# ANALYSING STRESS STRENGTH RELIABILITY OF LIFE TEST ON AEROSOL PARTICLES BY GOMPERTZ DISTRIBUTION USING NOVEL JOINT PROGRESSIVE CENSORING

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**Abstract**

*System reliability has become a popular subject of study in engineering due to its wide range of potential applications. When a reliability practitioner aims to observe a specific number of failed units, a balanced case of joint progressive censoring scheme is proposed to improve the efficiency of an experiment. This paper investigates parametric inference for system reliability using a novel case of joint progressive censoring, with a specific focus on two independent Gompertz populations. The system reliability parameter is estimated using both likelihood and Bayesian inferential approaches. The Metropolis-Hastings algorithm is employed to compute Bayesian estimates under different loss functions. Asymptotic confidence intervals and Bayesian credible intervals are also derived. A thorough simulation study is conducted to evaluate the performance of the proposed methods across various sample sizes. Additionally, to demonstrate the practical utility of the approach, virus-containing aerosol particles (VCAP) lifetime data under two velocities are analyzed.*

**Keywords:** Bayesian Estimates, Gompertz Populations, Likelihood Estimation, Loss Function, Stress-Strength Reliability, VCAP data.

## 1. Introduction

In recent years, the accuracy of estimation in practical experiments is greatly affected by sample size, and it may be impossible to conduct complete testing due to advancements in manufacturing design and technology because products are designed with high reliability and longer lifespan. Censoring techniques are frequently employed to reduce the time spent studying while simultaneously saving money. Type-I censoring and Type-II censoring are the most common ones. In the field of reliability tests, engineers prefer the progressive censoring scheme (ProgCeS) over the above-mentioned censoring schemes (CeS) due to the flexibility of this scheme in removing test items. The ProgCeS is an attractive and has received widespread attention, see, Mousa & Sagheer [1], Balakrishna [2], Huang & Wu [3], Lee and Seo [4] and the reference therein. Further, in industrial manufacturing and practical life, two or more populations need to perform life-test experiments. Therefore, a joint CeS is very useful to improve the efficiency of the comparative life experiment. Statistical inference under joint CeS discussed by different authors, see Rasouli and Balakrishnan [5], Mondal and Kundu [6], Asar and Belagh [7]. Also, for balance between total test time and number of failures need to statistical inference, balance case of joint CeS named BJProgCS is proposed by Mondal and Kundu [8].

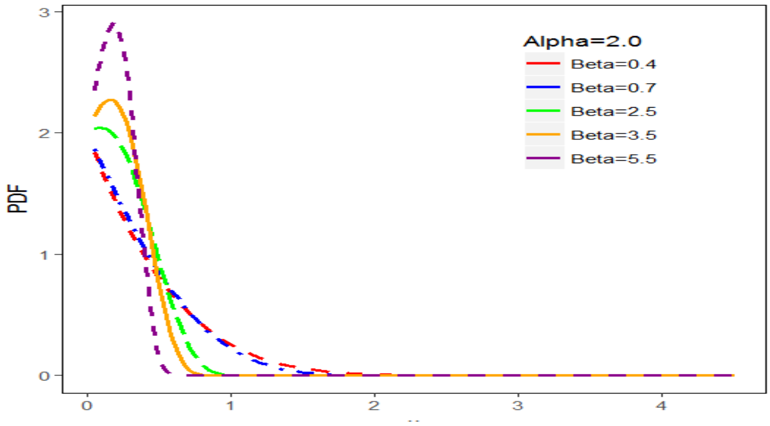


**Figure 1:** *The pdf plot of Gom Distribution*

Moreover, the Gompertz (Gom) distribution offers a flexible modeling tool for data exhibiting a wide range of shapes and is broadly used in numerous fields. The PDF, $f(.)$, and CDF $F(.)$ and hazard function $H(.)$, of the Gom distribution are given by:

$$f(x;\alpha,\beta) = \alpha \exp\left[\beta x - \frac{\alpha}{\beta}(e^{\beta x} - 1)\right]; \quad x > 0,\ \alpha > 0,\ \beta > 0 \tag{1}$$

$$F(x;\alpha,\beta) = 1 - \exp\left[-\frac{\alpha}{\beta}(e^{\beta x} - 1)\right], \tag{2}$$

and $H(t;\alpha,\beta) = \alpha e^{\beta t}$, respectively. In fact, the Gompertz PDF is unimodal or decreasing due to the parameter values (see Figure 1) . It is to be noticed that this distribution reduces to an exponential distribution, corresponding to $\alpha \to 0$. In the existing literature, statistical inferences on estimating unknown parameters of Gom have been provided by Shi and Wu [9], Wu et al. [10], Mazucheli et al. [11], Lee and Seo [12] and Asadi et al. [13].

Further, as the requirements of customers for product quality are continuously improved, the reliability model has become more important. Hence, how to evaluate reliability parameter is an important problem faced by system designers. In reliability studies, the life of a component with a random strength $X$ and a random stress $Y$ is described by the stress-strength model (SSReM). Suppose $X$ and $Y$ independently have Gom($\alpha$, $\beta$) and Gom($\lambda$, $\beta$) respectively. Then, the reliability is derived as follows:

$$\Im = \int_0^\infty \alpha e^{\beta t - \frac{\alpha}{\beta}(e^{\beta t}-1)} e^{-\frac{\lambda}{\beta}(e^{\beta t}-1)}\, dt = \frac{\lambda}{\alpha + \lambda}. \tag{3}$$

The stay of the article is classified as: Based on BJProgCS, likelihood and Bayes investigations for SSReM are developed in Sections 2 and 3, respectively. Section 4 evaluates the estimators' performance via simulations. Also, the VCAP real datasets are conducted to show the performance of all diverse methods considered in this paper. Finally, in Section 5, we conclude the study with a brief discussion.

## 2. Sampling and Likelihood Function

This section will employ model description and likelihood function to estimate the unknown model parameters. Approximate confidence for reliability parameter will also be constructed.

Presume two samples of size $m$ and $n$ from Population-1 and Population-2. The total number of observed failures ($k < \min(n, m)$) and progressive CeS $R_1, R_2, \ldots, R_{k-1}$, when $\sum_{i=1}^{k-1}(1 + R_i) <$

$\min(n, m)$ are the preassigned. After the first failure $D_1$ is observed, $R_1$ surviving units are randomly dropped from the remaining surviving units of Population-1 and $(R_1 + 1)$ surviving units are withdrawn from the surviving units of Population-2. The experiment is continual until the $k^{th}$ failure occurs. To learn more about the specifics of the afore mentioned censoring method, one might look at Sultana et al. [14], Lone et al. [15 & 16].

Following the BJProgCS and based on the observed data $(d, F)$, the likelihood function (LikF) can be expressed as $(T = (\alpha, \lambda, \beta))$:

$$\begin{aligned} L(\mathcal{T} \mid \text{data}) &\propto \alpha^{k_1}\lambda^{k_2}\prod_{i=1}^{k-1} e^{\beta d_i} \times \prod_{i=1}^{k-1}\left(e^{-\frac{\alpha}{\beta}(e^{\beta d_i}-1)}\right)^{(R_i+1)}\left(e^{-\frac{\lambda}{\beta}(e^{\beta d_i}-1)}\right)^{(R_i+1)} \\ &\times\left(e^{-\frac{\alpha}{\beta}(e^{\beta d_k}-1)}\right)^{(m-\sum_{i=1}^{k-1}(R_i+1))} \times\left(e^{-\frac{\lambda}{\beta}(e^{\beta d_k}-1)}\right)^{(n-\sum_{i=1}^{k-1}(R_i+1))} \\ &\propto \alpha^{k_1}\lambda^{k_2}e^{\beta\sum_{i=1}^{k} d_i} \times e^{\sum_{i=1}^{k-1}(R_i+1)\vartheta_i(\alpha,\beta)+(m-\sum_{i=1}^{k-1}(R_i+1))\vartheta_k(\alpha,\lambda)} \times \\ &e^{\sum_{i=1}^{k-1}(R_i+1)\tau_i(\lambda,\beta)+(n-\sum_{i=1}^{k-1}(R_i+1))\tau_k(\lambda,\beta)} \end{aligned} \tag{4}$$

Here,

$$\vartheta_i(\alpha,\beta) = -\frac{\alpha}{\beta}(e^{\beta d_i}-1), \vartheta_k(\alpha,\beta) = -\frac{\alpha}{\beta}(e^{\beta d_k}-1), \tau_i(\lambda,\beta) = -\frac{\lambda}{\beta}(e^{\beta d_i}-1), \tau_k(\lambda,\beta) = -\frac{\lambda}{\beta}(e^{\beta d_k}-1),$$

$$d = \{d_1, \ldots, d_k\}, k_1 = \sum_{i=1}^{k} F_i, k_2 = \sum_{i=1}^{k}(1 - F_i)$$

and

$$F_i = \begin{cases} 1, & \text{The failures } D_i \text{ are from } X \\ 0, & \text{The failures } D_i \text{ are from } Y \end{cases}$$

Subsequently, from (4), the log-likelihood becomes:

$$\begin{aligned} \log L \propto k_1 \ln\alpha + k_2 \ln\lambda + \beta\sum_{i=1}^{k} d_i + \sum_{i=1}^{k-1}(R_i+1)\vartheta_i(\alpha,\beta) + (m - \sum_{i=1}^{k-1}(R_i+1))\vartheta_k(\alpha,\beta) \\ + \sum_{i=1}^{k-1}(R_i+1)\tau_i(\lambda,\beta) + (n - \sum_{i=1}^{k-1}(R_i+1))\tau_k(\lambda,\beta). \end{aligned} \tag{5}$$

The likelihood equations are given by:

$$\frac{\partial \log L}{\partial \alpha} = \frac{k_1}{\alpha} - \frac{1}{\beta}\left[\sum_{i=1}^{k-1}(R_i+1)(e^{\beta d_i}-1) + (m - \sum_{i=1}^{k-1}(R_i+1))(e^{\beta d_k}-1)\right] = 0,$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{k_2}{\lambda} - \frac{1}{\beta}\left[\sum_{i=1}^{k-1}(R_i+1)(e^{\beta d_i}-1) + (n - \sum_{i=1}^{k-1}(R_i+1))(e^{\beta d_k}-1)\right] = 0$$

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{k} d_i + \frac{\alpha}{\beta}\left\{ \sum_{i=1}^{k-1}(R_i+1)\left[\frac{e^{\beta d_i}-1}{\beta} - d_i e^{\beta d_i}\right] + \left(m - \sum_{i=1}^{k-1}(R_i+1)\right)\left[\frac{e^{\beta d_k}-1}{\beta} - d_k e^{\beta d_k}\right] \right\} \\ + \frac{\lambda}{\beta}\left\{ \sum_{i=1}^{k-1}(R_i+1)\left[\frac{e^{\beta d_i}-1}{\beta} - d_i e^{\beta d_i}\right] + \left(n - \sum_{i=1}^{k-1}(R_i+1)\right)\left[\frac{e^{\beta d_k}-1}{\beta} - d_k e^{\beta d_k}\right] \right\}. \end{aligned}$$

Moreover, the $\log L$ is a concave for fixed $\beta$ as a function of $\alpha$ and $\lambda$, which obtains the unique maximum at the point $(\hat{\alpha}(\beta), \hat{\lambda}(\beta))$, where $\hat{\alpha}(\beta)$ and $\hat{\lambda}(\beta)$ are the MLEs of $\alpha$ and $\lambda$. Due to extremely complicated non-linear equations that cannot be solved analytically, the Newton-Raphson approach can be employed to solve the afore mentioned equations numerically.

Therefore, using the invariance property of the MLE, the maximum likelihood estimate of the SSReM parameter, denoted by $\hat{\Im}$ can be written as:

$$\hat{\Im} = \frac{\hat{\lambda}}{\hat{\alpha} + \hat{\lambda}}. \tag{6}$$

Now, in order to obtain the approximate confidence bounds (ApCB) of $\Im$, we use asymptotic normality property of MLEs. The variance-covariance of $\hat{T} = (\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ can be evaluated from the Fisher matrix $I(T)$, which is presented as follows:

$$I(T) = \left[ -\frac{\partial^2}{\partial \theta_i, \partial \theta_j} \log L \right],$$

For brevity, detailed derivations are not given. Utilizing the regularity condition, we have $\hat{I}m \sim N(\Im, v)$, (Rao et al. [17]). Where,

$$v = \left(\frac{\partial \Im}{\partial \alpha}\right)^2 I_{11}^{-1} + \left(\frac{\partial \Im}{\partial \lambda}\right)^2 I_{22}^{-1} + 2\left(\frac{\partial \Im}{\partial \alpha}\right)\left(\frac{\partial \Im}{\partial \lambda}\right) I_{12}^{-1} = \frac{\lambda^2}{(\alpha+\lambda)^4} I_{11}^{-1} + \frac{\alpha^2}{(\alpha+\lambda)^4} I_{22}^{-1} - \frac{2\alpha\lambda}{(\alpha+\lambda)^4} I_{12}^{-1}. \tag{7}$$

Here

$$I^{-1}(T) = \begin{bmatrix} \mathrm{Var}_{\hat{\alpha}} & \mathrm{Cov}_{\hat{\alpha},\hat{\beta}} & \mathrm{Cov}_{\hat{\alpha},\hat{\lambda}} \\ & \mathrm{Var}_{\hat{\lambda}} & \mathrm{Cov}_{\hat{\lambda},\hat{\beta}} \\ & & \mathrm{Var}_{\hat{\beta}} \end{bmatrix}$$

Thus, the $100(1-\gamma)$ % APCB of $\Im$ can be created by:

$$(\hat{\Im} - z_{\gamma/2}\hat{v}, \hat{\Im} + z_{\gamma/2}\hat{v}), \tag{8}$$

where $z_\gamma$ is $z_{\gamma/2}$, the $100\gamma^{\text{th}}$ percentile of the standard normal distribution $N(0,1)$.

## 3. Bayesian Estimates (BayE)

The objective of this section is to compute BayE of $\Im$, as well as its related HPD bounds under BJProgCS. The BayE of $\Im$ is attained when all model parameters have independent gamma prior distributions. The joint posterior distribution (JoPD) of parameters is acquired through the Bayes principle, as expressed below:

$$\pi(T \mid \mathbf{data}) \propto \alpha^{k_1+a_1-1}\lambda^{k_2+a_2-1}\beta^{a_3-1}e^{-\alpha b_1 - \lambda b_2 - \beta(b_3 - \sum_{i=1}^{k} d_i)}$$
$$\times e^{\sum_{i=1}^{k-1}(R_i+1),\vartheta_i(\alpha,\beta) + (m - \sum_{i=1}^{k-1}(R_i+1))\vartheta_k(\alpha,\beta) + \sum_{i=1}^{k-1}(R_i+1)\tau_i(\lambda,\beta) + (n - \sum_{i=1}^{k-1}(R_i+1))\tau_k(\lambda,\beta)} \tag{9}$$

The Bayesian estimator of $\Im$ is defined as $\hat{\Im}_{SELF}$ and, $\hat{\Im}_{LLF}$ respectively, where it minimizes the symmetric (squared error loss function (SELF)) and asymmetric (Linex loss function (LLF)) as: $L_{\text{SELF}}(T, \hat{T}) = (\hat{T} - T)^2$,

$$L_{\text{LLF}}(T, \hat{T}) = e^{k(\hat{T}-T)} - k(\hat{T} - T) - 1; \quad k \neq 0$$

From (9), the conditional posterior distributions (CP) of $\alpha$, $\lambda$ and $\beta$ are provided by:

$$\pi_1(\alpha \mid \lambda, \beta, \mathbf{data}) \propto \alpha^{k_1+a_1-1} \exp\left[-\alpha\left(b_1 + \sum_{i=1}^{k-1}(R_i+1)(\frac{e^{\beta d_i}-1}{\beta}) + (m - \sum_{i=1}^{k-1}(R_i+1))(\frac{e^{\beta d_k}-1}{\beta})\right)\right] \tag{10}$$

$$\pi_2(\lambda \mid \alpha, \beta, \mathbf{data}) \propto \lambda^{k_2+a_1-1} \exp\left[-\lambda\left(b_2 + \sum_{i=1}^{k-1}(R_i+1)(\frac{e^{\beta d_i}-1}{\beta}) + (n - \sum_{i=1}^{k-1}(R_i+1))(\frac{e^{\beta d_k}-1}{\beta})\right)\right] \tag{11}$$

and also,

$$\pi_3(\beta \mid \alpha, \lambda, \text{data}) \propto \beta^{a_3-1} \times e^{-\beta(b_3-\sum_{i=1}^{k} d_i)} \times e^{\sum_{i=1}^{k-1}(R_i+1)\,\vartheta_i(\alpha,\beta)+(m-\sum_{i=1}^{k-1}(R_i+1))\vartheta_k(\alpha,\beta)}$$
$$\times e^{\sum_{i=1}^{k-1}(R_i+1)\,\tau_i(\lambda,\beta)+\left(n-\sum_{i=1}^{k-1}(R_i+1)\right)\tau_k(\lambda,\beta)}. \quad (12)$$

Notice that the CP distribution of $\beta$ is very difficult to reduce analyticaly to some well-known distributions. We employ the Gibbs within the Metropolis–Hasting (GiMeHa) algorithm to BayE of $\Im$ (Metropolis et al. [18] and Hastings [19]). This is how the algorithm functions.

Step 1: Start with an initial parameter vector, $T^0 = (\hat{\alpha}_{\text{MLE}}, \hat{\beta}_{\text{MLE}}, \hat{\lambda}_{\text{MLE}})$ and then set $Q = 1$.
Step 2: Draw $\beta^Q$ using the GiMeHa technique with a normal proposal density.
Step 3: For a given $\beta^Q$, generate $\alpha^Q$ from
Gamma $\left(k_1 + a_1,\ (b_1 + \sum_{i=1}^{k-1}(R_i+1)(\frac{e^{\beta d_i}-1}{\beta}) + (m - \sum_{i=1}^{k-1}(R_i+1))(\frac{e^{\beta d_k}-1}{\beta}))\right)$.
Step 4: For a given $\beta^Q$, generate $\lambda^Q$ from
Gamma $\left(k_2 + a_2, (b_2 + \sum_{i=1}^{k-1}(R_i+1)(\frac{e^{\beta d_i}-1}{\beta}) + (n - \sum_{i=1}^{k-1}(R_i+1))(\frac{e^{\beta d_k}-1}{\beta}))\right)$.
Step 5: Find $\Im^{(Q)} = \frac{\lambda^{(Q)}}{\alpha^{(Q)}+\lambda^{(Q)}}$.
Step 6: Set $Q = Q + 1$.
Step 7: Repeat the iteration  times to obtain $\Im^{(1)}, \Im^{(2)}, \ldots, \Im^{()}$.
Step 8: Compute the Bayes estimates under the SELF and LLF loss functions as follows:

$$\hat{\Im}_{SELF} = \frac{1}{-\ _0} \sum_{i=_0+1} \Im^{(Q)}; \hat{\Re}_{LLF} = -\frac{1}{k} \ln \left[ \frac{1}{-\ _0} \sum_{i=_0+1} e^{-} k \Im^{(Q)} \right];$$

$_0$ is the burn-in period.
Step 10: Find the $100(1-\gamma)\%$ Bayesian Highest Posterior Density (HPD) interval for $(\Im)$ by sorting its simulated MCMC iterations as $\Im^{(Q)}; i = 1, \ldots, .$ Then, the HPD interval is specified as: $[\hat{\Im}_{((\frac{\gamma}{2}))}, \hat{\Im}_{((1-\frac{\gamma}{2}))}]$.

# 4. Simulation Studies and Real Data Analysis

## 4.1. Simulation Studies and Discussions

This section offers a simulation analysis based on the Monte-Carlo simulation approach to evaluate the performance of the approaches utilizing various choices of $n, m$ and $R_i; i = 1, 2, ..., k-1$. The efficiency of the estimators is measured in terms of MSEs, whereas the efficiency of the ApCBs is measured in terms of their lengths and coverage percentages (CoPs). The Monte Carlo simulation study has been performed with the initial guess of the parameters as $\alpha$=2, $\lambda$=1.5 and $\beta$=1. In the Bayesian technique, both non-informative (NonI) and informative $(I)$ priors are taken. For prior $I$, we take the values of hyper-parameters (HP) as $a_1 = 2$, $a_2 = 1.5$, $a_3 = 1$, $b_i = 1, i = 1, 2, 3$. Notice that the selection of HP values under prior $I$ is considered in such a way that the prior means always remain sufficiently close to the true parameter values. From Bayesian method, posterior samples are drawn from the JoPD by using the GiMeHa algorithm. To mitigate dependency on initial values, we generate 10000 sample sizes and discard 5000 as burn-in samples.

We compare the performance of the maximum likelihood, Bayesian estimates, and associated 95% APCB, HPD bounds respectively, in terms of the MSEs and the average length of intervals and CoPs. The following conclusions may well be drawn from the results shown in Table 1 and Table 2:

- All acquired points as well as interval results of $\Im$ behaved fine.
- The MSEs of $\Im$ are dropping as a result of increasing values of the sample size. So, the estimation accuracy is improved.
- The results reveal that the estimates of the reliability parameter under different CeS display diminishing MSE and bound length as observed failure times $m$ increase.

- With an increase in $k$, MSEs and the average lengths are getting low with fixed values of $n$ and $m$.
- The ApCBs and HPD intervals are quite precise, and the estimated values are always within these ranges.
- Left censoring scheme demonstrates less MSEs according to performance indicators in the majority of scenarios.
- Under the LLF and GiMeHa algorithm, the Bayes estimates show smaller MSEs.
- Moreover, as expected, the accuracy of Bayesian estimates under $I$ is superior to that under Non- $I$ , because the former combines the informative prior of unknown model parameters and the data.
- The CoPs of various bounds are close to 95% of the nominal level. This holds good under various sampling schemes.
- The length of all bounds becomes narrower as the values of $n$ and $k$ increase, with fixed other values.
- To summarize, we advise estimating the system reliability parameter via the GiMeHa sampler in the presence of data generated through the proposed mechanism.

## 4.2. Illustrative Data Analysis

To illustrate the usefulness of the suggested techniques, real VCAP data sets from Asadi et al. [20] has been considered in this section. The data consists of two sets of VCAP data under two simultaneous air velocities, with sample sizes of 19 and 15. Before further development, we demand to verify how well the Gom distribution model fits the real data set. The KS statistics and the related $p$-values are shown in Table 3. This suggests that the Gom distribution is a reasonable choice to interpret these datasets.

**Table 1:** *The MSEs, ApCBs, HPD and CoPs of the SSReM parameter under different CeSs with $n = m = 20$.*

| $k$ | $\rightarrow k = 12$ | $k = 12$ | $k = 12$ | $k = 14$ | $k = 14$ | $k = 14$ |
|---|---|---|---|---|---|---|
| $CeS$ | $\rightarrow R = (0_{(5)}, 8, 0)$ | $R = (8, 0_{(10)})$ | $R = (0_{(10)}, 8)$ | $R = (6, 0_{(6)}, 6)$ | $R = (6, 0_{(12)})$ | $R = (0_{(12)}, 6)$ |
| $\hat{\Im}_{MLE}$ | 0.022468 | 0.019875 | 0.024278 | 0.019870 | 0.017443 | 0.020936 |
| $\hat{\Im}_{Bayes_{SELF-}}$ | 0.019205 | 0.016799 | 0.021589 | 0.016745 | 0.014988 | 0.017654 |
| $\hat{\Im}_{Bayes_{LLF-I}}$ | 0.019112 | 0.016768 | 0.021532 | 0.016711 | 0.014981 | 0.017650 |
| $\hat{\Im}_{Bayes_{SELF-}}$ | 0.021875 | 0.019136 | 0.023885 | 0.019106 | 0.016790 | 0.019078 |
| $\hat{\Im}_{Bayes_{LLF-N}}$ | 0.021864 | 0.019109 | 0.023793 | 0.019086 | 0.016699 | 0.019033 |
| $\hat{\Im}_{ApCB}$ | 0.503243 | 0.489324 | 0.513894 | 0.487664 | 0.478450 | 0.489920 |
| $CoP_{ApCB}$ | 94.0 | 94.2 | 93.7 | 94.3 | 95.9 | 96.2 |
| $\hat{\Im}_{HPD}$ | 0.445680 | 0.419972 | 0.448091 | 0.425233 | 0.403396 | 0.428989 |
| $CoP_{HPD}$ | 94.5 | 95.6 | 96.0 | 96.1 | 95.3 | 94.0 |

**Table 2:** *The MSEs, ApCBs, HPD and CoPs of the SSReM parameter under different CeSs with $n = m = 60$.*

| $k$ | $\rightarrow k = 45$ | $k = 45$ | $k = 45$ | $k = 50$ | $k = 50$ | $k = 50$ |
|---|---|---|---|---|---|---|
| $CeS$ | $\rightarrow R = (0_{(5)}, 8, 0)$ | $R = (8, 0_{(10)})$ | $R = (0_{(10)}, 8)$ | $R = (6, 0_{(6)}, 6)$ | $R = (6, 0_{(12)})$ | $R = (0_{(12)}, 6)$ |
| $\hat{\Im}_{MLE}$ | 0.019864 | 0.017645 | 0.019977 | 0.018543 | 0.015890 | 0.018995 |
| $\hat{\Im}_{Bayes_{SELF-}}$ | 0.015894 | 0.015431 | 0.016008 | 0.013993 | 0.013214 | 0.014107 |
| $\hat{\Im}_{Bayes_{LLF-I}}$ | 0.015743 | 0.015401 | 0.015897 | 0.013974 | 0.013199 | 0.014094 |
| $\hat{\Im}_{Bayes_{SELF-}}$ | 0.019245 | 0.016760 | 0.019562 | 0.017966 | 0.015480 | 0.018329 |
| $\hat{\Im}_{Bayes_{LLF-N}}$ | 0.019231 | 0.016751 | 0.019523 | 0.017954 | 0.015469 | 0.018321 |
| $\hat{\Im}_{ApCB}$ | 0.473425 | 0.457811 | 0.478442 | 0.457843 | 0.442895 | 0.45985 |
| $CoP_{ApCB}$ | 95.9 | 94.4 | 96.4 | 94.1 | 94.8 | 96.2 |
| $\hat{\Im}_{HPD}$ | 0.426842 | 0.409996 | 0.428691 | 0.403172 | 0.396980 | 0.406641 |
| $CoP_{HPD}$ | 95.4 | 95.3 | 93.9 | 94.6 | 94.7 | 95.6 |

**Table 3:** *The data sets of VCAP data.*

| Data Sets | $\mathrm{MLE_s}$ | kS | p−value |
|---|---|---|---|
| Data Set I | Shape=2.5746, Scale=0.06665 | 0.1331 | 0.953 |
| Data Set II | Shape=8.0947, Scale=$3.028 \times 10^{-4}$ | 0.0909 | 0.997 |

Further, the empirical CDF (EmCDF) and probability-probability (P-P) plots along with associated fitted Gom distributions are provided in Figure 2 and Figure 3, respectively. The likelihood ratio test is then used to determine whether the scale parameters can be assumed to have the same value, and the $p - value$ is 0.8692. Therefore, we accept the null hypothesis. We further calculate the point and the interval estimates using the NR method, and due to the non-availability of prior information, the BayEs are calculated using Non$I$ priors. All the results are reported in Table 4. It has been noticed from Table 4 that the MLEs and BayE are extremely comparable. The HPD bounds perform better than ApCBs in terms of width, which yields that the results are consistent with the previously simulated results.

The trace plot of reliability model is displayed in Figure 4, where the number of iterations is 14,000, and that of burn-in is 5,000. Figure 4 shows simulation number generated by MCMC method based on $k = 15$, $R = (0_{(14)}, 4)$. Reasonably satisfactory behavior is observed from this plot.

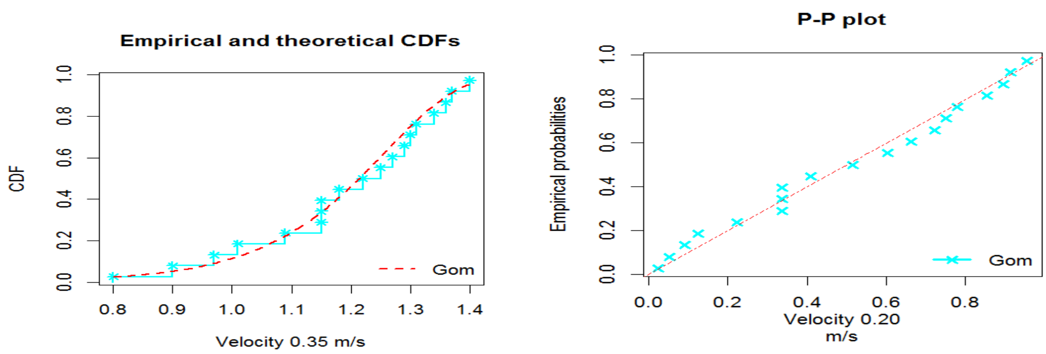


**Figure 2:** *The EmCDF and P-P plots for data set I.*

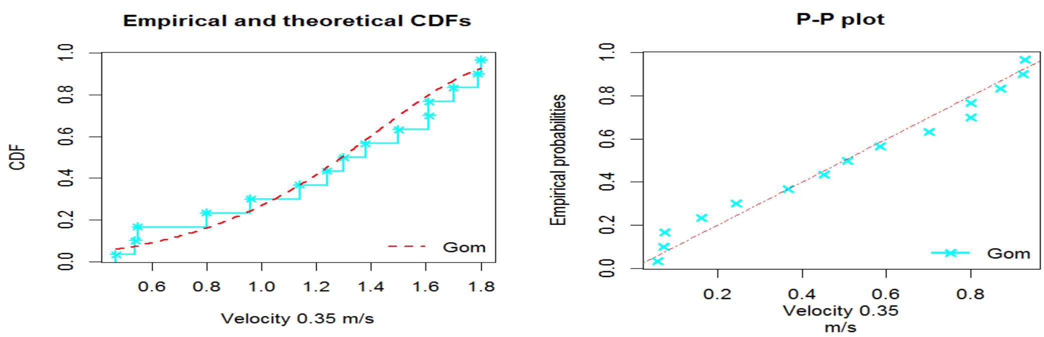


**Figure 3:** *The EmCDF and P-P plots for data set II.*

**Table 4:** *Different estimates of the SSReM parameter for VCMD data*

| $k$ | Censoring | $\hat{\Im}_{MLE}$ | $\hat{\Im}_{Bayes}$ | ApCB$_{\Im}$ | HPD$_{\Im}$ |
|---|---|---|---|---|---|
| $k = 15$ | $R = (0_{(14)}, 4)$ | 0.58965 | 0.59458 | 1.43280 | 0.89643 |
| $k = 15$ | $R = (4, 0_{(14)})$ | 0.59044 | 0.59874 | 1.32182 | 0.82364 |
| $k = 10$ | $R = (0_{(9)}, 5)$ | 0.61320 | 0.62018 | 1.46115 | 0.91536 |
| $k = 10$ | $R = (5, 0_{(9)})$ | 0.59906 | 0.60557 | 1.45219 | 0.89985 |

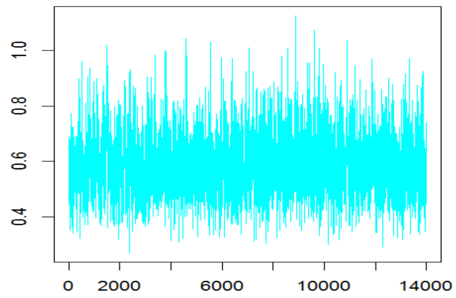


**Figure 4:** *Simulation MCMC generated number of the model parameter*

## 5. Conclusion

The problem of statistical inferences of comparative populations in competing duration needs to put a joint sample of units under test. Therefore, we looked at statistical inference of SSReM under BJProgCS.

For deriving the estimates of the SSReM parameter, we considered frequentist and Bayesian approaches.

Since MLEs of the model cannot be derived explicitly, Newton's iterative method has been implemented for this purpose. For the Bayesian approach, we suggested Non$I$ and $I$ priors and obtained the posterior distribution, which turned out to be intractable, so we proposed the GiMeHa technique to generate samples from the CP distributions of unknown parameters. Different interval estimates of Iare also computed. The effectiveness of the recommended procedures is compared through simulation studies and VCAP real data analysis. Comparing the joint CeS with progressive CeS and determining the optimal scheme may potentially be studied in the future.

## References

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