A COMPARATIVE STUDY OF GENERALIZED ARADHANA DISTRIBUTION WITH SIGNIFICANT STATISTICAL PROPERTIES AND APPLICATIONS

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Abstract

This study introduces a novel extension of the generalized Aradhana distribution, termed the length biased generalized Aradhana distribution. The proposed distribution is formulated by applying the length biased technique to the baseline generalized Aradhana distribution, enhancing its applicability in reliability and survival analysis. Fundamental statistical properties, including moments, order statistics, reliability functions and entropy are rigorously derived to characterize the distribution's behavior. The parameters of proposed model are estimated using the maximum likelihood estimation method to ensure statistical efficiency. To assess its practical utility, the distribution is applied to two real-world lifetime data sets, demonstrating its superior performance and robustness compared to existing models. These findings highlight the significance and applicability of the length biased generalized Aradhana distribution in modeling lifetime data.

Keywords: length biased distribution, generalized Aradhana distribution, reliability measures, entropy

I. Introduction

In recent years, the increasing popularity and advancements in distribution theory have driven a strong emphasis on developing new, precise and flexible distributions. These innovative models aim to better capture complexities of real-world data across various fields enhancing statistical accuracy and applicability. The core importance of distribution theory lies in its ability to enhance existing distributions by introducing shape parameter because it increases the compatibility and flexibility of the existing distribution. This shape parameter comprises the classical distribution in a more flexible situation while comparing with other distributions and is illustrated through advanced and reputed weighted technique. Fisher [7] constructed the weighted distributions to capture the ascertainment bias which was subsequently extended by Rao [15] in a cohesive manner while modeling statistical data in which standard distributions were not found appropriate. The weighted distributions were incorporated in such a manner to record the observations with equal probability. It is also observed that weighted distributions were applied in several experimental problems in case of biased samples that appear naturally in various situations. The weighted distribution reduced to the length biased distribution specifically if the weight function considers only the length of the units of interest. The concept of length biased distribution introduced by Cox [6] which is a particular case of weighted distributions. Sometimes, the length biased distribution may occur in numerous situations because it seems quite impossible to work with truly random sample from population of interest and hence length biased distributions may arised in clinical trials, reliability theory, population studies and survival analysis were a proper sampling frame is unavailable. Thus length biased distribution may arise if sample observations are unfairly selected and therefore if the observations are selected with probability proportional to their lengths resulting distributions is termed length biased distribution.

There are various authors who contributed remarkably to develop several important length biased probability distributions along with illustrations and applications from diversified fields. Al-Omari and Alsmairan [2], Almakhareez and Alzoubi [1], Rashwan [16], Mustafa and Khan [12], Ratnaparkhi and Nimbalkar [18], Salama et al. [21], Beghriche and Zeghdoudi [5], Perveen et al. [14], Ganaie et al. [8], Mustafa and Khan [11], Saghir et al. [19], Ganaie and Rajagopalan [9], Bashir and Khan [4], Rasool and Ahmad [17], Saghir and Khadim [20], Mudasir and Ahmad [13].

A generalized Aradhana distribution is a two parameteric recently proposed distribution introduced by Welday and Shanker [22] which includes Aradhana distribution as its particular case of it. Some of its structural properties have been derived and determined. Further, its parameters are estimated through maximum likelihood estimation.

II. Length Biased Generalized Aradhana (LBGA) Distribution

The probability density function of generalized Aradhana distribution is given by

$$f(x;\theta,\alpha)=\frac{\theta^3}{\left(\theta^2+2\alpha\theta+2\alpha^2\right)}\left(1+\alpha x\right)^2 e^{-\theta x}\quad ;x>0,\theta>0,\alpha\geq 0 \tag{1}$$

and cumulative distribution function of generalized Aradhana distribution is given by

$$F(x;\theta,\alpha)=1-\left(1+\frac{\alpha\theta x\left(2\theta+\alpha\theta x+2\alpha\right)}{\left(\theta^2+2\theta\alpha+2\alpha^2\right)}\right)e^{-\theta x}\quad ;\ x>0,\theta>0,\alpha\geq 0 \tag{2}$$

Consider the random variable *X* which constitutes non-negative condition with probability density function *f(x)*. Assume its weight function *w(x)* which will be non-negative, then probability density function of weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0$$

Here *w(x)* be the weight function which is non - negative then $E(w(x))=\int w(x)f(x)dx<\infty$.

In this study, we have considered the weight function as $w(x)=x$ to get the length biased version of generalized Aradhana distribution and its probability density function is clearly defined as

$$f_l(x;\theta,\alpha)=\frac{x\,f(x;\theta,\alpha)}{E(x)} \tag{3}$$

Hence $E(x)=\int_0^{\infty}x\,f(x;\theta,\alpha)dx$

$$E(x) = \frac{(\theta^2 + 6\alpha^2 + 4\alpha\theta)}{\theta(\theta^2 + 2\alpha\theta + 2\alpha^2)} \quad (4)$$

After applying the equation (1) and (4) in equation (3), we will obtain probability density function of length biased generalized Aradhana distribution as

$$f_l(x;\theta,\alpha) = \frac{x\theta^4}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}\left(1+\alpha x\right)^2 e^{-\theta x} \quad (5)$$

and the cumulative distribution function of length biased generalized Aradhana distribution will be obtained as

$$F_l(x;\theta,\alpha) = \int_0^x f_l(x;\theta,\alpha)dx$$

$$= \int_0^x \frac{x\theta^4}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}\left(1+\alpha x\right)^2 e^{-\theta x}dx \quad (6)$$

After simplification of equation (6), we will have determined the cumulative distribution function of length biased generalized Aradhana distribution as

$$F_l(x;\theta,\alpha) = \frac{1}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}\left(\theta^2\gamma(2,\theta x) + \alpha^2\gamma(4,\theta x) + 2\alpha\theta\gamma(3,\theta x)\right) \quad (7)$$

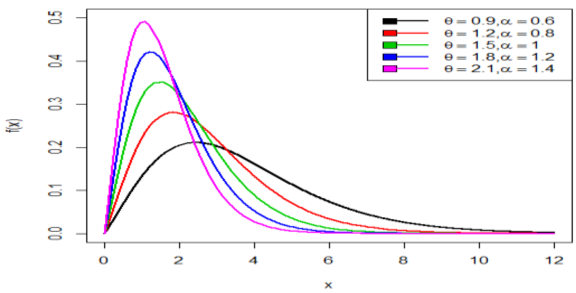

Figure 1: *Pdf plot of LBG Aradhana distribution*

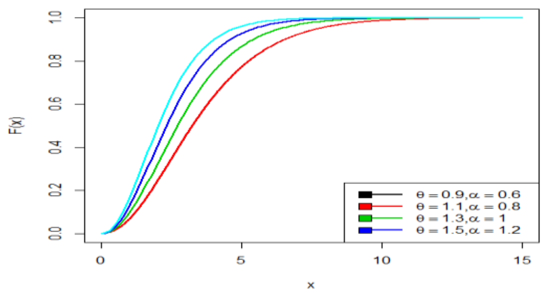

Figure 2: *Cdf plot of LBG Aradhana distribution*

III. Reliability Measures

In this portion, we will have determined some fundamental aspects like reliability function, hazard function, reverse hazard function and mills ratio of length biased generalized Aradhana distribution.

The reliability function of length biased generalized Aradhana distribution is given by

$$R(x) = 1 - F_l(x;\theta,\alpha)$$

$$R(x) = 1 - \frac{1}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}\left(\theta^2\gamma(2,\theta x) + \alpha^2\gamma(4,\theta x) + 2\alpha\theta\gamma(3,\theta x)\right)$$

The hazard rate function of length biased generalized Aradhana distribution is given by

$$h(x) = \frac{f_l(x;\theta,\alpha)}{1. - F_l(x;\theta,\alpha)}$$

$$h(x) = \frac{x\theta^4(1+\alpha x)^2 e^{-\theta x}}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right) - \left(\theta^2\gamma(2,\theta x) + \alpha^2\gamma(4,\theta x) + 2\alpha\theta\,\gamma(3,\theta x)\right)}$$

The reverse hazard rate function of length biased generalized Aradhana distribution is given by

$$h_r(x) = \frac{f_l(x;\theta,\alpha)}{F_l(x;\theta,\alpha)}$$

$$h_r(x) = \frac{x\theta^4(1+\alpha x)^2 e^{-\theta x}}{\left(\theta^2\gamma(2,\theta x) + \alpha^2\gamma(4,\theta x) + 2\alpha\theta\,\gamma(3,\theta x)\right)}$$

The Mills Ratio of length biased generalized Aradhana distribution is given by

$$M.R = \frac{1}{h_r(x)} = \frac{\left(\theta^2\gamma(2,\theta x) + \alpha^2\gamma(4,\theta x) + 2\alpha\theta\,\gamma(3,\theta x)\right)}{x\theta^4(1+\alpha x)^2 e^{-\theta x}}$$

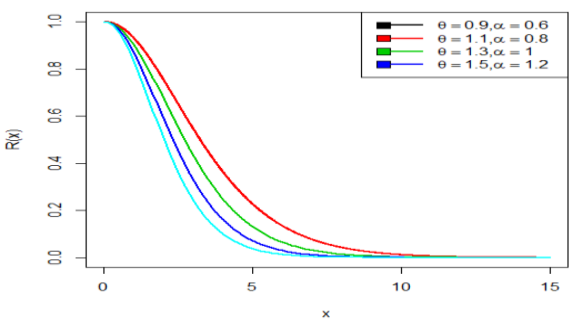

Figure 3: *Reliability plot of LBG Aradhana distribution*

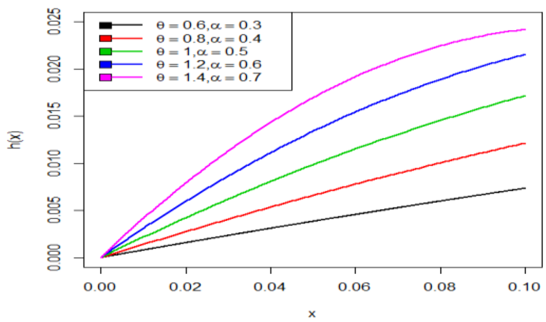

Figure 4: *Hazard plot of LBG Aradhana distribution*

IV. Statistical Properties

In this portion, we have determined and derived some properties of LBG Aradhana distribution.

I. Moments

Consider X be the random variable represents length biased generalized Aradhana distribution with parameters θ and α, then r^{th} order moment of LBG Aradhana distribution will be determined as

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \frac{x^{r+1}\,\theta^4}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}(1+\alpha x)^2 e^{-\theta x}dx \tag{8}$$

After the simplification of expression (8), we obtain

$$\mu_r' = \frac{\left(\theta^2\Gamma(r+2) + \alpha^2\Gamma(r+4) + 2\alpha\theta\,\Gamma(r+3)\right)}{\theta^r\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)} \tag{9}$$

Thus by applying *r* = 1, 2, 3 and 4 in equation (9), we will get the mean and other moments of length biased generalized Aradhana distribution

$$\mu_1' = \frac{\left(2\theta^2 + 24\alpha^2 + 12\alpha\theta\right)}{\theta\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}$$

$$\mu_2' = \frac{\left(6\theta^2 + 120\alpha^2 + 48\alpha\theta\right)}{\theta^2\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}$$

$$\mu_3' = \frac{\left(24\theta^2 + 720\alpha^2 + 240\alpha\theta\right)}{\theta^3\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}$$

$$\mu_4' = \frac{\left(120\theta^2 + 5040\alpha^2 + 1440\alpha\theta\right)}{\theta^4\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}$$

$$\text{Variance} = \frac{\left(2\theta^4 + 108\alpha^2\theta^2 + 24\alpha\theta^3 + 144\alpha^4 + 192\alpha^3\theta\right)}{\theta^2(\theta^2 + 6\alpha^2 + 4\alpha\theta)^2}$$

$$\text{S.D}(\sigma) = \frac{\sqrt{\left(2\theta^4 + 108\alpha^2\theta^2 + 24\alpha\theta^3 + 144\alpha^4 + 192\alpha^3\theta\right)}}{\theta(\theta^2 + 6\alpha^2 + 4\alpha\theta)}$$

II. Harmonic Mean

The harmonic mean of explored distribution will be obtained by

$$\text{H.M} = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \frac{\theta^4}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}\left(1 + \alpha^2x^2 + 2\alpha x\right)e^{-\theta x}dx \tag{10}$$

After the simplification of expression (10), we obtain

$$\text{H.M} = \frac{\theta\left(\theta + 2\alpha^2 + 2\alpha\theta\right)}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}$$

III. Moment generating function and characteristic function

The moment generating function of explored LBG Aradhana distribution will be obtained as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \frac{x\theta^4}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}\left(1 + \alpha x\right)^2 e^{-(\theta - t)x}dx \tag{11}$$

After simplification, we obtain from expression (11)

$$M_X(t) = \frac{1}{(\theta - t)^4 \left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)} \theta^4 \left((\theta - t)^2 + 4\alpha(\theta - t) + 6\alpha^2\right)$$

Particularly, characteristic function of length biased generalized Aradhana distribution is given by

$$\varphi_X(t) = \frac{1}{(\theta - it)^4 \left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)} \theta^4 \left((\theta - it)^2 + 4\alpha(\theta - it) + 6\alpha^2\right)$$

V. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are also termed as classical curves which were studied in order to depict the behavior of inequality in income and poverty. The explored curves can be stated as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x\, f_l(x;\theta,\alpha)dx$$

$$L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x\, f_l(x;\theta,\alpha)dx$$

$$\text{Here } \mu_1' = \frac{\left(2\theta^2 + 24\alpha^2 + 12\alpha\theta\right)}{\theta\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}$$

$$B(p) = \frac{\theta\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)}{p\left(2\theta^2 + 24\alpha^2 + 12\alpha\theta\right)} \int_0^q \frac{x^2\theta^4}{\left(\theta^2 + 6\alpha^2 + 4\alpha\theta\right)} \left(1 + \alpha^2 x^2 + 2\alpha x\right) e^{-\theta x} dx \tag{12}$$

After simplification, we obtain from expression (12)

$$B(p) = \frac{1}{p\left(2\theta^2 + 24\alpha^2 + 12\alpha\theta\right)} \left(\theta^2\gamma(3,\theta q) + \alpha^2\gamma(5,\theta q) + 2\alpha\theta\,\gamma(4,\theta q)\right)$$

$$L(p) = \frac{1}{\left(2\theta^2 + 24\alpha^2 + 12\alpha\theta\right)} \left(\theta^2\gamma(3,\theta q) + \alpha^2\gamma(5,\theta q) + 2\alpha\theta\,\gamma(4,\theta q)\right)$$

VI. Order Statistics

Order statistics is basically a fundamental aspect within the domain of statistical sciences deals with the application of ordered values and its functions. Suppose the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ of a random sample $X_1, X_2, \ldots, X_n$ from a continuous distribution with probability density function $f_x(x)$ and cumulative distribution function $F_x(x)$, then probability density function of r^{th} order statistics $X_{(r)}$ is defined as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x) \left(F_X(x)\right)^{r-1} \left(1 - F_X(x)\right)^{n-r} \tag{13}$$

By applying equation (5) and (7) in equation (13), we will determine the required probability density function of r^{th} order statistics $X_{(r)}$ of length biased generalized Aradhana distribution which is

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}\left(\frac{x\,\theta^4\left(1+\alpha x\right)^2 e^{-\theta x}}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)\left(\frac{\left(\theta^2\gamma(2,\theta x)+\alpha^2\gamma(4,\theta x)+2\alpha\theta\,\gamma(3,\theta x)\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)^{r-1}$$

$$\times\left(1-\frac{\left(\theta^2\gamma(2,\theta x)+\alpha^2\gamma(4,\theta x)+2\alpha\theta\gamma(3,\theta x)\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)^{n-r}$$

The pdf of higher order statistics $X_{(n)}$ of length biased generalized Aradhana distribution is given by

$$f_{x(n)}(x) = \frac{nx\,\theta^4\left(1+\alpha x\right)^2 e^{-\theta x}}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\left(\frac{\left(\theta^2\gamma(2,\theta x)+\alpha^2\gamma(4,\theta x)+2\alpha\theta\,\gamma(3,\theta x)\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)^{n-1}$$

The pdf of first order statistics $X_{(1)}$ of length biased generalized Aradhana distribution is given by

$$f_{x(1)}(x) = \frac{n\,x\,\theta^4\left(1+\alpha x\right)^2 e^{-\theta x}}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\left(1-\frac{\left(\theta^2\gamma(2,\theta x)+\alpha^2\gamma(4,\theta x)+2\alpha\theta\,\gamma(3,\theta x)\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)^{n-1}$$

VII. Maximum Likelihood Estimation and Fisher's Information Matrix

In this portion, we will describe the technique of parameter estimation to estimate the parameters of length biased generalized Aradhana distribution by applying maximum likelihood estimation. Suppose $X_1, X_2, \ldots, X_n$ be the random sample of size n from length biased generalized Aradhana distribution, then the likelihood function can be stated as

$$L(x) = \prod_{i=1}^{n} f_l(x;\theta,\alpha)$$

$$= \prod_{i=1}^{n}\left(\frac{x_i\;\theta^4}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\left(1+\alpha\,x_i\right)^2 e^{-\theta x_i}\right)$$

$$= \frac{\theta^{4n}}{\left(\theta^2+6\,\alpha^2+4\alpha\theta\right)^n}\prod_{i=1}^{n}\left(x_i\left(1+\alpha^2 x_i^{\,2}+2\alpha\,x_i\right)e^{-\theta x_i}\right)$$

The log likelihood function should be defined as

$$\log L = 4n\log\theta - n\log\left(\theta^2+6\,\alpha^2+4\alpha\theta\right)+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log\left(1+\alpha^2 x_i^{\,2}+2\alpha\,x_i\right)-\theta\sum_{i=1}^{n}x_i \tag{14}$$

Therefore by differentiating the log-likelihood equation (14) with respect to parameters θ and α. The following equations must have been satisfied

$$\frac{\partial\log L}{\partial\theta} = \frac{4n}{\theta} - n\left(\frac{\left(2\theta+4\alpha\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right) - \sum_{i=1}^{n}x_i = 0$$

$$\frac{\partial \log L}{\partial \alpha} = -n\left(\frac{(12\alpha+4\theta)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right) + \sum_{i=1}^{n}\left(\frac{\left(2\alpha x_i^2+2x_i\right)}{(1+\alpha^2 x_i^2+2\alpha x_i)}\right) = 0$$

It is quite clear to mention here that above system of non-linear equations are too complicated to solve those equations algebraically. Therefore, we apply the numerical technique like R and wolfram mathematics for estimating the required parameters of presented distribution.
In order to apply the asymptotic normality results for determining the confidence interval. We state that if $\hat{\eta} = (\hat{\theta}, \hat{\alpha})$ which constitutes the MLE of $\eta = (\theta, \alpha)$. The results should clearly interpreted as

$$\sqrt{n}\,(\hat{\eta}-\eta) \rightarrow N_2(0, I^{-1}(\eta))$$

Where $I^{-1}(\eta)$ is Fisher's information matrix.

$$I(\eta) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta\,\partial \alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha\,\partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) \end{pmatrix}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{4n}{\theta^2} - n\left(\frac{(\theta^2+6\alpha^2+4\alpha\theta)(2)-(2\theta+4\alpha)^2}{(\theta^2+6\alpha^2+4\alpha\theta)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = -n\left(\frac{(\theta^2+6\alpha^2+4\alpha\theta)(12)-(12\alpha+4\theta)^2}{(\theta^2+6\alpha^2+4\alpha\theta)^2}\right) + \sum_{i=1}^{n}\left(\frac{(1+\alpha^2 x_i^2+2\alpha x_i)(2x_i^2)-(2\alpha x_i^2+2x_i)^2}{(1+\alpha^2 x_i^2+2\alpha x_i)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta\,\partial \alpha}\right) = -n\left(\frac{(\theta^2+6\alpha^2+4\alpha\theta)\,(4)-(2\theta+4\alpha)\,(12\alpha+4\theta)}{(\theta^2+6\alpha^2+4\alpha\theta)^2}\right)$$

Since η is not known therefore we estimate $I^{-1}(\eta)$ is estimated by $I^{-1}(\hat{\eta})$ and it should be applied to get the asymptotic confidence interval for θ and α.

VIII. Entropy

The entropy is a scientific concept which is broadly associated with randomness or uncertainty in a system. The entropy is applied in vast areas and is defined as

I. Renyi Entropy

The term Renyi entropy is most fundamental component in quantum information and is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left(\int f_l^{\,\gamma}(x;\theta,\alpha)\,dx\right)$$

$$= \frac{1}{1-\gamma}\log\int_0^\infty\left(\frac{x\theta^4}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\left(1+\alpha x\right)^2 e^{-\theta x}\right)^{\gamma} dx \qquad (15)$$

After simplification of expression (15), we obtain

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left(\left(\frac{\theta^{3\gamma-k-1}}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)^{\gamma}}\right)\sum_{k=0}^{2\gamma}\binom{2\gamma}{k}\alpha^k\frac{\Gamma(\gamma+k+1)}{\gamma^{\gamma+k+1}}\right)$$

Particularly, the Tsallis entropy of explored distribution will be obtained by applying the expression

$$T_s(\xi)=\frac{1}{\xi-1}\left(1-\int_0^\infty f_l^{\xi}(x;\theta,\alpha)\,dx\right)$$

$$=\frac{1}{\xi-1}\left(1-\int_0^\infty\left(\frac{x\theta^4}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}(1+\alpha x)^2e^{-\theta x}\right)^{\xi}dx\right) \tag{16}$$

After simplification of expression (16), we obtain

$$T_s(\xi)=\frac{1}{\xi-1}\left(1-\left(\frac{\theta^{3\xi-j-1}}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)^{\xi}}\right)\sum_{j=0}^{2\xi}\binom{2\xi}{j}\alpha^j\frac{\Gamma(\xi+j+1)}{\xi^{\xi+j+1}}\right)$$

IX. Likelihood Ratio Test

Consider the random sample $X_1, X_2, \ldots, X_n$ of size n from the generalized Aradhana or length biased generalized Aradhana distribution. To examine its significance the hypothesis is to be tested.

$$H_o: f(x)=f(x;\theta,\alpha) \quad \text{against} \quad H_1: f(x)=f_l(x;\theta,\alpha)$$

To identify whether if the random sample of size n comes from the generalized Aradhana or length biased generalized Aradhana distribution the following test statistic procedure is applied.

$$\Delta=\frac{L_1}{L_o}=\prod_{i=1}^{n}\frac{f_l(x;\theta,\alpha)}{f(x;\theta,\alpha)}$$

$$=\prod_{i=1}^{n}\left(\frac{x_i\,\theta\left(\theta^2+2\alpha\theta+2\alpha^2\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)$$

$$=\left(\frac{\theta\left(\theta^2+2\alpha\theta+2\alpha^2\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)^n\prod_{i=1}^{n}x_i$$

The null hypothesis should be refused to accept if

$$\Delta=\left(\frac{\theta\left(\theta^2+2\alpha\theta+2\alpha^2\right)}{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}\right)^n\prod_{i=1}^{n}x_i>k$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k\left(\frac{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}{\theta(\theta^2+2\alpha\theta+2\alpha^2)}\right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, \text{Here } k^* = k\left(\frac{\left(\theta^2+6\alpha^2+4\alpha\theta\right)}{\theta(\theta^2+2\alpha\theta+2\alpha^2)}\right)^n$$

X. Application

In this portion, we have incorporated and fitted two real lifetime data sets to explore the flexibility and significance of length biased generalized Aradhana distribution and then potentiality has been accomplished in order to illustrate that length biased generalized Aradhana distribution provides a better fit over generalized Aradhana, quasi Aradhana, new quasi Aradhana, Aradhana, Shanker, Ishita, Suja and Lindley distributions.

Data set 1: The first data set which represents the failure stresses (in GPa) of 65 single carbon fibers of length 50 mm, respectively reported by Bader and Priest [3]. The observations of data set are
1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.84, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.27, 2.272, 2.28, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.39, 2.41, 2.43, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.62, 2.633, 2.67, 2.682, 2.699, 2.705, 2.735, 2.785, 3.02, 3.042, 3.116, 3.174

Data set 2: The second uncensored data set relates to breaking stress of carbon fibres in Gba which insists of (n =100) observations discussed by Mahmoud and Mandouh [10]. The following values are
0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.830, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.020, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306

To predict the model comparison criterion values in accompanied with unknown parameters are also estimated through analysis of R software. To capture and highlight the performance of length biased generalized Aradhana distribution over generalized Aradhana, quasi Aradhana, new quasi Aradhana, Aradhana, Shanker, Ishita, Suja and Lindley distributions, the criterions like *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected), *CAIC* (Consistent Akaike Information Criterion), Shannon entropy *H(X)* and *-2logL* have been applied. The distribution performs quite better if it has the smaller criterion values of *AIC, BIC, AICC, CAIC, H(X)* and *-2logL* as compared over other distributions. The criterion values are obtained as

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L, \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$CAIC = -2\log L + \frac{2kn}{n-k-1} \quad \text{and} \quad H(X) = -\frac{2\log L}{n}$$

Here $-2\log L$ is the maximized log-likelihood function, k is the number of parameters in the statistical model and n is the sample size.

Table 1: *shows MLE and S.E of performed distributions*

Data set 1			Data set 2		
Distribution	MLE	S.E	Distribution	MLE	S.E
LBGA	$\hat{\alpha} = 2417.554$ $\hat{\theta} = 1.78228$	$\hat{\alpha} = 1211.2035$ $\hat{\theta} = 0.1105221$	LBGA	$\hat{\alpha} = 8.037644$ $\hat{\theta} = 2.412787$	$\hat{\alpha} = 1.115002$ $\hat{\theta} = 1.206393$
GA	$\hat{\alpha} = 7.40697$ $\hat{\theta} = 1.33679$	$\hat{\alpha} = 2.105967$ $\hat{\theta} = 9.572945$	GA	$\hat{\alpha} = 7273.082$ $\hat{\theta} = 1.809438$	$\hat{\alpha} = 112.0043$ $\hat{\theta} = 0.000000$
QA	$\hat{\alpha} = 0.00100$ $\hat{\theta} = 1.33656$	$\hat{\alpha} = 0.4179194$ $\hat{\theta} = 0.1603995$	QA	$\hat{\alpha} = 0.00100$ $\hat{\theta} = 1.80915$	$\hat{\alpha} = 0.31270$ $\hat{\theta} = 0.17171$
NQA	$\hat{\alpha} = 1.175703$ $\hat{\theta} = 1.336786$	$\hat{\alpha} = 1.677723$ $\hat{\theta} = 9.572441$	NQA	$\hat{\alpha} = 8535.289$ $\hat{\theta} = 1.80930$	$\hat{\alpha} = 6653.122$ $\hat{\theta} = 0.10444$
Aradhana	$\hat{\theta} = 0.9843106$	$\hat{\theta} = 0.07340$	Aradhana	$\hat{\theta} = 1.2499381$	$\hat{\theta} = 0.076716$
Shanker	$\hat{\theta} = 0.70435$	$\hat{\theta} = 0.0578065$	Shanker	$\hat{\theta} = 0.891169$	$\hat{\theta} = 0.05958$
Ishita	$\hat{\theta} = 0.98371$	$\hat{\theta} = 0.0607588$	Ishita	$\hat{\theta} = 1.186687$	$\hat{\theta} = 0.059421$
Suja	$\hat{\theta} = 1.73751$	$\hat{\theta} = 0.0809330$	Suja	$\hat{\theta} = 2.023585$	$\hat{\theta} = 0.075263$
Lindley	$\hat{\theta} = 0.70672$	$\hat{\theta} = 0.0648246$	Lindley	$\hat{\theta} = 0.917731$	$\hat{\theta} = 0.068961$

Table 2: *shows interpretation, analysis and performance of fitted distributions of data set 1*

Distributions	-2logL	AIC	BIC	AICC	CAIC	H(X)
LBGA	144.0153	148.0153	152.3641	148.2088	148.2088	2.2156
GA	161.3134	165.3134	169.6621	165.5069	165.5069	2.4817
QA	161.3463	165.3463	169.695	165.5398	165.5398	2.4822
NQA	161.3122	165.3122	169.661	165.5057	165.5057	2.4817
Aradhana	197.0756	199.0756	201.25	199.1390	199.1390	3.0319
Shanker	209.7126	211.7126	213.887	211.7760	211.7760	3.2263
Ishita	203.2007	205.2007	207.3751	205.2641	205.2641	3.1261
Suja	184.4748	186.4748	188.6492	186.5382	186.5382	2.8380
Lindley	213.9985	215.9985	218.1729	216.0619	216.0619	3.2922

Table 3: *shows interpretation, analysis and performance of fitted distributions of data set 2*

Distributions	-2logL	AIC	BIC	AICC	CAIC	H(X)
LBGA	177.4824	181.4824	186.6927	181.6061	181.6061	1.7748
GA	198.6203	202.6203	207.8307	202.7440	202.7440	1.9862
QA	198.662	202.662	207.8723	202.7857	202.7857	1.9866
NQA	198.6278	202.6278	207.8381	202.7515	202.7515	1.9862
Aradhana	257.5579	259.5579	262.1631	259.5987	259.5987	2.5755
Shanker	275.4242	277.4242	280.0294	277.4650	277.4650	2.7542
Ishita	283.1591	285.1591	287.7643	285.1999	285.1999	2.8315
Suja	297.3904	299.3904	301.9956	299.4312	299.4312	2.9739
Lindley	277.1388	279.1388	281.744	279.1796	279.1796	2.7713

Obviously, it is observed and revealed from results given above in table 2 and table 3 that the length biased generalized Aradhana distribution have lesser *AIC*, *BIC*, *AICC*, *CAIC*, *H(X)* and *-2logL* values as compared to generalized Aradhana, quasi Aradhana, new quasi Aradhana, Aradhana, Shanker, Ishita, Suja and Lindley distributions. Hence, it is realized that length biased generalized Aradhana distribution provides a quite satisfactory fit as compared to generalized Aradhana, quasi Aradhana, new quasi Aradhana, Aradhana, Shanker, Ishita, Suja and Lindley distributions.

XI. Conclusion

In this paper, we have formulated a new class of generalized Aradhana distribution termed as length biased generalized Aradhana distribution which has been explored by incorporating specific length biased technique to its baseline distribution. Thus presented new distribution illustrated and accomplished with its distinguished statistical properties which are moments, mean and variance, standard deviation, survival function, hazard function, reverse hazard rate function, moment generating function, bonferroni and Lorenz curves have been described. Additionally Renyi entropy measure and order statistics has been discussed. Further, the parameters of specific new distribution are also estimated through the maximum likelihood estimation. Furthermore, the dominance and significance of proposed new distribution are analyzed and investigated by applying two real data sets. Hence, it is inspected and is clearly indicated from the result that the length biased generalized Aradhana distribution gives a better fit in comparison over generalized Aradhana, quasi Aradhana, new quasi Aradhana, Aradhana, Shanker, Ishita, Suja and Lindley distributions.

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