THE QUASI-STATIONARY DISTRIBUTION OF A COMPLEX TWO-UNIT REPAIRABLE SYSTEM

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Abstract

Like the first-busy-period in a queuing system, the time-to-system-failure is an important characteristic of a reliability system. In some reliability systems the time-to-system-failure may be sufficiently large to allow the residual life of the system to settle down to a state of statistical equilibrium. The conditional limit distribution of the residual lifetime, also known as the quasi-stationary distribution, plays a vital role in the statistical analysis of such systems. In this work, the quasi-stationary distribution for the residual lifetime of a two-unit complex reparable system is obtained using regenerative stochastic process. The results of the study are numerically illustrated assuming exponential lifetime distribution for the units.

Keywords: Quasi-stationary distribution, two-unit complex reparable system, warm standby, repair facility, residual lifetime.

I. Introduction

When the traffic intensity of a queueing system is less than or equal to unity, the first busy period of the system is finite with probability one. If such a period is sufficiently long, then the queue size process takes a long time to settle down to a state of equilibrium within the first busy period. The limiting distribution of the queue size process conditioned on it still being in the first busy period is termed the quasi-stationary distribution (QSD) of the process.

The reliability counterpart of QSD is equally interesting. It is well known that the time to system failure is an important characteristic of any reliability problem. In many reliability problems the behavior of the system after its first breakdown is of less interest either because the failure of a unit is irrevocable or because the system recovery is expensive. For instance, the performance of an aircraft is of great importance before, and not after, its first breakdown in mid-air. In such cases it would be of great interest to study the limiting distribution of the residual lifetime of the reliability system conditioned on its failure-free performance during the period.

Munda and Taneja [6] analyzes a system comprised of one operative unit, cold standby unit, and one warm standby unit. Various system performance measures have been defined by using the Markov process and regeneration point method.

Chaudhary, Sharma and Gupta [2] deals with the stochastic analysis of two non-identical units with failure and repair times of the unit taken as independent random variables of discrete nature having geometric distributions with different parameters.

Papageorgiou and Kokolakis [7] discusses the reliability analysis of a two-unit general parallel system with (n-2) warm standbys. A parallel (2, n-2) system is investigated where any one of them is replaced instantaneously upon its failure by one of the (n-2) warm standbys.

Kyprianou [4] introduced this concept of the QSD in the study of queue size process and the virtual waiting time process associated with the first busy period of G1/M/1 queue. Cavendar [1] derived the QSD of a birth death process.

Van Doorn and Pollett [10] have discussed a survey of results related to quasi-stationary distributions, which arise in the setting of stochastic dynamical systems that eventually evanesce, and which may be useful in describing the long-term behaviour of such systems before evanescence. Reinertsen [8] presents and discusses research related to residual life of non-repairable and repairable technical systems.

Li and Cao [5] have proved that the limiting distributions of the residual lifetimes of several repairable systems, conditional upon the event that the systems have not been down at any time in (0, t], are exponential irrespective of the distributions of the lifetimes and repair times of the individual units, provided that their Laplace transforms are rational functions of their arguments.

Kalpakam and Hameed [3] discussed the existence of QSD residual lifetime of a two-unit warm standby redundant system. Shankar Bhat [9] discussed the existence of QSD of a complex two unit system.

In the present work, we discuss the existence of QSD residual lifetime of a complex two-unit warm standby system in which the switching device is subject to failure. The section below describes the reliability system under study.

II. System Description

[01] The reliability system consists of two identical repairable units. At $t = 0$, one unit is placed online and the other is placed as a warm standby.

[02] Upon the failure of the online unit, the standby unit is placed online if found in operable condition. If the standby unit is under repair, the online failed unit queues up for repair. The failure time and repair time distributions of a unit in standby are assumed to be distributed exponential with parameters α and β respectively.

[03] The operable standby unit is instantaneously placed online using a switching device. This switching device has its own pattern of life and repair time distributions. The failure time and repair time distributions of the switching device are assumed exponential with parameters γ and μ respectively.

[04] The system is supported by two repair facilities one for the units and the other for the switching device.

[05] The repair of units is undertaken in FIFO order.

[06] The repair is perfect in the sense that the repaired units as well as switching device are treated as good as new.

A schematic representation of the system is depicted in Figure 1.

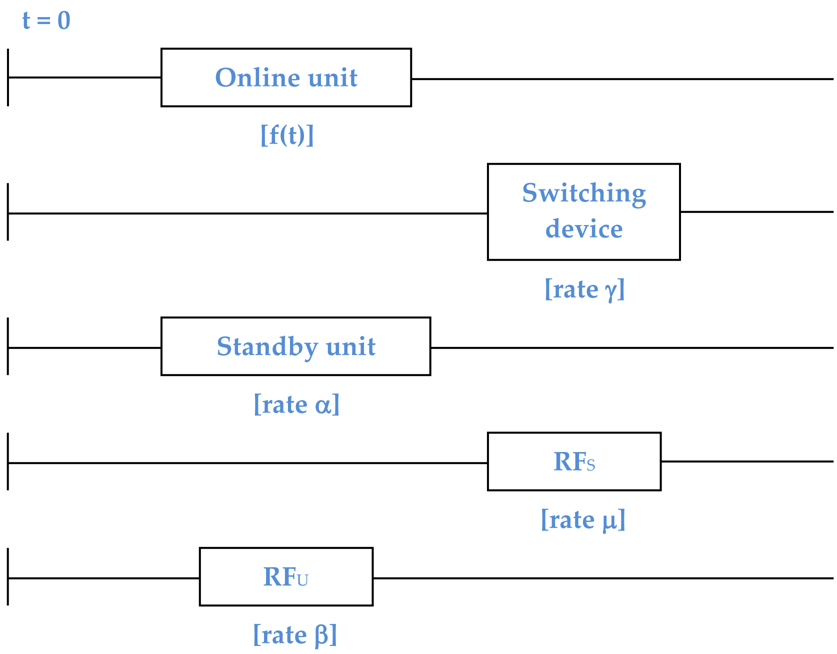

Figure 1: *Schematic Representation of the System*

III. Stochastic Behavior of the Standby Unit

The stochastic behavior of the standby unit, while the other unit is continuously operating online can be described through an alternating renewal process. Let [Y(t), t ≥ 0] be the process characterizing the state of the standby unit during failure-free performance of the unit in service. The random variable Y(t) is a two valued random variable which takes the value O if the standby unit is in operable condition and F when under repair. The process [Y(t), t ≥ 0] completely describes the subsystem and is an alternating renewal process.

Define

$$P_{ij}(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} P[Y(t) = j / Y(t) \text{ entered i in } (-\Delta, 0)],\ \text{i,j} = 0,F \quad (1)$$

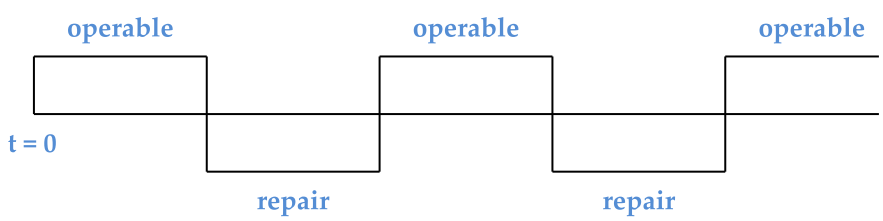

Figure 2: *Stochastic Behavior of the Standby Unit*

The functions $P_{ij}(t)$ can be easily obtained through renewal theoretic arguments

$$P_{OO}(t) = e^{-\alpha t} + [\alpha e^{-\alpha t} © \beta e^{-\beta t}] © P_{OO}(t) \quad (2)$$

$$P_{FO}(t) = \beta e^{-\beta t} © P_{OO}(t) \quad (3)$$

$$P_{OF}(t) = 1 - P_{OO}(t) \quad (4)$$

$$P_{FF}(t) = 1 - P_{FO}(t) \quad (5)$$

Observe that $P_{00}(t)$ is the p-function of the Kingman's regenerative phenomenon. Since the exponential distributions are compatible for Laplace Transforms, the expressions (2) to (3) can be easily evaluated upon successive operation of Laplace Transforms. Thus,

$$P_{00}(t) = \frac{1}{(\alpha+\beta)}\left[\beta + \alpha e^{-(\alpha+\beta)t}\right] \tag{6}$$

$$P_{F0}(t) = \frac{\beta}{(\alpha+\beta)}\left[1 - e^{-(\alpha+\beta)t}\right] \tag{7}$$

$$P_{0F}(t) = \frac{\alpha}{(\alpha+\beta)}\left[1 - e^{-(\alpha+\beta)t}\right] \tag{8}$$

$$P_{FF}(t) = \frac{1}{(\alpha+\beta)}\left[\alpha + \beta e^{-(\alpha+\beta)t}\right] \tag{9}$$

IV. Stochastic Behavior of the Switching Device

The switching device has its own failure and repair time distributions. The subsystem consisting of a switching device and a repair facility can be seen as a parallel system to the subsystem consisting the standby unit and the repair facility attached to it. The stochastic behavior of the switching device during a failure-free operation of the online unit can be identified with an alternating renewal process. Let [Z(t), t ≥ 0] be the process that describes the state of the switching device. The state space of the process is [0, F], where '0' indicates that the switching device is in operable condition and 'F' indicates that it is undergoing repair. The transitional probabilities of the process are given by the following function.

$$Q_{ij}(t) = \lim_{\Delta\to 0}\frac{1}{\Delta}\Pr[Z(t) = j \,/\, Z(t) \text{ entered i in } (-\Delta,0)] \quad i,j = 0,F \tag{10}$$

The expressions $Q_{ij}(t)$ can be obtained easily with the help of renewal theoretic arguments.

$$Q_{OO}(t) = e^{-\gamma t} + \{\gamma e^{-\gamma t} © \mu e^{-\mu t}\} © Q_{OO}(t) \tag{11}$$

$$Q_{FO}(t) = \mu e^{-\mu t} © Q_{OO}(t) \tag{12}$$

$$Q_{OF}(t) = 1 - Q_{OO}(t) \tag{13}$$

$$Q_{FF}(t) = 1 - Q_{FO}(t) \tag{14}$$

These functions are compatible for Laplace Transforms and are obtained explicitly as

$$Q_{00}(t) = \frac{1}{(\gamma+\mu)}\left[\mu + \gamma e^{-(\gamma+\mu)t}\right] \tag{15}$$

$$Q_{F0}(t) = \frac{\mu}{(\gamma+\mu)}\left[1 - e^{-(\gamma+\mu)t}\right] \tag{16}$$

$$Q_{0F}(t) = \frac{\gamma}{(\gamma+\mu)}\left[1 - e^{-(\gamma+\mu)t}\right] \tag{17}$$

$$Q_{FF}(t) = \frac{1}{(\gamma+\mu)}\left[\gamma + \mu e^{-(\gamma+\mu)t}\right] \tag{18}$$

V. The Quasi-Stationary Distribution

Let X(t) denotes the residual lifetime of the system at any time t and T denotes the time to system failure. The period T may be sufficiently large to allow the process to settle down to a state of statistical equilibrium within this period. Then the limit as $t \to \infty$ of conditional probability,

$$L(x,t) = Pr\{X(t) \le x / T > t\} \tag{19}$$

is termed as the QSD of X(t). The conditioned distribution L(x,t) can also be expressed as

$$L(x,t) = \frac{R(t) - R(x+t)}{R(t)} = 1 - \frac{R(x+t)}{R(t)} \tag{20}$$

$$L(x) = \lim_{t\to\infty} L(x,t) = 1 - \lim_{t\to\infty} \frac{R(x+t)}{R(t)} \tag{21}$$

Observing that the function R(t) is the reliability of the system, the above limit is studied by first writing the integral equations for R(t) using regenerative point method.

Define e_1-event as the time epoch at which the repair of an online failed unit commences; at this instant the standby unit begins to operate online and the switching device is found in operable condition. Since e_1-event is regenerative in nature, we have

$$L_1(x) = 1 - \lim_{t\to\infty} \frac{R_1(x+t)}{R_1(t)} \tag{22}$$

Define e_0-event as the time epoch at which the operation of a unit commences; at this instant the standby unit as well as the switching device are in operable condition. Thus, conditioning on e_0-event, we have

$$L_0(x) = 1 - \lim_{t\to\infty} \frac{R_0(x+t)}{R_0(t)} \tag{23}$$

Conditioned on ei-events at t = 0, the functions $R_1(t)$ can easily be derived.

$$R_1(t) = \overline{F}(t) + \int_0^t e_1(u)R_i(t-u)\text{du} \tag{24}$$
$$R_0(t) = \overline{F}(t) + \int_0^t e_0(u)R_i(t-u)\text{du} \tag{25}$$

where $e_1(u)$ is the pdf of time interval between consecutive e_1-events and $e_0(u)$ is the pdf of time interval between e_0-event and the first e_1-event.

The functions $e_i(t)_{i=0,1}$ are obtained using $P_{ij}(t)$ and $Q_{ij}(t)$ as

$$e_0(t) = f(t)\, P_{OO}(t)\, Q_{OO}(t) \tag{26}$$
$$e_1(t) = f(t)\, P_{FO}(t)\, Q_{OO}(t) \tag{27}$$

In view of (26) and (27), the functions (24) and (25) can also be expressed as

$$R_i(t) = \overline{F}(t) + e_i(t) © R_1(t) \quad i = 0,1 \tag{28}$$

Taking Laplace Transform on (28) we get

$$R_i^*(s) = \overline{F^*}(s) + e_i^*(s)R_1^*(s) \quad i = 0,1 \tag{29}$$

where $e_0^*(s) = \int_0^\infty f(t)P_{00}(t)Q_{00}(t)e^{-\text{st}}\text{dt}$

$e_1^*(s) = \int_0^\infty f(t)P_{\text{F0}}(t)Q_{00}(t)e^{-\text{st}}\text{dt}$

On simplification we get

$$R_1^*(s) = \frac{\overline{F^*}(s)}{1-e_1^*(s)} \tag{30}$$
$$R_0^*(s) = \overline{F^*}(s)\frac{[1-e_1^*(s)+e_0^*(s)]}{1-e_1^*(s)} \tag{31}$$

In order to find the nature of the resulting distribution of $L_i(x)$ defined in (22) and (23), it is essential to find the roots of the equation $1 - e_1^*(s) = 0$. When the lifetime of a unit operating online is assumed to be distributed exponential with parameter δ, the function $e_1^*(s)$ modifies itself into

$$e_1^*(s) = \delta \int_0^\infty e^{-(s+\delta)t} P_{F0}(t) Q_{00}(t) \mathrm{d}t \tag{32}$$

Since f(t) is exponential in nature and $P_{F0}(t)$ and $Q_{00}(t)$ are rational functions of their arguments, $e_1^*(s)$ is also a rational function. Thus,

$$e_1^*(s) = \frac{A(s)}{B(s)} \tag{33}$$

where A(s) and B(s) are polynomials of degree n and m (n < m) respectively. In view of (33), the expression in (30) modifies itself into

$$R_1^*(s) = \frac{B(s)}{(s+\delta)[B(s)-A(s)]} \tag{34}$$

Define $p_1^*(s) = 1 - sR_1^*(s)$, then

$$p_1^*(s) = -\frac{\delta[B(s)-A(s)]-sB(s)}{(s+\delta)[B(s)-A(s)]} \tag{35}$$

Observing that $R_1^*(s)$ and $p_1^*(s)$ are rational functions of their arguments, their singularities are essentially same. Moreover, all the singularities of $p_1^*(s)$ are poles.

Let s = η be the pole of $p_1^*(s)$ situated closest to the right of the s-plane. As per Widder [11], η is real and negative and is the abscissa of convergence of $p_1^*(s)$.

Define $\Psi(s) = 1 - e_1^*(s)$ then

$$\Psi(s) = 1 - \delta \int_0^\infty e^{-(s+\delta)t} P_{F0}(t) Q_{00}(t) \mathrm{d}t \tag{36}$$

Observing that

$$\delta \int_0^\infty e^{-(s+\delta)t} P_{F0}(t) Q_{00}(t) \mathrm{d}t < \delta \int_0^\infty e^{-(s+\delta)t} \mathrm{d}t \tag{37}$$

we get $\Psi(s) < \frac{\delta}{s+\delta}$ and at s = 0, ψ(s) < 1.

Observing that $P_{F0}(t)$ and $Q_{00}(t)$ is the probability that both the standby unit and the switching device are in operable condition at time t given that a repair has commenced initially, we get $\psi(s) > 0$ and $\Psi(-\delta) = 1 - e_1^*(-\delta) < 0$

This follows from (36) and also because –δ is a singularity of 1/(s+δ). Thus from (33) and (34), it can be seen that there exists at least one ξ of ψ(s) such that –δ < Re ξ < 0. Since η is a pole of $p_1^*(s)$ which is closest to the right of s-plane, we get –δ < η < 0. Thus $\Psi(\eta) = 1 - e_1^*(\eta) = 0$ and which implies that $e_1^*(\eta) = 1$. This means $e_1^*(s)$ is convergent for s = η and

$$\Psi'(\eta) = \int_0^\infty t e_1(t) e^{-\eta t} \mathrm{d}t$$

exists and is positive. Thus, η is a root of $1 - e_1^*(s) = 0$ with multiplicity one. This implies that η is a simple pole of $p_1^*(s)$ and also of $R_1^*(s)$. Consequently η is a zero of [B(s) – A(s)].

Observing the fact that the pole η of $p_1^*(s)$ is a simple pole and lies between –δ and 0, we write

$$R_1(t) = \frac{1}{\eta+\delta} \lambda e^{\eta t} + o(e^{\eta t}) \tag{38}$$

$$R_1(x+t) = \frac{1}{\eta+\delta} \lambda e^{\eta(x+t)} + o(e^{\eta t}) \tag{39}$$

where $\lambda = \lim_{s\to\eta} \frac{s-\eta}{1-e_1^*(s)}$

$$\lim_{t\to\infty}\frac{R_1(x+t)}{R_1(t)}=\lim_{t\to\infty}\frac{\lambda e^{\eta(x+t)}+o(e^{\eta t})}{\lambda e^{\eta t}+o(e^{\eta t})}=e^{\eta x} \quad (40)$$

Substituting (40) in (22) we get

$$L_1(x)=1-e^{\eta x},\quad \eta<0$$

Thus, the QSD of the residual lifetime of a complex two-unit standby redundant system given that an e_1-event has occurred at t = 0, is distributed exponential with parameter – η (η < 0) which is the closest negative root of the polynomial ψ(s) equated to zero. The section to follow clearly illustrates our result.

VI. Illustration

When the lifetime distribution of the unit operating online is assumed exponential with parameter δ, the derivation of the roots of the polynomial equated to zero can be easily obtained. Now consider

$$e_1(t)=\delta e^{-\delta t}P_{F0}(t)Q_{00}(t)$$
$$e_1^*(s)=\delta\int_0^\infty e^{-(s+\delta)t}P_{F0}(t)Q_{00}(t)\mathrm{d}t$$
$$P_{F0}(t)=\frac{\beta}{(\alpha+\beta)}[1-e^{-(\alpha+\beta)t}]$$
$$Q_{00}(t)=\frac{1}{(\gamma+\mu)}[\mu+\gamma e^{-(\gamma+\mu)t}]$$

Let $\Psi(s)=1-e_1^*(s)$

$$\Psi(s)=0$$
$$1-\frac{\delta\beta}{(\alpha+\beta)(\gamma+\mu)}\left[\frac{\mu}{s+a_1}+\frac{\gamma}{s+a_2}-\frac{\mu}{s+a_3}-\frac{\gamma}{s+a_4}\right]=0$$

Furthermore on simplification, we get

$$\sum_{i=1}^{5}k_i s^{5-i}=0$$

Observe that the above equation is a fourth degree polynomial in s equated to zero.

$$k_1=(\alpha+\beta)(\gamma+\mu)$$
$$k_2=k_1b_1$$
$$k_3=k_1(b_2+b_3+b_4)-a_1m_1$$
$$k_4=k_1[(a_1+a_2)b_3+(a_3+a_4)b_2]-a_1(t_1+t_2-t_3-t_4)$$
$$k_5=k_1a_1a_2a_3a_4-a_1m_2$$
$$b_1=a_1+a_2+a_3+a_4$$
$$b_2=a_1a_2$$
$$b_3=a_3a_4$$
$$b_4=(a_1+a_2)(a_3+a_4)$$
$$t_1=\beta\mu[a_2a_3+(a_2+a_3)a_4]$$
$$t_2=\beta\gamma[a_1a_3+(a_1+a_3)a_4]$$
$$t_3=\beta\mu[a_1a_2+(a_1+a_2)a_4]$$
$$t_4=\beta\gamma[a_1a_2+(a_1+a_2)a_3]$$
$$m_1=c_1\beta\mu+c_2\beta\gamma-c_3\beta\mu-c_4\beta\gamma$$
$$m_2=d_1\beta\mu+d_2\beta\gamma-d_3\beta\mu-d_4\beta\gamma$$
$$c_1=a_2+a_3+a_4$$
$$c_2=a_1+a_3+a_4$$
$$c_3=a_1+a_2+a_4$$
$$c_4=a_1+a_2+a_3$$

$$d_1 = a_2 a_3 a_4$$
$$d_2 = a_1 a_3 a_4$$
$$d_3 = a_1 a_2 a_4$$
$$d_4 = a_1 a_2 a_3$$
$$a_1 = \delta$$
$$a_2 = \delta + \gamma + \mu$$
$$a_3 = \delta + \alpha + \beta$$
$$a_4 = \delta + \gamma + \mu + \alpha + \beta$$

Once the values of the parameters α, β, γ, μ and δ are specified, the network of constants can be evaluated and all the four roots of the equation can be obtained through methods of numerical analysis. Table 1 provides the simple negative root closest to the origin for specified values of the parameters.

A coding is written for the purpose and the code seeks data about the parameters and then provides the negative root closest to zero. For instance, when (α, β, γ, μ, δ) = (20, 30, 1.5, 6, .5) then the negative root closest to zero is −0.2595 and $L(x) = 1\ e^{-0.2595t}$.

Table 1: *The negative root closest to zero of ψ(s) = 0*

(α, β)	(γ, μ)	Negative root closest to zero		
		δ = 0.5	δ = 0.6	δ = 0.7
(20, 30)	(1.5, 6)	-0.2595	-0.3113	-0.3631
	(2.0, 7)	-0.2663	-0.3195	-0.3727
	(2.5, 8)	-0.2712	-0.3254	-0.3796
	(3.0, 9)	-0.2749	-0.3299	-0.3848
	(3.5, 10)	-0.2778	-0.3333	-0.3889
(22, 35)	(1.5, 6)	-0.2537	-0.3043	-0.3549
(24, 40)		-0.2492	-0.2988	-0.3484
(26, 45)		-0.2455	-0.2944	-0.3432
(28, 50)		-0.2425	-0.2908	-0.3389
(30, 33)		-0.2400	-0.2877	-0.3354

VII. Conclusion

In this article, the quasi-stationary distribution for the residual lifetime of a two-unit complex reparable system is obtained using regenerative stochastic process. The results of the study are numerically illustrated assuming exponential lifetime distribution for the units. We observe that whatever be the lifetime and repair time distributions, provided that their Laplace transforms are rational, the quasi-stationary distribution is always exponential and is independent of the initial event.

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