ANALYSIS OF JOINT MULTIPLY TYPE-II CENSORED DATA USING THE GIBBS SAMPLER ALGORITHM

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Abstract

This paper introduces a systematic approach for analyzing data under joint multiply type-II censoring. The study assumes a one-parameter exponential lifetime distribution and focuses on estimating unknown parameters. The maximum likelihood method is used to obtain frequentist point estimates, while a Bayesian framework is adopted to draw the corresponding Bayes inferences. To effectively handle censored data, an extended Gibbs sampler algorithm is employed, treating the unknown observations as further unknowns and estimating them accordingly. This methodology ensures a comprehensive and robust inference process by simultaneously addressing parameter uncertainty and the challenges posed by the censored observations.

Keywords: Joint multiply type-II censoring, Non-informative prior, Maximum likelihood estimation, Bayes inferences, One-sample prediction, Gibbs sampler.

1. Introduction

A life testing experiment is a widely used experimentation process for assessing the reliability and lifespan of products, materials, or biological systems. It involves subjecting units to specific conditions and monitoring their performance over time until failure occurs, or a predefined endpoint is reached. To enhance efficiency, reduce testing costs, and expedite results, such experiments are often terminated before all failures are observed. A variety of censoring schemes have been introduced in the existing research to address these challenges. Censoring occurs when the exact failure times of some units are unknown, which can significantly impact the validity and precision of statistical inferences drawn from the data. A censoring scheme that balances time, cost, and efficiency is of great interest in life testing studies.

Among the various censoring schemes, type-I, type-II, and interval censoring are the most prevalent. Type-I censoring occurs when the study is concluded after a fixed duration, regardless of the number of failures observed. In contrast, type-II censoring involves stopping the experiment once the number of items failed reaches a fixed count, even if it extends beyond the expected duration. In interval censoring, the exact time of failure is not observed but is known to fall within a certain time interval (a, b), where $b > a$. Most existing research focuses on single-sample censored data under these schemes, with numerous studies addressing their individual aspects. However, there is a growing recognition of the need for a comprehensive framework to analyze data under joint censoring schemes.

Joint censoring schemes are particularly relevant in fields such as medical research, reliability engineering, and social sciences, where different censoring mechanisms may coexist in a single study ([4]). For instance, when comparing two products or treatments within the same setting, a two-sample joint censoring approach can be employed. This method entails conducting a life-testing experiment on independent samples from two different groups simultaneously, with the study concluding upon the happening of the targeted event, either at a predetermined time or after observing a specified number of failures. Researchers have introduced various joint censoring schemes to enhance the efficiency of life testing experiments involving multiple groups. Notable examples include the joint type-II progressive censoring, balanced joint type-II progressive censoring, joint type-I generalized hybrid censoring, and joint type-II generalized progressive hybrid censoring (see, for example, [8], [7] and [1], etc.). Among these, the joint multiply type-II censoring scheme (see [2]) is particularly useful for simultaneously evaluating two similar products under the same facility. It can also model medical studies involving two groups by tracking the occurrence of predefined events. This scheme offers flexibility by encompassing various censoring approaches, such as multiply type-II, type-II right, left type-II, mid, interval, doubly type-II, and left censoring, etc. as special cases. More details on these special cases can be found in [14], [13], and [6].

In reliability analysis, lifetime distributions are fundamental for evaluating product performance. Various lifetime models including the exponential, Weibull, gamma, and lognormal distributions, have been widely studied. The selection of suitable model depends on factors like the failure rate of the product. Among these, the exponential distribution holds particular significance due to its mathematical simplicity and suitability for scenarios where the failure rate remains constant. Extensive research has explored the exponential lifetime model under different censoring schemes, although the applicability of the exponential model has often been constrained by its underlying assumption of a constant failure rate. The Weibull, gamma and lognormal distributions, on the other hand, have monotonic failure rate and, therefore, have wider applicability across various settings relative to the exponential model. These models are, however, slightly more difficult for inferential developments, especially under conditions of complex censoring embedded in the data. One can refer to [5] for a detailed description of various lifetime models and the corresponding inferential developments. The description gives an idea of the complexity involved with these models, especially when dealing with many of the above mentioned censoring schemes.

Once a lifetime distribution is selected for a given problem involving any type of censoring mechanism, the next step is to infer the unknown parameters of the distribution. This requires the application of various inferential techniques. A substantial body of literature exists on this topic, encompassing both frequentist and Bayesian approaches. While both methods have their own merits, we prefer the Bayesian approach due to its inherent advantages, such as incorporating prior knowledge and providing a comprehensive probabilistic framework for inferential developments. The censored data, however, introduce unique challenges to Bayesian analysis, primarily due to the presence of often non-closed forms of corresponding cumulative distribution function terms in the likelihood. These complexities often lead to intractable posterior distributions and computational difficulties in the form of intractable integrations. To overcome these difficulties, various approximate techniques have been developed in the literature. These include a number of analytical approximation techniques and sometimes even numerical integration techniques. The simulation-based approaches are other alternatives that are often user-friendly and straightforward to implement. A few such procedures include sampling-importance-resampling, the Gibbs sampler algorithm and, more generally, the Metropolis-Hastings algorithm, etc. These latter methods facilitate efficient Bayesian implementation, making them particularly useful for analyzing censored lifetime data.

This paper employs the Gibbs sampler algorithm and its extension to address problems involving censored data (see [15], [11]). By leveraging this technique, the complexity of the likelihood function, which leads to an intractable posterior distribution, can be significantly reduced. The study focuses on estimating parameters under joint multiply type-II censoring using both maximum likelihood (ML) and Bayesian approaches. A non-informative prior is employed for the Bayesian inferences. Also, the inferences for the unknown censored observations are obtained by computing the highest predictive density intervals (HPrDI) and identifying the mean value from the generated samples corresponding to censored data based on the Gibbs outcome.

The layout of this paper is as follows: The next section outlines the joint multiply type-II censoring scheme and presents the modeling framework under the assumption of an exponential distribution. Section 3 details the inferential developments, with two subsections dedicated to deriving the ML estimates and the Bayes inferences for the model parameters, respectively. The discussion is also focused on predicting the unknown censored observations, treating them as a one-sample prediction problem. A step-wise systematic discussion is also provided to facilitate the practical ease of implementing the considered algorithm. To demonstrate the practical effectiveness of the proposed methodology, a real dataset of failure times from air conditioning systems in two planes, "7914" and "7913" (sourced from [9]), is analyzed, and the corresponding results are obtained. Finally, the paper concludes with a summary of key findings with an extensive list of relevant references.

2. The Joint Multiply Type-II Censoring Scheme and the Model Formulation

To begin with, let us first describe the joint multiply type-II censoring scheme considered in this paper. Let us consider a situation where the experimenter aims to investigate the failure times of a number of two similar products, referred to as Product 1 and Product 2, operated under the same environment. The initial step involves randomly selecting m items from Product 1 and n items from Product 2, ensuring that each item has an equal probability of being chosen. Subsequently, the experimenter proceeds with conducting a life testing experiment on both the products simultaneously. The experimenter defines k as the predetermined number of failures that he aims to observe with the assumption that any of the failures could occur from either Product 1 or Product 2. The experiment terminates upon observing the r_k^{th} failure. Subsequently, all surviving units are withdrawn from the testing process.

Assume further that a_k represents the number of failures out of k failures, from Product 1, and b_k represents the rest of the failures that are from Product 2. Let Y represents the failure times of items from Product 1 and W represents the failure times of items corresponding to Product 2. The density and the distribution function of Y are assumed to be $f(y)$ and $F(y)$, respectively. Similarly, the density and the distribution function of W are assumed to be $g(w)$ and $G(w)$, respectively. Consider two independent random samples: $Y_1, Y_2, \ldots, Y_m$ of size m from Product 1 and $W_1, W_2, \ldots, W_n$ of size n from Product 2. Let us next consider that $N = m + n$ represents the total number of items under testing from the two products together. Further assume that $X_1, X_2, X_3, \ldots, X_N$ are the pooled ordered failure times of the two products when considered in ascending order. It may be noted that if two or more observations have identical values (ties) during the ordering process, they are randomly shuffled at that specific point of time. Also, since one parameter exponential distribution with mean parameter θ_1 is assumed as the appropriate model for Product 1, one can write the corresponding density and distribution function, respectively, as

$$\begin{aligned} f(y) &= \frac{1}{\theta_1} exp\left(-\frac{y}{\theta_1}\right)\ ;\ y > 0,\ \theta_1 > 0 \\ F(y) &= 1 - exp\left(-\frac{y}{\theta_1}\right). \end{aligned} \tag{1}$$

Similarly, for Product 2, if one assumes an exponential model with mean parameter θ_2, the corresponding density and distribution functions can be written, respectively, as

$$\begin{aligned} g(w) &= \frac{1}{\theta_2} exp\left(-\frac{w}{\theta_2}\right)\ ;\ w > 0,\ \theta_2 > 0 \\ G(w) &= 1 - exp\left(-\frac{w}{\theta_2}\right). \end{aligned} \tag{2}$$

Assume further that we aim to observe only k failures with failure times $X_{r_1}, X_{r_2}, X_{r_3}, \ldots, X_{r_k}$ from the N items under test. Also suppose that u_i denotes the number of unobserved failures of

Product 1 in the interval $(X_{r_{i-1}}, X_{r_i})$ and v_i denote the number of unobserved failures of Product 2 in the same interval, for $i = 1, 2, \dots, k$. Here X_{r_0} is assumed to be 0. After the k^{th} failure is observed, let u_{k+1} denote the number of items from Product 1 that are still functioning and v_{k+1} denote the number of items from Product 2 that are still functioning in the experiment. Let $Z_i\,(i = 1, 2, 3, \dots, k)$ be the indicator variable with $Z_i = 1$ denoting that the i^{th} failure is from Product 1 and $Z_i = 0$, denoting that the i^{th} failure is from Product 2. So, under this assumption, the likelihood function with data comprising of (Z, X) can be written as

$$L = \frac{m!n!}{\prod_{i=1}^{k+1} u_i! v_i!} F(x_{r_1})^{u_1} G(x_{r_1})^{v_1} \prod_{i=2}^{k} [F(x_{r_i}) - F(x_{r_{i-1}})]^{u_i} \prod_{i=2}^{k} [G(x_{r_i}) - G(x_{r_{i-1}})]^{v_i}$$
$$\times [1 - F(x_{r_k})]^{u_{k+1}} [1 - G(x_{r_k})]^{v_{k+1}} \prod_{i=1}^{k} [f(x_{r_i})]^{z_i} \prod_{i=1}^{k} [g(x_{r_i})]^{1-z_i} \tag{3}$$

The likelihood function given above will lay the foundation for estimating the unknown parameters using different approaches. For further information on this censoring scheme, one may refer to [2]. The paper considers ML method of estimation as well as the Bayesian method for drawing inferences, the latter using vague priors for the parameters.

3. Inferential developments

3.1. Maximum likelihood method

Using (1) and (2) in (3), the likelihood function for the present situation can be written as

$$L = \frac{m!n!}{\prod_{i=1}^{k+1} u_i! v_i!} \left(1 - e^{-\frac{x_{r_1}}{\theta_1}}\right)^{u_1} \left(1 - e^{-\frac{x_{r_1}}{\theta_2}}\right)^{v_1} \prod_{i=2}^{k} \left(e^{-\frac{x_{r_{i-1}}}{\theta_1}} - e^{-\frac{x_{r_i}}{\theta_1}}\right)^{u_i} \prod_{i=2}^{k} \left(e^{-\frac{x_{r_{i-1}}}{\theta_2}} - e^{-\frac{x_{r_i}}{\theta_2}}\right)^{v_i}$$
$$\times \left(e^{-\frac{x_{r_k}}{\theta_1}}\right)^{u_{k+1}} \left(e^{-\frac{x_{r_k}}{\theta_2}}\right)^{v_{k+1}} \prod_{i=1}^{k} \left(\frac{1}{\theta_1} e^{-\frac{x_{r_i}}{\theta_1}}\right)^{z_i} \prod_{i=1}^{k} \left(\frac{1}{\theta_2} e^{-\frac{x_{r_i}}{\theta_2}}\right)^{1-z_i} \tag{4}$$

The ML estimates of θ_1 and θ_2 can be obtained by maximizing (4) with varying θ_1 and θ_2. This involves first taking the logarithm on both the sides of (4), followed by partial differentiation of the resulting expression with respect to θ_1 and θ_2 separately. By equating the partial derivatives to zero, one can obtain

$$\frac{\partial logL}{\partial \theta_1} = -\frac{u_1}{\theta_1^2} \left\{ \frac{x_{r_1} e^{\left(-x_{r_1}/\theta_1\right)}}{1 - e^{\left(-x_{r_1}/\theta_1\right)}} \right\} + \sum_{i=2}^{k} \frac{u_i}{\theta_1^2} \left\{ \frac{x_{r_{i-1}} e^{\left(-x_{r_{i-1}}/\theta_1\right)} - x_{r_i} e^{\left(-x_{r_i}/\theta_1\right)}}{e^{\left(-x_{r_{i-1}}/\theta_1\right)} - e^{\left(-x_{r_i}/\theta_1\right)}} \right\} + \frac{u_{k+1} x_{r_k}}{\theta_1^2}$$
$$+ \sum_{i=1}^{k} \frac{z_i x_{r_i}}{\theta_1^2} - \frac{a_k}{\theta_1} = 0,$$
$$\frac{\partial logL}{\partial \theta_2} = -\frac{v_1}{\theta_2^2} \left\{ \frac{x_{r_1} e^{\left(-x_{r_1}/\theta_2\right)}}{1 - e^{\left(-x_{r_1}/\theta_2\right)}} \right\} + \sum_{i=2}^{k} \frac{v_i}{\theta_2^2} \left\{ \frac{x_{r_{i-1}} e^{\left(-x_{r_{i-1}}/\theta_2\right)} - x_{r_i} e^{\left(-x_{r_i}/\theta_2\right)}}{e^{\left(-x_{r_{i-1}}/\theta_2\right)} - e^{\left(-x_{r_i}/\theta_2\right)}} \right\} + \frac{v_{k+1} x_{r_k}}{\theta_2^2}$$
$$+ \sum_{i=1}^{k} \frac{(1 - z_i) x_{r_i}}{\theta_2^2} - \frac{b_k}{\theta_2} = 0. \tag{5}$$

It can be seen that the two equations in (5) are not obtained in explicit forms and, therefore, a numerical solution appears to be the only possibility. Among various approaches, the Newton-Raphson method is particularly favoured as it was seen to efficiently determine the ML estimates for both θ_1 and θ_2. It may be noted that obtaining ML estimates is not the actual objective of the paper; rather, the same are obtained to aid in the Bayesian developments that follow in the next subsection. The ML estimates can also be used to provide a comparative study with the Bayesian estimates of the two parameters where censoring also plays a crucial role.

3.2. Bayesian method

This subsection focuses on the Bayesian reasoning with the ultimate objective of estimating the two parameters, θ_1 and θ_2. It may be noted that the Bayesian method begins by defining suitable prior distributions for the parameters. This requires the expertise of a statistician to accurately translate the beliefs and insights of subject-matter experts into a quantifiable form. In situations where limited or no information is available about the parameters and expert input is absent, a non-informative prior is often a suitable choice. There can be a variety of ways of specifying non-informative priors, but we consider the simplest and the most widely used Jeffreys' prior for the two scale parameters of considered exponential models (see also [14]). Obviously, the priors for θ_1 and θ_2 can be written as

$$\begin{aligned} \pi_1(\theta_1) &\propto \frac{1}{\theta_1}, \\ \pi_2(\theta_2) &\propto \frac{1}{\theta_2}. \end{aligned} \tag{6}$$

Next, by combining the likelihood in (4) with the priors in (6) using the Bayes theorem, the posterior distribution can be expressed up to proportionality as

$$\begin{aligned} p(\theta_1, \theta_2 | \underline{x}) \propto & \frac{1}{\theta_1^{a_k+1} \theta_2^{b_k+1}} \left(1 - e^{-\frac{x_{r_1}}{\theta_1}}\right)^{u_1} \left(1 - e^{-\frac{x_{r_1}}{\theta_2}}\right)^{v_1} \\ & \times \prod_{i=2}^{k} \left(e^{-\frac{x_{r_{i-1}}}{\theta_1}} - e^{-\frac{x_{r_i}}{\theta_1}}\right)^{u_i} \prod_{i=2}^{k} \left(e^{-\frac{x_{r_{i-1}}}{\theta_2}} - e^{-\frac{x_{r_i}}{\theta_2}}\right)^{v_i} \\ & \times \left(e^{-\frac{1}{\theta_1}\left(u_{k+1} x_{r_k} + \sum_{i=1}^{k} z_i x_{r_i}\right)}\right) \left(e^{-\frac{1}{\theta_2}\left(v_{k+1} x_{r_k} + \sum_{i=1}^{k} (1-z_i) x_{r_i}\right)}\right). \end{aligned} \tag{7}$$

Obviously, the posterior distribution given in (7) appears difficult to write in closed form mainly because of the involvement of censored data. It is essential to point out that, in the absence of censoring, the posterior distribution would have been much simpler to handle analytically. Now coming to the present form in (7), various alternative approaches such as numerical integration or analytical approximation techniques can be used. We, however, advocate the application of Markov chain Monte Carlo (MCMC) based approaches because of their simplicity and ease of drawing a variety of inferences in a routine manner. These approaches offer an additional advantage in the sense that inferences about the censored observations can also be obtained routinely, especially if someone employs the Gibbs sampler algorithm and considers these censored observations as further unknowns. Drawing inferences about unknown censored data can equivalently be considered as a one-sample prediction problem. As such, the paper first considers the possibility of applying the Gibbs sampler algorithm, the simplest form of MCMC based procedure, for analyzing the posterior in (7).

Before we proceed further, let us first briefly review the Gibbs sampler algorithm, especially that form of the algorithm which provides due consideration to censored data as well. This review will be done solely considering the problem that we have introduced. The Gibbs sampler algorithm, first proposed by [3], is a widely used MCMC approach that generates samples from the joint posterior distribution of unknown parameters by iteratively sampling from their often one-dimensional full conditionals. These full conditionals, up to a proportionality, are obtained from the joint posterior distribution. The latter also needs to be specified up to proportionality only by combining the likelihood function of the observed data with the prior distributions of the parameters using Bayes theorem. The Gibbs sampler algorithm proceeds in a cyclic manner, that is, at each iteration, a parameter is updated by drawing from its full conditional, using the most recent updated values of all the other unknowns. After a sufficient number of iterations, the Markov chain so obtained converges to a random sample from the posterior distribution. To reduce serial correlation between generated observations in a single long run of the chain and hence to obtain independent sample observations, a suitably spaced subset of the generated samples is retained for drawing the posterior based inferences. For further details, one may refer to [11], [15], etc.

The presence of censored data typically introduces the computational complexity in the Bayesian framework. However, the apparent advantage of the Gibbs sampler algorithm is that it routinely caters to the needs of censored data problems. The approach requires assuming that there is no censoring and reintroducing the censored observations as further unknowns in the implementation of the algorithm. Full conditionals are then defined for both the model parameters and the censored data observations. The full conditionals for the model parameters remain the same as in the uncensored case, while those for the censored observations correspond to their respective truncated parent sampling distributions. The truncation limits vary depending on each individual censored observation.

To clarify the algorithm in the present case, it may be noted that the unknowns consist of $(\boldsymbol{\theta}, \underline{x}^*)$, where $\boldsymbol{\theta} = (\theta_1, \theta_2)$, and $\underline{x}^*$ denotes the set of unknown censored observations. The likelihood function for this case, assuming that there is no censoring, can be written as

$$\begin{aligned} L^* &= m!n!\prod_{i=1}^{N} (f(x_i))^{z_i} \times \prod_{i=1}^{N} (g(x_i))^{1-z_i} \\ &= \frac{m!n!}{\theta_1^m \theta_2^n} \left(e^{-\Sigma_{i=1}^N \frac{z_i x_i}{\theta_1}} \right) \left(e^{-\Sigma_{i=1}^N \frac{(1-z_i)x_i}{\theta_2}} \right), \end{aligned} \tag{8}$$

although the likelihood function consists of unknown censored observations $\underline{x}^*$ as well. As such, the joint posterior distribution of θ_1 and θ_2, up to proportionality, can be written as

$$\begin{aligned} p^*(\theta_1, \theta_2 | \underline{x}', \underline{x}^*) &= p^*(\theta_1, \theta_2 | \underline{x}) \\ &\propto \frac{1}{\theta_1^{m+1} \theta_2^{n+1}} \left(e^{-\Sigma_{i=1}^N \frac{z_i x_i}{\theta_1}} \right) \left(e^{-\Sigma_{i=1}^N \frac{(1-z_i)x_i}{\theta_2}} \right). \end{aligned} \tag{9}$$

Here, $\underline{x} = (\underline{x}', \underline{x}^*)$, where $\underline{x}'$ denotes the set of observed data. Obviously, the posterior as given in (9) involves multiple unknown quantities including the two model parameters and the unknown censored observations, making it well-suited for routine application of the Gibbs sampler algorithm. So, from the equation (9), the full conditionals for the two parameters can be written as

$$\begin{aligned} p^*(\theta_1 | \underline{x}) &\propto \frac{1}{\theta_1^{m+1}} \left(e^{-\Sigma_{i=1}^N \frac{z_i x_i}{\theta_1}} \right), \\ p^*(\theta_2 | \underline{x}) &\propto \frac{1}{\theta_2^{n+1}} \left(e^{-\Sigma_{i=1}^N \frac{(1-z_i)x_i}{\theta_2}} \right). \end{aligned} \tag{10}$$

The full conditionals for the unknown censored observations are nothing but truncated exponential distributions with truncation in each case decided on the basis of the corresponding censored observation. Say, for instance, if a censored observation falls in the range x_c to x_c', the corresponding exponential distribution will be truncated below x_c and above x_c'. From the full conditionals given in (10), it can be observed that both θ_1 and θ_1 can be generated from inverse gamma (IG) distributions where $\theta_1 \sim \text{IG}(m, \Sigma_{i=1}^N z_i x_i)$ and $\theta_2 \sim \text{IG}(n, \Sigma_{i=1}^N (1-z_i) x_i)$, respectively. Next, to execute the Gibbs sampler algorithm, the initial value is needed for each unknown, which can be chosen using some properly selected initial estimate to achieve faster convergence of the iterating chain. Say, for instance, one can consider ML estimates for the two unknown model parameters and an appropriately selected value for each unknown censored observation. This appropriate choice for each censored observation can be taken, for example, as the most relevant available information. Say, for instance, x_c can be considered as the relevant information in the example discussed above. Thus, one can easily implement the Gibbs sampler algorithm for the considered censored data problem as discussed above.

In addition, one can also be interested in characteristics such as reliability and the hazard rate, etc. although these characteristics are not of prime concern in the paper. One can easily see that these characteristics are the functions of one of the two parameters depending on whether a particular characteristic corresponds to Product 1 or Product 2. For instance, the reliability function for Product 1 and 2 can be obtained using the distribution function given in (1) and (2), respectively. Obviously, inferences about these characteristics can be easily obtained once the posterior samples corresponding to the two model parameters are made available. Say, for example, a posterior sample of the reliability function at a specified mission time can simply be obtained by substituting the posterior sample of the parameter in the corresponding expression of reliability.

4. Numerical Illustration

This section presents a real data example to demonstrate the performance of the Gibbs sampler algorithm under the specified censoring scenario and simultaneously to provide the full posterior and predictive analysis of unknown censored observations. The dataset used in the section was proposed by [9]. The dataset consists of failure times of air-conditioning systems of a fleet of Boeing 720 jet airplanes. For the purpose of illustration, "7914" and "7913" jet airplanes were chosen where the air conditioning systems of plane "7914" were considered as items of Product 1 and the air conditioning systems of plane "7913" were considered as items of Product 2. The required dataset taken from [9] is given in Table 1 for a quick reference.

Table 1: *Failure times of air-conditioning systems in "7914" and "7913" airplanes*

Plane	Failure times
7914	50, 44, 102, 72, 22, 39, 3, 15, 197, 188, 79, 88, 46, 5, 5, 36, 22, 139, 210, 97, 30, 23, 13, 14
7913	97, 51, 11, 4, 141, 18, 142, 68, 77, 80, 1, 16, 106, 206, 82, 54, 31, 216, 46, 111, 39, 63, 18, 191, 18, 163, 24

Table 2: *Joint multiply type-II censored sample when* $k = 15$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
r_i	3	5	8	11	16	19	25	28	30	33	35	37	43	46	48	-
X_{r_i}	4	5	14	18	23	31	46	54	68	79	82	97	141	188	197	-
Z	0	1	1	0	1	0	0	0	0	1	0	1	0	1	1	-
u_i	1	1	1	1	2	1	4	1	0	1	0	1	2	0	0	1
v_i	1	0	1	1	2	1	1	1	1	1	1	0	3	2	1	2

The same dataset has been analyzed earlier by [7], [2], [10], [12], and several others, employing similar or different censoring schemes and assuming various failure time distributions. [9] initially assumed that the failure times of the air conditioning system approximately follow an exponential distribution, despite variability in the failure rates across different planes.

For illustration purposes, two schemes of joint multiply type-II censored samples are constructed by arranging the failure times of both the planes in ascending order. If, at any position in this ordered sequence, two or more failure times from the two planes are identical, a random arrangement of these observations is applied at that position. The first scheme considers a total of $k = 15$ observed number of failures, whereas the second scheme considers a total of $k = 22$ observed number of failures. It may be noted that the sample sizes for Product 1 and Product 2 are $m = 24$ and $n = 27$, respectively, and therefore the considered schemes offer nearly 71% and 57% censored observations. It may be further noted that the observed items, or those with exactly known failure times, are selected randomly to reflect naturally occurring variations. The

Table 3: *Joint multiply type-II censored sample when $k = 22$*

i	1	2	3	4	5	6	7	8	9	10	11	12
r_i	4	6	8	11	12	14	17	18	21	25	26	29
X_{r_i}	5	11	14	18	18	22	24	30	39	46	50	63
Z	1	0	1	0	0	1	0	1	1	0	1	0
u_i	1	1	1	1	0	0	2	0	1	2	0	0
v_i	2	0	0	1	0	1	0	0	1	1	0	2
i	13	14	15	16	17	18	19	20	21	22		
r_i	30	36	37	38	39	41	43	44	48	49	-	
X_{r_i}	68	88	97	97	102	111	141	142	197	206	-	
Z	0	1	1	0	1	0	0	0	1	0	-	
u_i	0	2	0	0	0	0	1	0	1	0	1	
v_i	0	3	0	0	0	1	0	0	2	0	1	

Table 4: *ML estimates of θ_1, θ_2 and reliability for Product i $(i = 1, 2)$*

Observed failures	$\hat{\theta_1}$	$\hat{\theta_2}$	$\hat{R_1}(100)$	$\hat{R_2}(100)$
$k = 15$	65.136	84.180	0.215	0.305
$k = 22$	65.203	78.971	0.216	0.282
$k = 51$ (no censoring)	64.125	76.815	0.210	0.272

considered joint multiply type-II censored schemes are shown schematically in Table 2 and Table 3 for clarity of the two schemes.

The ML estimates of the parameters are obtained by solving the likelihood equations given in (5) using the methodology outlined in subsection 3.1. These estimates are computed for both the considered schemes of joint multiply type-II censored samples and also for the case of a complete dataset with no censored data assumption. The ML estimates of the reliability at time t = 100 hours for Product i $(i = 1, 2)$ are also obtained for all the three schemes mentioned above. The corresponding results of ML estimates are reported in Table 4. It is worth noting that $\hat{R}_i(100)$ in Table 4 corresponds to the ML estimate of reliability at time t = 100 hours for Product i $(i = 1, 2)$.

Table 5: *Estimated posterior summaries of parameters*

Known values	**Parameter**	**Posterior estimates**			**0.95 HPDI**
		Mean	**Median**	**Mode**	
15	θ_1	67.786	65.523	61.374	(43.362, 94.428)
	θ_2	87.867	85.678	83.268	(56.102, 122.886)
22	θ_1	68.179	65.521	60.826	(43.419, 97.971)
	θ_2	81.693	79.452	75.624	(54.209, 113.763)
51	θ_1	66.534	64.910	60.453	(42.145, 94.107)
(no censoring)	θ_2	80.172	78.190	74.525	(54.771, 111.094)

It can be clearly seen from Table 4 that θ_1 remains relatively stable across different choices of k, whereas θ_2 exhibits greater variability. It suggests that losing information due to censoring will not have any appreciable impact on the estimate of θ_1 that corresponds to Product 1. On the other hand, θ_2 corresponding to Product 2 does lead to an overestimated value because of censoring. A similar thing is obviously reflected in the estimates of reliability as well, since the two estimates of reliability depend ultimately on θ_1 and θ_2. As a result, the reliability corresponding to Product 1 remains relatively stable across different values of k whereas the same corresponding to Product

2 leads to an overestimated value because of censoring.

In order to develop the sample based Bayesian inferences, the method outlined in subsection 3.2 is followed. The ML estimates of the parameters θ_1 and θ_2 serve as the initial values for running the algorithm described in that subsection. For unknown censored observations, the initial values are assigned in accordance with the suggestion given in subsection 3.2, ensuring that the observations fall within their respective failure time intervals.

In order to obtain the posterior samples and correspondingly draw the sample based inferences, a single extended run of the Gibbs sampler was performed. An initial burn-in period was used to eliminate transient behavior, after which convergence was assessed using the ergodic averages. Once convergence was established, every 10th sample was retained from the iterating chain to make the autocorrelation negligibly small, resulting in a posterior sample of size 1000 for each of the marginal posterior distributions of θ_1 and θ_2. Table 5 presents the estimated posterior mean, median, mode and highest posterior density interval with coverage probability 0.95 (0.95 HPDI) of the parameters under three different scenarios, that is, $k = 15$, $k = 22$, and $k = 51$ (case of no censoring).

Table 6: *Predictive estimates for unknown censored observations when* $k = 15$

Sr. No.	Observed failure times	Estimated predictive means	0.95 HPrDI
1	1	1.98	(0.23, 3.99)
2	3	2.02	(0.09, 3.82)
3	5	4.50	(4.01, 5.00)
4	11	9.26	(5.15, 13.50)
5	13	9.40	(5.18, 13.72)
6	15	16.01	(14.20, 17.98)
7	16	16.05	(14.04, 17.83)
8	18	20.41	(18.00, 22.77)
9	18	20.47	(18.00, 22.99)
10	22	20.50	(18.04, 22.78)
11	22	20.54	(17.96, 22.74)
12	24	26.72	(23.19, 30.71)
13	30	26.88	(23.46, 30.95)
14	36	37.90	(31.08, 45.20)
15	39	38.20	(31.10, 45.20)
16	39	38.22	(31.08, 44.98)
17	44	38.29	(31.12, 45.25)
18	46	38.50	(31.77, 46.00)
19	50	49.85	(46.05, 53.51)
20	51	49.87	(46.00, 53.52)
21	63	60.83	(54.36, 67.57)
22	72	73.26	(68.14, 78.44)
23	77	73.50	(68.49, 79.00)
24	80	80.46	(79.06, 81.93)
25	88	89.24	(82.25, 96.51)
26	97	116.02	(96.97, 136.95)
27	102	116.14	(97.24, 138.37)
28	106	116.77	(97.01, 138.48)
29	111	117.34	(97.01, 137.48)
30	139	117.72	(97.10, 140.43)
31	142	162.31	(141.41, 185.82)
32	163	162.52	(141.07, 184.79)
33	191	192.42	(188.29, 196.74)
34	206	263.01	(197.04, 439.93)
35	210	281.34	(197.13, 474.33)
36	216	285.38	(197.05, 401.49)

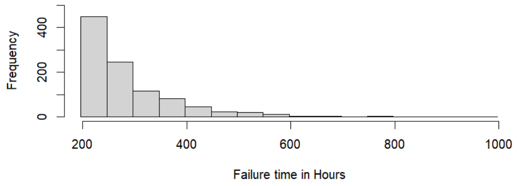
Table 7: *Predictive estimates for unknown censored observations when* $k = 22$

Sr.No.	Observed failure times	Estimated predictive means	0.95 HPrDI
1	1	2.43	(0.01 4.76)
2	3	2.45	(0.10, 4.79)
3	4	2.53	(0.00, 4.75)
4	5	7.91	(5.00, 10.68)
5	13	12.47	(11.13, 13.96)
6	15	16.00	(14.12, 17.84)
7	16	16.02	(14.14, 17.92)
8	18	19.93	(18.00, 21.84)
9	22	23.00	(22.00, 23.90)
10	23	23.06	(22.07, 23.99)
11	31	34.28	(30.11, 38.47)
12	36	34.36	(30.16, 38.65)
13	39	42.33	(39.00, 45.70)
14	44	42.50	(39.14, 45.76)
15	46	42.54	(39.09, 46.00)
16	51	56.24	(50.06, 62.29)
17	54	56.38	(50.54, 62.80)
18	72	77.25	(68.00, 86.62)
19	77	77.44	(68.10, 86.69)
20	79	77.50	(68.10, 86.95)
21	80	77.52	(68.22, 87.03)
22	82	77.66	(68.97, 87.85)
23	106	106.58	(102.26, 110.78)
24	139	125.00	(111.21, 139.78)
25	163	165.84	(142.02, 193.59)
26	188	166.08	(142.50, 193.62)
27	191	166.61	(142.01, 193.07)
28	210	276.22	(206.00, 419.13)
29	216	287.09	(206.08, 463.66)

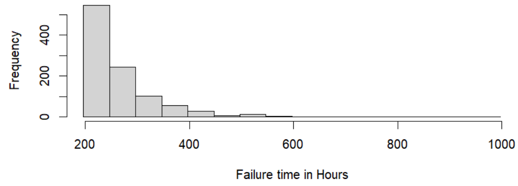
One can observe from Table 5 that the results are, in general, closely align to the corresponding ML estimates of the parameters given in Table 4. This similarity in the two estimates is expected, given that the prior distributions used are non-informative, allowing the posterior distribution to be largely dominated by the likelihood function. Also, all the three marginal posterior distributions exhibit positive skewness. In such cases, relying solely on the estimated posterior mean or median can be misleading, as these measures are sensitive to the skewness. In the very same sense, the ML estimator may also be suboptimal. The posterior mode is, of course, a better candidate as it represents the most probable value of the parameter.

After the inferential developments on the unknown parameters, predictive samples of size 1000 are generated corresponding to each unknown censored observation. The estimated predictive means for the unknown censored observations for the cases when $k = 15$ and $k = 22$ are reported in Table 6 and Table 7, respectively. The tables also provide the highest one-sample predictive density interval with coverage probability 0.95 (0.95 HPrDI), offering a measure of uncertainty in the predictive estimates of censored observations.

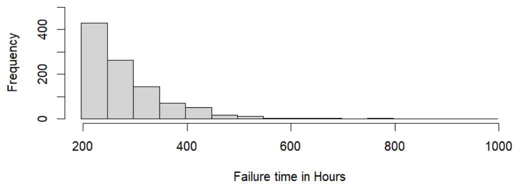
The predictive performance, summarized in Table 6 and Table 7, demonstrates the effectiveness of the proposed Bayesian methodology in predicting unknown censored observations. Overall, the predictive means closely align with the true values that are removed during execution for most of the observations and, as such, the proposed methodology can be considered satisfactory even if we have a good number of censored observations. This is particularly evident in the values arising in the middle of the ordered observations, where the predicted values deviate very little from the true observations and, as such, the predictions can be considered precise enough.



(1.1)

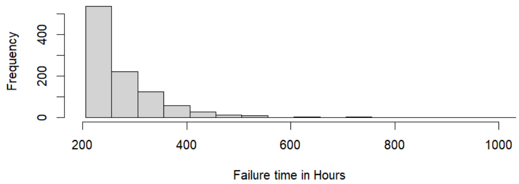


(1.2)

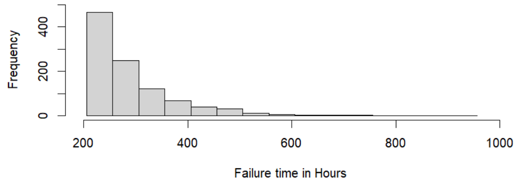


(1.3)

Figure 1: *Histograms showing the density estimates for X_{49}, X_{50}, and X_{51}, respectively, in (1.1), (1.2), and (1.3).*



(2.1)



(2.2)

Figure 2: *Histograms showing the density estimates for X_{50} and X_{51}, respectively, in (2.1) and (2.2).*

Even in cases where some deviation is observed, the true values are successfully captured within the corresponding 0.95 HPrD intervals. This indicates that the uncertainty associated with each prediction has been appropriately quantified to a great extent. The consistent inclusion of the true values within these predictive intervals underscores the reliability of the utilized technique and its strong capability for uncertainty estimation. Additionally, the methodology demonstrates adaptability across all the values, from small to large in magnitude, as evidenced by the varying widths of the predictive intervals.

Since the data are joint multiply type-II censored, one may be interested more in the observations that remain operational at the time of termination of the experiment than those that failed earlier. As such, to provide a more thorough idea, density estimates for such observations are shown in the form of histograms. It may be noted that there are three and two such functional units for the considered cases when $k = 15$ and $k = 22$. These functional units have true values of 206, 210, 216 and 210, 216, respectively, when $k = 15$ and $k = 22$. For $k = 15$, these functional units are denoted by X_{49}, X_{50}, and X_{51}. The corresponding density estimates in the form of histograms are shown in Figure 1. Similarly, for $k = 22$, the functional units are denoted by X_{50} and X_{51} with the corresponding predictive density estimates shown in Figure 2.

One can see that each figure reflects the gradual decline from a specific point, consistent with an exponential-like decay in failure likelihood over time. This behavior confirms that the model

Table 8: *Posterior estimates of reliability at t=100*

k		Mean	Median	Mode
15	Product 1	0.225	0.217	0.198
	Product 2	0.314	0.311	0.307
22	Product 1	0.226	0.217	0.198
	Product 2	0.289	0.284	0.277
51 (no censoring)	Product 1	0.219	0.214	0.193
	Product 2	0.282	0.278	0.266

correctly incorporates the censoring mechanism.

Additionally, the posterior estimates of reliability at $t = 100$ hours are obtained and presented in Table 8. The results of the posterior characteristics of reliability are somewhat similar to what we have obtained through ML estimation. It can be seen that the posterior distributions of reliability corresponding to Product 1 and Product 2 appear to be slightly skewed across all the three scenarios, $k = 15$, $k = 22$ and no censoring, suggesting the use of modal values as the preferred point estimates. Furthermore, the posterior estimates of reliability in the censored cases show minimal deviation from those obtained in the absence of censoring. This consistency reinforces the effectiveness of the proposed estimation procedure that suggests minimum loss of information because of heavy censoring.

5. Conclusion

In this paper, we have analyzed data subject to joint multiply type-II censoring under an exponential modeling assumption. To facilitate parameters estimation, we proposed the use of the Gibbs sampler algorithm for censored data problems. The problem proceeds mainly with two important objectives. First, the estimation of model parameters and, second, the prediction of unknown censored observations. It is shown on the basis of a real data example with artificially formulated censoring that the proposed methodology offers quite satisfactory results even if we have a very large proportion of censored observations, say, for example, more than 55% or 70% censoring. The predictive estimates for unknown censored observations are also found satisfactory in the sense that they closely represent the true observations.

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