FRACTIONAL INTEGRAL INEQUALITIES AND THEIR Q-EXTENSION

Farooq Ahmad[1], Ajjaz Maqbool[2], D.K. Jain[3], R. Vijaykrishnaraj[4], Aafaq A. Rather[5,*], D. Vedavathi Saraja[6], Syed M. Parveen[7], Raeesa Bashir[8], Showkat Ahmad Bhat[9]

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[1]Department of Mathematics, Government College for Women Nawakadal, Srinagar, India
[2]Department of Mathematics, Sheikh-ul-Alam Memorial Degree college Budgam, India
[3]Department of Engineering Mathematics Computing,Madhav Institute of Technology and Science.Gwalior-474005, Madhya Pradesh, India
[4]Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai, India
[5,*]Symbiosis Statistical Institute, Symbiosis International (Deemed University), Pune, India
[6,7]Department of Statistics, Maris Stella College, Vijayawada, India
[8]Department of Statistics, Cluster University, Jammu and Kashmir, India
[9]Symbiosis School of Economics, Symbiosis International (Deemed University), Pune, India
[1]sheikhfarooq85@gmail.com, [2]ajjazmaqbool013@gmail.com, [4]drvjmaths@gmail.com,
[5,*]aafaq7741@gmail.com, [6]sarajayoganand@gmail.com, [7]parveen@marisstella.ac.in,
[8]raisabashir52@gmail.com [9]showkat.bhat@sse.ac.in

Abstract

This paper aims to establish novel fractional integral inequalities for synchronous functions associated with the Chebyshev functional, incorporating the Gauss hypergeometric function. By employing advanced integral techniques, we derive refined bounds that extend and generalize existing results in the literature. The final section explores several special cases, particularly fractional integral inequalities involving Riemann-Liouville type fractional integral operators. Furthermore, we discuss potential applications of these findings in various mathematical and applied fields, highlighting their significance in fractional calculus and related domains.

Keywords: Synchronous functions, Fractional integral inequalities & Saigo operator.

1. Introduction

The most beneficial uses of fractional integral inequalities are in determining the uniqueness of solutions to fractional boundary value issues and fractional partial differential equations. Additionally, they offer upper and lower bounds for the solutions of the aforementioned equations. These factors have prompted a number of scholars working in the area of integral inequalities to investigate various extensions and generalisations by utilising fractional calculus operators. For instance, the book [1] and the publications [2-11] both contain references to such works.

Purohit and Raina [9] recently looked into some integral inequalities of the Chebyshev type [12] utilising Saigo fractional integral operators and established the q-extensions of the main findings. This study uses the fractional hypergeometric operator proposed by Curiel and Galue [13] to prove a few generalized integral inequalities for synchronous functions related to the Chebyshev functional. As special examples of our findings, the results attributed to Purohit and Raina [9] and Belarbi and Dahmani [2] are presented below.

2. The terms and definitions used in this paper are as follows

Definition 2.1 : Two functions f & g are said to be synchronous on [a, b] if

$$(f(x)-f(y))(g(x)-g(y))[a,\, b] \tag{1}$$

Definition 2.2: A real-valued function f(t) (t> 0) is said to be in the space C_μ ($\mu \in$ R) if there exists a real number p $> \mu$, such that f(t) $= \mathrm{t^p}\emptyset(\mathrm{t})$, where $\emptyset(\mathrm{t}) \in \mathrm{C}(0,\infty)$.

2.1 Saigo's Fractional operator

Useful and interesting generalization of both the Riemann-Liouville and Erdelyi-Kober fractional integration operators is introduced by Saigo [13], in terms of Gauss's hypergeometric function as given below

Let α, βand ηare complex numbers ∈ C and let $x \in$ R₊ the fractional Re (α)>0 and the fractional derivative Re(α)<0 of the first kind of a function

$$I_{0,x}^{\alpha,\beta,n} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1}\, 2F_1[\alpha+\beta,-n;1-\frac{t}{x}]\, f(t)\,dt,\ \mathrm{Re}(\alpha)>0 \tag{2}$$

$$= \frac{d^n}{dx^n} I_{0,x}^{\alpha+n,\beta+n,\eta-n} f(x),\ 0<\mathrm{Re}(\alpha)+\eta \le 1\ (n \in N_0) \tag{3}$$

$$I_{0,x}^{\alpha,\beta,n,\mu} f(x) = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma\alpha} \int_0^x \tau^{\mu}(x-t)^{\alpha-1}\, 2F_1[\alpha+\beta,-n;1-\frac{t}{x}]\, f(t)\,dt \tag{4}$$

2.2 Main Results

In this section, we obtain certain Chebyshev type integral inequalities involving the generalized fractional integral operator. The following lemma is used for our first result.

Lemma: let **α** > 0, μ >-1, β, η ∈ R; then the following image formula for the power function under the operator (1.2) holds true:

$$I_{0,x}^{\alpha,\beta,n}(x^{\tau-1}) = \frac{\Gamma(\tau)\Gamma(\tau-\beta+\eta)}{\Gamma(\tau-\beta)\Gamma(\tau+\alpha+\eta)} x^{\tau-\beta-\mu-1} \tag{5}$$

Theorem 1: Let f&g be two synchronous functions on$(0,\infty)$ then

$$I_{0,x}^{\alpha,\beta,n}(f(x)g(x)) \ge \frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}{\Gamma(1-\beta+\eta)} x^{\beta+\mu} \times I_{0,x}^{\alpha,\beta,n}(f(x))\, I_{0,x}^{\alpha,\beta,n}(g(x)) \tag{6}$$

For all $x>0, \alpha>\max\{0,-\beta-\mu\}, \beta<1, \mu>-1, \eta<0$.

Proof: Let f&g be two synchronous functions, then using definition1, for all $\tau, \rho \epsilon (0,t), t \geq 0$, we have

$$(f(\tau)-f(p))(g(\tau)-g(p))\geq 0 \tag{7}$$

which implies that

$$\left(f(\tau)g(\tau)+f(p)g(p)\right)\geq (f(\tau)g(p)+f(p)g(\tau)) \tag{8}$$

Consider

$$F(x,\tau)=\frac{x^{-\alpha-\beta}(x-t)^{\alpha-1}}{\Gamma\alpha}2F_1[\alpha+\beta,-n;1-\frac{\tau}{x}]f(t)dt \tag{9}$$

$$=\frac{(x-\tau)^{\alpha-1}}{\Gamma(ax)^{\alpha+\beta}}\times\frac{(\alpha+\beta)(-\eta)(x-\tau)^{\alpha}}{\Gamma(\alpha+1)x^{\alpha+\beta}}+\frac{(\alpha+\beta+\mu)(\alpha+\beta+1)(x-\tau)^{\alpha-1}}{\Gamma(\alpha+2)}\times\frac{(x-\tau)^{\alpha+1}}{x^{\alpha+\beta+2}} \tag{10}$$

Our observation is that each term of the above series is positive in view of the conditions stated with Theorem 1.

$$I_{0,x}^{\alpha,\beta,n}(f(x)g(x))\geq\frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}{\Gamma(1-\beta+\eta)}x^{\beta}\times I_{0,x}^{\alpha,\beta,n}(f(x))\,I_{0,x}^{\alpha,\beta,n}(g(x)) \tag{11}$$

In light of the circumstances outlined in theorem 2.2.1, we note that each term in the afore mentioned series is positive. On integrating from 0 to τ and using (2.1.2), we get

$$(I_{0,x}^{\alpha,\beta,\eta,\mu}f(x)g(x))+\left(f(p)g(p)I_{0,x}^{\alpha,\beta,\eta}\{1\}\right)\geq g(p)I_{0,x}^{\alpha,\beta,\eta}f(x)+(f(p)I_{0,x}^{\alpha,\beta,\eta}g(x)) \tag{12}$$

Theorem 2: let f & g be two synchronous functions on $(0,\infty)$, then

$$\begin{aligned}&\Rightarrow\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)x^{\beta+1}}I_{0,x}^{\delta,\gamma,\theta,\varepsilon}(f(x)g(x))\\&+\frac{\Gamma(1+\theta)\Gamma(1-\delta+\varepsilon)}{\Gamma(1-\delta)\Gamma(1+\theta+\gamma+\varepsilon)x^{\beta+\mu}}I_{0,x}^{\alpha,\beta,\eta,\mu}(f(x)g(x))\\&\geq(I_{0,x}^{\alpha,\beta,\eta,\mu}f(x)I_{0,x}^{\delta,\gamma,\theta,\varepsilon}g(x))+I_{0,x}^{\delta,\gamma,\theta,\varepsilon}g(x)(f(p)I_{0,x}^{\alpha,\beta,\eta,\mu}f(x))\end{aligned} \tag{13}$$

For all $x>0,\alpha>\max\{0,-\beta-\mu\},\gamma>Max(0,-\delta,-\theta),\mu,\theta>-1.$

Proof: To prove the above theorem, we use previous theorem , we have

$$\frac{x^{-\gamma-2\theta-\delta}p^{\delta}(x-p)^{\gamma-1}}{\Gamma\gamma}2F_1[\delta+\gamma+\theta,-\varepsilon;1-\frac{p}{x}]f(p)dp \tag{14}$$

which, given the circumstances mentioned with, continues to be positive as per above theorem and we obtain by integrating with regard to from 0 to t.

$$\begin{aligned}&(I_{0,x}^{\alpha,\beta,\eta,\mu}f(x)g(x))I_{0,x}^{\delta,\gamma,\theta,\varepsilon}\{1\}+I_{0,x}^{\delta,\gamma,\theta,\varepsilon}\left(f(x)g(x)I_{0,x}^{\alpha,\beta,\eta,\mu}\{1\}\right)\\&\geq I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\}I_{0,x}^{\delta,\gamma,\theta,\varepsilon}\{g(x)\}+I_{0,x}^{\delta,\gamma,\theta,\varepsilon}(g(x)I_{0,x}^{\alpha,\beta,\eta,\mu}f(x))\end{aligned} \tag{15}$$

3. q-Extension

Recently, q calculus has served as a bridge between mathematics and physics. Consequently, there has been a significant increase in activity in the field of q-calculus due to applications of q-calculus in mathematics, statistics, and physics. Most of the scientists in the world who use the q -calculation today are physicists. The q- calculus is a generalization of many topics, such as chain metaphysics, generating functions, complexity analysis, and particle physics. In short, q -calculation is a very popular topic today. One of the important branches of q -calculus in number theory is the q- class of special generating functions, such as q-Bernoulli numbers, q-Euler numbers and q-Genocchi numbers. Here we define a new class of multivariable integrals of type q (generalizations of beta integrals) called Saigo's q- Integral Operator.

The a Saigo's q- Integral Operator which is analogue to the well known Bernardi integral operator is investigated. In the following, we obtain a simple conceptual q extension of the results obtained in the first part of the paper, using some q-Saigo's q- Integral Operator.

3.1 Saigo's q- Integral Operator

A basic analogue of Saigo's fractional integral operator [29] is defined as

$$I_{0,x}^{\alpha,\beta,n} f(x) = \frac{x^{-\beta-1}}{\Gamma q^{(\alpha)}} \int_0^x \left(\frac{tq}{x};q\right)_{\alpha-1} \times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta},q)_m (q^{-n},q)_m}{(q^{\alpha},q)_m (q,q)_m} q^{(\eta-\beta)m} (-1)^m (-q)^{\binom{-m}{2}} \left(\frac{t}{x}-1\right)_m f(t)\, d_q(t)$$

$$f(x) = \frac{q^{-\binom{\alpha}{2}-\beta}}{\Gamma q^{(\alpha)}} \int_0^x \left(\frac{x}{t};q\right)_{\alpha-1} t^{-\beta-1} \times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta},q)_m (q^{-n},q)_m}{(q^{\alpha},q)_m (q,q)_m} q^{(\eta-\beta)m} (-1)^m (-q)^{\binom{-m}{2}} \left(\frac{x}{qt}-1\right)_m f(tq^{1-\alpha})\, d_q(t)$$

By making use q- integral definition, the above operators can be written as

$$I_{0,x}^{\alpha,\beta,n} f(x) = x^{-\beta} (1-q)^{\alpha} \times T \tag{16}$$

where, $T = \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta},q)_m (q^{-n},q)_m}{(q^{\alpha},q)_m (q,q)_m} q^{(\eta-\beta+1)m} \sum_{k=0}^{\infty} q^k \frac{(q^{\alpha+\beta},q)_k (q^{-n},q)_m}{(q,q)_m} f(xq^{k+m})$

and $K_q^{\alpha,\beta,n} f(x) = x^{-\beta} (q)^{\binom{-\alpha}{2}} (1-q)^{\alpha} \times T$

4. Main Results

In this section we obtain certain Chebyshev type integral inequalities involving the generalized q- fractional integral operator. The following lemma is used for our first result.
Theorem 3.1: Let f&g be two synchronous functions on$(0,\infty)$ then

$$I_q^{\alpha,\beta,n}(f(x)g(x)) \geq \frac{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)}{\Gamma_q(1-\beta+\eta)} x^{\beta+\mu} \times I_q^{\alpha,\beta,n}(f(x)) I_q^{\alpha,\beta,n}(g(x)) \tag{17}$$

For all $x>0, \alpha > \max\{0,-\beta-\mu\}, \beta<1, \mu>-1, \eta<0.$

Proof: Let f&g be two synchronous functions ,then using definition1, for all $\tau, \rho \epsilon (0,t), t \geq 0$, we have

$$\big((f(\tau)+f(p)g(\tau)\big)-g(p)) \geq 0 \tag{18}$$

which implies that

$$\big(f(\tau)g(\tau)+f(p)g(p)\big) \geq (f(\tau)g(p)+f(p)g(\tau)) \tag{19}$$

Consider

$$F(x,\tau) = \frac{x^{-\alpha-\beta-2\mu}\tau^{\mu}(x-t)^{\alpha-1}}{\Gamma q^{\alpha}} 2F_1[\alpha+\beta+\mu,-n;1-\frac{\tau}{x}] f(t)\,dt \tag{20}$$

$$= \frac{\tau^{\mu}(x-\tau)^{\alpha-1}}{\Gamma q^{\alpha}(x)^{\alpha+\beta+2\mu}} \times \frac{\tau^{\mu}(\alpha+\beta+\mu)(-\eta)(x-\tau)^{\alpha}}{\Gamma_q(\alpha+1)x^{\alpha+\beta+2\mu}} + \frac{\tau^{\mu}(\alpha+\beta+\mu)(\alpha+\beta+\mu+1)(x-\tau)^{\alpha-1}}{\Gamma_q(\alpha+2)} \times \frac{(x-\tau)^{\alpha+1}}{x^{\alpha+\beta+2\mu+2}}$$

Our observation is that each term of the above series is positive in view of the conditions stated with Theorem 3.1.

$$I_q^{\alpha,\beta,n}(f(x)g(x)) \geq \frac{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)}{\Gamma_q(1-\beta+\eta)} x^{\beta+\mu} \times I_q^{\alpha,\beta,n}(f(x)) I_q^{\alpha,\beta,n}(g(x)) \tag{21}$$

In light of the circumstances outlined in Theorem 3.1, we note that each term in the afore mentioned series is positive. On integrating from 0 to τ and using 2.1.2 , we get

$$(I_q^{\alpha,\beta,\eta} f(x)g(x)) + \left(f(p)g(p)I_q^{\alpha,\beta,\eta}\{1\}\right) \geq g(p)I_q^{\alpha,\beta,\eta} f(x) + (f(p)I_q^{\alpha,\beta,\eta} g(x)) \tag{22}$$

Theorem 3.2: let f & g be two synchronous functions on $(0,\infty)$, then

$$\begin{aligned} &\Rightarrow \frac{\Gamma_q(1+\mu)\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\mu+\alpha+\eta)x^{\beta+\mu}} I_q^{\delta,\gamma,\theta,\varepsilon}(f(x)g(x)) \\ &+ \frac{\Gamma_q(1+\theta)\Gamma_q(1-\delta+\varepsilon)}{\Gamma_q(1-\delta)\Gamma_q(1+\theta+\gamma+\varepsilon)x^{\beta+\mu}} I_q^{\alpha,\beta,\eta,\mu}(f(x)g(x)) \\ &\geq (I_q^{\alpha,\beta,\eta,\mu} f(x) I_q^{\delta,\gamma,\theta,\varepsilon} g(x)) + I_q^{\delta,\gamma,\theta,\varepsilon} g(x) I_q^{\alpha,\beta,\eta,\mu} f(x)) \end{aligned} \tag{23}$$

For all $x > 0, \alpha > \max\{0, -\beta-\mu\}, \gamma > Max(0, -\delta, -\theta), \mu, \theta > -1.$

Proof: To prove the above theorem, we use previous theorem , we have

$$\frac{x^{-\gamma-2\theta-\delta} p^{\delta}(x-p)^{\gamma-1}}{\Gamma_q \gamma} 2F_1[\delta+\gamma+\theta, -\varepsilon; 1-\frac{p}{x}] f(p)dp \tag{24}$$

which, given the circumstances mentioned with, continues to be positive as per above theorem and we obtain by integrating with regard to from 0 to t.

$$\begin{aligned} &(I_q^{\alpha,\beta,\eta,\mu} f(x)g(x)) I_q^{\delta,\gamma,\theta,\varepsilon}\{1\} + I_q^{\delta,\gamma,\theta,\varepsilon}\left(f(x)g(x)I_q^{\alpha,\beta,\eta,\mu}\{1\}\right) \\ &\geq I_q^{\alpha,\beta,\eta,\mu}\{f(x)\} I_q^{\delta,\gamma,\theta,\varepsilon}\{g(x)\} + I_q^{\delta,\gamma,\theta,\varepsilon}(g(x)I_q^{\alpha,\beta,\eta,\mu} f(x)) \end{aligned} \tag{25}$$

4.1 Special cases

Here, we take a quick look at a few implications of the findings in the preceding section. The operator (3) would instantly decrease to the thoroughly studied, Erdelyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, according to Curiel and Galué [13], given by the relationships below (see also [14])

$$\text{Case I: } I_{0,x}^{\alpha,\beta,n} f(x) = I_{0,x}^{\alpha,\beta,\eta,0} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-\tau)^{\alpha-1} 2F_1\left(\alpha+\beta, -\eta; 1-\frac{\tau}{x}\right) f(t)\, d(t) \tag{26}$$

Here we get, Erdelyi-Kober fractional operator.

$$I_{0,x}^{\alpha,n} f(x) = I_{0,x}^{\alpha,0,\eta,0} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-\tau)^{\alpha-1} \tau^n f(t)\, d(t) \tag{27}$$

Case II: Here we get, Riemann-Liouville fractional operator

$$R^{\alpha} f(x) = I_x^{\alpha,-\alpha,\eta,0} f(x) = \frac{1}{\Gamma\alpha} \int_0^x (x-\tau)^{\alpha-1} f(t)\, d(t) \tag{28}$$

Case III: Putting γ=α, δ = β, ξ = η & ϑ = μ in theorem 3.1 reduces into theorem (2)

$$(I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)g(x))I_{0,x}^{\delta,\gamma,\theta,\varepsilon} + I_{0,x}^{\delta,\gamma,\theta,\varepsilon}\left(f(x)g(x)I_{0,x}^{\alpha,\beta,\eta,\mu}\{1\}\right)$$
$$\geq I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\}I_{0,x}^{\delta,\gamma,\theta,\varepsilon}\{g(x)\} + I_{0,x}^{\delta,\gamma,\theta,\varepsilon}(g(x)I_{o,x}^{\alpha,\beta,\eta,\mu} f(x)) \tag{29}$$

Puttingγ=α, δ = β, ξ=η&ϑ = μ , we get

$$(I_{0,x}^{\alpha,\beta,\eta,\mu} f(x)g(x)) + \left(f(p)g(p)I_{0,x}^{\alpha,\beta,\eta,\mu}\right)$$
$$\geq g(p)I_{0,x}^{\alpha,\beta,\eta,\mu}\{f(x)\} + f(p)I_{0,x}^{\alpha,\beta,\eta,,\mu}(g(x) \tag{30}$$

5. Conclusion

Since the beginning of differential calculus, fractional calculus (FC) has been used in mathematical theory. However, throughout the past 20 years, FC's application has emerged as a result of the progress in chaos, demonstrating minute connections to FC tenets. FC has recently enjoyed success in the fields of science and engineering. Several scientific sectors are paying more for research topics. now pay attention to the FC ideas. There has been some work done. in the theory of dynamic systems, even if the suggested models Such algorithms are still being developed at a young stage.

Several case studies on FC based models were provided in the text. controls that highlight the benefits of utilising FC theory in a variety of scientific and engineering disciplines. This piece examined a number of physical systems, in particular.

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