-wise** as follows:*

$$\left[\int_{\mathbb{T}^{[x,y]}} Q(\tau)\nabla\tau\right]^{\alpha} = \int_{\mathbb{T}^{[x,y]}} Q_{\alpha}(\tau)\nabla\tau = \left\{\int_{\mathbb{T}^{[x,y]}} q(\tau)\nabla\tau : q \in S_{Q_{\alpha}}(\mathbb{T}^{[x,y]})\right\},$$

where $S_{Q_{\alpha}}(\mathbb{T}^{[x,y]})$ denotes the set of all ∇-integrable sectors of Q_{α} on $\mathbb{T}^{[x,y]}$.

Corollary 1. *Let $Q : \mathbb{T}^{[x,y]} \to \mathbb{E}^n$ be ∇_g-integrable and possess a ld-continuous nabla-measurable sector. Then the ∇_g-integral has the following properties:*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\left[\int_{s_0}^{s} Q(\tau)\nabla\tau\right]^{\alpha} = \int_{s_0}^{s} Q_{\alpha}(\tau)\, d\tau.$$

(ii) If $\mathbb{T}$ consists of isolated points, then

$$\int_{s_0}^{s} Q(\tau)\nabla\tau = \begin{cases} \sum_{s\in(a,b]} \nu(s)\, Q(s), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum_{s\in(b,a]} \nu(s)\, Q(s), & \text{if } a > b. \end{cases}$$

Remark 2. *Let $Q : \mathbb{T}^{[x,y]} \to \mathbb{E}^n$ have a ld-continuous nabla-measurable sector. If Q is ∇_g-differentiable on $\mathbb{T}^{[x,y]}$ and Q^{∇} is continuous on $\mathbb{T}^{[x,y]}$, then*

$$\int_{s_0}^{s} Q^{\nabla_g}(\tau)\nabla\tau = Q(s) \ominus_h Q(s_0). \tag{1}$$

(i) If Q is ∇-$GH1$ type differentiable at a left-dense point s, or ∇-$GH4$ type differentiable at ls-points s, then equation (1) is equivalent to:

$$Q(s) = Q(s_0) \oplus \int_{s_0}^{s} Q^{\nabla}(\tau)\nabla\tau.$$

(ii) If Q is ∇-$GH2$ type differentiable at a left-dense or nabla-$GH3$ type differentiable at ls-points s, then equation (1) is equivalent to:

$$Q(s) = Q(s_0) \ominus_h \left[(-1) \odot \int_{s_0}^{s} Q^{\nabla}(\tau)\nabla\tau\right].$$

4. Solutions of fuzzy nabla dynamic equations on time scales

We now establish the *variation of constants formula* to determine solutions of LFNDET's, using the concept of ∇_g-differentiability. This formula plays a fundamental role in constructing explicit solutions to non-homogeneous equations involving fuzzy-valued functions on arbitrary time scales.

Consider the first-order nonhomogeneous dynamic equation:

$$u^{\nabla}(s) = A(s) \odot u(s) \oplus Q(s), \tag{2}$$

with the corresponding homogeneous equation:

$$u^{\nabla}(s) = A(s) \odot u(s). \tag{3}$$

If Q is ld-continuous and the homogeneous equation (3) is regressive, then the non-homogeneous equation (2) is also regressive.

Now, consider the initial value problem:

$$u^{\nabla}(s) = A(s) \odot u(s) \oplus Q(s), \quad u(s_0) = u_0, \tag{4}$$

where $Q : \mathbb{T} \to \mathbb{E}_n$ is ld-continuous, $A : \mathbb{T} \to \mathbb{R}$ is a regressive function, and $u_0 \in \mathbb{E}^n$.

It is important to note that the equation (4) is not equivalent to the following two FNDET's:

$$u^{\nabla}(s) \oplus (-A(s)) \odot u(s) = Q(s), \quad u(s_0) = u_0, \tag{5}$$

and

$$u^{\nabla}(s) \oplus (-Q(s)) = A(s) \odot u(s), \quad u(s_0) = u_0. \tag{6}$$

Each of the equations (4), (5), and (6) yields different solutions, even under the same initial condition. For example, if (4) and (5) were to yield the same solution, we would have:

$$A(s) \odot u(s) \oplus Q(s) \oplus (-A(s)) \odot u(s) = Q(s),$$

which implies

$$A(s) \odot u(s) \oplus (-A(s)) \odot u(s) = \hat{0},$$

leading to $u(s) \in \mathbb{R}^n$, contradicting the fuzzy nature of $u(s)$. Thus, these equations are generally not equivalent in the fuzzy setting.

Moreover, consider the FNDET:

$$u^{\nabla}(s) \oplus (-Q(s)) \oplus (-A(s) \odot u(s)) = \hat{0}, \quad u(s_0) = u_0, \tag{7}$$

which, in the crisp case, is equivalent to one of the previous equations. However, no non-real (fuzzy) solution exists for (7) in general. In practical applications, fuzzy systems are derived from the fuzzification of crisp systems, and thus may yield multiple distinct solutions. These solutions provide different behaviors, allowing us to select one that best captures the imprecision of the system.

The following theorem establishes the variation of constants formula for equation (4) using ∇_g-differentiability.

We now present the variation of constants formula, which provides distinct solutions to LFNDET's using ∇_g-differentiability.

Theorem 6 (Variation of Constants Formula)**.** *Let $s_0 \in \mathbb{T}^{[x,y]}$, $u_0 \in \mathbb{E}^n$, and suppose the following LFNDET's is regressive:*

$$u^{\nabla}(s) = A(s) \odot u(s) \oplus Q(s), \quad u(s_0) = u_0, \tag{8}$$

where $A : \mathbb{T} \to \mathbb{R}$ is a regressive crisp function and $Q : \mathbb{T} \to \mathbb{E}_n$ is ld-continuous.

Define:

$$u_1(s) = \hat{e}_A(s,s_0) \odot \left[u_0 \oplus \int_{s_0}^{s} Q(\tau) \odot \hat{e}_A(s_0,\rho(\tau)) \nabla\tau \right], \tag{9}$$

$$u_2(s) = \hat{e}_A(s,s_0) \odot \left[u_0 \ominus_h \int_{s_0}^{s} (-Q(\tau)) \odot \hat{e}_A(s_0,\rho(\tau)) \nabla\tau \right], \tag{10}$$

provided the HK-Difference in (10) *exists.*

Then:

(a) *If $A(s) > 0$ for all $s \in \mathbb{T}^{[x,y]}$, then $u_1(s)$ is ∇-GH1 type differentiable at left-dense and ∇-GH4 type differentiable at ls-points. Moreover, $u_1(s)$ is a solution to equation* (8).

(b) *If $A(s) < 0$ and the HK-Difference in* (10) *exists, then $u_2(s)$ is ∇-GH4 type differentiable at both left-dense and ls-points. Moreover, $u_2(s)$ is a solution to equation* (8).

(c) *If $A(s) < 0$ and u_1 satisfies conditions (GL1) at left-dense and (GL3) at ls-points (or conditions (GL2) and (GL4), respectively), then u_1 also satisfies one of the following equivalent formulations:*

$$u^{\nabla}(s) \oplus (-A(s)) \odot u(s) = Q(s) \quad \text{or} \quad u^{\nabla}(s) \oplus (-Q(s)) = A(s) \odot u(s).$$

(d) *If $A(s) > 0$, the HK-Difference in* (10) *exists, and u_2 satisfies (GL1) and (GL3) (or (GL2) and (GL4)), then u_2 is also a solution to one of the above equivalent equations.*

Proof. We first prove part (a); the other cases follow similarly by algebraic manipulations involving the properties of ∇-differentiability and HK-Differences.

Let $A(s) > 0$ and $A(s) = \hat{e}_A(s;s_0)$. Since A is regressive, the nabla exponential function $\hat{e}_A(s,s_0)$ is well-defined. We know that $g^{\rho}(s) = (1-\nu(s)A(s))\hat{e}_A(s;s_0)$, $g^{\nabla}(s) = \alpha(s))\hat{e}_A(s;s_0)$. Therefore $g^{\rho}(s)g^{\nabla}(s) = A(s)(1-\nu(s)A(s))\hat{e}_A^2(s;s_0)$.
Let $h_1(s) = u_0 \oplus \int_{s_0}^{s} G(\tau) \odot \hat{e}_\alpha(s_0,\rho(\tau))\nabla\tau$ and $h_2(s) = u_0 \ominus_h \int_{s_0}^{s} (-G(\tau)) \odot \hat{e}_\alpha(s_0,\rho(\tau))\nabla\tau$

We differentiate $u_1(s)$ using the nabla product rule for fuzzy-valued functions:

$$(u_1)^{\nabla}(s) = [\hat{e}_A(s,s_0)]^{\nabla} \odot \left[u_0 \oplus \int_{s_0}^{s} Q(\tau) \odot \hat{e}_A(s_0,\rho(\tau)) \nabla\tau \right] \oplus \hat{e}_A^{\rho}(s,s_0) \odot Q(s).$$

But we know $(\hat{e}_A(s,s_0))^{\nabla} = A(s) \odot \hat{e}_A(s,s_0)$. Substituting, we get:

$$(u_1)^{\nabla}(s) = A(s) \odot \hat{e}_A(s,s_0) \odot \left[u_0 \oplus \int_{s_0}^{s} Q(\tau) \odot \hat{e}_A(s_0,\rho(\tau)) \nabla\tau \right] \oplus Q(s) \odot \hat{e}_A(s,s_0).$$

Now distribute $\hat{e}_A$ over the sum inside the brackets:

$$(u_1)^{\nabla}(s) = A(s) \odot u_1(s) \oplus Q(s).$$

Hence, $u_1(s)$ satisfies equation (8). The differentiability type follows from the fact that $\hat{e}_A$ is crisp and smooth, and the integral is a fuzzy function depending on ld-continuity.

Proofs of parts (b), (c), and (d) follow similar reasoning using the properties of $\ominus_h$, negative scalar multiplication, and the GL-conditions. ■

Theorem 7. *Let $s_0 \in \mathbb{T}^{[x,y]}$, $u_0 \in \mathbb{E}^n$, and suppose equation* (2) *is regressive. Define:*

$$u_1(s) = \hat{e}_A(s,s_0) \odot \left[u_0 \oplus \int_{s_0}^{s} Q(\tau) \odot \hat{e}_A(s_0,\rho(\tau)) \nabla\tau \right], \tag{11}$$

and

$$u_2(s) = \hat{e}_A(s,s_0) \odot \left[u_0 \ominus_h \int_{s_0}^{s} (-Q(\tau)) \odot \hat{e}_A(s_0,\rho(\tau)) \nabla\tau \right], \tag{12}$$

provided the HK-Difference exists in (12). *Then:*

(a) If $A(s) > 0$ and for all left-dense s in $\mathbb{T}^{[x,y]}$, then $u_1(s)$ is ∇-GH1 type differentiable and ∇-GH4 type differentiable at ls-points, and it is a solution to (4).

(b) If $A(s) < 0$ and the HK-Difference in (12) *exists, then $u_2(s)$ is ∇-GH4 type differentiable at both left-dense and ls-points, and it is a solution to* (4).

(c) If $A(s) < 0$ and u_1 satisfies conditions (GL1) at left-dense and (GL3) at ls-points (or GL2 and GL4, respectively), then u_1 is also a solution of (5) *or* (6).

(d) If $A(s) > 0$, the HK-difference in (12) *exists, and u_2 satisfies (GL1) and (GL3) (or GL2 and GL4), then u_2 is a solution of* (5) *or* (6)*, respectively.*

Proof. Let $A(s) = \hat{e}_A(s; s_0)$, $h^\rho(s) = (1 - \nu(s)A(s))\hat{e}_A(s; s_0)$, $h^\nabla(s) = A(s))\hat{e}_A(s; s_0)$.
So, $h^\rho(s))h^\nabla(s) = A(s)(1 - \nu(s)A(s))\hat{e}_A^2(s; s_0)$.
Let $q_1(s) = u_0 \oplus \int_{s_0}^{s} Q(\tau) \odot \hat{e}_\alpha(s_0, \rho(\tau))\nabla\tau$, and $q_2(s) = u_0 \ominus_h \int_{s_0}^{s}(-Q(\tau)) \odot \hat{e}_\alpha(s_0, \rho(\tau))\nabla\tau$.

We know that a constant function is nabla differentiable under any case of ∇_g-differentiability and in view of Theorem 5, it follows that $u_0 \oplus \int_{s_0}^{s} Q(\tau) \odot \hat{e}_\alpha(s_0, \rho(\tau))\nabla\tau$ is ∇-$GH1$ type differentiable function at ld-point s (or) $\nabla - GH4$ type differentiable function at ls-point s and

$$\left(u_0 \oplus \int_{s_0}^{s} Q(\tau) \odot \hat{e}_A(s_0, \rho(\tau))\nabla\tau\right)^{\nabla_g} = Q(s) \odot \hat{e}_\alpha(s_0, \rho(s)). \tag{13}$$

If the HK-Difference $u_0 \ominus_h \int_{s_0}^{s}(-G(\tau)) \odot \hat{e}_\alpha(s_0, \rho(\tau))\nabla\tau$ exists then it is $\nabla - GH4$ type differentiable function at ld-point s and

$$\left(u_0 \ominus \int_{s_0}^{s}(-Q(\tau)) \odot \hat{e}_\alpha(s_0, \rho(\tau))\nabla\tau\right)^{\nabla_g} = Q(s) \odot \hat{e}_\alpha(s_0, \rho(s)). \tag{14}$$

(a) Let $q(s) = \hat{e}_A(s, s_0)$ and $q_1(s) = s_0 \oplus \int_{s_0}^{s} h(\tau) \odot \hat{e}_\alpha(s_0, \rho(\tau))\nabla\tau$, since $A(s) > 0$ for $s \in \mathbb{T}^{[x,y]}$, we have $h^\rho(s)h^\nabla(s) > 0$ and from (13) the conditions in Theorem 5(a) follows and then $u_1^\nabla(s) = A(s) \odot u_1(s) \oplus h(s)$.
i.e. u_1 is a solution of (4).

(b) Since $A(s) < 0$ for $s \in \mathbb{T}^{[x,y]}$, we have $h(s)h^\nabla(s) < 0$. Also, since

$$q_2(s) = u_0 \ominus \int_{s_0}^{s}(-h(\tau)) \odot \hat{e}_A(s_0, \rho(\tau))\nabla\tau$$

is a ∇-GH4 type differentiable function at left-dense or at ls-points on $\mathbb{T}^{[x,y]}$, the conditions in Theorem 5(b) hold. Hence, we obtain

$$u_2^\nabla(s) = A(s) \odot u_2(s) \oplus g(s),$$

i.e., u_2 is a solution of (4).

(c) Since $A(s) < 0$, we have $h(s)h^\nabla(s) < 0$. Suppose u_1 follows the condition $(GL1)$ at left-dense and condition $(GL3)$ at ls-points. Then the assumptions of Theorem 5(d) are applicable. Therefore, we have

$$u_1^\nabla(s) = h^\nabla(s) \odot q_1(s) \ominus_h g^\rho(s) \odot (-1) \odot q_1^{\nabla_g}(s) = h(s) \ominus_h (-A(s)) \odot u_1(s).$$

That is,

$$u_1^\nabla(s) \oplus (-A(s)) \odot u_1(s) = h(s),$$

and hence u_1 is a solution of (5).

Similarly, if u_1 satisfies condition $(GL2)$ at left-dense or condition $(GL4)$ at ls-points, then the conditions of Theorem 5(f) are fulfilled. Thus, we have

$$Q^{\nabla_g}(s) \odot h^\rho(s) \ominus_h (-Q(s)) \odot g^\nabla(s).$$

Also,

$$u_1^{\nabla}(s) = q_1(s) \odot (-1) \odot h^{\nabla}(s) \ominus_h h^{\rho}(s) \odot q_1^{\nabla_g}(s) = A(s) \odot u_1(s) \ominus_h h(s).$$

That is,

$$u_1^{\nabla}(s) \oplus (-h(s)) = A(s) \odot u_1(s),$$

so u_1 is a solution of (6).

(d) If $A(s) > 0$, then $h(s) \odot h^{\nabla}(s) > 0$. Also, from (14), q_2 is a ∇-GH4-type differentiable function at left-dense and at ls-points. If u_2 satisfies condition $(GL1)$ at left-dense and condition $(GL3)$ at ls-points, then the conditions in Theorem 5(c) are satisfied. Therefore,

$$u_2^{\nabla}(s) = A(s) \odot u_2(s) \ominus (-h(s)),$$

which is equivalent to

$$u_2^{\nabla}(s) \oplus (-h(s)) = A(s) \odot u_2(s),$$

so u_2 is a solution of (6).

In the same manner, we can show that if u_2 satisfies condition $(GL2)$ at left-dense and condition $(GL4)$ at ls-points, then the conditions in Theorem 5(e) are fulfilled. Hence, u_2 is a solution of (5).

■

Remark 3. *Theorem 7 shows that the solution to the LFNDTEs is not unique. While this may seem limiting, it allows us to choose the solution that best fits the real system's behavior.*

The following example illustrates the best behavior of the system that allows us to choose the solutions that fits to the problem.

Example 2. *Consider the fuzzy nabla dynamic equation on the time scale* $\mathbb{T} = \mathbb{Z} \cap [0,3]$*:*

$$u^{\nabla}(s) = A(s) \odot u(s) \oplus G(s), \quad s \in \mathbb{T}^{[0,3]},$$

With the fuzzy initial condition:

$$u(0) = \tilde{u}_0 = [1,2,3],$$

where $A(s) = -1$ *,* $g(s) = [-1,0,1]$ *(a triangular fuzzy number), and* $\odot, \oplus$ *denote fuzzy scalar multiplication and addition under generalized Hukuhara arithmetic. so that we have two Possible fuzzy Solutions:*

Solution 1: (Oscillatory response)

$$u_1(s) = \hat{e}_{\alpha}(0,s) \odot \tilde{u}_0 \oplus \sum_{\tau=0}^{s-1} \hat{e}_{\alpha}(\rho(\tau),s) \odot \tilde{g}(\tau),$$

where $\hat{e}_{\alpha}(0,s) = (-1)^s$*. This leads to an oscillatory response due to alternating signs.*

Solution 2: (Cumulative response)

$$u_2(s) = \tilde{u}_0 \oplus \sum_{k=0}^{s-1} (-1)^k \odot \tilde{g}(k),$$

which corresponds to cumulative contributions from fuzzy input over time.

Both $u_1(s)$ *and* $u_2(s)$ *satisfy the same equation and initial condition. However, their behaviors differ:*

- $u_1(s)$*: Suitable for systems with oscillatory dynamics.*
- $u_2(s)$*: Suitable for systems with cumulative or additive effects.*

Thus, the non-uniqueness in Theorem 7 allows us to choose the most appropriate solution for modeling real-world behavior as illustrated igraphically in Figure 2.

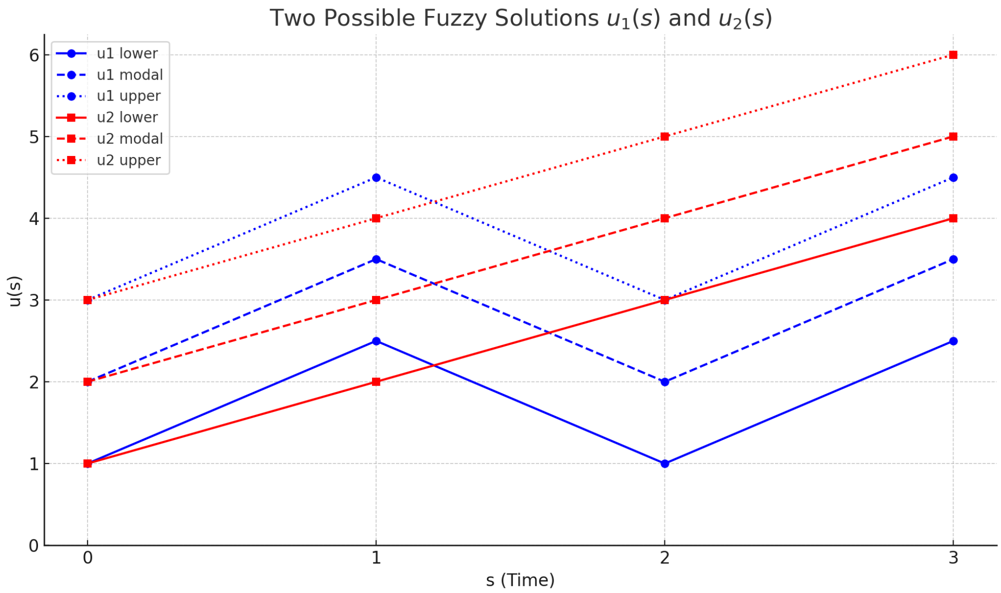

Figure 2: *Graphical comparison of oscillatory and cumulative fuzzy solutions*

5. Conclusions

In this paper, we develop the product rule for crisp and fuzzy functions on time scales, which provides multiple solution forms of LFNDET's and also investigates the solutions of LFNDET's using the variation of constants formula under different versions of the generalized Hukuhara nabla derivative. The numerical examples illustrated prove the theoretical results. The rich and well-structured behavior of these solutions highlights the effectiveness and significance of the generalized nabla derivative in the context of dynamic equations on time scales. The present study lays a strong foundation for a wide range of future research in controllability and Stability analysis in fuzzy environment.

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