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INFERENCES FOR ALPHA POWER TRANSFORMED INVERSE LINDLEY DISTRIBUTION BASED ON ORDER STATISTICS WITH APPLICATIONS

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Abstract

The alpha power-transformed inverse Lindley distribution (APTIL) consist of one shape and one scale parameter. In some cases, the moments of the APTIL distribution may not exist due to its heavy tail, so we instead focus on the concept of inverse moments. In this paper, order statistics is used to derive the expression for moments, product moments and moment generating function for this distribution. The entropy of using order statistics for the APTIL distribution is also derived. Furthermore, tabulated values of the inverse moments of order statistics are provided for various sample sizes and parameter combinations. Three classical estimation techniques, viz., maximum product spacing estimators, least squares estimators, and weighted least squares estimators, are considered for parameter estimation and also evaluate the model parameters based on Type-II censored data. A Monte Carlo simulation-based approach is employed to assess the accuracy and validity of these estimators and the ensuing results. Finally, an empirical study using available real-life data sets demonstrates the utility of the distribution.

Keywords: Alpha power transformed inverse Lindley distribution, Order statistics, inverse moments, Maximum likelihood estimator, Entropy

1. Introduction

Order statistics play a crucial role in various theoretical and practical applications, including reliability analysis, entropy estimation, censored data analysis, quality control, outlier detection, material strength evaluation, optimal selection methods, and income inequality studies. The significance of order statistics has increased substantially in recent years, reflecting their growing role in contemporary research and applications. A characterization of the probability distribution and goodness of fit test has been provided by Arnold et al. (2003). There are several domains where moments of order statistics are applicable, including auction theory, quality control, and reliability. However, in certain situations where moments do not exist, the concept of inverse moments is employed. Inverse moments are essential in numerous practical applications, such as Stein estimation (Woof, 1985), life testing problems (Mendenhall and Lehman, 1960), and risk evaluation of estimators (Fujioka, 2001). Additionally, they are significant in financial and insurance mathematics (Ramsay, 1993) and in assessing the power of the test statistics (Fujioka, 2001). Wong and Chen (1990) discussed the entropy of order statistics, while Park (1995) derived recurrence relations for their entropy. A general form of Shannon entropy, referred to as Renyi entropy, plays a vital role in cryptography, high-resolution scalar quantization, time-frequency resolution, and signal segmentation in the time-frequency plane. The Shannon entropy of a random variable Z quantifies the average reduction of uncertainty in another random variable X. Estimation methods for unknown parameters in different distributions have been extensively studied. Kundu and Raqab (2005) focused on the generalized Rayleigh distribution, while Alkasasbeh and Raqab (2009) examined the generalized logistic distribution. Additionally, the Rayleigh distribution with two parameters, weighted exponential distribution, and exponentiated Chen distribution were explored by Dey et al. (2014, 2015, 2017), respectively.

In the literature, numerous authors have introduced modified distributions that offer greater flexibility and applicability for modeling real-world datasets. Sharma et al. (2015) proposed the inverse Lindley distribution, a one-parameter model that is unimodal and exhibits an upside-down bathtub failure rate function. This distribution has demonstrated superior performance compared to the inverse Rayleigh distribution in modeling head and neck cancer datasets. Over time, various extensions of the inverse Lindley distribution have been developed. Alkarni (2015) introduced the extended inverse Lindley distribution, deriving key statistical properties such as moments, skewness, kurtosis, and quantile functions. Sharma et al. (2016) proposed the generalized inverse Lindley distribution and provided parameter estimates using different estimation methods. Barco et al. (2017) developed the inverse power Lindley distribution by combining the power Lindley and inverse Lindley distributions to create more flexible models. Jan et al. (2018) introduced the exponentiated inverse power Lindley distribution, a three-parameter generalization of the inverse Lindley distribution. Mahmoud (2019) further extended this framework by proposing the extended exponentiated inverse Lindley distribution, a four-parameter generalization that enhances model adaptability.

The Alpha Power Transformed Inverse Lindley (APTIL) distribution was introduced by Dey et al. (2019), building on a concept similar to that of Mahdavi and Kundu (2017). The authors have explored the characteristics and properties of the APTIL model highlighting the classical estimation procedures. They have also illustrated the applicability of the APTIL distribution with the help of real life dataset analysis. They described that the given model provide better results than some other well-known lifetime models and its ability of unimodal and upside-down bathtub-shaped hazard rates which are throughly applicable in many real-life problems, makes it a special distribution. This model can be used in various areas, such as biomedical studies, public health, reliability and survival analysis etc.. The core intention of this paper is to explore the distributions of the order statistics arising from this APTIL distribution and derive the expressions for the corresponding single and product moments, marginal moment generating function and entropy of order statistics. For details on the studies related to the moments of the order statistics, one may refer to the works of Balakrishnan et al. (2015), Kumar and Dey (2017), Kumar and Goyal (2019), Goyal et al. (2024), Kumar et al. (2024) among others. Ahsanullah and Alzaatreh (2018) derived the moments of order statistics and estimated the unknown parameters of the Log-logistic distribution using order statistics. Next, we want to find the estimates for the unknown parameters using various statistical procedures, viz., maximumum likelihood estimation (MLE) for different methods like least square estimator, weighted least square estimator, method of maximum product spacing estimator and also obtain the maximum likelihood estimator for type-II censored samples for the APTIL distribution. The probability density function (PDF), the cumulative distribution function (CDF) and the hazard function (HF) of a APTIL random variable

$X > 0$ with parameters $\beta > 0$ and $\omega > 0$ are respectively given by Equations (1.1), (1.2) and (1.3)

$$f(x;\beta,\omega) = \frac{\omega^2 (1+x)\log\beta}{x^3(\beta-1)(\omega+1)} e^{-\frac{\omega}{x}} \beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}, \; x>0,\, \beta>0,\, \omega>0,\, \beta\neq 1, \tag{1}$$

$$F(x;\beta,\omega) = \frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}-1}{\beta-1}, \quad x>0,\, \beta>0,\, \omega>0,\, \beta\neq 1, \tag{2}$$

$$h(x;\beta,\omega) = \frac{\omega^2 \log(\beta)\,(1+x)\, e^{-\frac{\omega}{x}}\, \beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}}{(\omega+1)\, x^3 \left(\beta - \beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}\right)}, \; x>0,\, \beta>0,\, \omega>0,\, \beta\neq 1, \tag{3}$$

Special Cases:

1. As $\beta \to 1$, the APTIL distribution reduces to the inverse Lindley distribution.
2. As $\beta = e$, the APTIL distribution reduces to the inverse Poisson-Lindley distribution.

Now the paper is further organized into the following: Section 2, covers some statistical properties like inverse moments, and the order statistic in which we found the mathematical formulation of inverse moments, the inverse product moments, and marginal moment generating function for each of these order statistics. In Section 3, we obtained the mathematical expression of entropy of the k^{th} order statistic. We briefly describe the different method of estimation for estimating the unknown parameters β and ω of the given distribution like, maximum likelihood estimation based on Type-II censored sample, least square and weighted least square estimation and maximum product spacing estimation in section 4. In Section 5, we obtained the numerical value to check whether our expression is correct using the R software. In Section 6, the practical applicability of the model is demonstrated. In Section 7, we conclude the paper.

2. Statistical Properties

In some situations, the moment of the APTIL distribution may not exist due to its heavy tail, so we instead focus on the concept of inverse moment. Here, the p^{th} $(p = 1, 2, \ldots)$ inverse moment of the distribution given in (1), can be easily computed as

$$E\left[\left(\frac{1}{X}\right)^p\right] = \frac{\omega^2\log\beta}{(\beta-1)(\omega+1)} \sum_{\gamma_1=0}^{\infty}\sum_{\gamma_2=0}^{\gamma_1} \binom{\gamma_1}{\gamma_2} \left(\frac{\omega}{\omega+1}\right)^{\gamma_2} \frac{(\log\beta)^{\gamma_1}(\gamma_2+p)!(\gamma_2+p+1+\omega\gamma_1+\omega)}{\gamma_1!\,(\omega\gamma_1+\omega)^{\gamma_2+p+2}}. \tag{4}$$

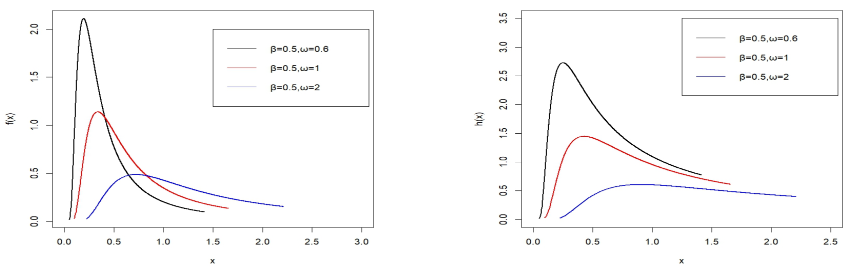


Figure 1: *PDF and HF plots for APTIL distribution*

From the Figure 1, we can see that the APTIL distribution has unimodal and upside-down bathtub-shaped hazard rates.

2.1. APTIL Distribution: Order Statistics and Their Properties

Let $X_1, X_2, \ldots, X_n$ be a random sample from the APTIL distribution with PDF and CDF denoted by (1) and (2) respectively, and an ordered random sample $X_{1:n}, \leq X_{2:n}, \leq, ..., \leq X_{k:n}$ from APTIL distribution, where $X_{k:n}$ is the k^{th} order statistic. The PDF of k^{th} order statistics and the joint PDF of the k^{th} and l^{th} order statistics are given by

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1}[1-F(x)]^{n-k} f(x), x > 0, \tag{5}$$

$$f_{X_{k,l:n}}(x,y) = C_{k,l:n} F(x)^{k-1}[F(y)-F(x)]^{l-k-1}[1-F(y)]^{n-l} f(x) f(y), \;\; x < y, \;\; k < l, \tag{6}$$

where, $C_{k,l:n} = \frac{n!}{(k-1)!(k-l-1)!(n-l)!}$.

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ represent the order statistics of a sample of size n from the APTIL distribution. The PDF of the k^{th} order statistics say $X_{k:n}$, $1 \leq k \leq n$ for APTIL distribution can be obtained by using Equations (1), (2) and (5)

$$f_{X_{k:n}}(x) = C_{k:n} \sum_{\gamma_1=0}^{n-k} \frac{\omega^2 \log\beta}{(\beta-1)(\omega+1)} (-1)^{\gamma_1} \binom{n-k}{\gamma_1} \left(\frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{\frac{-\omega}{x}}} - 1}{\beta - 1} \right)^{r+\gamma_1-1} \frac{(1+x)\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}}{x^3},$$

and the corresponding CDF $\left(F_{X_{k:n}}(x)\right)$, survival function $\left(S_{X_{k:n}}(x)\right)$ and hazard rate function $\left(H_{X_{k:n}}(x)\right)$ of k^{th} order statistic as follows:

$$\begin{aligned}
F_{X_{k:n}}(x) &= \sum_{\gamma_1=k}^{n} \sum_{\gamma_2=0}^{n-\gamma_1} \binom{n}{\gamma_1} \binom{n-\gamma_1}{\gamma_2} (-1)^{\gamma_2} \left(\frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{\frac{-\omega}{x}}} - 1}{\beta-1} \right)^{\gamma_2+\gamma_1}, \\
S_{X_{k:n}}(x) &= 1 - F_{X_{k:n}}(x) = 1 - \sum_{\gamma_1=k}^{n} \sum_{\gamma_2=0}^{n-\gamma_1} \binom{n}{\gamma_1} \binom{n-\gamma_1}{\gamma_2} (-1)^{\gamma_2} \left(\frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{\frac{-\omega}{x}}} - 1}{\beta-1} \right)^{\gamma_2+\gamma_1}, \\
H_{X_{k:n}}(x) &= \frac{f_{X_{k:n}}(x)}{S_{X_{k:n}}(x)}.
\end{aligned}$$

Similarly, by substituting PDF and CDF of APTIL distribution mentioned in Equations (1), (2) into Equation (6), the joint PDF of the k^{th} and l^{th} order statistics $(X_{k:n}, X_{l:n})$ of the APTIL distribution is

$$\begin{aligned}
f_{X_{k,l:n}}(x,y) &= C_{k,l:n} \sum_{\gamma_1=0}^{l-k-1} \sum_{\gamma_2=0}^{n-l} \binom{l-k-1}{\gamma_1} \binom{n-l}{\gamma_2} (-1)^{l-k-1-\gamma_1+\gamma_2} \left(\frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{\frac{-\omega}{x}}} - 1}{\beta-1} \right)^{l-2-\gamma_1} \\
&\quad \times \left(\frac{\beta^{\left(1+\frac{\omega}{(\omega+1)y}\right)e^{\frac{-\omega}{y}}} - 1}{\beta-1} \right)^{\gamma_1+\gamma_2} \left(\frac{\omega^2 \log\beta}{(\beta-1)(\omega+1)} \right)^2 \left(\frac{(1+x)}{x^3} e^{-\frac{\omega}{x}} \beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}} \right) \\
&\quad \times \left(\frac{(1+y)}{y^3} e^{-\frac{\omega}{y}} \beta^{\left(1+\frac{\omega}{(\omega+1)y}\right)e^{-\frac{\omega}{y}}} \right), \qquad x < y, \;\; k < l,
\end{aligned} \tag{7}$$

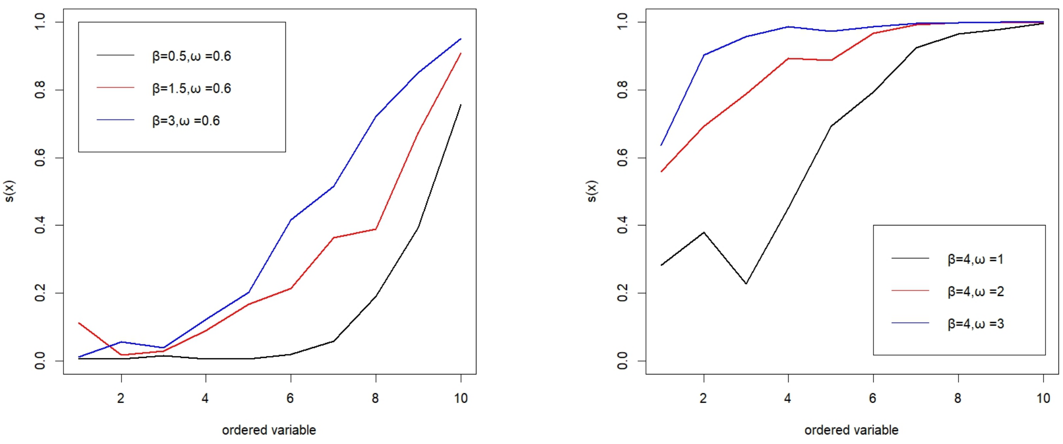


Figure 2: *Survival function of k^{th} order statistics of the APTIL distribution*

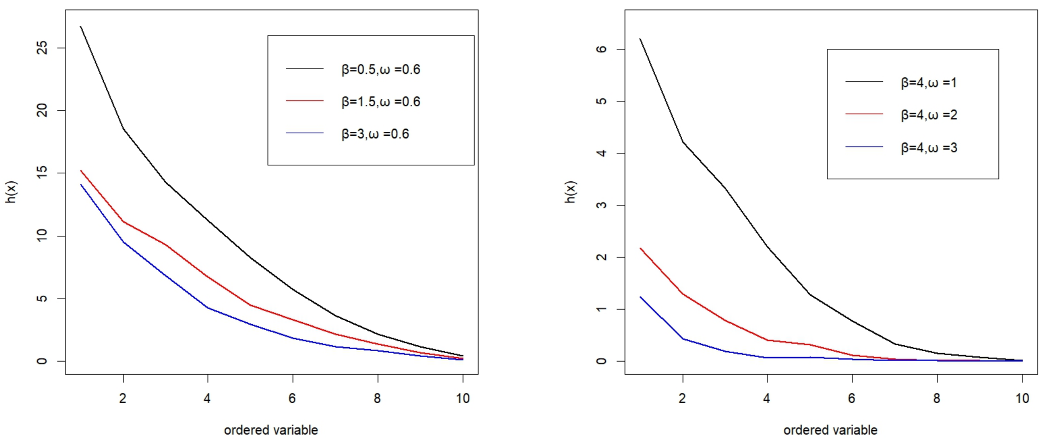


Figure 3: *Hazard rate function of k^{th} order statistics of the APTIL distribution*

From the Figure 2 and 3, we see that the survival function of the distribution increases with higher values of β and ω, indicating greater survival probabilities for larger ordered variables. Conversely, the hazard rate function decreases as the k^{th} ordered variable increases and is inversely proportional to both the shape parameter β and the scale parameter ω.

2.1.1 Single moment of the order statistics for APTIL distribution

The expectation is a crucial attribute of a random variable. This section deals with the $p^{th}(p = 1, 2, \ldots)$ moment of k^{th} order statistics for the APTIL model, which will facilitate the computation of given identity.

Theorem 1. Let equations (1) and (2) define the PDF and CDF of the APTIL model, then the p^{th} moment of the k^{th} order statistics say $\mu_{k:n}^{(p)} = E\left(\left(\frac{1}{X_{k:n}^{p}}\right)\right)$[1] is given by

[1]The moments of the k^{th} order exists if $\omega(z)$ exists, where z positive integers. For non positive integers gamma is not defined.

$$\mu_{k:n}^{p} = \frac{n!}{(k-1)!(n-k)!}\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_4=0}^{\gamma_3}\sum_{\gamma_1=0}^{n-k}\sum_{\gamma_2=0}^{k-1+\gamma_1}(-1)^{k-1-\gamma_2}\binom{n-k}{\gamma_1}\binom{k-1+\gamma_1}{\gamma_2}\binom{\gamma_3}{\gamma_4}\frac{\omega^{-p}}{(\omega+1)^{\gamma_4+1}}$$
$$\times\frac{(\log\beta)^{\gamma_3+1}}{\gamma_3!(\beta-1)^{k+\gamma_1}}(\gamma_2+1)^{\gamma_3}(\gamma_3+1)^{-p-2-\gamma_4}\left(\Gamma(\gamma_4+2+p)+\omega(\gamma_3+1)\Gamma(\gamma_4+1+p)\right), \quad (8)$$

Proof: Appendix 1 provides the proof of the theorem.

Now, the first and second expectations are two essential and broadly used characteristic of a random variable. Based on Theorem 1, the first two inverse moments of the *pth* order statistics say $\frac{1}{X_{k:n}^{p}}$ are respectively given by

$$E(X_{k:n}^{-1}) = \frac{n!}{(k-1)!(n-k)!}\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_4=0}^{\gamma_3}\sum_{\gamma_1=0}^{n-k}\sum_{\gamma_2=0}^{k-1+\gamma_1}(-1)^{k-1-\gamma_2}\binom{n-k}{\gamma_1}\binom{k-1+\gamma_1}{\gamma_2}\binom{\gamma_3}{\gamma_4}$$
$$\times\frac{(\log\beta)^{\gamma_3+1}}{\gamma_3!(\beta-1)^{k+\gamma_1}}\frac{(\gamma_2+1)^{\gamma_3}\left(\Gamma(\gamma_4+3)+\omega(\gamma_3+1)\Gamma(\gamma_4+2)\right)}{\omega(\gamma_3+1)^{3+\gamma_4}(\omega+1)^{\gamma_4+1}},$$

$$E(X_{k:n}^{-2}) = \frac{n!}{(k-1)!(n-k)!}\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_4=0}^{\gamma_3}\sum_{\gamma_1=0}^{n-k}\sum_{\gamma_2=0}^{k-1+\gamma_1}(-1)^{k-1-\gamma_2}\binom{n-k}{\gamma_1}\binom{k-1+\gamma_1}{\gamma_2}\binom{\gamma_3}{\gamma_4}$$
$$\times\frac{(\log\beta)^{\gamma_3+1}}{\gamma_3!(\beta-1)^{k+\gamma_1}}\frac{(\gamma_2+1)^{\gamma_3}\left(\Gamma(\gamma_4+4)+\omega(\gamma_3+1)\Gamma(\gamma_4+3)\right)}{\omega^2(\gamma_3+1)^{4+\gamma_4}(\omega+1)^{\gamma_4+1}}.$$

These two equations have been used to calculate the values of expectations and variances reported in Table 1 and 2 in Section 5. Let $X_{1:n}\le X_{2:n}\le\ldots\le X_{n:n}$ represent the order statistics of a sample of n size from the APTIL distribution. Then, moment generating function of k^{th} order statistic is given by

$$M_{X_{k:n}}(t) = \frac{n!}{(k-1)!(n-k)!}\sum_{\gamma_4=0}^{\infty}\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_5=0}^{\gamma_4}\sum_{\gamma_1=0}^{n-k}\sum_{\gamma_2=0}^{k-1+\gamma_1}(-1)^{k-1-\gamma_2}\binom{n-k}{\gamma_1}\binom{k-1+\gamma_1}{\gamma_2}\binom{\gamma_4}{\gamma_5}\frac{t^{\gamma_3}}{\gamma_3!}$$
$$\times\frac{(\log\beta)^{\gamma_4+1}}{\gamma_4!(\beta-1)^{k+\gamma_1}}\frac{\omega^{\gamma_3}}{(\omega+1)^{\gamma_5+1}}(\gamma_2+1)^{\gamma_4}(\gamma_4+1)^{\gamma_3-2-\gamma_5}\left(\Gamma(\gamma_5+2-\gamma_3)+\omega(\gamma_4+1)\right.$$
$$\left.\Gamma(\gamma_5+1-\gamma_3)\right).$$

2.1.2 Product moment of the order statistics for APTIL distribution

In this section, the product moment $\mu_{k,l:n}^{(p,q)} = E\left[\frac{1}{X_{k:n}^{p}}\frac{1}{X_{k:n}^{q}}\right]$ of two order statistics are investigated.

Theorem 2: Let equations (1) and (2) define the PDF and CDF of the APTIL model. For $1\le k\le l\le n$ and $p,q>0$. The product moments of k^{th} and l^{th} order statistic are given by

$$\mu_{k,l:n}^{(p,q)} = \frac{n!\psi(\beta,\omega)}{(k-1)!(l-k-1)!(n-l)!}\frac{(\log\beta)^{\gamma_5+\gamma_6+2}}{(\beta-1)^{l+\gamma_2}}\frac{\omega^{\gamma_7+\gamma_8+\gamma_9+4}}{(\omega+1)^{\gamma_7+\gamma_8+2}}\frac{(\gamma_3+1)^{\gamma_5}(\gamma_4+1)^{\gamma_6}(\gamma_6+1)^{\gamma_9}}{\gamma_5!\gamma_6!\gamma_9!}$$
$$\times\left[\frac{(\gamma_7+\gamma_8+\gamma_9+3+p+q+\omega(\gamma_5+1))\Gamma(\gamma_7+\gamma_8+\gamma_9+3+p+q)}{(\gamma_8+\gamma_9+2+q)(\omega(\gamma_5+1))^{\gamma_7+\gamma_8+\gamma_9+4+p+q}}\right.$$
$$\left.+\frac{(\gamma_7+\gamma_8+\gamma_9+2+p+q+\omega(\gamma_5+1))\Gamma(\gamma_7+\gamma_8+\gamma_9+2+p+q)}{(\gamma_8+\gamma_9+1+q)(\omega(\gamma_5+1))^{\gamma_7+\gamma_8+\gamma_9+3+p+q}}\right],$$

where

$$\psi(\beta,\omega) = \sum_{\gamma_9=0}^{\infty}\sum_{\gamma_6=0}^{\infty}\sum_{\gamma_5=0}^{\infty}\sum_{\gamma_1=0}^{l-k-1}\sum_{\gamma_2=0}^{n-l}\sum_{\gamma_3=0}^{l-2-\gamma_1}\sum_{\gamma_4=0}^{\gamma_1+\gamma_2}\sum_{\gamma_7=0}^{\gamma_5}\sum_{\gamma_8=0}^{\gamma_6}(-1)^{k-1-\gamma_1-\gamma_3-\gamma_4+\gamma_9}\binom{l-k-1}{\gamma_1}\binom{n-l}{\gamma_2}$$
$$\times\binom{l-2-\gamma_1}{\gamma_3}\binom{\gamma_5}{\gamma_7}\binom{\gamma_6}{\gamma_8}\binom{\gamma_1+\gamma_2}{\gamma_4}.$$

Proof: Appendix 2 provides the proof of the theorem.

3. Entropy

Shannon entropy is essential to information theory and is broadly applied in multiple fields. Entropy is used to measure the variation of the ambiguity of the random variable X. If X has probability density function $f(\cdot)$, then the Renyi entropy is defined (Renyi's, 1961) as

$$H_\delta(x) = \frac{1}{1-\delta}\log\left(\int_{-\infty}^{\infty} f(x)^\delta dx\right), \delta > 0, \delta \neq 1.$$

Now, The Renyi entropy of the k^{th} order statistic (Baratpour et al.,2008) of APTIL distribution is defined as

$$\begin{aligned} H_\delta(X_{k:n}) &= \frac{1}{1-\delta}\log\left(\int_{-\infty}^{\infty} f_{X_{k:n}}(x)^\delta dx\right), \delta > 0, \delta \neq 1 \\ &= \frac{1}{1-\delta}\log\left(\int_0^\infty C_{k:n}^\delta \left(\frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}-1}{\beta-1}\right)^{(k-1)\delta}\left(1-\frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}-1}{\beta-1}\right)^{(n-k)\delta}\right. \\ &\times \left.\left(\frac{\log\beta}{\beta-1}\frac{\omega^2}{\omega+1}\frac{1+x}{x^3}e^{-\frac{\omega}{x}}\beta^{\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}\right)^\delta dx\right) \\ &= \frac{1}{1-\delta}\log\left(\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_4=0}^{\gamma_3}\sum_{\gamma_1=0}^{\delta(n-k)}\sum_{\gamma_2}^{\delta(k-1)+\gamma_1}\sum_{\gamma_5=0}^{\delta}(-1)^{\delta(k-1)-\gamma_2}\binom{\delta(n-k)}{\gamma_1}\binom{\delta(k-1)+\gamma_1}{\gamma_2}\right. \\ &\times \binom{\gamma_3}{\gamma_4}\binom{\delta}{\gamma_5}\frac{(\log\beta)^{\delta+\gamma_3}}{(\beta-1)^{\delta k+\gamma_1}}\frac{\omega^{\gamma_5-\delta+1}}{\gamma_3!(\omega+1)^{\delta+\gamma_4}}(\delta+\gamma_2)^{\gamma_3}C_{k:n}^\delta(\delta+\gamma_3)^{\gamma_5-\gamma_4-3\delta+1} \\ &\times \left.\Gamma(3\delta+\gamma_4-\gamma_5-1)\right). \end{aligned} \tag{9}$$

Special Cases:

1. If put $k = 1$ in Equation (9), we get

$$H_\delta(X_{1:n}) = \frac{1}{1-\delta}\log\left(\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_4=0}^{\gamma_3}\sum_{\gamma_1=0}^{\delta(n-1)}\sum_{\gamma_2}^{\gamma_1}\sum_{\gamma_5=0}^{\delta}(-1)^{-\gamma_2}\binom{\delta(n-1)}{\gamma_1}\binom{\gamma_1}{\gamma_2}\binom{\gamma_3}{\gamma_4}\binom{\delta}{\gamma_5}\right.$$
$$\left.\frac{(\log\beta)^{\delta+\gamma_3}}{(\beta-1)^{\delta+\gamma_1}}\frac{\omega^{\gamma_5-\delta+1}}{\gamma_3!(\omega+1)^{\delta+\gamma_4}}(\delta+\gamma_2)^{\gamma_3}n^\delta(\delta+\gamma_3)^{\gamma_5-\gamma_4-3\delta+1}\Gamma(3\delta+\gamma_4-\gamma_5-1)\right).$$

2. If put $k = n$ in Equation (9), we get

$$H_\delta(X_{n:n}) = \frac{1}{1-\delta}\log\left(\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_4=0}^{\gamma_3}\sum_{\gamma_2}^{\delta(n-1)}\sum_{\gamma_5=0}^{\delta}(-1)^{\delta(n-1)-\gamma_2}\binom{\delta(n-1)}{\gamma_2}\binom{\gamma_3}{\gamma_4}\binom{\delta}{\gamma_5}\right.$$
$$\left.\frac{(\log\beta)^{\delta+\gamma_3}}{(\beta-1)^{\delta n}}\frac{\omega^{\gamma_5-\delta+1}}{\gamma_3!(\omega+1)^{\delta+\gamma_4}}(\delta+\gamma_2)^{\gamma_3}n^\delta(\delta+\gamma_3)^{\gamma_5-\gamma_4-3\delta+1}\Gamma(3\delta+\gamma_4-\gamma_5-1)\right).$$

4. Various Estimation Techniques

4.1. Maximum likelihood estimation of unknown parameters from Type II censored data

The maximum likelihood estimates (MLEs) of the unknown parameters for complete data was obtained by Dey et al. (2019). In this section, we obtained MLEs of the unknown parameters

(β, ω) of the APTIL distribution. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{k:n}$ be Type-II censored sample from the distribution, then the likelihood function is defined as

$$L(t, \beta, \omega) = \frac{n!}{(n-k)!} \prod_{j=1}^{k} \left(\frac{\log \beta}{\beta - 1} \frac{\omega^2}{\omega + 1} \frac{1 + x_{j:n}}{x_{j:n}^3} e^{-\frac{\omega}{x_{j:n}}} \beta^{\left(1 + \frac{\omega}{(\omega+1)x_{j:n}}\right) e^{-\frac{\omega}{x_{j:n}}}} \right) \times \left[1 - \left(\frac{\beta^{\left(1 + \frac{\omega}{(\omega+1)x_{k:n}}\right) e^{-\frac{\omega}{x_{k:n}}}} - 1}{\beta - 1} \right) \right]^{n-k},$$

and the corresponding log-likelihood function is defined by Equation

$$\log L = \log n! - \log(n-k)! + k \log \log \beta - k \log(\beta - 1) - k \log(\omega + 1) + \sum_{j=1}^{k} \log \left(\frac{1 + x_{j:n}}{x_{j:n}^3} \right) + (2k \log \omega) + \sum_{j=1}^{k} \left(1 + \frac{\omega}{(\omega+1)x_{j:n}} \right) e^{-\frac{\omega}{x_{j:n}}} \log \beta - \sum_{j=1}^{k} \frac{\omega}{x_{j:n}} + (n-k) \log \left(\frac{\beta - \beta^{\left(1 + \frac{\omega}{(\omega+1)x_{k:n}}\right) e^{-\frac{\omega}{x_{k:n}}}}}{\beta - 1} \right).$$

The unknown parameters β and ω can be estimated using the maximum likelihood method by solving the following two nonlinear equations simultaneously,

$$\frac{d \log L}{d\omega} = \sum_{j=1}^{k} \log \beta \left(\frac{e^{-\frac{\omega}{x_{j:n}}}}{x_{j:n}} \left(\frac{1}{(\omega+1)} - \frac{\omega}{(\omega+1)^2} \right) - \frac{e^{-\frac{\omega}{x_{j:n}}} \left(1 + \frac{\omega}{(\omega+1)x_{j:n}} \right)}{x_{j:n}} \right) - \sum_{j=1}^{k} \frac{1}{x_{j:n}} + \frac{2k}{\omega}$$
$$+ \frac{\log \beta (n-k) \left(\beta^{\left(1 + \frac{\omega}{(\omega+1)x_{k:n}}\right) e^{-\frac{\omega}{x_{k:n}}}} \right)}{\left(\beta - \beta^{\left(1 + \frac{\omega}{(\omega+1)x_{k:n}}\right) e^{-\frac{\omega}{x_{k:n}}}} \right)} \left(\frac{e^{-\frac{\omega}{x_{k:n}}}}{x_{k:n}} \left(\frac{1}{(\omega+1)} - \frac{\omega}{(\omega+1)^2} \right) - \frac{e^{-\frac{\omega}{x_{k:n}}} \left(1 + \frac{\omega}{(\omega+1)x_{k:n}} \right)}{x_{k:n}} \right)$$
$$- \frac{k}{\omega + 1} = 0,$$

$$\frac{d \log L}{d\beta} = \frac{k}{\beta \log(\beta)} - \frac{k}{\beta - 1} + \frac{1}{\beta} \sum_{j=1}^{k} \left(1 + \frac{\omega}{(\omega+1)x_{j:n}} \right) e^{-\frac{\omega}{x_{j:n}}} + (n-k) \left(\frac{\beta - 1}{\beta - \beta^{\left(1 + \frac{\omega}{(\omega+1)x_{k:n}}\right) e^{-\frac{\omega}{x_{k:n}}}}} \right)$$
$$\times \left(\frac{1 - \left(\frac{\omega}{(\omega+1)x_{k:n}} e^{-\frac{\omega}{x_{k:n}}} \beta^{\left(1 + \frac{\omega}{(\omega+1)x_{k:n}}\right) e^{-\frac{\omega}{x_{k:n}}} - 1} \right)}{(\beta - 1)^2} - \frac{\beta - \beta^{\left(1 + \frac{\omega}{(\omega+1)x_{k:n}}\right) e^{-\frac{\omega}{x_{k:n}}}}}{(\beta - 1)^2} \right) = 0.$$

4.2. Method of maximum product spacing estimators

This method was proposed by Cheng and Amin (1979, 1983) as an alternative way to estimate the MLE in continuous univariate distributions. Let $x_{1:n} \leq x_{2:n} \leq ... \leq x_{n:n}$ be a random sample of n size from APTIL distribution.

We define the uniform spacings of a random sample as:

$$D_j(\beta, \omega) = F(x_{j:n}, \beta, \omega) - F(x_{j-1:n}, \beta, \omega), \tag{10}$$

where $F(x_{0:n}, \beta, \omega) = 0$, $F(x_{n+1:n}, \beta, \omega) = 1$ and $\sum_{j=1}^{n+1} D_j(\beta, \omega) = 1$.

The maximum product spacing estimator $(\hat{\beta}, \hat{\omega})$ of unknown parameters β and ω are obtained by maximizing the geometric mean of the uniform spacings in terms of β and ω

$$G(\beta, \omega) = \left[\prod_{j=1}^{n+1} D_j(\beta, \omega) \right]^{\frac{1}{n+1}},$$

or, Alternatively, this can be achieved through the maximization of the function

$$H(\beta, \omega) = \frac{1}{n+1} \sum_{j=1}^{n+1} \log[D_j(\beta, \omega)]. \tag{11}$$

Now, differentiate equation (11) with respect to β and ω respectively,

$$\frac{dH(\beta, \omega)}{d\beta} = \frac{1}{n+1} \sum_{j=1}^{n+1} \frac{1}{D_j(\beta, \omega)} \left[\Delta_1(x_{j:n}|\beta, \omega) - \Delta_1(x_{j-1:n}|\beta, \omega) \right], \tag{12}$$

$$\frac{dH(\beta, \omega)}{d\omega} = \frac{1}{n+1} \sum_{j=1}^{n+1} \frac{1}{D_j(\beta, \omega)} \left[\Delta_2(x_{j:n}|\beta, \omega) - \Delta_2(x_{j-1:n}|\beta, \omega) \right], \tag{13}$$

Where

$$\Delta_1(x_{j:n}|\beta, \omega) = \frac{\beta^{\left[-1+(1+(\frac{\omega}{(1+\omega)x_j})e^{\frac{-\omega}{x_j}})\right]} e^{\frac{-\omega}{x_j}} \left(1 + \frac{\omega}{(1+\omega)x_j}\right)}{\beta - 1} - \frac{\beta^{\left[(1+(\frac{\omega}{(1+\omega)x_j})e^{\frac{-\omega}{x_j}})\right]} e^{\frac{-\omega}{x_j}}}{(\beta-1)^2},$$

and

$$\Delta_2(x_{j:n}|\beta, \omega) = \frac{\beta^{\left(1+\frac{\omega}{(\omega+1)x_j}\right)e^{-\frac{\omega}{x_j}}} \log\beta \left(e^{-\omega/x_j} \left[\frac{1}{(\omega+1)} - \frac{\omega}{x_j(\omega+1)^2} \right] - \frac{e^{-\omega/x_j}\left(1+\frac{\omega}{(\omega+1)x_j}\right)}{x_j} \right)}{\beta - 1}.$$

Solve these non-linear equations (12) and (13), we get the $\hat{\beta}$, $\hat{\omega}$ of the parameters β and ω.

4.3. Ordinary and Weighted Least Square Estimator

Swain et al. (1988) introduced methods for estimating the parameters of beta distributions using ordinary least square and weighted least square estimators (WLSEs). Let $x_{1:n} \leq x_{2:n} \leq ... \leq x_{n:n}$ be a random sample of size n from APTIL distribution.

The ordinary least square estimator (OLSEs) of the parameters $\hat{\beta}_{LSE}$, $\hat{\omega}_{LSE}$ of unknown parameters β and ω are obtained by maximizing the negative of the function $S(\beta, \omega)$.

$$S(\beta, \omega) = \sum_{j=1}^{n} \left[F(x_{j:n}, \beta, \omega) - \frac{j}{n+1} \right]^2, \tag{14}$$

These estimators for the distribution can be determined by solving the two non-linear equations.

$$\frac{dS(\beta, \omega)}{d\beta} = \sum_{j=1}^{n} \left[F(x_{j:n}, \beta, \omega) - \frac{j}{n+1} \right] \Delta_1(x_{j:n}|\beta, \omega) = 0, \tag{15}$$

$$\frac{dS(\beta, \omega)}{d\omega} = \sum_{j=1}^{n} \left[F(x_{j:n}, \beta, \omega) - \frac{j}{n+1} \right] \Delta_2(x_{j:n}|\beta, \omega) = 0. \tag{16}$$

Now, The WLSEs $\hat{\beta}_{WLSE}$, $\hat{\omega}_{WLSE}$ of unknown parameters β and ω are obtained by maximizing the negative of the function $W(\beta, \omega)$.

$$W(\beta,\omega) = \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left[F(x_{j:n},\beta,\omega) - \frac{j}{n+1} \right]^2, \tag{17}$$

Now, differentiate with respect to β and ω, Also the WLSEs can be obtained by solving the two non-linear equaitons

$$\frac{dW(\beta,\omega)}{d\beta} = \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left[F(x_{j:n},\beta,\omega) - \frac{j}{n+1} \right] \Delta_1(x_{j:n}|\beta,\omega) = 0, \tag{18}$$

$$\frac{dW(\beta,\omega)}{d\omega} = \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left[F(x_{j:n},\beta,\omega) - \frac{j}{n+1} \right] \Delta_2(x_{j:n}|\beta,\omega) = 0. \tag{19}$$

5. Numerical Tabulated Values

In this section, we obtained the numerical values of the simple inverse moments, ordered inverse moments, and entropy under different parameter configurations. Table 1 shows that the values of inverse moments are decreasing concerning both β and ω. Table no. 2 shows that the inverse ordered moments increase when the sample sizes increase and decrease when the order of the sample increases. In Table 3, we reported the uncertainty of the random variables (entropy) based on order statistics. We see that entropy is reduced concerning n and δ, and increased concerning r, ω, and β. The average values of the point estimators, absolute bias (Abias), and mean square errors (MSEs) of the MPSEs, LSEs, and WLSEs are obtained in Table 4-5. It can be seen that the estimators demonstrate the consistency property, which states that the MSEs fall with increasing sample size. Also, Abias of $\hat{\beta}$ and $\hat{\omega}$ drop as sample size n grows.The Abias of $\hat{\beta}$ and $\hat{\omega}$ increases when the given values of β and ω increase, respectively. Additionally, the MSEs of the $\hat{\omega}$ are smaller than the $\hat{\beta}$ across all estimation techniques. In Table 6, we obtained the average value of the estimators based on Type-II censored data for different sample sizes up to 20. R software is used to obtain the numerical value in the section.

6. Data Analysis

This section demonstrates the application of the APTIL distribution to model two complete datasets. The dataset comprises survival times of cancer patients who received both radiotherapy and chemotherapy, as documented by Sharma et al. (2015), along with laboratory animals infected with virulent tubercle bacilli (VCB), as described by Bjerkedal (1960).

We compare the fit of the APTIL model against the inverse Lindley (IL), generalized inverse Lindley (GIL), and exponentiated inverse Lindley (EIL) models. The parameters of the given APTIL model are estimated using the maximum likelihood method. The goodness-of-fit for each model is evaluated using $-2L$ (where L represents the log-likelihood function at the maximum likelihood estimates), the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC) and the Kolmogorov-Smirnov (K-S) statistic, etc. The probability density functions (PDFs) of the models being compared are as follows:

$$IL : f(y,\omega) = \frac{\omega^2\,(1+y)e^{-\frac{\omega}{y}}}{(\omega+1)\,y^3},$$

$$GIL : f(y,\beta,\omega) = \frac{\omega^2\,\beta(1+y^\beta)e^{-\frac{\omega}{y^\beta}}}{(\omega+1)\,y^{2\beta+1}},$$

$$EIL : f(y,\beta,\omega) = \frac{\omega^2\,\beta(1+y)e^{-\frac{\omega}{y}}}{(\omega+1)\,y^3} \left[1+\frac{\omega}{(\omega+1)y}\right]^{\beta-1}.$$

Table 1: ***Simple inverse moments for different values of parameters.***

ω	β	$E(X^{-1})$	Simu.	$E(X^{-2})$	Simu.	$E(X^{-3})$	Simu.	$E(X^{-4})$	Simu.
1	0.1	2.314548	2.331223	7.704317	7.771673	32.68911	32.70861	168.3211	168.0245
	0.2	2.073209	2.034595	6.529134	6.528879	26.84380	26.18420	135.7008	134.8664
	0.3	1.928075	1.917395	5.854024	5.895169	23.58125	23.53371	117.8160	116.4582
	0.4	1.824470	1.816186	5.386630	5.394053	21.36384	21.22291	105.7997	105.0310
	0.5	1.744264	1.756829	5.032941	5.021223	19.70951	19.82699	96.9118	96.91610
	1.5	1.363571	1.355178	3.452668	3.444808	12.57426	12.65432	59.4707	59.71060
	2.0	1.270962	1.265415	3.093743	3.055214	11.03344	11.09712	51.5297	51.51160
	2.5	1.201935	1.246439	2.833117	2.886154	9.925360	9.919440	45.91030	45.00080
	5.0	1.005357	1.013699	2.125455	2.114904	7.006350	7.034690	31.34890	31.72369
	10	0.838151	0.843284	1.568548	1.558635	4.813250	4.805650	20.74810	20.52640
	50	0.558882	0.559042	0.756664	0.753257	1.881790	1.896040	7.231200	7.321900
2	0.1	1.048285	1.045695	1.636890	1.632499	3.204340	3.191229	8.270195	8.252589
	0.2	0.934639	0.9325277	1.380429	1.383598	2.715235	2.732706	6.648657	6.614443
	0.3	0.866495	0.862905	1.233720	1.237202	2.379165	2.353065	5.762054	5.767440
	0.4	0.817968	0.811374	1.132404	1.138817	2.151287	2.167789	5.167429	5.150019
	0.5	0.780471	0.783008	1.055904	1.050696	1.981576	1.990138	4.728195	4.767613
	1.5	0.603377	0.607440	0.715969	0.710153	1.253991	1.259122	2.884162	2.888481
	2.0	0.560539	0.561522	0.639253	0.632687	1.096489	1.099304	2.494614	2.412339
	2.5	0.528679	0.532921	0.583684	0.586809	0.984184	0.984960	2.219369	2.220244
	5.0	0.438306	0.439433	0.433480	0.434357	0.689175	0.680692	1.508024	1.509796
	10	0.361934	0.633851	0.316161	0.319214	0.469388	0.468313	0.992091	0.995838
	50	0.235820	0.238021	0.147422	0.141941	0.178502	0.174052	0.340021	0.349247

Table 2: *Inverse ordered moments for* $\beta = 1.5$ *and* $\omega = 1$.

n	k	$E(X_{k:n}^{-1})$	Simu.	$E(X_{k:n}^{-2})$	Simu.	$E(X_{k:n}^{-3})$	$E(X_{k:n}^{-4})$
1	1	1.363571	1.391780	3.452678	3.502368	12.58426	59.7407
2	1	2.010400	2.051238	5.944973	5.856776	23.31445	114.298
	2	0.761741	0.728368	0.960353	0.949521	1.854076	4.64344
3	1	2.431213	2.346852	7.929712	7.745784	32.79313	165.522
	2	1.168776	1.140782	1.975494	1.930773	4.357072	11.8497
	3	0.490723	0.493803	0.452798	0.482469	0.602577	1.040294
4	1	2.742270	2.707621	9.594681	9.421134	41.35207	213.824
	2	1.498041	1.454515	2.934806	2.923782	7.116332	20.6156
	3	0.839511	0.823645	1.016182	0.971225	1.597813	3.08382
	4	0.374461	0.376432	0.265003	0.219342	0.270832	0.359116
5	1	2.988658	2.923374	11.03727	11.32705	49.19754	259.683
	2	1.756717	1.772059	3.824311	3.964862	9.970175	30.38680
	3	1.110027	1.160500	1.600550	1.679393	2.835567	5.95881
	4	0.659166	0.650886	0.626603	0.594682	0.772643	1.167160
	5	0.303284	0.373556	0.174603	0.214931	0.145379	0.157104
6	1	3.192471	3.193420	12.31509	12.31837	56.46846	303.4548
	2	1.969593	1.964130	4.648214	4.643644	12.84293	40.82870
	3	1.330966	1.332808	2.176504	2.178386	4.224657	9.503093
	4	0.889087	0.889453	1.024596	1.013474	1.446477	2.414532
	5	0.544205	0.546930	0.427607	0.422068	0.435725	0.543484
	6	0.255099	0.255863	0.124003	0.125899	0.087310	0.079828
7	1	3.366156	3.368594	13.46526	13.44433	63.26377	345.4096
	2	2.150362	2.147629	5.414017	5.412450	15.69657	51.72626
	3	1.517670	1.516302	2.733705	2.746321	5.708834	13.58479
	4	1.082027	1.085734	1.433570	1.434607	2.245734	4.060830
	5	0.744382	0.748583	0.717865	0.708529	0.847020	1.179813
	6	0.464135	0.465456	0.311504	0.312752	0.271208	0.288952
	7	0.220261	0.223928	0.092751	0.090561	0.056613	0.044974
8	1	3.517409	3.506901	14.51329	14.52638	69.65698	385.7631
	2	2.307382	2.314942	6.129064	6.129839	18.51136	62.93498

n	k	$E(X_{k:n}^{-1})$	Simu.	$E(X_{k:n}^{-2})$	Simu.	$E(X_{k:n}^{-3})$	$E(X_{k:n}^{-4})$
8	3	1.679303	1.679057	3.268878	3.268596	7.252214	18.10010
	4	1.248283	1.241471	1.841749	1.834859	3.136535	6.059281
	5	0.915772	0.922539	1.025392	1.023524	1.354972	2.062366
	6	0.641549	0.645641	0.533350	0.530055	0.542249	0.650282
	7	0.404996	0.408295	0.237556	0.237371	0.180861	0.168509
	8	0.193869	0.193878	0.072065	0.071588	0.038917	0.027326
9	1	3.651320	3.644169	15.47748	15.39416	75.70446	424.6892
	2	2.446126	2.439801	6.799766	6.808035	21.27713	74.35425
	3	1.821777	1.825339	3.781605	3.777954	8.831169	22.96754
	4	1.394354	1.394907	2.243426	2.244367	4.094303	8.365233
	5	1.065693	1.068704	1.339653	1.329791	1.939324	3.176840
	6	0.795834	0.792929	0.773987	0.775974	0.887491	1.170780
	7	0.564406	0.562387	0.413033	0.414537	0.369628	0.390029
	8	0.359451	0.358772	0.187419	0.183222	0.126927	0.105217
	9	0.173171	0.172704	0.057646	0.057665	0.027916	0.017591
10	1	3.771425	3.781958	16.37148	16.38612	81.45062	462.3311
	2	5.270374	2.575892	7.431547	7.414166	23.98898	85.91212
	3	1.949133	1.924260	4.272650	4.279779	10.42974	28.12275
	4	1.524614	1.529267	2.635844	2.640944	5.101183	10.93872
	5	1.198965	1.197620	1.657498	1.664207	2.583984	1.505009
	6	0.932421	0.936639	1.024507	1.024196	1.294663	1.848669
	7	0.704776	0.704908	0.606968	0.603754	0.616042	0.718864
	8	0.504248	0.501662	0.329918	0.326356	0.264022	0.249101
	9	0.323252	0.323361	0.151794	0.150796	0.092653	0.069246
	10	0.156496	0.158029	0.047185	0.046646	0.020723	0.011850

Table 3: ***Entropy based on ordered statistic for different values of parameters.***

				$H_\delta(X_{k:n})$	$H_\delta(X_{k:n})$		$H_\delta(X_{k:n})$	$H_\delta(X_{k:n})$					$H_\delta(X_{k:n})$	$H_\delta(X_{k:n})$		$H_\delta(X_{k:n})$	$H_\delta(X_{k:n})$
δ	ω	n	k	β	β	ω	β	β	δ	ω	n	k	β	β	ω	β	β
				0.5	1.5		0.5	1.5					0.5	1.5		0.5	1.5
2	1	1	1	0.566016	0.983094	2	1.441281	1.874449	3	1	1	1	0.362866	0.764917	2	1.228919	1.646012
		2	1	-0.135989	0.246265		0.720812	1.118697			2	1	-0.273459	0.097624		0.577321	0.962754
			2	1.149889	1.634683		2.063844	2.566560				2	0.936543	1.412741		1.843326	2.338296
		3	1	-0.481382	-0.121843		0.364686	0.739056			3	1	-0.595965	-0.245181		0.245632	0.609722
			2	0.605059	0.758941		1.197879	1.671258				2	0.161715	0.608681		1.049365	1.515824
			3	1.510850	2.021782		2.443791	2.970939				3	1.292717	1.797606		2.220009	2.742552
		4	1	-0.701641	-0.358653		0.137178	0.494235			4	1	-0.804659	-0.469384		0.030573	0.378815
			2	-0.123826	0.307564		0.755447	1.206316				2	-0.242556	0.182984		0.632716	1.077415
			3	0.599675	1.084983		1.512504	2.017256				3	0.453208	0.933360		1.361591	1.861784
			4	1.774751	2.299498		2.719045	3.257917				4	1.553867	2.074008		2.493617	3.029387
		5	1	-0.859900	-0.529844		-0.026488	0.317032			5	1	-0.956012	-0.632673		-0.125561	0.210208
			2	-0.400436	0.013471		0.469246	0.902171				2	-0.506796	-0.090784		0.359653	0.787017
			3	0.126992	0.592328		1.025834	1.511668				3	0.005987	0.466849		0.901226	1.382761
			4	0.824517	1.327734		1.750463	2.271946				4	0.676091	1.175161		1.598196	2.116398
			5	1.983316	2.516566		2.935119	3.480610				5	1.760665	2.290247		2.708742	3.252008
4	1	1	1	0.259625	0.654236	2	1.121323	1.530249	5	1	1	1	0.195645	0.585811	2	1.054813	1.458796
		2	1	-0.347959	0.017552		0.499937	0.879068			2	1	-0.395811	-0.033644		0.450395	0.825719
			2	0.828393	1.300370		1.731665	2.222673				2	0.761528	1.230938		1.662724	2.151326
		3	1	-0.659983	-0.314031		0.179453	0.538351			3	1	-0.701824	-0.358595		0.136341	0.492102
			2	0.084342	0.527885		0.969473	1.432427				2	0.034784	0.476286		0.918426	1.379263
			3	1.182215	1.684043		2.106686	2.626819				3	1.113955	1.614014		2.036739	2.555423
		4	1	-0.863275	-0.531855		-0.029772	0.314021			4	1	-0.901992	-0.572877		-0.069497	0.271608
			2	-0.308651	0.114007		0.564681	1.006274				2	-0.351306	0.069615		0.521201	0.959492
			3	0.374274	0.851883		1.280453	1.778359				3	0.331095	0.793387		1.248487	1.726132
			4	1.442004	1.959814		2.379451	2.913628				4	1.368747	1.889275		2.311420	2.842035
		5	1	-1.011296	-0.691367		-0.182338	0.149527			5	1	-1.059585	-0.728887		-0.204637	0.111071
			2	-0.560595	-0.158418		0.298095	0.723395				2	-.060679	-0.202019		0.257137	0.680027
			3	-0.264616	0.553374		0.872713	1.329425				3	-0.104941	0.352372		0.787506	1.265550
			4	0.783015	1.095762		1.137290	2.058786				4	0.545149	1.040940		1.006431	1.979750
			5	1.651690	2.174688		2.640668	3.138845				5	1.578300	2.104890		2.52332	3.06577

Table 4: ***Parameter estimates (β, ω) and average bias with their associated MSEs (in parentheses) values for LSE, WLSE, and MPSE methods for different sample size.***

β	ω	n	LSE				WLSE				MPSE			
			$\hat{\beta}$ (MSE)	$\hat{\omega}$ (MSE)	$Abias(\hat{\beta})$	$Abias(\hat{\omega})$	$\hat{\beta}$ (MSE)	$\hat{\omega}$ (MSE)	$Abias(\hat{\beta})$	$Abias(\hat{\omega})$	$\hat{\beta}$ (MSE)	$\hat{\omega}$ (MSE)	$Abias(\hat{\beta})$	$Abias(\hat{\omega})$
0.5	1	25	0.637386	0.975585	0.237987	0.101824	0.616275	0.995538	0.260177	0.114827	0.698037	0.971588	0.397927	0.155466
			(0.092579)	(0.017561)			(0.105501)	(0.023888)			(0.196783)	(0.039136)		
		50	0.609423	0.976824	0.195808	0.072025	0.590378	0.999206	0.213446	0.082310	0.649748	0.970567	0.288038	0.114304
			(0.065617)	(0.008520)			(0.072633)	(0.011945)			(0.124681)	(0.020299)		
		100	0.582075	0.978794	0.148608	0.054627	0.565123	0.989177	0.161078	0.059489	0.605188	0.975144	0.217581	0.083261
			(0.044035)	(0.004643)			(0.046532)	(0.006093)			(0.074684)	(0.010564)		
		200	0.552606	0.985619	0.100785	0.041366	0.543132	0.991719	0.113691	0.043795	0.564618	0.984897	0.157162	0.060981
			(0.025782)	(0.002641)			(0.027502)	(0.003135)			(0.041276)	(0.005696)		
1.5	1	25	1.905604	0.974577	0.652596	0.106630	1.336341	1.065713	0.359367	0.143569	2.066333	0.982984	1.0854	0.181821
			(0.706241)	(0.020519)			(0.236062)	(0.039367)			(1.666633)	(0.056958)		
		50	1.843058	0.970749	0.549995	0.074901	1.325943	1.054899	0.321701	0.103841	1.950562	0.978026	0.896248	0.138122
			(0.549203)	(0.009122)			(0.190441)	(0.019975)			(1.163314)	(0.031867)		
		100	1.761556	0.974634	0.419861	0.056233	1.318710	1.047398	0.300888	0.078161	1.837731	0.978454	0.704062	0.102236
			(0.386169)	(0.004887)			(0.161448)	(0.010768)			(0.746056)	(0.016849)		
		200	1.668748	0.982078	0.281873	0.044522	1.322490	1.041600	0.292825	0.061516	1.719594	0.984513	0.523868	0.077184
			(0.228348)	(0.003030)			(0.142932)	(0.006222)			(0.440508)	(0.009504)		
5	1	25	6.336132	0.965381	1.613710	0.106121	6.351245	0.966245	1.736097	0.103350	6.580865	0.990788	3.003141	0.173545
			(5.564003)	(0.021493)			(5.989269)	(0.019601)			(13.03988)	(0.057943)		
		50	6.202758	0.964856	1.437081	0.070188	6.183371	0.966311	1.550269	0.069881	6.548007	0.975345	2.725966	0.133489
			(4.936744)	(0.008437)			(5.203307)	(0.008253)			(10.80154)	(0.033449)		
		100	5.976685	0.969497	1.153179	0.052524	5.922517	0.971595	1.222183	0.052348	6.205078	0.976962	2.254750	0.104387
			(3.925176)	(0.004456)			(3.978750)	(0.004328)			(7.743268)	(0.020084)		
		200	5.700874	0.977424	0.832392	0.041629	5.625562	0.981081	0.859995	0.040664	5.888415	0.978980	1.697956	0.078459
			(2.722126)	(0.002744)			(2.652198)	(0.002563)			(4.836741)	(0.011203)		

Table 5: ***Parameter estimates (β, ω) and average bias with their associated MSEs (in parentheses) values for LSE, WLSE, and MPSE methods for different sample size.***

β	ω	n	LSE				WLSE				MPSE			
			$\hat{\beta}$ (MSE)	$\hat{\omega}$ (MSE)	$Abias(\hat{\beta})$	$Abias(\hat{\omega})$	$\hat{\beta}$ (MSE)	$\hat{\omega}$ (MSE)	$Abias(\hat{\beta})$	$Abias(\hat{\omega})$	$\hat{\beta}$ (MSE)	$\hat{\omega}$ (MSE)	$Abias(\hat{\beta})$	$Abias(\hat{\omega})$
0.5	2	25	0.638882	1.972788	0.259959	0.230225	0.616498	2.000746	0.274173	0.255499	0.711213	1.939852	0.398558	0.330856
			(0.108036)	(0.093138)			(0.116257)	(0.117912)			(0.214034)	(0.174596)		
		50	0.604535	1.960607	0.208851	0.161835	0.580101	1.997143	0.223901	0.185229	0.645494	1.944217	0.290703	0.244174
			(0.072373)	(0.043693)			(0.078317)	(0.062302)			(0.127908)	(0.091682)		
		100	0.576255	1.960745	0.153265	0.120029	0.559081	1.985559	0.170572	0.132093	0.601691	1.953117	0.220355	0.178802
			(0.045095)	(0.022600)			(0.049974)	(0.030005)			(0.077051)	(0.049104)		
		200	0.550372	1.973254	0.109226	0.089986	0.537889	1.987824	0.120818	0.096547	0.565249	1.968135	0.160842	0.129992
			(0.027338)	(0.012448)			(0.028971)	(0.015610)			(0.043184)	(0.025832)		
1.5	2	25	1.917391	1.961460	0.714866	0.240787	1.815394	2.016646	0.748448	0.287442	2.084332	1.973753	1.126338	0.394566
			(0.830152)	(0.107883)			(0.910942)	(0.158919)			(1.803469)	(0.271302)		
		50	1.824188	1.951354	0.588530	0.169041	1.686914	2.010717	0.581895	0.209202	1.988509	1.948221	0.934984	0.297445
			(0.593426)	(0.049448)			(0.582744)	(0.084241)			(1.261134)	(0.144138)		
		100	1.744089	1.954090	0.455460	0.126933	1.619094	2.003510	0.421048	0.153755	1.842560	1.956033	0.726977	0.223902
			(0.415277)	(0.026031)			(0.327590)	(0.043731)			(0.795395)	(0.078744)		
		200	1.661623	1.968009	0.315977	0.095642	1.574445	1.996413	0.280953	0.110909	1.730847	1.961936	0.528972	0.164719
			(0.252087)	(0.014442)			(0.145785)	(0.020639)			(0.447243)	(0.041942)		
5	2	25	6.363081	1.932593	1.736845	0.225954	6.333086	1.935449	1.904326	0.232242	6.635368	2.007516	3.255765	0.402176
			(6.019899)	(0.096361)			(6.572107)	(0.098715)			(15.11342)	(0.320577)		
		50	6.214566	1.926994	1.546195	0.152766	6.130379	1.937618	1.685373	0.154465	6.590782	1.964840	2.928085	0.309377
			(5.271518)	(0.040251)			(5.555693)	(0.040804)			(12.18044)	(0.177384)		
		100	5.979247	1.932987	1.236184	0.113084	5.937988	1.944444	1.371404	0.114479	6.259217	1.959632	2.425413	0.244034
			(4.142691)	(0.020326)			(4.410698)	(0.020823)			(8.67794)	(0.107961)		
		200	5.667356	1.952429	0.855571	0.088937	5.635294	1.959508	0.962237	0.089892	5.882566	1.962738	1.847117	0.183401
			(2.774001)	(0.012422)			(2.859480)	(0.012411)			(5.367904)	(0.059379)		

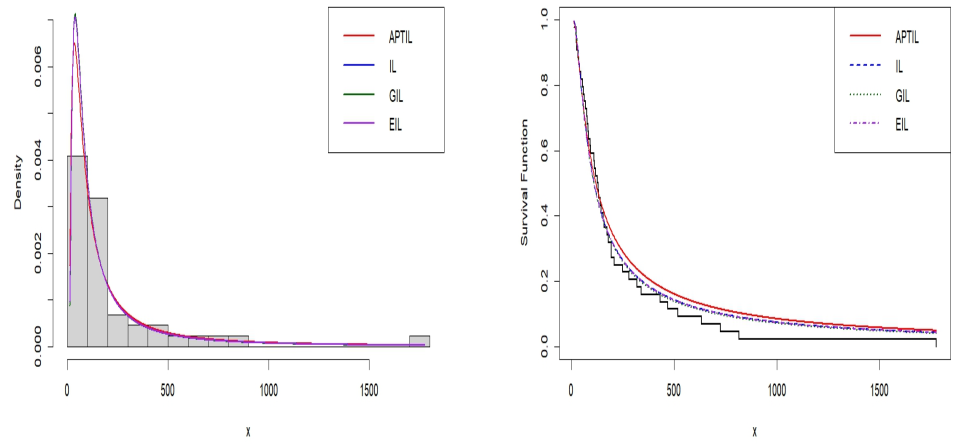
Table 6: *Average value and average Abias of the estimate and asscoiated MSEs (in parentheses) based on Type-II censored sample.*

β	ω	n	k	$\hat{\beta}$	$\hat{\omega}$	$bias(\hat{\beta})$	$bias(\hat{\omega})$	β	ω	$\hat{\beta}$	$\hat{\omega}$	$bias(\hat{\beta})$	$bias(\hat{\omega})$
2	2	5	1	2.790050	2.454869	0.797004	0.454868	2	3	3.123014	3.550028	1.213014	0.550028
				(2.228371)	(0.799882)					(5.392636)	(1.359068)		
			2	3.123735	2.306525	1.123735	0.306525			3.595945	3.391116	1.595945	0.391115
				(4.077838)	(0.466996)					(8.703422)	(0.865754)		
			3	2.514182	1.740898	0.514181	0.279108			2.769508	2.705041	0.769509	0.294959
				(0.750596)	(0.235947)					(2.156670)	(0.376933)		
			4	2.171652	1.740898	0.711652	0.259102			2.336716	2.603423	0.336716	0.396576
				(0.089397)	(0.199420)					(0.349001)	(0.487257)		
			5	2.032492	1.752091	0.032927	0.247908			2.063930	2.637320	0.065122	0.362679
				(0.005382)	(0.191177)					(0.018063)	(0.390866)		
0.5	1	20	1	0.802458	1.064635	0.302458	0.062463	1.5	1	1.781518	1.127389	0.281518	0.127389
				(0.298688)	(0.172691)					(0.239425)	(0.054648)		
			2	0.793376	1.048744	0.293376	0.048743			1.806608	1.107338	0.306607	0.107389
				(0.273455)	(0.009591)					(0.262285)	(0.037906)		
			3	0.797748	1.041419	0.297748	0.041419			1.828743	1.087333	0.328742	0.093425
				(0.276233)	(0.006794)					(0.325842)	(0.028448)		
			4	0.810955	1.032140	0.310955	0.032139			1.879684	1.072602	0.379683	0.088521
				(0.310601)	(0.004196)					(0.425071)	(0.024868)		
			5	0.811176	1.025800	0.311176	0.026108			1.921654	1.056325	0.421653	0.083093
				(0.303562)	(0.003056)					(0.460824)	(0.022317)		
			6	0.829626	1.016446	0.329626	0.017774			2.004831	1.039661	0.504831	0.083317
				(0.338018)	(0.001926)					(0.572395)	(0.022806)		
			7	0.850027	1.004636	0.350027	0.009101			2.104177	1.016852	0.604177	0.081103
				(0.387530)	(0.001296)					(0.720625)	(0.022901)		
			8	0.936181	0.983426	0.436181	0.019494			2.328127	0.981741	0.828127	0.100794
				(0.609032)	(0.003253)					(1.107095)	(0.028560)		
			9	1.069133	0.942251	0.569132	0.059001			2.427752	0.934527	0.927752	0.117319
				(0.938765)	(0.014236)					(1.419301)	(0.029612)		
			10	1.119406	0.896001	0.619461	0.104698			2.087746	0.909165	0.587746	0.114038
				(0.883747)	(0.029747)					(0.771543)	(0.029529)		

Table Continued...

Average value and average Abias of the estimate and asscoiated MSEs (in parentheses) based on Type-II censored sample.

β	ω	n	k	$\hat{\beta}$	$\hat{\omega}$	$bias(\hat{\beta})$	$bias(\hat{\omega})$	β	ω	$\hat{\beta}$	$\hat{\omega}$	$bias(\hat{\beta})$	$bias(\hat{\omega})$
0.5	1	20	11	1.050574	0.874865	0.550574	0.125134	1.5	1	1.786126	0.901026	0.286125	0.099249
				(0.605585)	(0.036527)					(0.213238)	(0.026931)		
			12	0.936244	0.881583	0.436244	0.118461			1.693016	0.906343	0.193016	0.093656
				(0.396349)	(0.032982)					(0.106864)	(0.025609)		
			13	0.825256	0.888847	0.325256	0.111155			1.631462	0.916451	0.131462	0.083549
				(0.246175)	(0.031717)					(0.049882)	(0.020465)		
			14	0.740543	0.896861	0.240544	0.103139			1.592876	0.919981	0.098276	0.080019
				(0.153904)	(0.029732)					(0.027359)	(0.018383)		
			15	0.681176	0.903957	0.181176	0.096043			1.572537	0.923384	0.072536	0.076615
				(0.100784)	(0.026622)					(0.014759)	(0.016728)		
			16	0.620422	0.912274	0.120422	0.087725			1.551152	0.927165	0.051152	0.076615
				(0.053929)	(0.022591)					(0.007293)	(0.015175)		
			17	0.569343	0.920849	0.069343	0.079151			1.535271	0.929231	0.035271	0.070768
				(0.015463)	(0.018257)					(0.003487)	(0.014556)		
			18	0.540318	0.929294	0.040318	0.070705			1.523008	0.929032	0.023008	0.070967
				(0.005115)	(0.014735)					(0.001564)	(0.015211)		
			19	0.519504	0.935767	0.019504	0.064233			1.511859	0.930826	0.011858	0.069173
				(0.001231)	(0.012284)					(0.000431)	(0.014179)		
			20	0.503568	0.938633	0.003841	0.061366			1.501700	0.933863	0.001755	0.066133
				(0.000078)	(0.011617)					(0.000015)	(0.013063)		



(a) *The relative histogram and fitted density graph* **(b)** *The fitted and estimated survival function*

Figure 4: *The fitted and empirical survival functions for data 1*

Table 7: ***Goodness of fit Statistics for different data***

Data 1	Model	$\hat{\beta}$	$\hat{\omega}$	$-2L$	AIC	BIC	K-S
	APTIL(β,ω)	2.197908	62.17588	558.8457	562.8457	566.4141	0.08469
	IL(ω)		77.67544	688.2096	690.2096	691.9938	0.08884
	GIL(β,ω)	1.013325	81.73121	559.1425	563.1425	566.7108	0.09273
	EIL(β,ω)	2.799310	28.33204	559.1604	563.1604	566.7288	0.08922
Data 2	APTIL(β,ω)	0.032191	2.808439	230.8168	234.8168	239.3701	0.14570
	IL(ω)		1.576856	239.5689	241.5689	243.8456	0.19414
	GIL(β,ω)	1.071339	1.548701	238.6981	242.6981	247.2514	0.18107
	EIL(β,ω)	0.694550	2.109365	239.4938	243.4938	248.0471	0.19900

The value of the MLEs of the parameters, $-2L$, AIC, BIC, and K-S displayed in Table no. 7. Among all the competing models, the APTIL distribution stands out with the lowest values of $-2L$, AIC, BIC, and K-S statistics show that the APTIL model is the best fit for explaining the survival times for both groups. The relative histogram with the fitted densities, together with the plots of the fitted survival and empirical survival functions, are presented in Figures 4 and 5.

7. Conclusion

The APTIL distribution model has one scale and one shape parameter. The APTIL distribution is useful for an upside-down bathtub-shaped hazard rate. This paper derives the single moments, product moments, and entropy based on order statistics. We also obtained the estimates of parameters in different ways: maximum product spacing estimator, ordinary and weighted least square estimator. The APTIL model is more flexible than the IL, GIL, and EIL distributions in cases like the survival outcomes of individuals diagnosed with head and neck cancer and laboratory animals infected with virulent tubercle bacilli. We hope that our results and the findings of the estimation procedure will contribute to further research and applications of the APTIL model based on order statistics.

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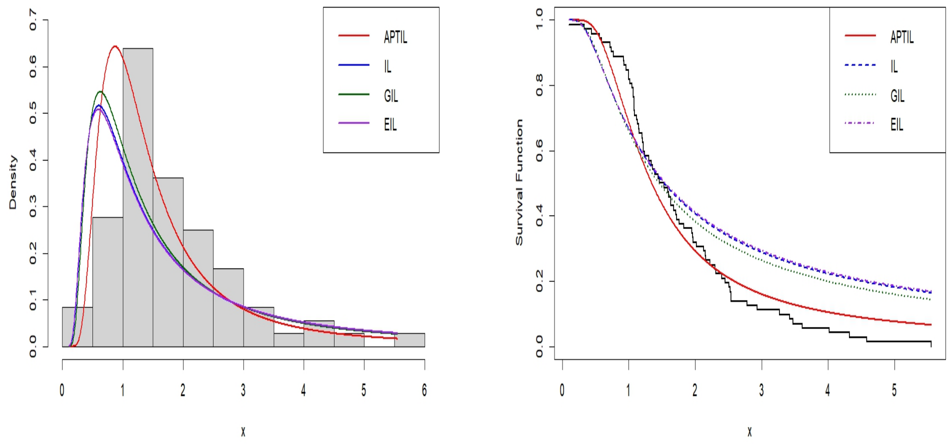

(a) *The relative histogram and fitted density graph* **(b)** *The fitted and estimated survival function*

Figure 5: *The fitted and empirical survival functions for data 2*

Conflicts of Interest: The author declare that there are no conflicts of interest regarding the publication of this paper.

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Appendix The proof of the theorem 1

From equation (5),

$$
\begin{aligned}
E\left[\left(\frac{1}{X_{k:n}}\right)^p\right] &= C_{k:n}\int_0^\infty \left(\frac{1}{x}\right)^p F(x)^{k-1}[1-F(x)]^{n-k}f(x)dx \\
&= C_{k:n}\sum_{\gamma_1=0}^{n-k}\sum_{\gamma_2=0}^{k-1+\gamma_1}(-1)^{k-1-\gamma_2}\binom{n-k}{\gamma_1}\binom{k-1+\gamma_1}{\gamma_2}\frac{\log\beta}{\beta-1}\frac{\omega^2}{\omega+1}\frac{1}{(\beta-1)^{k-1+\gamma_1}} \\
&\times \int_0^\infty \left(\frac{1}{x}\right)^p \frac{1+x}{x^3}e^{-\frac{\omega}{x}}\beta^{\gamma_2+1\left(1+\frac{\omega}{(\omega+1)x}\right)e^{-\frac{\omega}{x}}}dx \\
&= C_{k:n}\sum_{\gamma_3=0}^{\infty}\sum_{\gamma_4=0}^{\gamma_3}\sum_{\gamma_1=0}^{n-k}\sum_{\gamma_2=0}^{k-1+\gamma_1}(-1)^{k-1-\gamma_2}\binom{n-k}{\gamma_1}\binom{k-1+\gamma_1}{\gamma_2}\binom{\gamma_3}{\gamma_4}\frac{(\log\beta)^{\gamma_3+1}}{\gamma_3!} \\
&\times \frac{\omega^{\gamma_4+2}}{(\omega+1)^{\gamma_4+1}}\frac{(\gamma_2+1)^{\gamma_3}}{(\beta-1)^{k+\gamma_1}}\int_0^\infty x^{-p-3-\gamma_4}(1+x)e^{-\frac{\omega(\gamma_3+1)}{x}}dx.
\end{aligned}
$$

Now, put $y=\frac{\omega(\gamma_3+1)}{x}$. The theorem's result is obtained using the definition of the gamma function

The proof of the theorem 2

Consider, For $x<y$

$$
\begin{aligned}
E\left[\left(\frac{1}{X_{k:n}}\right)^p\left(\frac{1}{X_{l:n}}\right)^q\right] &= \int_0^\infty\int_x^\infty \left(\frac{1}{x}\right)^p\left(\frac{1}{y}\right)^q f_{X_{k:n},X_{l:n}}(x,y)dxdy \\
&= \sum_{\gamma_1=0}^{l-k-1}\sum_{\gamma_2=0}^{n-l}C_{k,l:n}\binom{l-k-1}{\gamma_1}\binom{n-l}{\gamma_2}(-1)^{l-k-1-\gamma_1+\gamma_2} \\
&\times \int_0^\infty\int_x^\infty \left(\frac{1}{x}\right)^p\left(\frac{1}{y}\right)^q [F(x)]^{l-2-\gamma_1}[F(y)]^{\gamma_1+\gamma_2}f(x)f(y)dydx \\
&= \eta(\beta,\omega)\sum_{\gamma_9=0}^{\infty}(-1)^{\gamma_9}\left(\frac{(\omega(\gamma_6+1))^{\gamma_9}}{\gamma_9!}\right)\left[\frac{1}{\gamma_8+\gamma_9+2+q}\int_0^\infty(1+x)\right. \\
&\times e^{\frac{-\omega(\gamma_5+1)}{x}}x^{-p-q-\gamma_7-\gamma_8-\gamma_9-5}dx+\frac{1}{\gamma_8+\gamma_9+1+q}\int_0^\infty(1+x)e^{\frac{-\omega(\gamma_5+1)}{x}} \\
&\times \left. x^{-p-q-\gamma_7-\gamma_8-\gamma_9-4}dx\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\eta(\beta,\omega) &= \sum_{\gamma_6=0}^{\infty}\sum_{\gamma_5=0}^{\infty}\sum_{\gamma_1=0}^{l-k-1}\sum_{\gamma_2=0}^{n-l}\sum_{\gamma_3=0}^{l-2-\gamma_1}\sum_{\gamma_4=0}^{\gamma_1+\gamma_2}\sum_{\gamma_7=0}^{\gamma_5}\sum_{\gamma_8=0}^{\gamma_6}C_{k,l:n}\binom{l-k-1}{\gamma_1}\binom{n-l}{\gamma_2}\binom{l-2-\gamma_1}{\gamma_3} \\
&\times \binom{\gamma_1+\gamma_2}{\gamma_4}\binom{\gamma_5}{\gamma_7}\binom{\gamma_6}{\gamma_8}(-1)^{k-1-\gamma_1-\gamma_3-\gamma_4}\left(\frac{1}{(\beta-1)^{l-2+\gamma_2}}\right)\left(\frac{\log\beta}{\beta-1}\right)^2\left(\frac{\omega^2}{\omega+1}\right)^2 \\
&\times \frac{(\log\beta)^{\gamma_5+\gamma_6}}{\gamma_5!\gamma_6!}(\gamma_3+1)^{\gamma_5}(\gamma_4+1)^{\gamma_6}\left(\frac{\omega}{\omega+1}\right)^{\gamma_7+\gamma_8}.
\end{aligned}
$$