

IN FUZZY MENER SPACE, CERTAIN FPT USING OCCASIONALLY WEAKLY COMPATIBLE MAPPING THAT USES A-CLAS FUNCTION

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Abstract

In this paper, we present new common fixed point theorems in the setting of fuzzy Menger spaces by considering self-mappings that satisfy the condition of occasional weak compatibility. The analysis is enriched through the incorporation of A-class functions, which serve as control functions in the formulation of contractive type conditions. This approach allows for the generalization and extension of several known results in the field of fuzzy fixed point theory. The structure of fuzzy Menger spaces, combined with the flexibility of A-class functions, enables the establishment of sufficient conditions for the existence and uniqueness of common fixed points among a family of self-mappings. To support the theoretical results, we also provide illustrative examples that demonstrate the applicability and effectiveness of the proposed theorems. The findings contribute to the broader understanding of fixed point results in fuzzy environments and highlight the utility of occasionally weakly compatible mappings under generalized contractive settings.

Keywords: Fuzzy Menger Space, Ccommon Fixed Point, Occasionally Weakly, A-Class Function.

I. Introduction

Fixed point theory in fuzzy settings has attracted considerable interest due to its wide range of applications in mathematical modeling under uncertainty. The notion of fuzzy Menger space, introduced by Kramosil and Michalek [4] and later extended by Schweizer and Sklar [15], provides a suitable framework for analyzing convergence and continuity in fuzzy environments. Within this context, various generalizations of compatibility among mappings have been studied to establish fixed point results under relaxed conditions.

One such generalization is the concept of occasionally weakly compatible (OWC) mappings, introduced by Al-Thagafi and Shahzad [1], which weakens the usual commutativity assumption and allows the mappings to commute at their coincidence points. Additionally, the use of A-class functions, as explored in the works of Singh and others [26], has enabled the formulation of broader contractive conditions beyond classical Lipschitz-type inequalities. Shrivastava, K. et al.[28] prove certain popular fixed point theorems for new rational expression eight mapping through occasionally weakly compatible maps interval-valued fuzzy metric space.

Motivated by these developments, this paper investigates new common fixed point theorems for self-mappings in fuzzy Menger spaces using the framework of occasionally weakly compatible

mappings and contractive conditions defined via A-class functions. Our results extend and generalize several existing theorems in fuzzy fixed point theory.

II. Preliminaries

we give some basic definitions which are useful for main result in this paper.

Definition 2.1[25] Let R represent the real's and R^+ represent the reals that are not negative. If a mapping $F: R \rightarrow R^+$ is non-decreasing and left continuous, $\inf\{F(t): t \in R\} = 0$ and $\sup\{F(t): t \in R\} = 1$, it is considered a distribution function.

Definition 2.2[25] A binary relation $t: [0,1] \times [0,1] \rightarrow [0,1]$ is called t -norm if it satisfies the following properties:

- (i) $t(0,0) = 0$ and $t(a,1) = a \quad \forall a \in [0,1]$;
- (ii) $t(a,b) = t(b,a) \quad \forall a, b \in [0,1]$;
- (iii) $t(t(a,b),c) = t(a,t(b,c)) \quad \forall a, b, c, d \in [0,1]$;
- (iv) $t(a,b) \leq t(c,d)$, whenever $a \leq c$ and $b \leq d \quad \forall a, b, c, d \in [0,1]$

Definition 2.3[25] An ordered pair (Ω, F_α) consisting of a non-empty set X and F_α form $\Omega \times \Omega$ into the collection of all fuzzy distribution functions, $F_\alpha \in R, \forall \alpha \in [0,1]$ is called fuzzy probabilistic metric space (FPM-space). The fuzzy distribution function is expressed by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(t)$ is the value of $F_{\alpha(x,y)}$ at t in R and following properties are met :

- (i) $F_{\alpha(x,y)}(0) = 0, \forall x, y \in \Omega$
- (ii) $F_{\alpha(x,y)}(t) = 1, \forall t > 0 \Leftrightarrow x = y$
- (iii) $F_{\alpha(x,y)}(t) = F_{\alpha(y,x)}(t), \forall x, y \in \Omega$
- (iv) if $F_{\alpha(x,y)}(s) = 1$ and $F_{\alpha(y,z)}(r) = 1 \Rightarrow F_{\alpha(x,z)}(s+r) = 1, \quad \forall x, y, z \in \Omega, s, r > 0$.

Definition 2.4[25] A fuzzy Menger Space is a triplet (Ω, F_α, T) , Where (Ω, F_α, t) is a fuzzy Probabilistic Metric Space (FPM -Space) and T is a t-norm such that for all $x, y, z \in \Omega$ and all $t_1, t_2 \geq 0$ and $\alpha \in [0,1]$, $F_{\alpha(x,z)}(s+t) \geq t(F_{\alpha(x,y)}(s), F_{\alpha(y,z)}(t))$.

Definition 2.5[25] Let (Ω, F_α, t) be a fuzzy menger space, if $x \in \Omega, \varepsilon > 0$ and $\lambda \in (0,1)$ then (ε, λ) - neighborhood of x , called $U_x(\varepsilon, \lambda)$ defined by $U_x(\varepsilon, \lambda) = \{y \in \Omega : F_{\alpha(x,y)}(\varepsilon) > (1 - \lambda)\}$
 An (ε, λ) - topology in Ω is the topology induced by the family $\{U_x(\varepsilon, \lambda) : x \in \Omega, \varepsilon > 0, \alpha \in [0,1] \text{ and } \lambda \in (0,1)\}$ of neighborhood.

Definition 2.6[25] Let (Ω, F_α, t) be a fuzzy menger space, Where T is a continuous t-norm then

- (i) A sequence $\{x_n\}$ in Ω is said to be convergent if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $x_n \in U_x(\varepsilon, \lambda) \quad \forall n \geq N$ or equivalently $F_\alpha(x_n, x, \varepsilon) > 1 - \lambda$ for all $n \geq N$ and $\alpha \in [0,1]$.
- (ii) A sequence $\{x_n\}$ in Ω is said to be a Cauchy Sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $F_\alpha(x_n, x_m, \varepsilon) > 1 - \lambda$ for all $n, m \geq N$ and $\alpha \in [0,1]$.
- (iii) A Menger space (Ω, F_α, t) is said to be Complete if every Cauchy Sequence in X converges to a point in Ω for all $\alpha \in [0,1]$.

Lemma 2.7[20] Let $(X, F_{\alpha}, *)$ be a fuzzy Menger space and $x, y \in X$. If there exists a constant $k \in (0, 1)$ such that $F_{\alpha(x,y)}(kt) \geq F_{\alpha(x,y)}(t)$ for all $t > 0$ then $x = y$

Lemma 2.8[20] Let $\{y_n\}$ be a sequence in fuzzy Menger space $(X, F_{\alpha}, *)$ with continuous t-norm and $t * t \geq t$, for all $t \in [0, 1]$ such that $F_{\alpha(y_n, y_{n+1})}(kt) \geq \min(F_{\alpha(y_{n-1}, y_n)}(t), F_{\alpha(y_n, y_{n+1})}(t))$.

For all $t > 0$ and $n \in \mathbb{N}$. then $\{y_n\}$ is a Cauchy sequence in X .

Lemma 2.9[20] Let $\{x_n\}$ be a sequence in fuzzy Menger space $(X, F_{\alpha}, *)$ with continuous t-norm and $t * t \geq t$, if there exists in a constant $k \in (0, 1)$ such that $F_{\alpha(x_n, x_{n+1})}(kt) \geq F_{\alpha(x_n, x_{n+1})}(t)$

For all $t > 0$ and $n = 1, 2, \dots$, then $\{x_n\}$ is a Cauchy sequence in X .

Lemma 2.10[25] Let (Ω, d) is a metric space then d induces a mapping $F_{\alpha}: \Omega \times \Omega \rightarrow L$ defined by $F_{\alpha(p,q)} = H_{\alpha}(x - d(p, q): p, q \in R$ for all $\alpha \in [0, 1]$. Further if $t: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is defined by $t(a, b) = \text{Min}\{a, b\}$ then (Ω, F_{α}, t) is a fuzzy Menger space. It is complete if and only if (Ω, d) is complete.

Definition 2.11[11] Let (Ω, F_{α}, t) is fuzzy Menger space then the mappings $A, B: \Omega \rightarrow \Omega$ are said to be (E, A) like property if there exists a sequence $\{x_n\} \in \Omega$ such that $Ax_n, Bx_n \rightarrow z$ for some $z \in A(\Omega) \cup B(\Omega)$ as $n \rightarrow \infty$.

Definition 2.12[25] Let (Ω, F_{α}, t) is fuzzy Menger space then the mappings $A, B: \Omega \rightarrow \Omega$ are said to be property (E, A) if there exists a sequence $\{x_n\} \in \Omega$ such that $Ax_n, Bx_n \rightarrow z$ for some $z \in \Omega$ as $n \rightarrow \infty$.

Definition 2.13[11] Let (Ω, F_{α}, t) is fuzzy Menger space then the mappings $A, B: \Omega \rightarrow \Omega$ are said to be CLR_B - property if there exists a sequence $\{x_n\} \in \Omega$ such that $Ax_n, Bx_n \rightarrow z$ for some $z \in \Omega$ as $n \rightarrow \infty$.

Definition 2.14[3] Let (Ω, F_{α}, t) is fuzzy Menger space then the mappings $A, B: \Omega \rightarrow \Omega$ are said to be weakly compatible if these are commuting at their coincidence points.

Definition 2.15[3] Let (Ω, F_{α}, t) is fuzzy Menger space then the mappings $A, B: \Omega \rightarrow \Omega$ are said to be weakly compatible if there is a coincidence point at which the mapping commutes.

Definition 2.16[6] (A-Class function) : A mapping $\hat{A}: (0, 1) \times (0, 1) \rightarrow R^+$ is continuous and hold the following conditions:

$$(A_a) \quad \hat{A}(u, v) \geq u;$$

$$(A_b) \quad \hat{A}(u, v) = u \Rightarrow \text{either } u = 1 \text{ or } v = 1;$$

$$(A_c) \quad \hat{A}(1, 1) = 1;$$

Then \hat{A} is called \hat{A} -class function.

Definition 2.17[6] Let φ be a class of mapping $\varphi: [0, 1] \rightarrow [0, 1]$:

$$(\varphi_a) \quad \varphi \text{ is continuous and non-decreasing on } [0, 1];$$

$$(\varphi_b) \quad \varphi(x) > x \text{ for all } x \in (0, 1);$$

$$(\varphi_c) \quad \varphi(0) = 0 \text{ and } \varphi(1) = 1 \text{ for all } x \in (0, 1);$$

Definition 2.18[6] Let $\omega: [0, 1] \rightarrow [0, 1]$ is monotonic increasing function with, $\omega(t) > t$ and $\omega(0) = 0$ and $\omega(1) = 1$.

The given theorem is proved by S. D. Diwan and Raja [2] and applied the concepts of expansion mappings and CLR_B-property and extracted fixed point results

Theorem[3]. Given a complete fuzzy PM-space $(\Omega, F\alpha, t)$ with continuous t-norm and t defined by $t(a, b) = \min\{a, b\}$ for $a, b \in [0, 1]$, let $G, H, P,$ and Q be self-mapping. They must also meet the following requirements:

- (1) $G(\Omega) \subseteq Q(\Omega), H(\Omega) \subseteq P(\Omega)$;
- (2) (G, P) satisfies CLR_p property or (H, Q) satisfies CLR_Q property;
- (3) (G, P) and (H, Q) are weakly compatible.
- (4) One of the $G(\Omega), H(\Omega), P(\Omega)$ or $Q(\Omega)$ is a closed subset of Ω .
- (5) there exists a constant $k > 1$ such that $(F_{\alpha(Gx, Hy)}(kt) \leq F_{\alpha(Px, Qy)}(t)$ for all $x, y \in \Omega, \alpha \in [0, 1]$ and $t > 0$. Then G, H, P and Q have a UCFP in Ω .

In this paper we generalized as a weaker condition for this we use the occasionally weakly compatible such as A-Class Function, to prove a few fixed point results.

III. Main Results

Theorem 3.1 Let G, P, H, Q, S and T are six self-mapping on a fuzzy Menger space (X, F_{α^*}) , where $*$ is the min t – norm and satisfying the following Conditions:

- (i) $G(X) \subseteq TH(X), P(X) \subseteq SQ(X)$
- (ii) $\varphi\{F_{\alpha(Gx, Py)}(t)\} \geq \mathring{A}[\varphi(F_{\alpha(x, y)}(t)), \omega(F_{\alpha(x, y)}(t))]$
 Where, $F_{\alpha(x, y)}(t) = \omega[\text{Min}\{F_{\alpha(Gx, SQx)}(t), F_{\alpha(Py, THy)}(t), F_{\alpha(Gx, THy)}(t), F_{\alpha(SQx, Py)}(2t)\}]$
 For all $x, y \in X, t > 0, \varphi \in Z. (G, SQ)$ and (P, TH) are Weakly compatible;
- (iii) (G, SQ) and (P, TH) Share one of the CLR_{SQ}, CLR_{TH} and $CLR_{(SQ)(TH)}$ – property.
- (iv) Then G, P, H, Q, S and T have a Ucfp in X .

Proof:

Case (I) - Since (G, SQ) share CLR_{SQ} –property. so, by the definition, there exist sequences $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Gx_n = \lim_{n \rightarrow \infty} SQx_n = z, \quad z \in SQ(X)$$

Since $G(X) \subseteq TH(X)$ then exist sequences $\{y_n\}$ in X such that $Gx_n = THy_n$

$$i.e \lim_{n \rightarrow \infty} Gx_n = \lim_{n \rightarrow \infty} THy_n = z$$

Now we claim $\lim_{n \rightarrow \infty} Py_n = z$

Putting $x = x_n$ and $y = y_n$ in (2) then

$$\varphi\{F_{\alpha(Gx_n, Py_n)}(t)\} \geq \mathring{A}[\varphi(F_{\alpha(x_n, y_n)}(t)), \omega(F_{\alpha(x_n, y_n)}(t))]$$

$$\text{Where, } F_{\alpha(x_n, y_n)}(t) = \omega[\text{Min}\{F_{\alpha(Gx_n, SQx_n)}(t), F_{\alpha(Py_n, THy_n)}(t), F_{\alpha(Gx_n, THy_n)}(t), F_{\alpha(SQx_n, Py_n)}(2t)\}]$$

$$F_{\alpha(x_n, y_n)}(t) = \omega\left[\text{Min}\left\{\begin{array}{l} F_{\alpha(Gx_n, SQx_n)}(t), F_{\alpha(Py_n, THy_n)}(t), F_{\alpha(Gx_n, THy_n)}(t), \\ F_{\alpha(SQx_n, Gx_n)}(t) * F_{\alpha(Gx_n, Py_n)}(t) \end{array}\right\}\right]$$

On taking limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, y_n)}(t) = \omega\left[\text{Min}\left\{\begin{array}{l} F_{\alpha(Gx_n, SQx_n)}(t), F_{\alpha(Py_n, THy_n)}(t), F_{\alpha(Gx_n, THy_n)}(t), \\ \lim_{n \rightarrow \infty} F_{\alpha(SQx_n, Gx_n)}(t) * \lim_{n \rightarrow \infty} F_{\alpha(Gx_n, Py_n)}(t) \end{array}\right\}\right]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, y_n)}(t) = \omega\left[\text{Min}\left\{\begin{array}{l} \lim_{n \rightarrow \infty} F_{\alpha(z, z)}(t), F_{\alpha(Py_n, z)}(t), F_{\alpha(z, z)}(t), \\ F_{\alpha(z, z)}(t) * F_{\alpha(z, Py_n)}(t) \end{array}\right\}\right]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, y_n)}(t) = \omega\left[\text{Min}\left\{\begin{array}{l} 1, F_{\alpha(Py_n, z)}(t), 1, \\ 1 * F_{\alpha(z, Py_n)}(t) \end{array}\right\}\right]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, y_n)}(t) = \omega[F_{\alpha(Py_n, z)}(t)]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, y_n)}(t) > [F_{\alpha(Py_n, z)}(t)]$$

Form (2)

$$\varphi \left\{ \lim_{n \rightarrow \infty} F_{\alpha(Gx_n, Py_n)}(t) \right\} \geq \dot{A} \left[\varphi \left(\lim_{n \rightarrow \infty} F_{\alpha(x_n, x_n)}(t) \right), \omega \left(\lim_{n \rightarrow \infty} F_{\alpha(x_n, y_n)}(t) \right) \right]$$

$$\varphi \{ F_{\alpha(Gx_n, Py_n)}(t) \} > \dot{A} \left[\varphi \left(F_{\alpha(Py_n, z)}(t) \right), \omega \left(F_{\alpha(Py_n, z)}(t) \right) \right] \geq \varphi [F_{\alpha(Py_n, z)}(t)]$$

By A- class function,

$$\text{Either } \varphi \left(\omega \left(F_{\alpha(Py_n, z)}(t) \right) \right) = 1 \text{ or } \omega \left(F_{\alpha(Py_n, z)}(t) \right) = 1$$

$$\omega \left(F_{\alpha(Py_n, z)}(t) \right) = 1 \Rightarrow \lim_{n \rightarrow \infty} Py_n = z.$$

In the same manner, we can prove

Case (II) - If the pair (P, TH) Share CLR_{TH} - property

Case (III) - Now (G, SQ) and (P, TH) Share $CLR_{(SQ)(TH)}$ - property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Gx_n = \lim_{n \rightarrow \infty} SQx_n = \lim_{n \rightarrow \infty} Py_n = z = \lim_{n \rightarrow \infty} THy_n = z, \quad z \in SQ(X) \cap TH(X)$$

Where,

$$z \in SQ(X) \cap TH(X) \text{ then } \exists \text{ two point } u \text{ and } v \in X \text{ such that } z = Squ \text{ and } z = THv.$$

Now to claim $Gu = z = Squ$ and $Pv = z = THv$

Putting $x = x_n$ and $y = v$ in (2) then

$$\varphi \{ F_{\alpha(Gx_n, Pv)}(t) \} \geq \dot{A} \left[\varphi \left(F_{\alpha(x_n, v)}(t) \right), \omega \left(F_{\alpha(x_n, v)}(t) \right) \right]$$

$$\text{Where, } F_{\alpha(x_n, v)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(Gx_n, SQx_n)}(t), F_{\alpha(Pv, THv)}(t), F_{\alpha(Gx_n, THv)}(t), F_{\alpha(SQx_n, Pv)}(2t) \right\} \right]$$

$$F_{\alpha(x_n, v)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(Gx_n, SQx_n)}(t), F_{\alpha(Pv, THv)}(t), F_{\alpha(Gx_n, THv)}(t), \right. \right. \\ \left. \left. F_{\alpha(SQx_n, Gx_n)}(t) * F_{\alpha(Gx_n, Pv)}(t) \right\} \right]$$

On taking limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, v)}(t) = \omega \left[\text{Min} \left\{ \lim_{n \rightarrow \infty} F_{\alpha(z, z)}(t), F_{\alpha(Pv, z)}(t), F_{\alpha(z, z)}(t), \right. \right. \\ \left. \left. F_{\alpha(z, z)}(t) * F_{\alpha(z, Pv)}(t) \right\} \right]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, v)}(t) = \omega \left[\text{Min} \left\{ 1, F_{\alpha(Pv, z)}(t), 1, \right. \right. \\ \left. \left. 1 * F_{\alpha(z, Pv)}(t) \right\} \right]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, v)}(t) = \omega [F_{\alpha(Pv, z)}(t)]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(x_n, v)}(t) > [F_{\alpha(Pv, z)}(t)]$$

Form (2)

$$\varphi \left\{ \lim_{n \rightarrow \infty} F_{\alpha(Gx_n, Pv)}(t) \right\} \geq \dot{A} \left[\varphi \left(\lim_{n \rightarrow \infty} F_{\alpha(x_n, v)}(t) \right), \omega \left(\lim_{n \rightarrow \infty} F_{\alpha(x_n, v)}(t) \right) \right]$$

$$\varphi \{ F_{\alpha(Gx_n, Pv)}(t) \} > \dot{A} \left[\varphi \left(F_{\alpha(Pv, z)}(t) \right), \omega \left(F_{\alpha(Pv, z)}(t) \right) \right] \geq \varphi [F_{\alpha(Pv, z)}(t)]$$

By A- class function,

$$\text{Either } \left(\omega \left(F_{\alpha(Pv, z)}(t) \right) \right) = 1 \text{ or } \omega \left(F_{\alpha(Pv, z)}(t) \right) = 1$$

$$\Rightarrow \omega \left(F_{\alpha(Pv, z)}(t) \right) = 1 \Rightarrow Pv = z.$$

Thus $Pv = z = THv$

But v is a point of coincidence of the pair (P, TH) .

So that $Pv = z = THv = Gu = Squ$.

Now (G, SQ) and (P, TH) are Weakly compatible then $Gz = SQz = Pz = THz$.

Now put $x = z$ and $y = v$ in (2) the

$$\varphi \{ F_{\alpha(Gz, Pv)}(t) \} \geq \dot{A} \left[\varphi \left(F_{\alpha(z, v)}(t) \right), \omega \left(F_{\alpha(z, v)}(t) \right) \right]$$

$$\text{Where, } F_{\alpha(z, v)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(Gz, SQz)}(t), F_{\alpha(Pz, THv)}(t), F_{\alpha(Gz, THv)}(t), F_{\alpha(SQz, Pv)}(2t) \right\} \right]$$

$$F_{\alpha(z, v)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(Pz, Pz)}(t), F_{\alpha(z, z)}(t), F_{\alpha(Gz, z)}(t), \right. \right. \\ \left. \left. F_{\alpha(z, Gz)}(t) * F_{\alpha(Pv, Pv)}(t) \right\} \right]$$

$$\lim_{n \rightarrow \infty} F_{\alpha(z,v)}(t) = \omega \left[\text{Min} \left\{ 1, 1, F_{\alpha(z,Gz)}(t), \right. \right. \\ \left. \left. F_{\alpha(z,Gz)}(t) * 1 \right\} \right]$$

$$F_{\alpha(z,v)}(t) = \omega [F_{\alpha(Gz,z)}(t)]$$

$$F_{\alpha(z,v)}(t) > [F_{\alpha(Gz,z)}(t)]$$

Form (2)

$$\varphi\{F_{\alpha(Gz,Pv)}(t)\} \geq \dot{A} \left[\varphi \left(\lim_{n \rightarrow \infty} F_{\alpha(z,v)}(t) \right), \omega \left(\lim_{n \rightarrow \infty} F_{\alpha(z,v)}(t) \right) \right]$$

$$\varphi\{F_{\alpha(z,Pv)}(t)\} > \dot{A} \left[\varphi \left(F_{\alpha(Gz,z)}(t) \right), \omega \left(F_{\alpha(Gz,z)}(t) \right) \right] \geq \varphi[F_{\alpha(Gz,z)}(t)]$$

By A- class function,

$$\text{Either } \left(\omega \left(F_{\alpha(Gz,z)}(t) \right) \right) = 1 \text{ or } \omega \left(F_{\alpha(Gz,z)}(t) \right) = 1$$

$$\Rightarrow \omega \left(F_{\alpha(Gz,z)}(t) \right) = 1 \Rightarrow Gz = z.$$

Thus $Gz = z = SQz$.

In the same manner we can get $Pz = z = TH$.

Now put $x = Qz$ and $y = z$ in (2) then

$$\varphi\{F_{\alpha(GQz,Pz)}(t)\} \geq \dot{A} \left[\varphi \left(F_{\alpha(Qz,z)}(t) \right), \omega \left(F_{\alpha(Qz,z)}(t) \right) \right]$$

Where, $F_{\alpha(Qz,z)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(GQz,SQz)}(t), F_{\alpha(Pz,THz)}(t), F_{\alpha(GQz,THz)}(t), F_{\alpha(SQz,Pz)}(2t) \right\} \right]$

$$F_{\alpha(Qz,z)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(GQz,SQz)}(t), F_{\alpha(Pz,THz)}(t), F_{\alpha(GQz,THz)}(t), \right. \right. \\ \left. \left. F_{\alpha(SQz,GQz)}(t) * F_{\alpha(GQz,Pz)}(t) \right\} \right]$$

$$F_{\alpha(Qz,z)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(Qz,Qz)}(t), F_{\alpha(z,z)}(t), F_{\alpha(Qz,z)}(t), \right. \right. \\ \left. \left. F_{\alpha(Qz,z)}(t) * F_{\alpha(Qz,Qz)}(t) \right\} \right]$$

$$F_{\alpha(Qz,z)}(t) = \omega \left[\text{Min} \left\{ 1, 1, F_{\alpha(Qz,z)}(t), \right. \right. \\ \left. \left. F_{\alpha(Qz,z)}(t) * 1 \right\} \right]$$

$$F_{\alpha(Qz,z)}(t) = \omega [F_{\alpha(Qz,z)}(t)]$$

$$F_{\alpha(Qz,z)}(t) > \omega [F_{\alpha(Qz,z)}(t)]$$

Thus

$$\varphi\{F_{\alpha(Gz,z)}(t)\} > \dot{A} \left[\varphi \left(F_{\alpha(Gz,z)}(t) \right), \omega \left(F_{\alpha(Gz,z)}(t) \right) \right] \geq \varphi[F_{\alpha(Gz,z)}(t)]$$

By A- class function,

$$\text{Either } \left(\omega \left(F_{\alpha(Gz,z)}(t) \right) \right) = 1 \text{ or } \omega \left(F_{\alpha(Gz,z)}(t) \right) = 1$$

$$\Rightarrow \omega \left(F_{\alpha(Gz,z)}(t) \right) = 1 \Rightarrow Gz = z.$$

Thus $Gz = z \Rightarrow SQz = Sz = z$.

Again put $x = z$ and $y = Hz$ in (2) then

$$\varphi\{F_{\alpha(Gz,PHz)}(t)\} \geq \dot{A} \left[\varphi \left(F_{\alpha(z,HZ)}(t) \right), \omega \left(F_{\alpha(z,HZ)}(t) \right) \right]$$

Where, $F_{\alpha(z,HZ)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(Gz,SQz)}(t), F_{\alpha(PHz,THHz)}(t), F_{\alpha(Gz,THHz)}(t), F_{\alpha(SQz,PHz)}(2t) \right\} \right]$

$$F_{\alpha(z,HZ)}(t) = \omega \left[\text{Min} \left\{ F_{\alpha(z,z)}(t), F_{\alpha(Hz,HZ)}(t), F_{\alpha(z,HZ)}(t), \right. \right. \\ \left. \left. F_{\alpha(z,HZ)}(t) * F_{\alpha(Hz,HZ)}(t) \right\} \right]$$

$$F_{\alpha(z,HZ)}(t) = \omega \left[\text{Min} \left\{ 1, 1, F_{\alpha(Hz,z)}(t), \right. \right. \\ \left. \left. F_{\alpha(Hz,z)}(t) * 1 \right\} \right]$$

$$F_{\alpha(z,HZ)}(t) = \omega [F_{\alpha(Hz,z)}(t)]$$

$$F_{\alpha(z,HZ)}(t) > F_{\alpha(Hz,z)}(t)$$

Thus

$$\varphi\{F_{\alpha(z,HZ)}(t)\} > \dot{A} \left[\varphi \left(F_{\alpha(z,HZ)}(t) \right), \omega \left(F_{\alpha(z,HZ)}(t) \right) \right] \geq \varphi[F_{\alpha(z,HZ)}(t)]$$

By A- class function,

$$\text{Either } \left(\omega \left(F_{\alpha(z,HZ)}(t) \right) \right) = 1 \text{ or } \omega \left(F_{\alpha(z,HZ)}(t) \right) = 1$$

$$\Rightarrow \omega \left(F_{\alpha(z,Hz)}(t) \right) = 1 \Rightarrow Hz = z.$$

Hence $THz = z \Rightarrow Tz = z$

Therefore $Pz = THz = z \Rightarrow Pz = z.$

Combining all the above results then we get $Gz = Sz = Qz = Hz = Pz = Tz = z.$

Hence z is a *cfp* of G, P, H, Q, S and $T.$

Uniqueness:

Let $\sigma (\sigma \neq z)$ be another common fixed point of G, P, H, Q, S and T then

$G\sigma = P\sigma = H\sigma = Q\sigma = S\sigma = T\sigma = \sigma$ and $Gz = Pz = Hz = Qz = Sz = Tz = z$

$$\varphi \{F_{\alpha(G\sigma, Pz)}(t)\} \geq \hat{A} \left[\varphi \left(F_{\alpha(\sigma, z)}(t) \right), \omega \left(F_{\alpha(\sigma, z)}(t) \right) \right]$$

Where, $F_{\alpha(\sigma, z)}(t) = \omega \left[\text{Min} \{F_{\alpha(G\sigma, SQ\sigma)}(t), F_{\alpha(Pz, THz)}(t), F_{\alpha(G\sigma, THz)}(t), F_{\alpha(SQ\sigma, Pz)}(2t)\} \right]$

$F_{\alpha(\sigma, z)}(t) = \omega \left[\text{Min} \{F_{\alpha(G\sigma, SQ\sigma)}(t), F_{\alpha(Pz, THz)}(t), F_{\alpha(G\sigma, THz)}(t), F_{\alpha(SQ\sigma, Pz)}(2t)\} \right]$

$F_{\alpha(\sigma, z)}(t) = \omega \left[\text{Min} \{F_{\alpha(G\sigma, SQ\sigma)}(t), F_{\alpha(Pz, THz)}(t), F_{\alpha(G\sigma, THz)}(t), F_{\alpha(SQ\sigma, Gz)}(t) * F_{\alpha(Gz, Pz)}(t)\} \right]$

$F_{\alpha(\sigma, z)}(t) = \omega \left[\text{Min} \{F_{\alpha(\sigma, \sigma)}(t), F_{\alpha(z, z)}(t), F_{\alpha(\sigma, z)}(t), F_{\alpha(\sigma, z)}(t) * F_{\alpha(z, z)}(t)\} \right]$

$F_{\alpha(\sigma, z)}(t) = \omega \left[\text{Min} \{1, 1, F_{\alpha(\sigma, z)}(t), F_{\alpha(\sigma, z)}(t) * 1\} \right]$

$F_{\alpha(\sigma, z)}(t) = \omega \left[F_{\alpha(\sigma, z)}(t) \right]$

$F_{\alpha(\sigma, z)}(t) > F_{\alpha(\sigma, z)}(t)$

Thus

$$\varphi \{F_{\alpha(\sigma, z)}(t)\} > \hat{A} \left[\varphi \left(F_{\alpha(\sigma, z)}(t) \right), \omega \left(F_{\alpha(\sigma, z)}(t) \right) \right] \geq \varphi \left[F_{\alpha(\sigma, z)}(t) \right]$$

By A- class function,

Either $\left(\omega \left(F_{\alpha(\sigma, z)}(t) \right) \right) = 1$ or $\omega \left(F_{\alpha(\sigma, z)}(t) \right) = 1$

$\Rightarrow \omega \left(F_{\alpha(\sigma, z)}(t) \right) = 1 \Rightarrow \sigma = z.$

Hence z is a unique common fixed point of G, P, H, Q, S and T in $X.$

Example: Let (Ω, F_α, T) be a complete Fuzzy *PM*- space. Define a mapping G, H, P and $Q: \Omega \rightarrow \Omega$ Such that

$$G(\alpha) = \begin{cases} \frac{3}{4}, & \text{for } 0 \leq \alpha \leq 1 \\ \frac{\alpha+2}{4}, & \text{for } 1 < \alpha \leq 2 \end{cases}, \quad H(\alpha) = \begin{cases} \alpha, & \text{for } 0 \leq \alpha < 1 \\ \frac{1}{2}, & \text{for } \alpha = 1 \\ \frac{2\alpha}{5}, & \text{for } 1 < \alpha \leq 2 \end{cases}$$

$$P(\alpha) = \begin{cases} \alpha, & \text{for } 0 \leq \alpha \leq 1 \\ \frac{3\alpha}{4}, & \text{for } 1 < \alpha \leq 2 \end{cases}, \quad Q(\alpha) = \begin{cases} \frac{3\alpha}{2}, & \text{for } 0 \leq \alpha \leq 1 \\ \frac{\alpha+1}{5}, & \text{for } 1 \leq \alpha \leq 2 \end{cases}$$

Let $\hat{A}(u, v)$ be A-Class function: $X = [0, 2]$ and a mapping $\hat{A}: (0, 1) \times (0, 1) \rightarrow R^+$ is defined by

$d(u, v) = |u - v|$ for $u, v \in X$, $F_{\alpha(u, v)}(t) = \frac{t}{t + |u - v|}$ be a *FMS*.

Since, $G(X) \subseteq T(X)$, $P(X) \subseteq S(X)$. The two sequences $\{x_n\}$ and $\{y_n\}$ define as

$x_n = 1 + \frac{2}{n+3}$, $y_n = 1 + \frac{1}{2n}$ in $X.$

Such that $\lim_{n \rightarrow \infty} G(x_n) = \lim_{n \rightarrow \infty} P(x_n) = \frac{3}{4} = P \left(\frac{3}{4} \right)$

and $\lim_{n \rightarrow \infty} H(y_n) = \lim_{n \rightarrow \infty} Q(y_n) = \frac{2}{5} = Q(1).$

Since $\frac{3}{4} \in P(X) \Rightarrow$ hence the pair (G, P) satisfy Clr_P -property

And $\frac{2}{5} \in Q(X) \Rightarrow$ then the pair (H, Q) satisfy Clr_Q - property.

IV. Conclusion

In this paper, we have established new fixed point and common fixed point theorems for self-mappings in the framework of fuzzy Menger spaces. By utilizing the notion of occasionally weakly

compatible mappings and incorporating A-class functions as control functions for contractive conditions, we have successfully generalized and extended several existing results in the literature. The combination of these generalized compatibility and contraction conditions allows for broader applicability in fuzzy settings, where classical assumptions may not hold.

Furthermore, illustrative examples provided in the paper validate the main results and demonstrate the effectiveness of the proposed framework. These results contribute to the ongoing development of fixed point theory in fuzzy metric spaces and open new directions for further investigation, particularly in exploring fixed point properties under other generalized contractive conditions and in different fuzzy or probabilistic settings.

We believe that the methods and techniques introduced in this study will serve as a foundation for future research in both theoretical and applied aspects of fuzzy analysis and fixed point theory.

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