RT&A, No 3 (86)
Volume 20, September 2025

ALPHA-POWER-BURR-HATKE-EXPONENTIAL MODEL: PROPERTIES, SIMULATIONS AND APPLICATIONS BASED ON UNCENSORED AND PROGRESSIVE TYPE-II-CENSORED SAMPLES

Adubisi O. D., Joshua T., David A. A.

•

Department of Statistics, Federal University Wukari, Nigeria.
adubisiobinna@fuwukari.edu.ng

joshua@fuwukari.edu.ng

david.aji@fuwukari.edu.ng

Abstract

This study introduced a novel two-parameter model known as the alpha power Burr Hatke exponential (APBHE) model, characterized by constant, increasing-constant, upside-down bathtub, decreasing and increasing failure shapes. Structural properties and basic reliability functions are derived. The Simulation study carried out for both uncensored and progressive type-II censored samples indicated that the maximum likelihood estimation (MLE) performed quite well in producing good parameter estimates at finite sample sizes and tend to the true parameter value quicker than the maximum spacing product (MPS) method with minimum bias. Specifically, the censored schemes simulation disclosed that the MSE and bias values decrease as the sample size increases for the various censoring proportions. To demonstrate the flexibility and relevance of the APBHE model, a real-life bladder-cancer dataset is examined and the APBHE model achieved the best performance when compared with other competing models. Additionally, the log-APBHE model and log-APBHE regression model functions are presented for further explorations.

Keywords: Alpha-power transform, Lambert function, Progressive censored data, Burr-Hatke exponential model, Estimations

1. Introduction

The development of novel flexible statistical distributions capable of effectively describing real-world data has attracted a lot of attention in recent years. The features seen in many different forms of data are frequently difficult for many classical probability distributions, such the normal or exponential distributions, to fully convey, particularly in domains like reliability analysis, survival studies, insurance, and finance [1, 2]. Because of this, scientists are investigating more generalized versions of these distributions in an effort to increase their accuracy and flexibility. Traditional models such as the exponential, Weibull, or gamma distributions frequently have problems when dealing with data having large tails, skewness, or a range of hazard rates. The exponential distribution is useful for simulating life spans or time-to-failure in reliability theory, but it requires a constant hazard rate, which may be unrealistic in many practical cases [3]. To overcome this, numerous generalizations and expansions to the exponential distribution have been proposed. Among these, the Burr-Hatke exponential distribution introduced by [4] noted for its versatility in modeling diverse forms of data especially in actuarial science and dependability research, has received attention for its capacity to describe data with different tail behaviours and skewness [5, 6, 7, 8, 9, 10].

The alpha-power (AP) transformation, introduced by [11], has provided a new method to increase the flexibility of existing distributions. This transformation allows for the generation of new families of distributions that can better capture tail behaviour and accommodate varying shapes of hazard functions. The alpha-power transformation has been successfully applied to various baseline distributions, such as the Weibull, gamma, and Lomax distributions, resulting in more flexible models that are capable of accurately describing various datasets [11]. The power of the AP transformation lies in its ability to create a more adaptable hazard rate function. For instance, the transformed distributions can model increasing, decreasing, or bathtub-shaped hazard functions, which are commonly encountered in reliability studies and survival analysis. By incorporating an additional parameter, the alpha-power transformation provides greater flexibility in fitting a wide range of data distributions while maintaining the desirable properties of the original distribution [12, 13, 14, 15, 16, 17, 18, 19, 20]. Building upon these developments, we apply the alpha-power transformation to the Burr-Hatke-Exponential distribution, the resulting Alpha-Power-Burr-Hatke-Exponential (APBHE) model introduces an additional parameter that allows for even greater flexibility in modeling various types of data. Thus, the APBHE model can provide a better fit for data in fields such as finance, insurance, survival analysis, and reliability engineering, where data often exhibit heavy tails, skewness, and varying hazard rates.

This article is sectionalized as follows: Section two, the proposed model is introduced, and its behaviour is investigated using various representation graphically. Section three, a variety of structural characteristics are derived. Estimation methods are discussed for both uncensored and progressive type II censored samples in Sections four and five, for determining the APBHE model parameters. Section six presents the log-APBHE model and log-APBHE regression model functions. The simulation study is described and executed, with the findings presented in Section seven. The applications based on uncensored and progressive type II censored schemes are presented in Section eight while conclusion is in Section nine.

2. Model Genesis

Here, a non-negative random variable (r.v) X is said to follow the Burr-Hatke exponential (BHE) model with cumulative distribution function (CDF) given by

$$G(x) = 1 - \frac{e^{-\eta x}}{1+\eta x}, \tag{1}$$

where $\eta > 0$. The corresponding probability density function (PDF) to Eq. (1) is

$$g(x) = e^{-\eta x} \frac{2+\eta x}{(1+\eta x)^2}. \tag{2}$$

The CDF and PDF of the alpha power (AP) transformation are specified as

$$F(x) = \frac{\alpha^{G(x)} - 1}{\alpha - 1},\ \alpha > 0,\ \alpha \neq 1, \tag{3}$$

and

$$f(x) = \frac{\log \alpha}{\alpha - 1} g(x)\, \alpha^{G(x)},\ \alpha > 0,\ \alpha \neq 1. \tag{4}$$

respectively. By inserting Eqs. (1) and (2) into Eqs. (3) and (4), the CDF and PDF of the alpha power Burr-Hatke Exponential (APBHE) model are specified as

$$F(x;\alpha,\eta) = \frac{\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)} - 1}{\alpha - 1},\ \alpha > 0,\ \alpha \neq 1, \tag{5}$$

and

$$f(x;\alpha,\eta) = \frac{\log \alpha}{\alpha - 1} \eta e^{-\eta x} \frac{2+\eta x}{(1+\eta x)^2} \alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)},\ \alpha > 0,\ \alpha \neq 1. \tag{6}$$

respectively. The survival (Reliability) function (SF) and hazard (failure) rate (HRF) of the APBHE model take the forms

$$S(x;\alpha,\eta) = 1 - \left[\frac{\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)} - 1}{\alpha - 1}\right], \tag{7}$$

and

$$h(x;\alpha,\eta) = \frac{\frac{\log\alpha}{\alpha-1}\eta e^{-\eta x}\frac{2+\eta x}{(1+\eta x)^2}\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)}}{1-\left[\frac{\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)}-1}{\alpha-1}\right]}, \tag{8}$$

More so, the cumulative and reversed HRFs of the APBHE model are

$$H(x;\alpha,\eta) = -\log\left\{1 - \left[\frac{\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)} - 1}{\alpha - 1}\right]\right\}, \tag{9}$$

and

$$r(x;\alpha,\eta) = \frac{\frac{\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)}-1}{\alpha-1}}{1-\left[\frac{\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)}-1}{\alpha-1}\right]}. \tag{10}$$

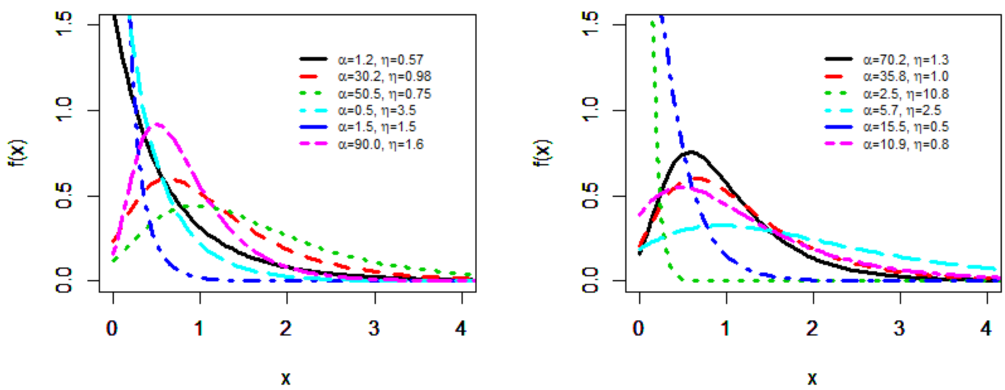

Figure 1: *PDF plots for the APBHE model*

The special cases of the APBHE model are the Burr-Hatke exponential (BHE) by [4], alpha power Burr-Hatke (APBH) by (new), Burr-Hatke (BH) by [21] and exponential (E) distributions. The PDF plots of the APBHE model constructed using various combinations of the parameter values depicted in Figure 1, exhibit right-skewed and reversed-J shapes. The HRF plots depicted in Figure 2 exhibit constant, increasing-constant, upside-down bathtub, decreasing and increasing failure shapes.

3. Structural properties

Some useful properties of the APBHE model are presented here.

3.1. Quantile function

If the r.v $X \sim \text{APBHE}(\alpha,\eta)$, then the quantile function by inverting Eq. (5) is derived as follows:

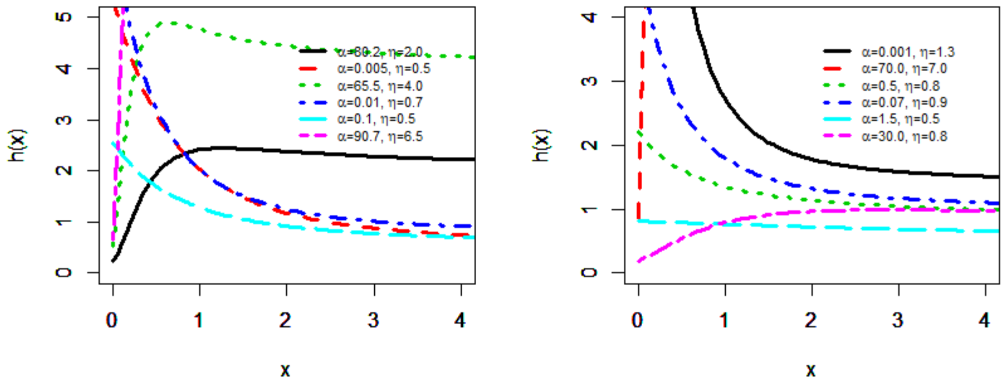

Figure 2: *HRF plots for the APBHE model*

$$u = \frac{\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)} - 1}{\alpha - 1}, \tag{11}$$

By manipulating Eq. (11) algebraically, we have

$$(1+\eta x)\, e^{1+\eta x} = e\left\{\frac{log\left[\frac{\alpha}{1+u(\alpha-1)}\right]}{log\alpha}\right\}^{-1}, \tag{12}$$

By introducing the Lambert function specified as $W\left(xe^{x}\right) = x$, the preceding equation takes the form

$$W\left[(1+\eta x)\, e^{1+\eta x}\right] = W\left(e\left\{\frac{log\left[\frac{\alpha}{1+u(\alpha-1)}\right]}{log\alpha}\right\}^{-1}\right), \tag{13}$$

with $Q(u) = x$, the quantile function of the APBHE model is given by

$$Q(u) = \frac{1}{\eta}\left(W\left\{\frac{elog\alpha}{log\left[\frac{\alpha}{1+u(\alpha-1)}\right]}\right\} - 1\right). \tag{14}$$

where $u \sim uniform(0,1)$.

3.2. Moments

The rth raw moment of the APBHE r.v, say $\mu'_r = \mathrm{E}\left(x^r\right) = \int_0^\infty x^r f\left(x\right) dx$, can be specified as

$$\mu'_r = \int_0^\infty x^r \frac{\log\alpha}{\alpha-1}\eta e^{-\eta x}\frac{2+\eta x}{(1+\eta x)^2}\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)}dx, \tag{15}$$

An expression for the rth moment of the APBHE model can be derived. Utilizing power-series expansion $\alpha^z = \sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}z^i$, and binomial theorem on $\alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)}$ gives

$$\mu'_r = \int_0^\infty x^r \eta\frac{2+\eta x}{\alpha-1}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^j(\log\alpha)^{i+1}}{i!}\binom{i}{j}\frac{e^{-(j+1)\eta x}}{(1+\eta x)^{j+2}}dx, \tag{16}$$

Applying the Taylor series expansion specified as $z^{-\varphi} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\varphi^{(k)}\left(z-1\right)^k$ on $\frac{e^{-(j+1)\eta x}}{(1+\eta x)^{j+2}}$ in the preceeding function gives

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (\log\alpha)^{i+1} \eta^{k+1}}{i!k!(\alpha-1)} \binom{i}{j} (j+2)^{(k)} \times \int_0^{\infty} (2+\eta x)\, x^{r+k} e^{-(j+1)\eta x} dx, \tag{17}$$

Thus, the rth raw moment of the APBHE model is specified as

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (\log\alpha)^{i+1} \eta^{k+1}}{i!k!(\alpha-1)} \binom{i}{j} (j+2)^{(k)} \times \left\{ \frac{2\Gamma(r+k+1)}{[(j+1)\eta]^{r+k+1}} + \frac{\eta\Gamma(r+k+2)}{[(j+1)\eta]^{r+k+2}} \right\}. \tag{18}$$

where $\Gamma(\cdot)$ is the gamma function.
The first four raw moments, coefficient of variation (CV), standard deviation (SD) and dispersion index (DI) from Eq. (18) using selected APBHE parameter values: $A = (\alpha = 0.5, \eta = 0.5)$, $B = (\alpha = 20, \eta = 2.5)$, $C = (\alpha = 2.5, \eta = 3.5)$ and $D = (\alpha = 3.5, \eta = 15)$ are reported in Table 1.

Table 1: *Descriptive statistics of the APBHE model*

μ_r'	A	B	C	D
μ_1'	0.982729	0.441738	0.214102	0.053848
μ_2'	2.416545	0.317158	0.092116	0.005592
μ_3'	10.25578	0.321278	0.061238	0.000881
μ_4'	63.17229	0.429446	0.056130	0.000190
CV	1.225655	0.790789	1.004752	0.963606
SD	1.204487	0.349321	0.215119	0.051888
DI	1.476286	0.276240	0.216141	0.050000

The skewness and kurtosis 3D plots of the APBHE model in Figure 3 depicts that the model can have increasing, decreasing, increasing-decreasing and decreasing-increasing values.

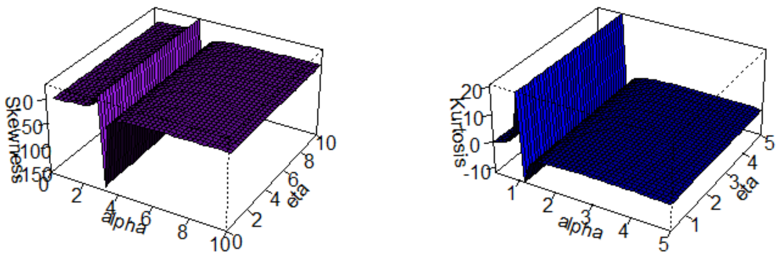

Figure 3: *Skewness and Kurtosis 3D plots for the APBHE model*

More so, the rth incomplete moment of the APBHE r.v, say $m_r(t) = \int_0^t x^r f(x)\, dx$, is specified as

$$m_r(t) = \int_0^t x^r \frac{\log\alpha}{\alpha-1} \eta e^{-\eta x} \frac{2+\eta x}{(1+\eta x)^2} \alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)} dx, \tag{19}$$

Utilizing the very paths that led to Eq. (18) on Eq. (19), we have

$$m_r(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (\log\alpha)^{i+1} \eta^{k+1}}{i!k!(\alpha-1)} \binom{i}{j} (j+2)^{(k)} \times \left\{ \frac{2\gamma(r+k+1,(j+1)\eta t)}{[(j+1)\eta]^{r+k+1}} + \frac{\eta\gamma(r+k+2,(j+1)\eta t)}{[(j+1)\eta]^{r+k+2}} \right\}. \tag{20}$$

where $\gamma(\cdot)$ denote the incomplete gamma function and the first incomplete moment of the APBHE model is obtained by setting $r = 1$ in Eq. (20).

3.3. Moments generating function

The moment generating function (MGF) of the APBHE model, say $M_X(t)$ is specified as

$$M_X(t) = \mathrm{E}\left(e^{tx}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)\, dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r', \tag{21}$$

By inserting Eq. (18) into Eq. (21), the MGF takes the form

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (\log \alpha)^{i+1} \eta^{k+1} t^r}{i!k!r!(\alpha-1)} \binom{i}{j} (j+2)^{(k)}. \tag{22}$$

3.4. Entropy

A measure of variation of the uncertainty in a system is the Rényi entropy defined as

$$I_\lambda(X) = \frac{1}{1-\lambda} \log \int_0^{\infty} f(x)^{\lambda}\, dx,\ \lambda > 0 \text{ and } \lambda \neq 1. \tag{23}$$

Inserting Eq. (6) into the preceding definition, the Rényi entropy of the APBHE model is specified as

$$I_\lambda(X) = \frac{1}{1-\lambda} \log \int_0^{\infty} \left(\frac{\log \alpha}{\alpha-1} \eta e^{-\eta x} \frac{2+\eta x}{(1+\eta x)^2} \alpha^{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)} \right)^{\lambda} dx, \tag{24}$$

and following the same paths that led to Eq. (18), we have

$$I_\lambda(X) = \frac{1}{1-\lambda} \log \left(\sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} (\log \alpha)^{i+\lambda} \lambda^i \eta^{k+\lambda}}{i!k!(\alpha-1)^{\lambda}} \binom{i}{j} (j+2\lambda)^{(k)} \times \left\{ \frac{2^{\lambda}\Gamma(k+1)}{[(j+\lambda)\eta]^{k+1}} + \frac{\eta^{\lambda}\Gamma(k+\lambda+1)}{[(j+\lambda)\eta]^{k+\lambda+1}} \right\} \right). \tag{25}$$

where $\Gamma(\cdot)$ is the gamma function. Table 2 reports the Rényi entropy results for selected values of the APBHE model parameters and various order of entropy (λ).

Table 2: *Rényi entropy results for selected parameter values of the APBHE model*

$\lambda = 0.5$			$\lambda = 1.3$			$\lambda = 2.0$		
α	η	$\mathrm{I}_\lambda(X)$	α	η	$\mathrm{I}_\lambda(X)$	α	η	$\mathrm{I}_\lambda(X)$
0.5	0.7	1.174	0.5	0.7	0.444	0.5	0.7	0.190
	1.5	0.412		1.5	-0.318		1.5	-0.572
	2.2	0.029		2.2	-0.701		2.2	-0.955
1.3	0.7	1.363	1.3	0.7	0.758	1.3	0.7	0.549
	1.5	0.601		1.5	-0.004		1.5	-0.213
	2.2	0.218		2.2	-0.387		2.2	-0.596
1.7	0.7	1.409	1.7	0.7	0.837	1.7	0.7	0.643
	1.5	0.647		1.5	0.075		1.5	-0.119
	2.2	0.264		2.2	-0.308		2.2	-0.502
2.3	0.7	1.458	2.3	0.7	0.921	2.3	0.7	0.742
	1.5	0.695		1.5	0.158		1.5	-0.020
	2.2	0.312		2.2	-0.224		2.2	-0.403

3.5. Order statistics

If $x_1, x_2, \ldots, x_n$ be a random sample from the APBHE r.v with $x_{j:n}$ as the order statistics (O.S). The pdf of the j$^{\text{th}}$ O.S, say $f_{j:n}(x)$ is defined as

$$f_{j:n}(x) = \frac{1}{\mathrm{B}(j, n-j+1)} g(x) [G(x)]^{j-1} [1 - G(x)]^{n-j}, \tag{26}$$

Expanding the last part of the preceding function, the PDF of the j$^{\text{th}}$ O.S takes the form

$$f_{j:n}(x) = \frac{1}{\mathrm{B}(j, n-j+1)} \sum_{l=0}^{n-j} (-1)^l \binom{n-j}{l} [G(x)]^{j+l-1} g(x), \tag{27}$$

By inserting Eqs. (5) and (6) into Eq. (27), then utilizing the power-series and binomial theorem. The PDF of j$^{\text{th}}$ O.S of the APBHE model is specified as

$$\begin{aligned} f_{j;n}(x) = {} & \tfrac{(2+\eta x)}{B(j,n-j+1)} \textstyle\sum_{l=0}^{n-j} \sum_{i=0}^{j+l-1} \sum_{k,m,g=0}^{\infty} \frac{(-1)^{l+i+m} (\log\alpha)^{k+1} (i+1)^k \eta^{g+1}}{k! g! (\alpha-1)^{j+l}} \\ & \times \binom{n-j}{l} \binom{j+l-1}{i} \binom{k}{m} (m+2)^{(g)} x^g e^{-(m+1)\eta x}. \end{aligned} \tag{28}$$

where $\mathrm{B}(\cdot)$ is the beta function. The minimum and maximum order statistics of the APBHE model can be derived by inserting $j = 1$ and $j = n$ in Eq. (28).

4. Estimators for Uncensored samples

Here, the estimators for the APBHE parameters using uncensored (complete) datasets are presented.

4.1. Maximum likelihood estimator

Let $x_1, x_2, \ldots, x_n$ be the random observed values of size (n) from APBHE model. the Maximum Likelihood estimates (MLE) are obtained from the log-likelihood function of Eq. (6) is specified as

$$\begin{aligned} L(\alpha, \eta) = {} & n \log \eta - \eta \textstyle\sum_{i=1}^{n} x_i + n \log\left(\frac{\log\alpha}{\alpha-1}\right) + \sum_{i=1}^{n} \left(1 - \frac{e^{-\eta x_i}}{1+\eta x_i}\right) \log\alpha \\ & + \textstyle\sum_{i=1}^{n} \log\left[\frac{2+\eta x_i}{(1+\eta x_i)^2}\right], \end{aligned} \tag{29}$$

The partial derivative of Eq. (29) with respect to α and η are specified as

$$\frac{\partial L(\alpha, \eta)}{\partial \alpha} = \frac{n\left(\frac{1}{\alpha} - \frac{\log\alpha}{(\alpha-1)}\right)}{\log\alpha} + \frac{\sum_{i=1}^{n}\left(1 - \frac{e^{-\eta x_i}}{1+\eta x_i}\right)}{\alpha}. \tag{30}$$

and

$$\begin{aligned} \tfrac{\partial L(\alpha,\eta)}{\partial \eta} = {} & \tfrac{n}{\eta} - \textstyle\sum_{i=1}^{n} \left[x_i - \left(\frac{x_i e^{-\eta x_i}}{1+\eta x_i} + \frac{x_i e^{-\eta x_i}}{(1+\eta x_i)^2}\right) \log\alpha\right] \\ & - \textstyle\sum_{i=1}^{n} \frac{x_i (3+\eta x_i)}{(2+\eta x_i)(1+\eta x_i)}. \end{aligned} \tag{31}$$

respectively. The ML estimates $\hat{\alpha}_{\mathrm{ML}}$ and $\hat{\eta}_{\mathrm{ML}}$ are found by maximizing Eq. (29) using the (*Optim function)* in R-programming software.

4.2. Maximum product spacing

The Maximum product spacing (MPS) estimation of unknown parameters with ordered sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ from APBHE model are considered. The MPS estimates $\hat{\alpha}_{\text{MPS}}$ and $\hat{\eta}_{\text{MPS}}$ can be found by maximizing relative to α and η, the logarithm of geometric mean of the spacing specified as

$$MPS(\alpha, \eta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\alpha, \eta), \tag{32}$$

where

$$\begin{aligned} D_i(\alpha, \eta) &= F_n\left(x_{(i)} \,|\alpha, \eta\right) - F_n\left(x_{(i-1)} \,|\alpha, \eta\right) \\ &= \frac{\alpha^{\left(1-\frac{e^{-\eta x_{(i)}}}{1+\eta x_{(i)}}\right)}-1}{\alpha-1} - \frac{\alpha^{\left(1-\frac{e^{-\eta x_{(i-1)}}}{1+\eta x_{(i-1)}}\right)}-1}{\alpha-1} \end{aligned} \tag{33}$$

with $F\left(x_{(0)} \,|\alpha, \eta\right) = 0$, $F\left(x_{(n+1)} \,|\alpha, \eta\right) = 1$ and $\sum_{i=1}^{n+1} D_i(\alpha, \eta) = 1$. The MPS estimates $\hat{\alpha}_{\text{MPS}}$ and $\hat{\eta}_{\text{MPS}}$ are found by maximizing Eq. (32) using the (*Optim function)* in R-programming software.

5. Estimators for progressive type II censored samples

Here, the estimators for the estimating the APBHE parameters using PTIIC samples are presented. The progressive Type-II censoring (PTIIC) scheme is described as follows: In a life testing experiment consisting n components where the number of observed failures (m) and $R_1, \ldots, R_m$ the censoring scheme are pre-determined. At the incidence of the first failure, R_1 of the remaining $n-1$ surviving units are randomly withdrawn (or censored). Subsequently, at the incidence of the second failure, R_2 of the remaining $n - R_1 - 1$ units are randomly withdrawn, and the process lingers until the $(m-1)^{th}$ failure is observed. Finally, at the incidence of the mth failure, all the remaining $R_m = n - m - R_1 - \cdots - R_{m-1}$ surviving units are withdrawn from the experiment. The progressive type-II censoring scheme consist of m and $R_1, \ldots, R_m$ given that $R_1 + \ldots + R_m = n - m$. More so, this scheme contain specially the conventional Type-II-right censoring when $R_1 = \cdots = R_{m-1} = 0, R_m = n - m$ and complete sample settings when $m = n, R_1 = \cdots = R_{m-1} = 0$, respectively.

5.1. Maximum likelihood estimator

Suppose that $X_{1:m:n} < \ldots < X_{m:m:n}$ is a PTIIC sample with $R_1, \ldots, R_m$ from the APBHE model, the likelihood function is given by

$$L(\alpha, \eta) = \Phi \prod_{i=1}^{m} f\left(x_{i:m:n}; \alpha, \eta\right) \left[1 - F\left(x_{i:m:n}; \alpha, \eta\right)\right]^{R_i} \tag{34}$$

where $\Phi = n(n - R_1 - 1) \cdots \left(n - \sum_{i=1}^{m-1} R_i - (m-1)\right)$, and R_i is the number of withdrawers at the i$^{\text{th}}$ censoring time. By inserting Eqs. (5) and (6) into Eq. (34) and taking the natural logarithm, the log-likelihood function of the APBHE model is specified as

$$\begin{aligned} L(\alpha, \eta) &= m \log \eta - \eta \textstyle\sum_{i=1}^{m} x_{i:m:n} + m \log\left(\frac{\log \alpha}{\alpha-1}\right) \\ &+ \textstyle\sum_{i=1}^{m} \left(1 - \frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right) \log \alpha + \sum_{i=1}^{m} \log\left[\frac{2+\eta x_{i:m:n}}{(1+\eta x_{i:m:n})^2}\right] \\ &+ \textstyle\sum_{i=1}^{m} R_i \log\left\{1 - \left[\frac{\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}-1}{\alpha-1}\right]\right\} \end{aligned} \tag{35}$$

The first partial derivative of Eq. (35) relative to α and η are specified as

$$\frac{\partial L(\alpha,\eta)}{\partial\alpha} = \frac{m}{\alpha\log\alpha} + \frac{m}{\alpha-1} + \frac{\sum_{i=1}^{m}\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}{\alpha}$$
$$+\sum_{i=1}^{m}\frac{R_i\left[-\frac{\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}{\alpha(\alpha-1)}+\frac{\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}-1}{(\alpha-1)^2}\right]}{\left\{1-\left[\frac{\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}-1}{\alpha-1}\right]\right\}} \tag{36}$$

and

$$\frac{\partial L(\alpha,\eta)}{\partial\eta} = \frac{m}{\eta} - \sum_{i=1}^{m}\left[x_{i:m:n} - \left(\frac{x_{i:m:n}e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}} + \frac{x_{i:m:n}e^{-\eta x_{i:m:n}}}{(1+\eta x_{i:m:n})^2}\right)\log\alpha\right]$$
$$+\sum_{i=1}^{m}\frac{x_{i:m:n}}{(2+\eta x_{i:m:n})} - 2\sum_{i=1}^{m}\frac{x_{i:m:n}}{(1+\eta x_{i:m:n})}$$
$$+\sum_{i=1}^{m}\left(-\frac{R_i\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}\left(\frac{x_{i:m:n}e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}+\frac{x_{i:m:n}e^{-\eta x_{i:m:n}}}{(1+\eta x_{i:m:n})^2}\right)\log\alpha}{(\alpha-1)\left\{1-\left[\frac{\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}-1}{\alpha-1}\right]\right\}}\right) \tag{37}$$

The ML estimates $\hat{\alpha}_{\text{ML}}$ and $\hat{\eta}_{\text{ML}}$ are found by solving the non-linear functions in Eqs. (36) and (37) simultaneously or utilizing the R-programming software (*Optim function)* in maximizing Eq. (35).

5.2. Maximum product spacing estimator

Suppose that $X_{1:m:n} < \ldots < X_{m:m:n}$ is a progressive type II censored sample from the APBHE model. The random sample uniform spacings are specified as

$$D_i\left(x_{i:m:n};\alpha,\eta\right) = \begin{cases} F\left(x_{1:m:n};\alpha,\eta\right) & if\ i=1, \\ F\left(x_{i:m:n};\alpha,\eta\right) - F\left(x_{i-1:m:n};\alpha,\eta\right) & if\ i=2,\ldots,m, \\ 1-F\left(x_{m:m:n};\alpha,\eta\right) & if\ i=m. \end{cases} \tag{38}$$

The MPS estimator based on the PTIIC sample [22], are found by maximizing the product of spacing specified as

$$P\left(x_{i:m:n};\alpha,\eta\right) = \Phi\prod_{i=1}^{m} D\left(x_{i:m:n};\alpha,\eta\right)\left[1-F\left(x_{i:m:n};\alpha,\eta\right)\right]^{R_i} \tag{39}$$

where Φ is a constant term. By utilizing the logarithm of the product of spacing with the CDF of the APBHE model, we found

$$L\left(x_{i:m:n};\alpha,\eta\right) = \log\left[\frac{\alpha^{\left(1-\frac{e^{-\eta x_{1:m:n}}}{1+\eta x_{1:m:n}}\right)}-1}{\alpha-1}\right] + \log\left[1-\frac{\alpha^{\left(1-\frac{e^{-\eta x_{m:m:n}}}{1+\eta x_{m:m:n}}\right)}-1}{\alpha-1}\right]$$
$$+\sum_{i=2}^{m}\log\left[\frac{\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}-1}{\alpha-1} - \frac{\alpha^{\left(1-\frac{e^{-\eta x_{i-1:m:n}}}{1+\eta x_{i-1:m:n}}\right)}-1}{\alpha-1}\right] \tag{40}$$
$$+\sum_{i=1}^{m}R_i\log\left[1-\frac{\alpha^{\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)}-1}{\alpha-1}\right]$$

The first partial derivative of Eq. (40) relative to α and η are specified as

$$\begin{aligned}\frac{\partial L(x_{i:m:n};\alpha,\eta)}{\partial\alpha} &= \frac{\left[\frac{\alpha^{\varphi_{1:m:n}}\varphi_{1:m:n}}{\alpha}-\frac{\alpha^{\varphi_{1:m:n}}}{\alpha-1}\right]}{\alpha^{\varphi_{1:m:n}}-1}+\frac{\left[\frac{\alpha^{\varphi_{m:m:n}}-1}{(\alpha-1)^2}-\frac{\alpha^{\varphi_{m:m:n}}\varphi_{m:m:n}}{\alpha(\alpha-1)}\right]}{1-\frac{\alpha^{\varphi_{m:m:n}}-1}{(\alpha-1)}} \\ &+\sum_{i=2}^{m}\frac{\frac{\alpha^{\varphi_{i:m:n}}\varphi_{i:m:n}}{\alpha(\alpha-1)}-\frac{\alpha^{\varphi_{i:m:n}}-1}{(\alpha-1)^2}-\frac{\alpha^{\varphi_{i-1:m:n}}\varphi_{i-1:m:n}}{\alpha(\alpha-1)}+\frac{\alpha^{\varphi_{i-1:m:n}}-1}{(\alpha-1)^2}}{\frac{\alpha^{\varphi_{i:m:n}}-1}{\alpha-1}-\frac{\alpha^{\varphi_{i-1:m:n}}-1}{\alpha-1}} \\ &+\sum_{i=1}^{m}\frac{R_i\left[\frac{\alpha^{\varphi_{i:m:n}}-1}{(\alpha-1)^2}-\frac{\alpha^{\varphi_{i:m:n}}\varphi_{i:m:n}}{\alpha(\alpha-1)}\right]}{1-\frac{\alpha^{\varphi_{i:m:n}}-1}{\alpha-1}}\end{aligned} \tag{41}$$

and

$$\begin{aligned}\frac{\partial L(x_{i:m:n};\alpha,\eta)}{\partial\eta} &= \frac{\alpha^{\theta_1}\gamma_1\ln(\alpha)}{\alpha^{\theta_1}-1}-\frac{\alpha^{\theta_m}\gamma_m\ln(\alpha)}{(\alpha-1)\left(1-\frac{\alpha^{\theta_m}-1}{\alpha-1}\right)} \\ &+\sum_{i=2}^{m}\frac{\frac{\alpha^{\theta_i}\gamma_i\ln(\alpha)}{\alpha-1}-\frac{\alpha^{\theta_{i-1}}\gamma_{i-1}\ln(\alpha)}{\alpha-1}}{\frac{\alpha^{\theta_i}-1}{\alpha-1}-\frac{\alpha^{\theta_{i-1}}-1}{\alpha-1}}+\sum_{i=1}^{m}\left(-\frac{R_i\alpha^{\theta_i}\gamma_i\ln(\alpha)}{(\alpha-1)\left(1-\frac{\alpha^{\theta_i}-1}{\alpha-1}\right)}\right)\end{aligned} \tag{42}$$

where
$\varphi_{1:m:n}=\left(1-\frac{e^{-\eta x_{1:m:n}}}{1+\eta x_{1:m:n}}\right)$, $\varphi_{m:m:n}=\left(1-\frac{e^{-\eta x_{m:m:n}}}{1+\eta x_{m:m:n}}\right)$, $\varphi_{i:m:n}=\left(1-\frac{e^{-\eta x_{i:m:n}}}{1+\eta x_{i:m:n}}\right)$, $\varphi_{i-1:m:n}=\left(1-\frac{e^{-\eta x_{i-1:m:n}}}{1+\eta x_{i-1:m:n}}\right)$, $\theta_1=1-\frac{e^{-\eta x_1}}{\eta x_1+1}$, $\gamma_1=\left(\frac{x_1e^{-\eta x_1}}{\eta x_1+1}+\frac{e^{-\eta x_1}x_1}{(\eta x_1+1)^2}\right)$, $\theta_m=1-\frac{e^{-\eta x_m}}{\eta x_m+1}$, $\theta_i=1-\frac{e^{-\eta x_i}}{\eta x_i+1}$, $\theta_{i-1}=1-\frac{e^{-\eta x_{i-1}}}{\eta x_{i-1}+1}$, $\gamma_m=\left(\frac{x_me^{-\eta x_m}}{\eta x_m+1}+\frac{e^{-\eta x_m}x_m}{(\eta x_m+1)^2}\right)$, $\gamma_{i-1}=\left(\frac{x_{i-1}e^{-\eta x_{i-1}}}{\eta x_{i-1}+1}+\frac{e^{-\eta x_{i-1}}x_{i-1}}{(\eta x_{i-1}+1)^2}\right)$. The MPS estimates $\hat{\alpha}_{\text{MPS}}$ and $\hat{\eta}_{\text{MPS}}$ are found by solving the non-linear functions in Eqs. (41) and (42) simultaneously or utilizing the R-programming software (*Optim function)* in maximizing Eq. (40).

6. Log-APBHE Regression model

Let X represent a random variable which follows the APBHE model. Based on the transformation $Y=\log(X)$ and $\eta=e^{-\frac{\mu}{\sigma}}$, the density of Y is specified as

$$f(y,\alpha,\mu,\sigma)=\frac{\log(\alpha)\,e^{\frac{y-\mu}{\sigma}}e^{-e^{\frac{y-\mu}{\sigma}}}}{\sigma(\alpha-1)}\frac{\left(2+e^{\frac{y-\mu}{\sigma}}\right)}{\left(1+e^{\frac{y-\mu}{\sigma}}\right)^2}\alpha^{\left(1-\frac{e^{-e^{\frac{y-\mu}{\sigma}}}}{1+e^{\frac{y-\mu}{\sigma}}}\right)} \tag{43}$$

where $\alpha,\sigma>0$, $y,\mu\in\Re$. Give that the random variable $X\sim APBHE(\alpha,\eta)$, then $Y\sim LogAPBHE(\alpha,\mu,\sigma)$. The survival function for the log-APBHE model is specified as

$$S(y,\alpha,\mu,\sigma)=1-\frac{\alpha^{\left(1-\frac{e^{-e^{\frac{y-\mu}{\sigma}}}}{1+e^{\frac{y-\mu}{\sigma}}}\right)}-1}{\alpha-1} \tag{44}$$

Let $z=(y-\mu)/\sigma$, $z\in\Re$, then the standardized log-APBHE density is specified as

$$f(z,\alpha)=\frac{\log(\alpha)\,e^ze^{-e^z}}{\sigma(\alpha-1)}\frac{(2+e^z)}{(1+e^z)^2}\alpha^{\left(1-\frac{e^{-e^z}}{1+e^z}\right)} \tag{45}$$

The explanatory vector associated with the ith response variable y_i, $i=1,\ldots,n$ is $\boldsymbol{X}_i=(x_{i1},\ldots,x_{im})^T$. The regression model based on the APBHE density function is given by

$$y_i=\boldsymbol{x}_i^T\boldsymbol{\beta}+\sigma z_i,\ i=1,\ldots,n, \tag{46}$$

where z_i is the error which follows the Eq. (45), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)^T, \sigma, \alpha > 0$ are unknown parameters and $\mu_i = \boldsymbol{X}_i^T \boldsymbol{\beta}$. The log-APBHE density and survival functions of y_i are specified as

$$f\left(y_i \, \alpha, \sigma, \boldsymbol{\beta}^T\right) = \frac{\log(\alpha) \, e^{z_i} e^{-e^{z_i}}}{\sigma(\alpha - 1)} \frac{(2 + e^{z_i})}{(1 + e^{z_i})^2} \alpha^{\left(1 - \frac{e^{-e^{z_i}}}{1 + e^{z_i}}\right)}, \tag{47}$$

and

$$S(y_i, \alpha, \mu, \sigma) = 1 - \frac{\alpha^{\left(1 - \frac{e^{-e^{z_i}}}{1 + e^{z_i}}\right)} - 1}{\alpha - 1}, \tag{48}$$

Here, F and C denote the group of individuals for which y_i is the log-lifetime and log-censoring with r and n-r observations, respectively. The log-likelihood (ℓ) for $\boldsymbol{\theta} = \left(\alpha, \sigma, \boldsymbol{\beta}^T\right)^T$ is specified as

$$\begin{aligned} \ell(\boldsymbol{\theta}) = r \log\left(\tfrac{\log(\alpha)}{\sigma(\alpha-1)}\right) + \textstyle\sum_{i \in F} z_i - \sum_{i \in F} e^{z_i} + \sum_{i \in F} \log(2 + e^{z_i}) \\ -2 \textstyle\sum_{i \in F} \log(1 + e^{z_i}) + \sum_{i \in F} \left(1 - \frac{e^{-e^{z_i}}}{1 + e^{z_i}}\right) \log \alpha \\ + \textstyle\sum_{i \in C} \log\left(1 - \frac{\alpha^{\left(1 - \frac{e^{-e^{z_i}}}{1 + e^{z_i}}\right)} - 1}{\alpha - 1}\right) \end{aligned} \tag{49}$$

where $z_i = \left(y_i - \boldsymbol{X}_i^T \boldsymbol{\beta}\right) / \sigma$ and the number of failures is represented by r. Maximizing the preceding function using the R-programme (*optim function*), the MLE $\hat{\theta}$ can be found.

7. Simulation for Uncensored and PTIIC

The Monte-Carlo simulation (MCS) is executed for the parameter (Pa.) estimates of the APBHE model. The R-programming packages are utilized in achieving the MCS steps. The complete and PTIIC samples are generated from APBHE model using Eq. (14) with $\alpha = 0.1$ and $\eta = 0.3$ and various sample sizes under the following schemes (SCHE) via the algorithm developed by [23].

SCHE1: $R_1 = n - m$ and $R_i = 0$ for $i \neq 1$,

SCHE2: $R_{m/2} = n - m$ and $R_i = 0$ for $i \neq (m/2)$,

SCHE3: $R_1 = R_m = (n - m)/2$ and $R_i = 0$ for $i \neq$ (1 and m)

SCHE4: $R_m = n - m$ and $R_i = 0$ for $i \neq m$

The average (AVE), biases (Bias), mean square errors (MSEs), 95% asymptotic confidence interval (Asy CI) and average interval length (AIL) are computed based on 1,000 PTIIC iterations are reported in Tables (3) and (4). Referring to Table (3), the MSE and Bias decrease as (m,n) increases for the estimates. However, in comparison the MLE estimates are more efficient than the MPS estimates according to their MSE values as seen in Table (3).

Table 3: *AVE and MSE (Bias) for APBHE model.*

(n, m)	SCHE	Pa.	MLE		MPS	
			AVE	MSE (Bias)	AVE	MSE (Bias)
(20, 20)	Complete	α	0.100	0.000 (0.000)	0.052	0.002 (-0.048)
		η	0.343	0.002 (0.043)	0.260	0.002(-0.040)
(20, 18)	1	α	0.113	0.000 (0.013)	0.071	0.001 (-0.029)
		η	0.342	0.002 (0.042)	0.186	0.013 (-0.114)
	2	α	0.121	0.000 (0.021)	0.070	0.001 (-0.030)
		η	0.350	0.003 (0.050)	0.200	0.010 (-0.100)
	3	α	0.122	0.001 (0.022)	0.088	0.000 (-0.012)
		η	0.350	0.002 (0.050)	0.254	0.002 (-0.046)
	4	α	0.127	0.001 (0.027)	0.095	0.000 (-0.005)
		η	0.353	0.003 (0.053)	0.305	0.000 (0.005)

Continued on the next page...

Table 3: (contd. . .)

(n, m)	SCHE	Pa.	MLE		MPS	
			AVE	MSE (Bias)	AVE	MSE (Bias)
(20, 14)	1	α	0.156	0.003 (0.056)	0.139	0.002 (-0.039)
		η	0.364	0.004 (0.064)	0.104	0.039 (-0.196)
	2	α	0.185	0.007 (0.185)	0.137	0.001 (0.037)
		η	0.385	0.007 (0.085)	0.121	0.032 (-0.179)
	3	α	0.188	0.008 (0.088)	0.157	0.003 (0.057)
		η	0.385	0.007 (0.085)	0.244	0.003 (-0.056)
	4	α	0.196	0.009 (0.096)	0.067	0.001 (-0.003)
		η	0.390	0.008 (0.390)	0.178	0.015 (-0.124)
(30, 30)	Complete	α	0.065	0.001 (-0.035)	0.040	0.126 (0.355)
		η	0.341	0.002 (0.041)	0.278	0.004 (-0.060)
(30, 26)	1	α	0.074	0.001 (-0.026)	0.068	0.000 (-0.022)
		η	0.330	0.001 (0.030)	0.173	0.001 (-0.032)
	2	α	0.082	0.000 (-0.018)	0.066	0.001(-0.034)
		η	0.339	0.002 (0.039)	0.188	0.013 (-0.112)
	3	α	0.088	0.000 (-0.012)	0.091	0.000 (-0.009)
		η	0.344	0.002 (0.044)	0.274	0.001 (-0.026)
	4	α	0.100	0.000 (0.000)	0.091	0.000 (-0.009)
		η	0.356	0.003 (0.056)	0.274	0.001 (0.026)
(30, 18)	1	α	0.115	0.000 (0.015)	0.165	0.004 (0.065)
		η	0.344	0.002 (-0.044)	0.073	0.052 (-0.227)
	2	α	0.150	0.003 (0.050)	-	-
		η	0.376	0.006 (0.076)	-	-
	3	α	0.146	0.002 (0.046)	0.175	0.006 (0.075)
		η	0.368	0.005 (0.068)	0.250	0.003 (-0.050)
	4	α	0.166	0.004 (0.066)	0.171	0.005 (0.071)
		η	0.382	0.007 (0.082)	0.380	0.006 (0.080)
(50, 50)	Complete	α	0.042	0.003 (-0.058)	0.029	0.005 (-0.071)
		η	0.340	0.002 (0.040)	0.294	0.000 (-0.006)
(50, 42)	1	α	0.048	0.003 (-0.052)	0.063	0.001 (-0.037)
		η	0.337	0.001 (0.037)	0.164	0.019 (-0.136)
	2	α	0.054	0.002 (-0.048)	0.060	0.002 (-0.040)
		η	0.347	0.002 (0.047)	0.180	0.015 (-0.120)
	3	α	0.075	0.001 (-0.025)	0.094	0.000 (-0.006)
		η	0.381	0.006 (0.081)	0.312	0.000 (0.012)
	4	α	0.098	0.000 (-0.002)	0.098	0.000 (-0.002)
		η	0.409	0.012 (0.109)	0.403	0.011 (0.103)
(50, 26)	1	α	0.077	0.001 (-0.023)	-	-
		η	0.333	0.001 (0.033)	-	-
	2	α	0.106	0.000 (0.006)	0.185	0.007 (0.085)
		η	0.365	0.004 (0.065)	0.062	0.057 (-0.238)
	3	α	1.481	1.906 (1.381)	0.191	0.008 (0.091)
		η	1.370	1.146 (1.070)	0.270	0.001 (-0.030)
	4	α	0.174	0.006 (0.074)	0.181	0.007 (0.081)
		η	0.421	0.015 (0.121)	0.424	0.015 (0.124)

Table 4: *Asy CI and AILs for APBHE model.*

(n, m)	SCHE	Pa.	MLE		MPS	
			Asy CI		Asy CI	
			(Lower, Upper)	AIL	(Lower, Upper)	AIL
(20, 20)	Complete	α	(-0.400, 0.590)	0.991	(-0.180, 0.280)	0.458
		η	(-0.220, 0.910)	1.129	(-0.100, 0.620)	0.722
(20, 18)	1	α	(-0.480, 0.710)	1.189	(-0.150, 0.290)	0.448
		η	(-0.260, 0.940)	1.206	(-0.014, 0.387)	0.400
	2	α	(-0.510, 0.750)	1.261	(-0.140, 0.280)	0.423
		η	(-0.260, 0.960)	1.229	(-0.007, 0.408)	0.410
	3	α	(-0.460, 0.640)	1.097	(-0.220, 0.400)	0.627
		η	(-0.380, 1.060)	1.596	(-0.055, 0.602)	0.657
	4	α	(-0.860, 1.120)	1.441	(-0.410, 0.600)	1.015
		η	(-0.610, 1.310)	1.923	(-0.260, 0.870)	1.127
(20, 14)	1	α	(-0.650, 0.960)	1.613	(-0.290, 0.570)	0.858
		η	(-0.280, 1.010)	1.291	(-0.021, 0.228)	0.249
	2	α	(-0.740, 1.110)	1.842	(-0.260, 0.530)	0.786
		η	(-0.290, 1.060)	1.358	(-0.015, 0.257)	0.272
	3	α	(-1.500, 1.900)	3.340	(-0.630, 0.940)	1.570
		η	(-0.860, 1.640)	2.500	(-0.210, 0.700)	0.913
	4	α	(-2.100, 2.500)	4.532	(-0.980, 1.310)	2.289
		η	(-1.300, 2.100)	3.350	(-0.570, 1.280)	1.850
(30, 30)	Complete	α	(-0.220, 0.350)	0.566	(-0.110, 0.190)	0.301
		η	(-0.130, 0.810)	0.939	(-0.041, 0.597)	0.638
(30, 26)	1	α	(-0.290, 0.430)	0.722	(-0.091, 0.226)	0.317

Continued on the next page. . .

Table 4: (contd...)

(n, m)	SCHE	Pa.	MLE Asy CI (Lower, Upper)	AIL	MPS Asy CI (Lower, Upper)	AIL
		η	(-0.180, 0.840)	1.029	(0.034, 0.312)	0.278
	2	α	(-0.310, 0.470)	0.785	(-0.162, 0.326)	0.488
		η	(-0.190, 0.870)	1.055	(-0.009, 0.654)	0.663
	3	α	(-0.490, 0.640)	1.097	(-0.623, 0.847)	0.627
		η	(-0.380, 1.060)	1.441	(-0.566, 1.443)	0.657
	4	α	(-0.590, 0.790)	1.386	(-0.380, 0.570)	0.945
		η	(-0.490, 1.210)	1.700	(-0.240, 0.920)	1.165
(30, 18)	1	α	(-0.490, 0.720)	1.204	(-0.280, 0.610)	0.883
		η	(-0.260, 0.950)	1.208	(-0.005, 0.151)	0.156
	2	α	(-0.590, 0.890)	1.482	-	- / -
		η	(-0.270, 1.020)	1.291	-	- / -
	3	α	(-1.400, 1.700)	3.094	(-0.710, 1.060)	1.772
		η	(-1.000, 1.800)	2.796	(-0.230, 0.730)	0.954
	4	α	(-2.000, 2.300)	4.365	(-1.200, 1.500)	2.651
		η	(-1.500, 2.200)	3.686	(-0.710, 1.470)	2.187
(50, 50)	Complete	α	(-0.1200, 0.200)	0.316	(-0.065, 0.122)	0.187
		η	(-0.046, 0.727)	0.773	(0.016, 0.571)	0.555
(50, 42)	1	α	(-0.140, 0.240)	0.385	(-0.042, 0.168)	0.210
		η	(-0.077, 0.750)	0.828	(0.070, 0.260)	0.188
	2	α	(-0.160, 0.270)	0.430	(-0.037, 0.158)	0.195
		η	(-0.079, 0.772)	0.851	(0.080, 0.280)	0.198
	3	α	(-0.280, 0.430)	0.712	(-0.160, 0.350)	0.514
		η	(-0.210, 0.980)	1.191	(0.013, 0.610)	0.597
	4	α	(-0.390, 0.590)	0.978	(-0.310, 0.500)	0.808
		η	(-0.290, 1.110)	1.409	(-0.170, 0.980)	1.148
(50, 26)	1	α	(-0.290, 0.450)	0.742	-	- / -
		η	(-0.180, 0.850)	1.033	-	- / -
	2	α	(-0.380, 0.590)	0.964	(-0.200, 0.570)	0.767
		η	(-0.190, 0.920)	1.114	(0.009, 0.115)	0.107
	3	α	(-1.100, 1.400)	2.425	(-0.700, 1.100)	1.774
		η	(-0.810, 1.610)	2.423	(-0.210, 0.750)	0.956
	4	α	(-1.600, 2.000)	3.575	(-1.100, 1.500)	2.656
		η	(-1.200, 2.000)	3.195	(-0.730, 1.580)	2.316

8. Applications

8.1. Uncensored dataset

Here, the fitness of the APBHE model is proved empirically via application to a real-dataset utilizing *AdequacyModel* package in R-programming software in comparison to alpha power Burr Hatke (APBH), inverse power Burr Hatke (IPBH), inverse pereto (IPE), pereto type-I (PE1), generalized exponential (GIE), type 1 half logistic skew-t (TIHLST), Exponentiated Inverse Rayleigh (EIR) and Burr hatke exponential (BHE) models.

The dataset discussed by [24] is the remission times (in months) for a random sample of 128 bladder cancer patients. [25] utilized the dataset in demonstrating the stability of their developed model. The data are: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88,5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63 and 22.69.

Some descriptive graphs for the data are depicted in Figure 4 and it is deduced that these dataset is right-skewed and leptokurtic in nature.

Table (5) reports the MLEs and standard errors (SEs) from the fitted models to the bladder cancer dataset. Also, the negative log-likelihood (-LL′), Consistent Akaike information criterion (CAIC′), Hannan-Quinn information criterion (HQIC′), Bayesian information criterion (BIC′), Akaike information criterion (AIC′) and Kolmogorov-Smirnov (KS′) and Pvalue of the fitted

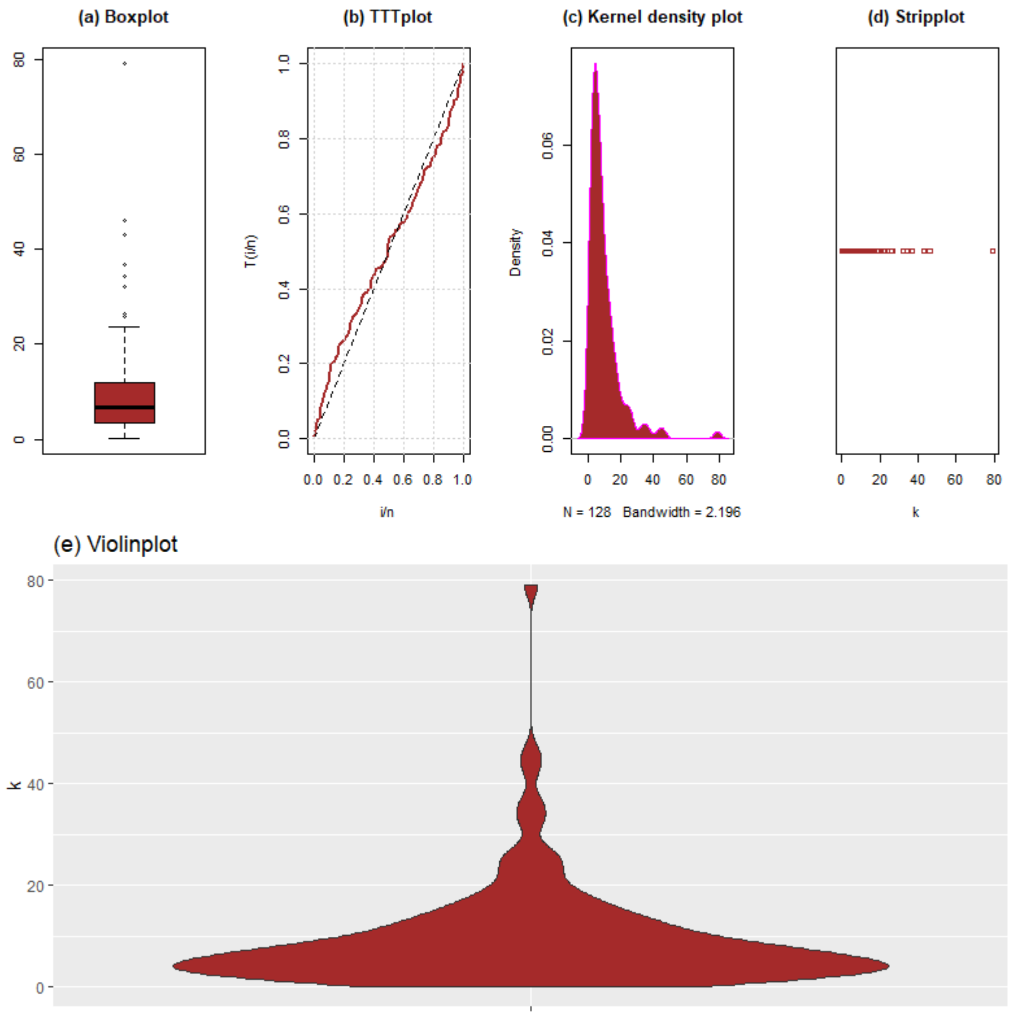

Figure 4: *Box-plot, TTT, kernel-density, strip, and Violin plots of bladder-cancer data.*

Table 5: *MLE (SEs) for the fitted models to the bladder-cancer uncensored dataset*

Model	MLEs (SEs)	
APBHE (α, η)	4.299 (2.648)	0.091 (0.014)
APBH (α, η)	136.300 (46.780)	0.037 (0.010)
IPBH (α, η)	1.722 (0.221)	0.803 (0.044)
IPE (α, η)	2.413 (0.582)	2.056 (0.646)
PE1 (η)	-	0.567 (0.050)
TIHLST (α, η)	0.528 (0.058)	15.321 (5.267)
EIR (α, η)	0.528 (0.058)	15.321 (5.267)
BHE (η)	-	0.062 (0.006)

models are reported in Table (6), the APBHE model has the highest -LL' and lowest values for the other selection criteria, especially KS' (high Pvalue), hence indicating that it provides the best fit to the bladder-cancer data among the fitted models.

More so, it is clear from Figure 5 that the APBHE model provides the best fit. Furthermore, the profile log-likelihood function plots with some parameter values (with fixed MLEs of other parameters) for the dataset are displayed in Figure 6. The likelihood ratio (LR) tests for the APBHE model against the BHE and IPBH models is carried-out. According to the empirical

Table 6: *Selection criteria for the fitted models to the bladder-cancer uncensored dataset*

Model	-LL′	AIC′	BIC′	CAIC′	HQIC′	KS′	Pvalue
APBHE	-413.8	831.7	837.4	831.8	834.0	0.06	0.70
BHE	-416.0	834.0	836.9	834.1	835.2	0.10	0.04
APBH	-415.7	835.4	841.1	835.5	837.8	0.10	0.03
IPBH	-442.7	889.4	895.1	889.5	891.7	0.10	0.02
IPE	-426.3	856.6	862.3	856.7	858.9	0.10	0.05
PE1	-426.5	855.0	857.9	855.0	856.2	3.00	2E-16
TIHLST	-446.8	897.5	903.2	897.6	899.8	0.20	6E-06
EIR	-507.0	1018.0	1023.7	1018.1	1020.3	0.40	3E-14

findings as reported in Table (7), the null hypotheses are rejected with low p-values, indicating that the APBHE model fits the dataset much better than the other models. This shows that the APBHE model provides the best fit to the bladder-cancer dataset.

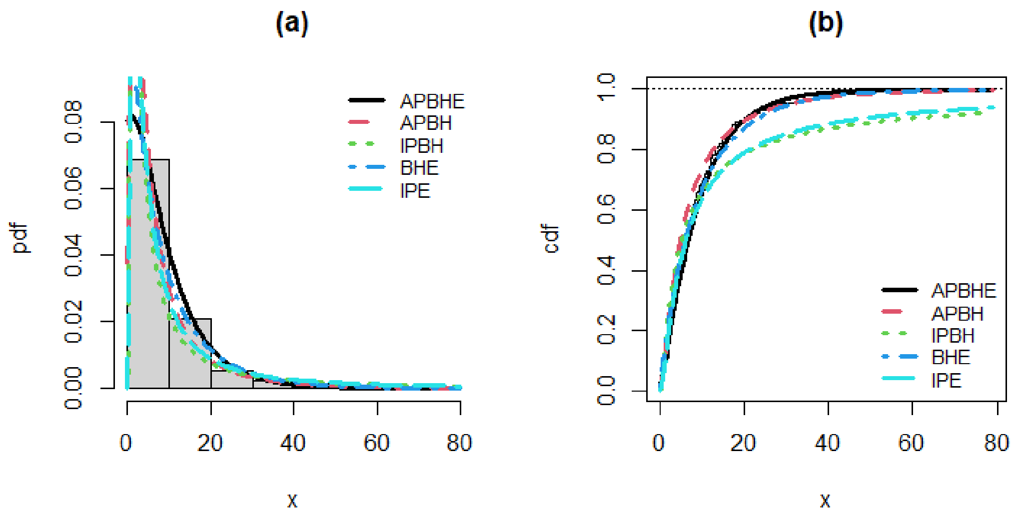

Figure 5: *Some Estimated PDFs (a) and CDFs (b) for the bladder-cancer uncensored dataset.*

Table 7: *LR tests for the bladder-cancer uncensored dataset*

Model	Hypotheses	LR	p-values
APBHE vs BHE	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	4.40	0.036
APBHE vs IPBH	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	58.0	<0.0001

Table 8: *Findings for MLE and MPS to the bladder-cancer uncensored dataset*

Model	pa	MLE	MPS
APBHE	α	4.301	3.129
	η	0.091	0.083

Table (8) reports the estimates of the APBHE parameter (pa) using the bladder-cancer dataset based on the MLE with (KS′ = 0.06, pvalue = 0.75) and MPS with (KS′ = 0.064, pvalue = 0.7) . For the sake of comparison, the MLE is quite appropriate for fitting the uncensored dataset based on the KS′ and larger Pvalues.

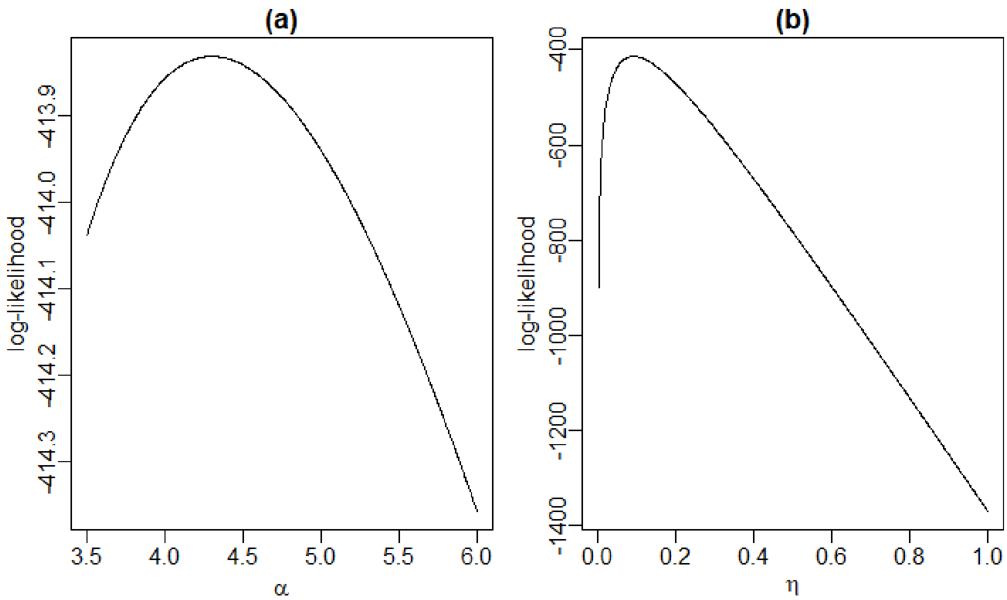

Figure 6: *Profile log-likelihood functions for bladder-cancer uncensored dataset.*

8.2. Censored dataset

The PTIIC using the bladder-cancer dataset via the different estimation methods is carried-out for the APBHE model. The average estimates and Asy CI using both the MLE and MPS at m = 90 are computed. However, the MPS performed poorly in the various PTIIC schemes as the estimates are all the same (*not presented*) . In conclusion, the MLE offer the best estimates and Asy CI values for the APBHE model in the PTIIC schemes as presented in Table (9).

Table 9: *PTIIC results for MLE to the bladder-cancer uncensored dataset*

(n, m)	SCHE	Pa.	MLE	Asy CI MLE
(128, 90)	1	α	4.776	(-0.670, 10.220)
		η	0.069	(0.047, 0.091)
	2	α	6.900	(-0.770, 14.580)
		η	0.081	(0.058, 0.104)
	3	α	6.000	(-0.740, 12.760)
		η	0.076	(0.054, 0.099)
	4	α	7.800	(-0.830, 16.330)
		η	0.085	(0.061, 0.108)

9. Conclusion

We propose a novel two-parameter model, known as the alpha power Burr-Hatke exponential (APBHE) model, which is an extension of Burke-Hatke exponential (BHE) model. This is driven due to the scares use of the BHE in statistical modeling and life testing especially in the field of medical sciences. Thus, some structural properties of the APBHE model including the log-APBHE model and the log-APBHE regression model are provided. The MLE and MPS estimation methods are utilized in estimating the APBHE parameters based on uncensored and PTIIC censoring schemes, and the simulation is utilized to ascertain the APBHE model output. However, the MLE estimates have more relative efficiency than the MPS for the parameter estimates of the APBHE model. This result is consistent with existing literature on comparison between the MLE and

MPS estimators using different existing and newly developed distributions. Furthermore, the empirical findings from the real-life dataset application inform that the novel APBHE consistently provides the best fit relative to other competing models considered.

References

[1] Adubisi, O. D., Abdulkadir, A. and Chiroma, H. (2021). A Two Parameter Odd Exponentiated Skew-T Distribution With J-Shaped Hazard Rate Function. *J. of Stat. Mod. & Analy. (JOSMA)*, 3:26–46.

[2] Adubisi, O. D. and Adubisi, C. E. (2023). Different Estimation Methods for Topp-Leone-(A) Model: Inference and Applications to Complete and Censored Datasets. *J. of Reliability: Theory & App. (RT&A)*, 18:100–117.

[3] Shanker, R., Hagos, F. and Sujatha, S. (2015). On modeling of Lifetimes data using exponential & Lindley distributions. *Biometrics & Biostat. Inter. J.*, 2:1–9.

[4] Yadav, A. S., Altun, E. and Yousof, H. M. (2019). Burr Hatke exponential distribution: A decreasing failure rate model, statistical inference and applications. *Annals of Data Sci.*, 8:241–260.

[5] Abouelmagd, T. H. M. (2018). The Logarithmic Burr Hatke exponential distribution for modeling reliability and medical data. *Inter. J. of Stat. & Pro.*, 7:73–85.

[6] El-Morshedy, M., Aji0hani, H. M., Eliwa, M. S., Nassar, M., Shakhatreh, M. K. and Afify, A. Z. (2021). The exponentiated Burr Hatke distribution and its discrete version: Reliability properties with CSALT model inference and application. *Maths.*, 9:2277.

[7] Afify, A. Z., Ajohani, H. M., Alghamdi, A. S., Gemeay, A. M. and Sarg, A. M. (2021). A new two-parameter Burr-Hatke distribution: properties and Bayesian and non-Bayesian inference with applications. *J. of Maths.*, 2021:1061083.

[8] Almuqrin, M. A. (2023). A new flexible distribution with applications to engineering data. *Alexandria Eng. J.*, 69:371-382.

[9] Gao, X. and Gui, W. (2023). Statistical inference of Burr-Hatke exponential distribution with partially accelerated life test under progressively type II censoring. *Maths.*, 11:1-21.

[10] Anyiam, K. E., Alghamdi, F. M., Nwaigwe, C. C., Aljohani, H. M. and Obulezi, O. J. (2024). A new extension of Burr-Hatke exponential distribution with engineering and biomedical applications. *Alexandria Eng. J.* 10:e38293.

[11] Mahdavi, A. and Kundu, D. (2017). A new method of generating distributions with an application to exponential distribution. *comm. in stat.-theory & meth.*, 46:6543-6557.

[12] Eissa, Y. F. and Sonar, D. C. (2023). Alpha power transformed extended power Lindley distribution. *J. of Stat.-theory & App.*, 22:1-18.

[13] Klakattawi, H. S. and Aljuhani, W. H. (2021). A new technique for generating distribution based on a combination of two techniques: Alpha power transformation and exponentiated T-X distributions family. *Symmetry*, 13:412.

[14] Bleed, S. O., Attwa, R. A., Ali, R. F. M. and Rahwan, T. (2024). On alpha power transformation generalized Pareto distribution and some properties. *J. of applied maths.*, 2024:6270350.

[15] Ehiwario, J. C. , Igabari, J. N. and Ezimadu, P.E. (2023). The alpha power Topp-Leone distribution: properties, simulations and applications. *J. of applied maths. & phy.* , 11:316-331.

[16] Dugasa, S. J., Goshu, A. T. and Arero, B. G. (2024). Alpha power transformation of the Lindley probability distribution. *J. of Prob. & Stat.*, 2024:9068114.

[17] Madlool, H. and AL-Kadim, K. A. (2024). Alpha power survival transformation exponential distribution. *J. of comp. analy. & app.*, 33:299-305.

[18] Oda, A. M. and AL-Kadim, K. A. (2024). Alpha power Quantile transformation exponential distribution. *J. of comp. analy. & app.*, 33:353-358.

[19] Abdulkadir, A., Adubisi, O. D. and Madaki, R. (2024). A new family of distributions based on amalgamation of two methods with an application to the Rayleigh model. *European. J. of Stat.* 4:1-19.

[20] Abdulkadir, A., Adubisi, O. D. and Madaki, R. (2024). Novel extended Weibull regression model for investigating the survival times of breast cancer patients. *Mikailalsys J. of Maths. & Stat.*, 2:135-155.

[21] Maniu, A. I. and Voda, V. G. (2008). Generalized Burr-Hatke equation as generator of a homographic failure rate. *J. of applied quant. meth.* , 3:215-222.

[22] Ng, H. K. T., Luo, L. and Duan, Y. H. (2012). Parameter estimation of three-parameter Weibull distribution based on progressive type-II censored samples. *J. of stat. comp. & simul.* , 82:1661-1678.

[23] Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive type-II censored samples. *The American Stat.* 9:229-230.

[24] Lee, T. E. and Wang, J. W. (2003). *Statistical methods for survival data analysis, Wiley, New York 2003*.

[25] Anzagra, L., Abubakari, A. G. and Nasiru, S. (2023). Chen Burr-Hatke exponential distribution: properties, regression and biomedical applications. *Comp. J. of Maths. & stat. sci.*, 2:80-105.