COMPARISON OF CLASSICAL AND BAYESIAN APPROACHES FOR ESTIMATING THE SCALE PARAMETER OF THE INVERSE POWER RAYLEIGH DISTRIBUTION

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Abstract

This manuscript investigates the parameter estimation of the Inverse Power Rayleigh Distribution (IPRD) using both Classical and Bayesian approaches. Parameters are estimated via Maximum Likelihood Estimation (MLE) and Bayesian methods under three prior assumptions: Jeffreys prior, extended Jeffreys prior, and quasi prior. Bayesian estimators, along with their associated risks, are evaluated using various symmetric and asymmetric loss functions, including the squared error loss function, Al-Bayyatis new loss function, precautionary loss function and entropy loss function. The performance of the estimators is assessed through simulated data based on the mean squared error (MSE) criterion. Results indicate that the Bayesian approach generally provides more accurate estimates with lower MSE compared to the classical MLE method.

Keywords: Inverse Rayleigh distribution, Loss functions, Maximum Likelihood estimation, Bayesian estimation.

1. Introduction

The inverse Rayleigh distribution (IRD) holds significant importance in reliability analysis and life testing studies. If X follows an inverse Rayleigh (IR) distribution, its probability density function (PDF) is expressed as:

$$g(x) = \frac{2\theta}{x^3}\, e^{-\frac{\theta}{x^2}}; \quad x > 0, \theta > 0. \tag{1}$$

The corresponding cumulative distribution function (CDF) is:

$$G(x) = e^{-\frac{\theta}{x^2}}; \quad x > 0, \theta > 0. \tag{2}$$

The inverse Rayleigh distribution (IRD) was first introduced by [21], who explored various properties of the scale parameter of the maximum likelihood estimator (MLE) for this distribution. The IRD has found extensive applications in longevity experiments and reliability studies. Over time, numerous researchers have investigated estimation methods and prediction approaches for the IRD. Abdel-Monem [1] developed several estimators and prediction results for the IRD. Soliman *et al.* [20] studied the estimation of the IRD based on lower record values and proposed Bayes estimators under the squared error and zero-one loss functions. Dey [6] focused on Bayesian estimation for the scale parameter and the reliability function of the IRD using squared error and linex loss functions. Sindhu *et al.* [19] extended this work to left-censored data and studied Bayes estimators under both symmetric and asymmetric loss functions. Prakash [16] discussed Bayesian estimation for the IRD under squared error and linex loss functions, while Fan [7] investigated

Bayesian estimators under various loss functions, including squared error, linex and entropy loss functions. Rasheed *et al.* [23] compared classical estimators with Bayesian estimators under the generalized squared error loss function. Several other researchers have explored the inverse Rayleigh distribution, including Abdullah and Aref [2], Fatima *et al.* [8], Sharma *et al.* [17], Noor *et al.* [14], Sharma and Kumar [18], Kumar *et al.* [9], Nuzhat *et al.* [3], Yanuar *et al.* [22], and Mir and Ahmad [10, 12, 11, 13].

Recently, Bhat *et al.* [5] introduced the Inverse Power Rayleigh Distribution (IPRD), a generalized form of the IRD. The PDF and CDF of the IPRD are defined as follows:

$$g(x;\alpha,\theta) = \frac{\alpha}{\theta^2} x^{-(2\alpha+1)} \exp\left(-\frac{x^{-2\alpha}}{2\theta^2}\right), \quad \alpha,\theta > 0, \tag{3}$$

$$G(x;\alpha,\theta) = \exp\left(-\frac{x^{-2\alpha}}{2\theta^2}\right), \quad \alpha,\theta > 0. \tag{4}$$

2. Methods

In this section, we estimate the parameters using maximum likelihood estimation as well as Bayesian estimation methods under three different priors : Jeffreys prior, extended Jeffreys prior and quasi prior and four loss functions: squared error loss function, Al-Bayyatis new loss function, precautionary loss function and entropy loss function.

2.1. Prior and Loss Function

As part of the analysis, we employ three different priors.

1. Jeffreys prior

$$g(\theta) \propto \sqrt{I(\theta)}$$

where,

$$I(\theta) = -nE\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right]$$

is the Fisher information matrix.

2. Extended Jeffreys prior [4]

$$g_1(\theta) \propto [I(\theta)]^{c_1}, \quad c_1 \in \mathbb{R}^+$$

where

$$I(\theta) = -nE\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right]$$

is the Fisher information matrix.

3. Quasi prior

$$g_2(\theta) = \frac{1}{\theta^d}, \quad \theta > 0,\ d > 0.$$

In this paper, we utilize the following loss functions to facilitate a comprehensive comparison in Bayesian analysis:

a) Squared error loss function (self)

$$L_{self}(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2$$

b) Al-Bayyatis new loss function (abnlf) [15]

$$L_{abnlf}(\hat{\theta},\theta) = \theta^{c_2}(\hat{\theta}-\theta)^2, \quad c_2 \in \mathbb{R}^+$$

c) Precautionary loss function (plf)

$$L_{plf}(\hat{\theta},\theta) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$$

d) Entropy loss function (elf)

$$L_{elf}(\hat{\theta}, \theta) = \left(\frac{\hat{\theta}}{\theta} - \log \left(\frac{\hat{\theta}}{\theta} \right) - 1 \right)$$

Here, θ and $\hat{\theta}$ denote the true and estimated values of the parameter, respectively.

2.2. Maximum Likelihood Estimation

The likelihood function corresponding is

$$L(x; \alpha, \theta) = \frac{\alpha^n}{\theta^{2n}} \prod_{a=1}^{n} x_a^{-(2\alpha+1)} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2} \right). \tag{5}$$

The log-likelihood function is obtained as

$$l(x; \alpha, \theta) = n \log \alpha - 2n \log \theta - (2\alpha + 1) \sum_{a=1}^{n} \log x_a - \frac{1}{2\theta^2} \sum_{a=1}^{n} x_a^{-2\alpha}. \tag{6}$$

To determine the MLE of θ, we differentiate Eq. (6)

$$\frac{\partial l(x; \alpha, \theta)}{\partial \theta} = -\frac{2n}{\theta} + \frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{\theta^3} = 0. \tag{7}$$

Simplifying and solving for θ, we obtain the MLE as

$$\hat{\theta} = \sqrt{\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2n}}. \tag{8}$$

2.3. Bayesian Method of Estimation

2.3.1 Posterior Density under Jeffreys Prior

For this model with the PDF specified in Eq. (3) and the likelihood function in Eq. (5), the Jeffreys prior for θ is given by

$$g(\theta) \propto \frac{1}{\theta}, \quad \theta > 0, \tag{9}$$

where θ is the scale parameter. Applying Bayes theorem, the posterior density is expressed as:

$$\pi_1(\theta \mid x) \propto L(x \mid \theta) g(\theta), \tag{10}$$

where $L(x \mid \theta)$ represents the likelihood function.

Substituting Eq. (9) and Eq. (5) into Eq. (10), we obtain:

$$\pi_1(\theta \mid x) \propto \frac{\alpha^n}{\theta^{2n+1}} \prod_{a=1}^{n} x_a^{-(2\alpha+1)} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2} \right) \tag{11}$$

$$= k^{-1} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2} \right), \tag{12}$$

where the normalization constant k^{-1} is given by

$$k^{-1} = \int_0^{\infty} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2} \right) d\theta. \tag{13}$$

Solving the integral,

$$k = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha} \right)^n}{\Gamma(n)\, 2^{n-1}}. \tag{14}$$

Thus, the posterior distribution is given by

$$\pi_1(\theta \mid x) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha} \right)^n}{\Gamma(n)\, 2^{n-1}} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2} \right). \tag{15}$$

2.3.2 Posterior Density under an Extension of Jeffreys Prior

For the model in Eq. (3), the extended Jeffreys prior for θ becomes

$$g_1(\theta) \propto \frac{1}{\theta^{2c_1}}, \quad \theta > 0. \tag{16}$$

Applying Bayes theorem, the posterior density is

$$\pi_2(\theta \mid x) \propto L(x \mid \theta) g_1(\theta). \tag{17}$$

Substituting Eq. (5) and Eq. (16) into Eq. (17), we get

$$\pi_2(\theta \mid x) \propto \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) \tag{18}$$

$$= k^{-1} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right), \tag{19}$$

$$k^{-1} = \int_0^\infty \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{20}$$

Solving this integral, we find

$$k = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}}. \tag{21}$$

Therefore, the posterior distribution under the extended Jeffreys prior is

$$\pi_2(\theta \mid x) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right). \tag{22}$$

2.3.3 Posterior Density under Quasi Prior

The quasi prior for θ is specified as

$$g_2(\theta) = \frac{1}{\theta^d}, \quad \theta > 0,\ d > 0. \tag{23}$$

Using Bayes theorem, the posterior density is given by

$$\pi_3(\theta \mid x) \propto L(x \mid \theta) g_2(\theta), \tag{24}$$

where $L(x \mid \theta)$ represents the likelihood function.

Substituting Eq. (5) and Eq. (23) into Eq. (24), we have

$$\pi_3(\theta \mid x) \propto \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) \tag{25}$$

$$= k^{-1} \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right), \tag{26}$$

$$k^{-1} = \int_0^\infty \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{27}$$

Solving the integral, we obtain

$$k = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) \cdot 2^{\frac{2n+d-3}{2}}}. \tag{28}$$

Thus, the posterior distribution under the quasi prior is given by

$$\pi_3(\theta \mid x) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) 2^{\frac{2n+d-3}{2}}} \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right). \tag{29}$$

2.4. Bayesian Estimation by Using Jeffreys Prior under Different Loss Functions

2.4.1 Squared error loss function (self)

The risk function (RF) for $L_{self}(\hat{\theta},\theta)$ is

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi_1(\theta \mid x)\, d\theta. \tag{30}$$

Substituting Eq. (15) into Eq. (30), we obtain:

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{31}$$

$$= c\frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \int_0^\infty \left(\hat{\theta}^2+\theta^2-2\hat{\theta}\theta\right) \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{32}$$

$$R(\hat{\theta}) = c\frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \left\{\int_0^\infty \hat{\theta}^2 \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta + \int_0^\infty \theta^2 \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \right.$$
$$\left. -2\hat{\theta}\int_0^\infty \theta \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \right\}. \tag{33}$$

Minimizing $R(\hat{\theta})$ with respect to $\hat{\theta}$, we derive:

$$\hat{\theta} = \frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n)} \sqrt{\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2}}. \tag{34}$$

2.4.2 Al-Bayyatis new loss function (abnlf)

The RF for $L_{abnlf}(\hat{\theta}$ is

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2}(\hat{\theta}-\theta)^2 \pi_1(\theta \mid x)\, d\theta. \tag{35}$$

By substituting Eq. (15) into Eq. (35), we get:

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2}(\hat{\theta}-\theta)^2 \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{36}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \int_0^\infty \theta^{c_2} \left(\hat{\theta}^2+\theta^2-2\hat{\theta}\theta\right) \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{37}$$

$$\begin{aligned} R(\hat{\theta}) = \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \Bigg\{ & \hat{\theta}^2 \int_0^\infty \theta^{c_2} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & + \int_0^\infty \theta^{c_2+2} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & - 2\hat{\theta} \int_0^\infty \theta^{c_2+1} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \Bigg\}. \end{aligned} \tag{38}$$

Minimizing $R(\hat{\theta})$ yields:

$$\hat{\theta} = \frac{\Gamma\left(n-c_2/2-1/2\right)}{\Gamma\left(n-c_2/2\right)} \sqrt{\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2}}. \tag{39}$$

2.4.3 Precautionary loss function (plf)

The RF for $L_{plf}(\hat{\theta},\theta)$ is

$$R(\hat{\theta}) = \int_0^\infty \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}} \pi_1(\theta \mid x)\, d\theta. \tag{40}$$

Substituting Eq. (15) into Eq. (40), we get:

$$R(\hat{\theta}) = \int_0^\infty \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}} \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{41}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \int_0^\infty \frac{(\hat{\theta}^2+\theta^2-2\hat{\theta}\theta)}{\hat{\theta}} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{42}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \hat{\theta} \Bigg\{ \int_0^\infty \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta + \int_0^\infty \frac{\theta^2}{\hat{\theta}} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta - 2\int_0^\infty \theta \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \Bigg\}. \tag{43}$$

Minimizing the risk results in:

$$\hat{\theta} = \sqrt{\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2(n-1)}}. \tag{44}$$

2.4.4 Entropy loss function (elf)

The RF for $L_{elf}(\hat{\theta},\theta)$ is

$$R(\hat{\theta}) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right) \pi_1(\theta|x)\, d\theta. \tag{45}$$

Substituting Eq. (15) in Eq. (45), we get

$$R(\hat{\theta}) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right) \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{46}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right) \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{47}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^n x_a^{-2\alpha}\right)^n}{\Gamma(n)\, 2^{n-1}} \Bigg\{ \int_0^\infty \frac{\hat{\theta}}{\theta} \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta - \int_0^\infty \log\left(\frac{\hat{\theta}}{\theta}\right) \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta - \int_0^\infty \frac{1}{\theta^{2n+1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \Bigg\}. \tag{48}$$

Minimizing this risk function with respect to $\hat{\theta}$ yields the optimal Bayes estimate:

$$\hat{\theta} = \frac{\Gamma(n)}{\Gamma(n+1/2)} \sqrt{\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2}}. \tag{49}$$

2.5. Bayesian Estimation by Using Extension of Jeffreys Prior under Different Loss Functions

2.5.1 Squared error loss function (self)

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta} - \theta)^2 \pi_2(\theta|x)\, d\theta. \tag{50}$$

Substituting Eq. (22) into Eq. (50), we obtain:

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta} - \theta)^2 \frac{(\sum_{a=1}^n x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{51}$$

$$R(\hat{\theta}) = c\frac{(\sum_{a=1}^n x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \int_0^\infty \left(\hat{\theta}^2 + \theta^2 - 2\hat{\theta}\theta\right) \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{52}$$

$$\begin{aligned} R(\hat{\theta}) = {} & c\frac{(\sum_{a=1}^n x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \Bigg\{ \hat{\theta}^2 \int_0^\infty \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & + \int_0^\infty \theta^2 \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta - 2\hat{\theta} \int_0^\infty \theta \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \Bigg\}. \end{aligned} \tag{53}$$

$$\hat{\theta} = \frac{\Gamma\left(n+c_1-1\right)}{\Gamma(n+c_1-\frac{1}{2})} \sqrt{\left(\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2}\right)}. \tag{54}$$

2.5.2 Al-Bayyatis new loss function (abnlf)

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2} (\hat{\theta} - \theta)^2 \pi_2(\theta|x)\, d\theta. \tag{55}$$

Substituting Eq. (15) into Eq. (55), we get:

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2} (\hat{\theta} - \theta)^2 \frac{(\sum_{a=1}^n x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{56}$$

$$R(\hat{\theta}) = \frac{(\sum_{a=1}^n x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \int_0^\infty \theta^{c_2} \left(\hat{\theta}^2 + \theta^2 - 2\hat{\theta}\theta\right) \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{57}$$

$$\begin{aligned} R(\hat{\theta}) = {} & \frac{(\sum_{a=1}^n x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \Bigg\{ \hat{\theta}^2 \int_0^\infty \theta^{c_2} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & + \int_0^\infty \theta^{c_2+2} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & - 2\hat{\theta} \int_0^\infty \theta^{c_2+1} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta \Bigg\}. \end{aligned} \tag{58}$$

$$\hat{\theta} = \frac{\Gamma\left(n+c_1-c_2/2-1\right)}{\Gamma\left(n+c_1-c_2/2-1/2\right)} \sqrt{\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2}}. \tag{59}$$

2.5.3 Precautionary loss function (plf)

$$R(\hat{\theta}) = \int_0^\infty \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}} \pi_2(\theta|x)\, d\theta. \tag{60}$$

Using Eq. (22) in Eq. (60), we get:

$$R(\hat{\theta}) = \int_0^\infty \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}} \frac{(\sum_{a=1}^n x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^n x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{61}$$

$$R(\hat{\theta}) = \frac{(\sum_{a=1}^{n} x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \int_0^\infty \frac{(\hat{\theta}^2+\theta^2-2\hat{\theta}\theta)}{\hat{\theta}} \quad \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \quad (62)$$

$$\begin{aligned} R(\hat{\theta}) = \frac{(\sum_{a=1}^{n} x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \Bigg\{ & \int_0^\infty \hat{\theta} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & + \frac{1}{\hat{\theta}} \int_0^\infty \theta^2 \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & - 2\int_0^\infty \theta \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \Bigg\}. \end{aligned} \quad (63)$$

$$\hat{\theta} = \sqrt{\frac{\Gamma(n+c_1-3/2)}{\Gamma(n+c_1-1/2)} \quad \frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2}}. \quad (64)$$

2.5.4 Entropy loss function (elf)

$$R(\hat{\theta}) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right) \pi_2(\theta|x)\, d\theta. \quad (65)$$

Substituting Eq. (22) into Eq. (65), we get:

$$R(\hat{\theta}) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right) \frac{(\sum_{a=1}^{n} x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \quad \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \quad (66)$$

$$R(\hat{\theta}) = \frac{(\sum_{a=1}^{n} x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right) \quad \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \quad (67)$$

$$\begin{aligned} R(\hat{\theta}) = \frac{(\sum_{a=1}^{n} x_a^{-2\alpha})^{n+c_1-1/2}}{\Gamma(n+c_1-1/2)\, 2^{n+c_1-3/2}} \Bigg\{ & \int_0^\infty \hat{\theta} \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & + \frac{1}{\hat{\theta}} \int_0^\infty \theta^2 \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \\ & - 2\int_0^\infty \theta \frac{1}{\theta^{2n+2c_1}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \Bigg\}. \end{aligned} \quad (68)$$

$$\hat{\theta} = \frac{\Gamma(n+c_1-1/2)}{\Gamma(n+c_1)} \sqrt{\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2}}. \quad (69)$$

2.6. Bayesian Estimation by Using Quasi Prior under Different Loss Functions

2.6.1 Squared error loss function (self)

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi_3(\theta|x)\, d\theta. \quad (70)$$

Using Eq. (29) in Eq. (70), we get

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) 2^{\frac{2n+d-3}{2}}} \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \quad (71)$$

$$R(\hat{\theta}) = c\frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) 2^{\frac{2n+d-3}{2}}} \int_0^\infty \left(\hat{\theta}^2+\theta^2-2\hat{\theta}\theta\right) \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \quad (72)$$

$$R(\hat{\theta}) = c\frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right)2^{\frac{2n+d-3}{2}}}\left\{\hat{\theta}^2\int_0^\infty \frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta \right.$$
$$+\int_0^\infty \theta^2\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta \tag{73}$$
$$\left. -2\hat{\theta}\int_0^\infty \theta\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta\right\}.$$

$$\hat{\theta} = \frac{\Gamma\left(n+d/2-1\right)}{\Gamma\left(n+d/2-1/2\right)}\sqrt{\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2}}. \tag{74}$$

2.6.2 Al-Bayyatis new loss function (abnlf)

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2}(\hat{\theta}-\theta)^2\pi_3(\theta|x)\,d\theta. \tag{75}$$

Substituting Eq. (29) in Eq. (75), we get

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2}(\hat{\theta}-\theta)^2\frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right)2^{\frac{2n+d-3}{2}}}\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta. \tag{76}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right)2^{\frac{2n+d-3}{2}}}\int_0^\infty \theta^{c_2}\left(\hat{\theta}^2+\theta^2-2\hat{\theta}\theta\right)\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta. \tag{77}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right)2^{\frac{2n+d-3}{2}}}\left\{\theta^{c_2}\int_0^\infty \hat{\theta}^2\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta \right.$$
$$+\int_0^\infty \theta^2\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta \tag{78}$$
$$\left. -2\hat{\theta}\int_0^\infty \theta\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta\right\}.$$

$$\hat{\theta} = \frac{\Gamma\left(n+d/2-c_2/2-1\right)}{\Gamma\left(n+d/2-c_2/2-1/2\right)}\sqrt{\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2}}. \tag{79}$$

2.6.3 Precautionary loss function (plf)

$$R(\hat{\theta}) = \int_0^\infty \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\pi_3(\theta|x)\,d\theta. \tag{80}$$

Using Eq. (29) in Eq. (80), we get

$$R(\hat{\theta}) = \int_0^\infty \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right)2^{\frac{2n+d-3}{2}}}\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta. \tag{81}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right)2^{\frac{2n+d-3}{2}}}\int_0^\infty \frac{\left(\hat{\theta}^2+\theta^2-2\hat{\theta}\theta\right)}{\hat{\theta}}\frac{1}{\theta^{2n+d}}\exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right)d\theta. \tag{82}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) 2^{\frac{2n+d-3}{2}}} \left\{ \hat{\theta} \int_0^\infty \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \right.$$
$$+ \frac{1}{\hat{\theta}} \int_0^\infty \theta^2 \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \tag{83}$$
$$\left. - 2 \int_0^\infty \theta \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \right\}.$$

$$\hat{\theta} = \sqrt{\frac{\Gamma(n + d/2 - 3/2)}{\Gamma(n + d/2 - 1/2)} \; \frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2}}. \tag{84}$$

2.6.4 Entropy loss function (elf)

$$R(\hat{\theta}) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1 \right) \pi_3(\theta|x) \, d\theta. \tag{85}$$

Substituting Eq. (29) in Eq. (85), we get

$$R(\hat{\theta}) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1 \right) \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) 2^{\frac{2n+d-3}{2}}} \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{86}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) 2^{\frac{2n+d-3}{2}}} \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1 \right) \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta. \tag{87}$$

$$R(\hat{\theta}) = \frac{\left(\sum_{a=1}^{n} x_a^{-2\alpha}\right)^{\frac{2n+d-1}{2}}}{\Gamma\left(\frac{2n+d-1}{2}\right) 2^{\frac{2n+d-3}{2}}} \left\{ \int_0^\infty \frac{\hat{\theta}}{\theta} \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \right.$$
$$- \int_0^\infty \log\left(\frac{\hat{\theta}}{\theta}\right) \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \tag{88}$$
$$\left. - \int_0^\infty \frac{1}{\theta^{2n+d}} \exp\left(-\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2\theta^2}\right) d\theta \right\}.$$

$$\hat{\theta} = \frac{\Gamma(n + d/2 - 1/2)}{\Gamma(n + d/2)} \sqrt{\frac{\sum_{a=1}^{n} x_a^{-2\alpha}}{2}}. \tag{89}$$

3. Results

This section presents the results of a simulation study designed to evaluate and compare the performance of classical and Bayesian estimators for the IPRD. The performance of these estimators is assessed using the MSE criterion, which serves as a measure of the accuracy of the estimates. The study covers different sample sizes ($n = 25$, $n = 50$ and $n = 100$) to explore the effect of sample size on the performance of the estimators. The simulation process is repeated 100,000 times for each scenario to ensure reliable and consistent results.

The following parameter values are used in the simulation study:

- $\alpha = 0.5, 1.5, 2.0$
- $\theta = 0.5$
- $C_1 = 0.5, 2.5, 4.5$

- $C_2 = 0, 1.5, 2.5$
- $d = 0.5, 1, 1.5$

These parameter combinations are chosen to cover a broad range of possible scenarios, allowing for a comprehensive evaluation of the estimators. By examining the MSE for each case, we can determine how the Bayesian and classical methods perform under varying conditions, including different sample sizes and parameter settings. The results are presented in Tables 1, 2 and 3.

Table 1: *Estimates and (MSE) using Jeffreys prior under different loss functions using generated data.*

		$\boldsymbol{\theta}_{\text{mle}}$	$\boldsymbol{\theta}_{\text{self}}$	$\boldsymbol{\theta}_{\text{plf}}$	$\boldsymbol{\theta}_{\text{elf}}$	$\boldsymbol{\theta}_{\text{abnlf}}$		
n	α					$C_2 = 0$	$C_2 = 1.5$	$C_2 = 2.5$
25	0.5	0.81616	0.82869	0.83302	0.82028	0.82869	0.84181	0.85091
		(0.09973)	(0.10804)	(0.11090)	(0.10258)	(0.10804)	(0.11683)	(0.12314)
	1.5	1.20336	1.22179	1.22817	1.20139	1.22179	1.24113	1.25455
		(0.49471)	(0.52098)	(0.53023)	**(0.49194)**	(0.52098)	(0.54928)	(0.56943)
	2.0	1.55286	1.57665	1.58488	1.54065	1.57665	1.60161	1.61892
		(1.10852)	(1.15918)	(1.17697)	**(1.08294)**	(1.15918)	(1.21346)	(1.25191)
50	0.5	0.78828	0.79419	0.79622	0.79019	0.79419	0.80081	0.80447
		(0.08307)	(0.08655)	(0.08775)	(0.08421)	(0.08655)	(0.09108)	(0.09270)
	1.5	1.02439	1.03215	1.03488	1.02295	1.03215	1.04010	1.04550
		(0.27498)	(0.28319)	(0.28590)	**(0.27348)**	(0.28319)	(0.29171)	(0.29775)
	2.0	1.19598	1.20504	1.20812	1.19397	1.20504	1.21432	1.22063
		(0.48438)	(0.49710)	(0.50143)	**(0.48159)**	(0.49710)	(0.51024)	(0.51983)
100	0.5	0.79846	0.80147	0.80248	0.79946	0.80147	0.80452	0.80658
		(0.08908)	(0.09089)	(0.09150)	(0.08877)	(0.09089)	(0.09273)	(0.09398)
	1.5	1.11112	1.11538	1.11678	1.11008	1.11538	1.11962	1.12247
		(0.37353)	(0.37869)	(0.38043)	**(0.37220)**	(0.37869)	(0.38396)	(0.38705)
	2.0	1.38955	1.39479	1.39655	1.38129	1.39479	1.40009	1.40366
		(0.79130)	(0.80065)	(0.80386)	**(0.77667)**	(0.80065)	(0.81016)	(0.81660)

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Table 2: *Estimates and (MSE) using Jeffreys prior under different loss functions using generated data.*

		θ_{mle}	θ_{self}	θ_{plf}	θ_{elf}	θ_{abnlf}		
n	α					$C_2 = 0$	$C_2 = 1.5$	$C_2 = 2.5$
25	0.5	0.81230	0.82331	0.82723	0.81817	0.82331	0.83414	0.84302
		(0.09086)	(0.09966)	(0.10204)	(0.09514)	(0.09966)	(0.10735)	(0.11225)
	1.5	1.20336	1.22179	1.22817	1.20139	1.22179	1.24113	1.25455
		(0.49471)	(0.52098)	(0.53023)	**(0.49194)**	(0.52098)	(0.54928)	(0.56943)
	2.0	1.55286	1.57665	1.58488	1.54065	1.57665	1.60161	1.61892
		(1.10852)	(1.15918)	(1.17697)	**(1.08294)**	(1.15918)	(1.21346)	(1.25191)
50	0.5	0.79031	0.79713	0.79865	0.79206	0.79713	0.80219	0.80576
		(0.07453)	(0.07782)	(0.07883)	(0.07567)	(0.07782)	(0.08136)	(0.08279)
	1.5	1.02439	1.03215	1.03488	1.02295	1.03215	1.04010	1.04550
		(0.27498)	(0.28319)	(0.28590)	**(0.27348)**	(0.28319)	(0.29171)	(0.29775)
	2.0	1.18598	1.19504	1.19812	1.18397	1.19504	1.20432	1.21063
		(0.45438)	(0.46710)	(0.47143)	**(0.45159)**	(0.46710)	(0.48024)	(0.48983)
100	0.5	0.79846	0.80147	0.80248	0.79946	0.80147	0.80452	0.80658
		(0.08908)	(0.09089)	(0.09150)	(0.08877)	(0.09089)	(0.09273)	(0.09398)
	1.5	1.11112	1.11538	1.11678	1.11008	1.11538	1.11962	1.12247
		(0.37353)	(0.37869)	(0.38043)	**(0.37220)**	(0.37869)	(0.38396)	(0.38705)
	2.0	1.38955	1.39479	1.39655	1.38129	1.39479	1.40009	1.40366
		(0.79130)	(0.80065)	(0.80386)	**(0.77667)**	(0.80065)	(0.81016)	(0.81660)

Table 3: *Estimates and (MSE) using Quasi prior under different loss functions using generated data.*

n	α	$\boldsymbol{\theta}_{\text{mle}}$	$\boldsymbol{\theta}_{\text{self}}$	$\boldsymbol{\theta}_{\text{plf}}$	$\boldsymbol{\theta}_{\text{elf}}$	$\boldsymbol{\theta}_{\text{abnlf}}$ $C_2 = 0$	$\boldsymbol{\theta}_{\text{abnlf}}$ $C_2 = 1.5$	$\boldsymbol{\theta}_{\text{abnlf}}$ $C_2 = 2.5$
25	0.5	0.81619	0.83299	0.83739	0.81445	0.83299	0.84632	0.85557
		(0.09997)	(0.11088)	(0.11383)	(0.09888)	(0.11088)	(0.11994)	(0.12643)
	1.5	1.20336	1.22179	1.22817	1.20139	1.22179	1.24113	1.25455
		(0.49471)	(0.52098)	(0.53023)	**(0.49194)**	(0.52098)	(0.54928)	(0.56934)
	2.0	1.55286	1.56859	1.57669	1.55282	1.56859	1.59316	1.61020
		(1.10852)	(1.14188)	(1.15927)	**(1.10844)**	(1.14188)	(1.19500)	(1.23254)
50	0.5	0.78822	0.79622	0.79826	0.78218	0.79622	0.80238	0.80657
		(0.08307)	(0.08774)	(0.08896)	(0.07963)	(0.08774)	(0.09143)	(0.09398)
	1.5	1.02439	1.03215	1.03479	1.02295	1.03215	1.04010	1.04550
		(0.27498)	(0.28319)	(0.28600)	**(0.27348)**	(0.28319)	(0.29171)	(0.29758)
	2.0	1.19598	1.20199	1.20505	1.19597	1.20199	1.21120	1.21746
		(0.48438)	(0.49279)	(0.49709)	**(0.48437)**	(0.49279)	(0.50581)	(0.51475)
100	0.5	0.79846	0.80248	0.80350	0.79046	0.80248	0.80554	0.80760
		(0.08908)	(0.09150)	(0.09211)	(0.08437)	(0.09150)	(0.09335)	(0.09462)
	1.5	1.11119	1.11538	1.11679	1.11008	1.11538	1.11962	1.12247
		(0.37355)	(0.37869)	(0.38043)	**(0.37220)**	(0.37869)	(0.38393)	(0.38747)
	2.0	1.38955	1.39304	1.39479	1.38955	1.39304	1.39832	1.40187
		(0.79130)	(0.79751)	(0.80065)	**(0.79130)**	(0.79751)	(0.80697)	(0.81337)

4. Discussion

This paper presents an analysis of the IPRD using both classical and Bayesian approaches, incorporating various prior distributions and loss functions. Bayesian estimators are derived under different priors and evaluated using multiple loss functions. A simulation study is conducted to compare the performance of Bayesian estimators with MLE based on MSE. The results indicate that, in most cases, Bayesian estimators with ELF consistently exhibit lower MSE than MLE, leading to the conclusion that the Bayesian method is more efficient for parameter estimation in this distribution.

References

[1] A. Abdel-Menem. *Estimation and Prediction for inverse Rayleigh life distribution*. PhD thesis, M. SC. Thesis Faculty of Education, Ain Shames University, 2003.

[2] H. Abdullah and R. Aref. Bayesian approach in estimation of scale parameter of inverse rayleigh distribution. *Mathematics and Statistics Journal*, 2(1):8–13, 2016.

[3] N. Ahad, S. Ahmad, and J. Reshi. Bayesian analysis of extended maxwell-boltzmann distribution using simulated and real-life data sets. *Reliability: Theory & Applications*, 19(3 (79)):675–688, 2024.

[4] H. Al-Kutubi. *On comparison estimation procedures for parameter and survival function exponential distribution using simulation*. PhD thesis, Ph. D. Thesis, Baghdad University, College of Education (Ibn-Al-Haitham ¦, 2005.

[5] A. Bhat, S. Ahmad, and S. Mudasir. A novel extension of power rayleigh distribution with properties. *International Journal of Statistics and Reliability Engineering*, 9(3):518–529, 2022.

[6] S. Dey. Bayesian estimation of the parameter and reliability function of an inverse rayleigh distribution. *Malaysian Journal of Mathematical Sciences*, 6(1):113–124, 2012.

[7] G. Fan. Research article bayes estimation for inverse rayleigh model under different loss functions. *Research Journal of Applied Sciences, Engineering and Technology*, 9(12):1115–1118, 2015.

[8] K. Fatima and S. Ahmad. Weighted inverse rayleigh distribution. *International Journal of Statistics and Systems*, 12(1):119–137, 2017.

[9] R. Kumar, H. Sharma, R. Gupta, and A. Kaur. Bayesian and e-bayesian estimation of exponentiated inverse rayleigh distribution using conjugate prior. *Reliability: Theory & Applications*, 19(3 (79)):704–716, 2024.

[10] A. A. Mir and S. Ahmad. Modeling and analysis of sine power rayleigh distribution: Properties and applications. *Reliability: Theory & Applications*, 19(1 (77)):703–716, 2024.

[11] A. A. Mir and S. Ahmad. Exploring an extended rayleigh distribution: Modeling and applications in real life scenarios. *Reliability: Theory & Applications*, 20(1 (82)):928–941, 2025.

[12] A. A. Mir and S. Ahmad. A new extended rayleigh distribution with applications of covid-19 data. *Austrian Journal of Statistics*, 54(2):69–84, 2025.

[13] A. A. Mir, S. U. Rasool, S. Ahmad, A. Bhat, T. M. Jawa, N. Sayed-Ahmed, and A. H. Tolba. A robust framework for probability distribution generation: Analyzing structural properties and applications in engineering and medicine. *Axioms*, 14(4):281, 2025.

[14] F. Noor, A. Sajid, M. Ghazal, I. Khan, M. Zaman, and I. Baig. Bayesian estimation of rayleigh distribution in the presence of outliers using progressive censoring. *Hacettepe Journal of Mathematics and Statistics*, 49(6):2119–2133, 2020.

[15] J. G. Norstrom. The use of precautionary loss functions in risk analysis. *IEEE Transactions on reliability*, 45(3):400–403, 1996.

[16] G. Prakash. Bayes estimation in the inverse rayleigh model. *Electronic Journal of Applied Statistical Analysis*, 6(1):67–83, 2013.

[17] A. Sharma, P. Kumar, and B. Devi. Entropy estimation of inverse rayleigh probability distribution and its order statistics. *International Journal of Electronics Engineering*, 11(2):508–513, 2019.

[18] H. Sharma and P. Kumar. Estimation of different entropies of inverse rayleigh distribution under multiple censored data. *Reliability: Theory & Applications*, 19(1 (77)):29–39, 2024.

[19] T. N. Sindhu, M. Aslam, and N. Feroze. Bayes estimation of the parameters of the inverse rayleigh distribution for left censored data. In *ProbStat Forum*, volume 6, pages 42–59, 2013.

[20] A. Soliman, E. A. Amin, and A. A. Abd-El Aziz. Estimation and prediction from inverse rayleigh distribution based on lower record values. *Applied Mathematical Sciences*, 4(62):3057–3066, 2010.

[21] V. G. Voda. On the inverse rayleigh distributed random variable. *Rep. Statis. App. Res. JUSE*, 19(4):13–21, 1972.

[22] F. Yanuar, M. Iqbal, D. Devianto, A. Zetra, Y. Asdi, R. Ilahi, and R. F. Sani. Bayesian estimation under different loss functions for the case of inverse rayleigh distribution. *Kuwait Journal of Science*, 52(1):100343, 2025.

[23] H. Zaki and I. Jabir. A comparison of the classical estimators with the bayes estimators of one parameter inverse rayleigh distribution. *International Journal*, 3(8):738–749, 2015.